2010

Heteroclinic Solutions to an Asymptotically Autonomous Second-Order Equation

Gregory S. Spradlin
Embry-Riddle Aeronautical University, spradlig@erau.edu

Electronic Journal of Differential Equations, Vol. 2010(2010), No. 137, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

HETEROCLINIC SOLUTIONS TO AN ASYMPTOTICALLY AUTONOMOUS SECOND-ORDER EQUATION

GREGORY S. SPRADLIN

Abstract. We study the differential equation $\ddot{x}(t) = a(t)V'(x(t))$, where V is a double-well potential with minima at $x = \pm 1$ and $a(t) \to l > 0$ as $|t| \to \infty$. It is proven that under certain additional assumptions on a, there exists a heteroclinic solution x to the differential equation with $x(t) \to -1$ as $t \to -\infty$ and $x(t) \to 1$ as $t \to \infty$. The assumptions allow $l - a(t)$ to change sign for arbitrarily large values of $|t|$, and do not restrict the decay rate of $|l - a(t)|$ as $|t| \to \infty$.

1. Introduction

Consider the autonomous second-order differential equation

$$\ddot{x}(t) = lV'(x(t)), \tag{1.1}$$

$$x(t) \to -1 \text{ as } t \to -\infty, \quad x(t) \to 1 \text{ as } t \to \infty. \tag{1.2}$$

where $l > 0$, $V \in C^2(\mathbb{R}, [0, \infty))$, $V(-1) = V(1) = 0$, and $V > 0$ on $(-1, 1)$. The presence of l seems superfluous at this point; however, we will use it later. It is easy to show that (1.1)-(1.2) has a solution: multiply both sides of (1.1) by $\dot{x}(t)$ and integrate, and conclude that $\frac{1}{2}\dot{x}(t)^2 - lV(x(t))$ is constant. Assuming that $V(x) \le c(1 \pm x)^2$ for some $c > 0$ in a neighborhood of -1 and 1 respectively, then setting the constant equal to zero, we find that (1.1)-(1.2) has a solution, which solves the first-order equation $\dot{x}(t) = \sqrt{2lV(x(t))}$. That solution is unique if we impose the condition $x(0) = 0$. From now on, we will refer to the unique solution of (1.1)-(1.2) with $x(0) = 0$ as ω.

The function ω can also be characterized as the unique (modulo translation) minimizer of the functional

$$F_l(u) = \int_{-\infty}^{\infty} \frac{1}{2}\dot{u}(t)^2 - lV(u(t))\,dt \tag{1.3}$$

over the affine space

$$W = \{u \in W^{1,2}_{\rm loc}(\mathbb{R}) : u + 1 \in W^{1,2}((-\infty, 0]),\ u - 1 \in W^{1,2}([0, \infty))\}. \tag{1.4}$$

2000 *Mathematics Subject Classification.* 34B40, 34C37.
Key words and phrases. Heteroclinic; non-autonomous equation; bounded solution; variational methods.

Submitted May 21, 2009. Published September 23, 2010.

An interesting problem is to replace l by a nonconstant, positive coefficient function $a(t)$ and find conditions on a under which

$$\ddot{x}(t) = a(t)V'(x(t)) \tag{1.5}$$

with (1.2) has solutions. We must assume something: note that if a is continuous and increasing, then if x solves (1.1)-(1.2), then $\frac{1}{2}\dot{x}(t)^2 - a(t)V(x(t)) \to 0$ as $|t| \to \infty$, but

$$\begin{aligned} \frac{d}{dt}(\frac{1}{2}\dot{x}(t)^2 - a(t)V(x(t))) &= \ddot{x}(t)\dot{x}(t) - a(t)V'(x(t))\dot{x}(t) - \dot{a}(t)V(x(t)) \\ &= -\dot{a}(t)V(x(t)) < 0. \end{aligned} \tag{1.6}$$

This is impossible.

There are many results concerning equations like (1.5) in which the analogue of $a(t)$ is periodic, and homoclinic, heteroclinic, and multitransition solutions of the equations are found. See [6], [10]. There seems to be fewer results for the case

- (A1) $a(t) \to l > 0$ as $|t| \to \infty$

In [2, Chapter 2, Thm. 2.2], a solution is found for when $0 < a(t) \le l$ for all $t \in \mathbb{R}$. In [5] (Section 5, Example 1) a solution is found when the coefficient $a(t)$ is definitively increasing with respect to $|t|$. In [8], a solution is found in the case $l \le a(t) \le L$ and L is suitably bounded from above. This result is a specific case of the result proven in this paper and is described more precisely later. In [9], a solution is found when $a(t)$ is increasing on $[t_0, \infty)$ and decreasing on $(-\infty, t_0]$ for some $t_0 > 0$ and $l - a(t)$ decays to zero slowly enough as $|t| \to \infty$. In this paper, we find conditions on a which allow $l - a(t)$ to change sign for arbitrarily large $|t|$ and do not require any assumptions on the monotonicity of a or the decay rate of $l - a(t)$ as $|t| \to \infty$. In more related work, in [7] heteroclinic orbits to a nonautonomous differential equation are found that connect stationary points of different energy levels. In [4], heteroclinic solutions connecting nonconsecutive equilibria of a triple-well potential are found for a fourth-degree ordinary differential equation.

Let V satisfy

- (V1) $V \in C^2(\mathbb{R}, \mathbb{R})$;
- (V2) $V(x) \ge 0$ for all $x \in \mathbb{R}$;
- (V3) $V(-1) = V(1) = 0$;
- (V4) $V > 0$ on $(-1, 1)$;
- (V5) $V''(-1) > 0$, $V''(1) > 0$.

Let

$$\xi_- = \min\{x : x > -1, V'(x) = 0\}, \quad \xi_+ = \max\{x : x < 1, V'(x) = 0\}. \tag{1.7}$$

Note that ξ_- and ξ_+ are well-defined by (V3)-(V5). Define

$$\nu = \min\Big(\int_{-1}^{\xi_-} \sqrt{V(x)}\,dx, \int_{\xi_+}^{1} \sqrt{V(x)}\,dx\Big) > 0. \tag{1.8}$$

Let $a : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying (A1) and

- (A2) $0 < \underline{l} \le a(t) \le L \equiv l + 4\nu\sqrt{l\underline{l}}/\int_{-1}^{1}\sqrt{V(x)}\,dx$ for all $t \in \mathbb{R}$

We will prove the following result.

Theorem 1.1. *Let V and a satisfy* (V1)-(V5), (A1)-(A2). *Then* (1.5), (1.2) *has a solution taking values in* $(-1, 1)$.

Note: if V is even and $V > 0$ on $(-1, 0)$, then $L = l + 2\sqrt{l\underline{l}}$ in (A2). If $\underline{l} = l$, we obtain the result of [8]. Due to a dearth of counterexamples, it is not known whether the upper bound on a in (A2) is really necessary.

This paper is organized as follows: Section 2 lays out the variational methods used in the proof and an outline of the proof. Section 3 contains the proofs of some subordinate propositions and lemmas, with the most involved proposition concerning the convergence of Palais-Smale sequences of the functional associated with (1.5). Section 4 wraps up the proof of Theorem 1.1.

2. Variational Method and Outline of Proof

Define the functional $F : W^{1,2}_{\text{loc}}(\mathbb{R}) \to [0, \infty]$ by

$$F(x) = \int_{-\infty}^{\infty} \frac{1}{2}\dot{x}(t)^2 + a(t)V(x(t))\, dt. \tag{2.1}$$

By (V1)-(V3), $F(x) < \infty$ for all $x \in W$. $F : W \to \mathbb{R}^+$ is Fréchet differentiable with

$$F'(x)u = \int_{-\infty}^{\infty} \dot{x}(t)\dot{u}(t) + V'(x(t))u(t)\, dt \tag{2.2}$$

for all $x \in W$, $u \in W^{1,2}(\mathbb{R})$. Critical points of $F : W \to \mathbb{R}^+$ are solutions of (1.5), (1.2). We will show via a minimax argument that F has at least one critical point.

Define

$$\mathcal{B} = F_l(\omega) > 0, \tag{2.3}$$

where F_l is as in (1.3). A *Palais-Smale sequence* for F is a sequence $(x_n) \subset W$ with $F(x_n)$ convergent and $\|F'(x_n)\| \to 0$ as $n \to \infty$, where $\|F'(x)\|$ is defined by the operator norm

$$\|F'(x)\| = \sup\{F'(x)u : u \in W^{1,2}(\mathbb{R}),\ \|u\|_{W^{1,2}(\mathbb{R})} = 1\}. \tag{2.4}$$

We will use the usual norm on $W^{1,2}(\mathbb{R})$,

$$\|u\|_{W^{1,2}(\mathbb{R})} = \Big(\int_{-\infty}^{\infty} \dot{u}(t)^2 + u(t)^2\, dt\Big)^{1/2}. \tag{2.5}$$

The $W^{1,2}(\mathbb{R})$-norm will be denoted simply by $\|\cdot\|$ for the rest of this article. We will prove the following proposition.

Proposition 2.1. *Let $(x_n) \subset W$ with $F'(x_n) \to 0$ and*

$$F(x_n) \to b \in [0, \mathcal{B}) \cup (\mathcal{B}, \mathcal{B} + 2\nu\sqrt{2\underline{l}}). \tag{2.6}$$

Then, there exists $\bar{x} \in W$ solving (1.5), (1.2) *and a subsequence of (x_n) (also called (x_n)) with $\|x_n - \bar{x}\| \to 0$ as $n \to \infty$.*

It is interesting that the conclusion of Proposition 2.1 fails precisely when $b = \mathcal{B}$. To verify this, define the translation operator τ by $\tau_a u(t) = u(t - a)$ for any $u : \mathbb{R} \to \mathbb{R}$ and $a, t \in \mathbb{R}$. Then the Palais-Smale sequence $(x_n) = (\tau_n \omega)$ satisfies $F(x_n) \to \mathcal{B}$ and $F'(x_n) \to 0$ as $n \to \infty$, but $x_n \to -1$ pointwise.

We use a minimax argument similar to that in [8]. Define

$$\Gamma = \{\gamma \in C(\mathbb{R}, W) : \|\tau_t\omega - \gamma(t)\| \to 0 \text{ as } |t| \to \infty\} \tag{2.7}$$

and

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in \mathbb{R}} F(\gamma(t)). \tag{2.8}$$

Clearly $c \geq \mathcal{B}$. We will show in Section 4 that $c < \mathcal{B}+2\nu\sqrt{2\underline{l}}$. There are two cases to consider: $c = \mathcal{B}$ and $c > \mathcal{B}$. If $c > \mathcal{B}$, then a standard deformation argument shows that there exists a Palais-Smale sequence (x_n) with $F(x_n) \to c$ and $F'(x_n) \to 0$ as $n \to \infty$. Applying Proposition 2.1, there exists a solution $\bar{x}$ of (1.5), (1.2) and a subsequence of (x_n) (also denoted (x_n)) with $\|x_n - \bar{x}\| \to 0$ as $n \to \infty$. If $c = \mathcal{B}$, then for every $n \in \mathbb{N}$, there exists $\gamma_n \in \Gamma$ with $\sup_{t\in\mathbb{R}} F(\gamma_n(t)) < \mathcal{B} + 1/n$. Choose $t_n \in \mathbb{R}$ with $\gamma_n(t_n)(0) = 0$ and let $x_n = \gamma_n(t_n)$. Since $(F(x_n))$ is bounded, we will show there exists a subsequence (also called (x_n)) and $x \in W^{1,2}_{\rm loc}(\mathbb{R})$ such that (x_n) converges to x locally uniformly and weakly in $W^{1,2}([-T, T])$ for all $T > 0$. We will show in Section 4 that in fact $x \in W$ and $F(x) \leq \mathcal{B}$. If x is a critical point of F, then Theorem 1.1 is proven. Otherwise, let $\mathcal{W}(y)$ denote the gradient of F at y for all $y \in W$; that is, for all $y \in W$ and $\varphi \in W^{1,2}(\mathbb{R})$,

$$(\mathcal{W}(y), \varphi)_{W^{1,2}(\mathbb{R})} = F'(y)\varphi, \tag{2.9}$$

where $(\cdot, \cdot)$ is the standard inner product on $W^{1,2}(\mathbb{R})$,

$$(f, g)_{W^{1,2}(\mathbb{R})} = \int_{-\infty}^{\infty} \dot{f}(t)\dot{g}(t) + f(t)g(t)\, dt. \tag{2.10}$$

Let η denote the solution of the gradient vector flow induced by the initial value problem:

$$\frac{d}{ds}\eta(s, u) = -\mathcal{W}(\eta(s, u));\ \eta(0, u) = u. \tag{2.11}$$

We will show in Section 4 that η is well-defined on $[0, \infty) \times W$.

Recall that we have $x \in W$ with $F(x) \leq \mathcal{B}$ and $F'(x) \neq 0$. Since F is nonnegative, there exists a sequence $(s_n) \subset \mathbb{R}^+$ with $F'(\eta(s_n, x)) \to 0$ as $n \to \infty$. By Proposition 2.1, there exists $\bar{x}$ satisfying (1.5), (1.2).

3. Palais-Smale Sequences

In this section, we prove Proposition 2.1 and some subsidiary lemmas and propositions leading up to it. Although the full strength of Proposition 2.1 is not necessary to prove Theorem 1.1, the strong convergence of Palais-Smale sequences that it implies is interesting and may be useful for other problems. From now on we assume that

$$V(x) > 0 \quad \text{for all } |x| > 1, \quad \text{and} \quad \lim_{|x|\to\infty} V(x) = \infty. \tag{3.1}$$

We may make this assumption because the solution we will find to (1.5) takes values in $(-1, 1)$.

Lemma 3.1. *If $x \in W^{1,2}_{\rm loc}(\mathbb{R})$ with $F(x) < \infty$, then $x(t) \to -1$ or $x(t) \to 1$ as $t \to -\infty$, and $x(t) \to 1$ or $x(t) \to -1$ as $t \to \infty$. In fact, $x + 1 \in W^{1,2}((-\infty, 0])$ or $x - 1 \in W^{1,2}((-\infty, 0])$, and $x + 1 \in W^{1,2}([0, \infty))$ or $x - 1 \in W^{1,2}([0, \infty))$.*

Proof. Suppose the lemma is false. Then there exist $x \in W^{1,2}_{\rm loc}(\mathbb{R})$ with $F(x) < \infty$, $\delta > 0$ and a sequence (t_n) with $|t_n| \to \infty$ as $n \to \infty$

$$x_n(t) \in (-\infty, -1 - \delta) \cup (-1 + \delta, 1 - \delta) \cup (1 + \delta, \infty). \tag{3.2}$$

Let

$$d = \inf\{V(x) : x \in (-\infty, -1-\delta/2) \cup (-1+\delta/2, 1-\delta/2) \cup (1+\delta/2, \infty)\} > 0. \tag{3.3}$$

Assume without loss of generality that $t_n \to \infty$, and taking a subsequence if necessary, that $t_{n+1} \geq t_n + 1$ for all n. If $x(t) \in (-\infty, -1 - \delta/2) \cup (-1 + \delta/2, 1 -$

$\delta/2) \cup (1+\delta/2, \infty)$ for all $t \in [t_n, t_n+1]$, then $\int_{t_n}^{t_n+1} V(x(t))\,dt \geq \delta$. Otherwise, there exists $t^* \in [t_n, t_{n+1}]$ with $|x(t_n) - x(t^*)| \geq \delta/2$, and by the Cauchy-Schwarz inequality,

$$\begin{aligned} \delta/2 \leq |x(t_n) - x(t^*)| &\leq \int_{t_n}^{t^*} |\dot{x}(t)|\,dt \\ &\leq \sqrt{t^* - t_n}\Big(\int_{t_n}^{t^*} \dot{x}(t)^2\,dt\Big)^{1/2} \leq \Big(\int_{t_n}^{t^*} \dot{x}(t)^2\,dt\Big)^{1/2}, \end{aligned} \tag{3.4}$$

$$\int_{t_n}^{t_n+1} \dot{x}(t)^2\,dt \geq \int_{t_n}^{t^*} \dot{x}(t)^2\,dt \geq \delta^2/4. \tag{3.5}$$

Either way,

$$\int_{t_n}^{t_n+1} \frac{1}{2}\dot{x}(t)^2 + a(t)V(x(t))\,dt \geq \min(\delta^2/8, d\underline{l}), \tag{3.6}$$

and

$$F(x) \geq \sum_{n=1}^{\infty} \int_{t_n}^{t_n+1} \frac{1}{2}\dot{x}(t)^2 + a(t)V(x(t))\,dt \geq \sum_{n=1}^{\infty} \min(\delta^2/8, d\underline{l}) = \infty, \tag{3.7}$$

which is a contradiction. So $x(t) \to -1$ or $x(t) \to 1$ as $t \to \infty$. Similarly, $x(t) \to -1$ or $x(t) \to 1$ as $t \to -\infty$.

By (V5), there exists $\epsilon > 0$ with $V(x) \geq \epsilon(x+1)^2$ for all $x \in (-1-\epsilon, -1+\epsilon)$ and $V(x) \geq \epsilon(x-1)^2$ for all $x \in (1-\epsilon, 1+\epsilon)$. So if $x(t) \to 1$ as $t \to \infty$, there exists $T > 0$ such that

$$\int_T^{\infty} (x(t)-1)^2\,dt \leq \int_T^{\infty} V(x(t))/\epsilon\,dt \leq \frac{1}{\epsilon\underline{l}} \int_T^{\infty} a(t)V(x(t))\,dt \leq \frac{F(x)}{\epsilon\underline{l}} < \infty \tag{3.8}$$

and $x - 1 \in W^{1,2}([0,\infty))$. Similar arguments apply to the cases $x(t) \to -1$ as $t \to \infty$, $x(t) \to 1$ as $t \to -\infty$, and $x(t) \to -1$ as $t \to -\infty$. □

Next we show that Palais-Smale sequences are bounded in $W^{1,2}_{\rm loc}(\mathbb{R})$.

Lemma 3.2. *Let $A, T > 0$. There exists $B > 0$ such that if $x \in W^{1,2}_{\rm loc}(\mathbb{R})$ with $F(x) \leq A$, then $\|x\|_{W^{1,2}([-T,T])} \leq B$.*

Proof. Clearly $\int_{-T}^{T} \dot{x}(t)^2\,dt \leq 2A$, so it suffices to find an upper bound on $|x|$ over $[-T,T]$. Let $C > 0$ such that $V(x) > C$ for all $|x| \geq A/2T$. Since $\int_{-T}^{T} V(x(t))\,dx \leq A$, there exists $t^* \in [T,T]$ with $V(t^*) \leq A/2T$ and $|x(t^*)| \leq C$. For any $s \in [-T,T]$,

$$\begin{aligned} |x(s)| &\leq |x(t^*)| + |\int_{t^*}^{s} \dot{x}(t)\,dt| \\ &\leq |x(t^*)| + \sqrt{|s-t^*|}\Big|\int_{t^*}^{s} \dot{x}(t)^2\,dt\Big|^{1/2} \\ &\leq C + \sqrt{2T}\cdot\sqrt{2A}. \end{aligned} \tag{3.9}$$

□

For $\Omega \subset \mathbb{R}$, define

$$F_\Omega(x) = \int_\Omega \frac{1}{2}\dot{x}(t)^2 + a(t)V(x(t))\,dt. \tag{3.10}$$

Then we have the following lemma.

Lemma 3.3. *If $x_0, x_1 \in (-1,1)$, $t_0 < t_1$, and $x \in W^{1,2}([t_0,t_1])$ with $x(t_0) = x_0$ and $x(t_1) = x_1$, then*

$$F_{[t_0,t_1]}(x) \geq \sqrt{2\underline{l}}\Big|\int_{x_0}^{x_1} \sqrt{V(x)}\,dx\Big|. \tag{3.11}$$

Proof. Let $\omega_{\underline{l}}$ denote the unique solution in W of the differential equation

$$\ddot{x}(t) = \underline{l}V'(x(t)) \tag{3.12}$$

satisfying $\omega_{\underline{l}}(0) = 0$. Then $\omega_{\underline{l}}$ minimizes the functional

$$F_{\underline{l}}(u) = \int_{-\infty}^{\infty} \frac{1}{2}\dot{u}(t)^2 + \underline{l}V(u(t))\,dt \tag{3.13}$$

over W. By the argument following (1.2),

$$\dot{\omega}_{\underline{l}}(t) = \sqrt{2\underline{l}V(\omega_{\underline{l}}(t))}. \tag{3.14}$$

Let $x_0, x_1 \in (-1,1), t_0 < t_1$, and $x \in W^{1,2}([t_0,t_1])$ with $x(t_0) = x_0$ and $x(t_1) = x_1$. Assume $x_0 < x_1$. Now

$$\int_{t_0}^{t_1} \frac{1}{2}\dot{x}(t)^2 + \underline{l}V(x(t))\,dt \geq \int_{t_0}^{t_1} \frac{1}{2}\dot{\omega}_{\underline{l}}(t)^2 + \underline{l}V(\omega_{\underline{l}}(t))\,dt; \tag{3.15}$$

otherwise, we could replace $\omega_{\underline{l}}|_{[\omega_{\underline{l}}^{-1}(x_0),\omega_{\underline{l}}^{-1}(x_1)]}$ by $x|_{[t_0,t_1]}$ to obtain $\tilde{\omega} \in W$ with $F_{\underline{l}}(\tilde{\omega}) < F_{\underline{l}}(\omega_{\underline{l}})$, contradicting the optimality of $\omega_{\underline{l}}$. $\tilde{\omega}$ is defined by

$$\begin{aligned}
&\tilde{\omega}(t)\\
&= \begin{cases} \omega_{\underline{l}}(t), & t \leq \omega_{\underline{l}}^{-1}(x_0);\\ x(t - \omega_{\underline{l}}^{-1}(x_0) + t_0), & \omega_{\underline{l}}^{-1}(x_0) \leq t \leq \omega_{\underline{l}}^{-1}(x_0) + t_1 - t_0;\\ \omega_{\underline{l}}(t + (\omega_{\underline{l}}^{-1}(x_1) - \omega_{\underline{l}}^{-1}(x_0)) - (t_1 - t_0)), & t \geq \omega_{\underline{l}}^{-1}(x_0) + t_1 - t_0. \end{cases}
\end{aligned} \tag{3.16}$$

Now by (3.14)-(3.15),

$$F_{[t_0,t_1]}(x) \geq \int_{t_0}^{t_1} \dot{\omega}_{\underline{l}}(t)^2 = \int_{t_0}^{t_1} \dot{\omega}_{\underline{l}}(t)\sqrt{2\underline{l}V(\omega_{\underline{l}}(t))}\,dt = \int_{x_0}^{x_1} \sqrt{2\underline{l}V(x(t))}\,dt. \tag{3.17}$$

For the case $x_0 > x_1$, define x_R, the reversal of x on $[t_0,t_1]$, by $x_R(t) = x(t_0 + t_1 - t)$. Then $x_R(t_0) = x_1$ and $x_R(t_1) = x_0$ so by the first case,

$$\begin{aligned}
F_{[t_0,t_1]}(x) &\geq \int_{t_0}^{t_1} \frac{1}{2}\dot{x}(t)^2 + \underline{l}V(x(t))\,dt = \int_{t_0}^{t_1} \frac{1}{2}\dot{x_R}(t)^2 + \underline{l}V(x_R(t))\,dt\\
&\geq \int_{x_1}^{x_0} \sqrt{2\underline{l}V(x_R(t))}\,dt = \Big|\int_{x_0}^{x_1} \sqrt{2\underline{l}V(x(t))}\,dt\Big|.
\end{aligned} \tag{3.18}$$

□

Recall that ξ_- and ξ_+ from (1.7), and assume from now on that

$$\begin{aligned}
V(x) = V(-1 + (-1 - x)) \quad &\text{for all } x \in [-1 - (\xi_- + 1), -1],\\
V(x) = V(1 - (x - 1)) \quad &\text{for all } x \in [1, 1 + (1 - \xi_+)].
\end{aligned} \tag{3.19}$$

Again, we may assume this because our solution of (1.5), (1.2) will take values in $(-1,1)$. To prove Proposition 2.1, we will use the following result.

Proposition 3.4. *If* $(x_n) \subset W$ *with* $F'(x_n) \to 0$,

$$F(x_n) \to b < 2\mathcal{B} + \sqrt{2\underline{l}} \int_{-1}^{1} \sqrt{V(x)}\, dx, \tag{3.20}$$

and $x_n \to \bar{x} \in W$ *locally uniformly and weakly in* $W^{1,2}([-T,T])$ *for all* $T > 0$ *as* $n \to \infty$, *then* $\bar{x}$ *solves* (1.5) *and* $\|x_n - \bar{x}\| \to 0$ *as* $n \to \infty$.

Proof. Let (x_n) and $\bar{x}$ be as in the Proposition statement. To prove $\bar{x}$ solves (1.5), let $\varphi \in C_0^\infty$. Then

$$\begin{aligned} 0 = \lim_{n\to\infty} F'(x_n)\varphi &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \dot{x}_n(t)\dot{\varphi}(t) + V'(x_n(t))\varphi(t)\, dt \\ &= \int_{-\infty}^{\infty} \dot{\bar{x}}(t)\dot{\varphi}(t) + V'(\bar{x}(t))\varphi(t)\, dt = F'(\bar{x})\varphi, \end{aligned} \tag{3.21}$$

and $\bar{x}$ is a weak solution of (1.5). Next we show that for any $T > 0$, $\|x_n - \bar{x}\|_{W^{1,2}([-T,T])} \to 0$ as $n \to \infty$. Let $T > 0$. Since $x_n \to \bar{x}$ uniformly on $[-T,T]$, $\int_{-T}^{T}(x_n(t) - \bar{x}(t))^2\, dt \to 0$ as $n \to \infty$. We must therefore show that $\int_{-T}^{T}(\dot{x}_n(t) - \dot{\bar{x}}(t))^2\, dt \to 0$ as $n \to \infty$. Since $\dot{x}_n \to \dot{\bar{x}}$ weakly in $L^2([-T,T])$,

$$\begin{aligned} &\limsup_{n\to\infty} \int_{-T}^{T} (\dot{x}_n(t) - \dot{\bar{x}}(t))^2\, dt \\ &= \limsup_{n\to\infty} \int_{-T}^{T} \dot{x}_n(t)^2 - 2\int_{-T}^{T} \dot{x}_n(t)\dot{\bar{x}}(t)\, dt + \int_{-T}^{T} \dot{\bar{x}}(t)^2\, dt \\ &= \limsup_{n\to\infty} \int_{-T}^{T} \dot{x}_n(t)^2 - \dot{\bar{x}}(t)^2\, dt, \end{aligned} \tag{3.22}$$

and it suffices to prove $\lim_{n\to\infty} \int_{-T}^{T} \dot{x}_n(t)^2 - \dot{\bar{x}}(t)^2\, dt = 0$. Define $(u_n) \subset W^{1,2}(\mathbb{R})$ by

$$u_n(t) = \begin{cases} 0 & t \le -T-1 \\ (x_n(-T) - \bar{x}(-T))(t + T + 1) & -T-1 \le t \le -T \\ x_n(t) - \bar{x}(t) & -T \le t \le T \\ (x_n(T) - \bar{x}(T))(-t + T + 1) & T \le t \le T+1 \\ 0 & t \ge T+1 \end{cases} \tag{3.23}$$

Clearly, (u_n) is bounded in $W^{1,2}(\mathbb{R})$. Since $u_n \to 0$ uniformly on $[-T-1, T+1]$,

$$\begin{aligned} 0 &= \lim_{n\to\infty} F'(x_n)u_n + F'(\bar{x})u_n \\ &= \lim_{n\to\infty} (x_n, u_n)_{W^{1,2}([-T-1,T+1])} + (\bar{x}, u_n)_{W^{1,2}([-T-1,T+1])} \\ &\quad - \int_{-T-1}^{T+1} a(t)V'(x_n(t))u_n(t)\, dt - \int_{-T-1}^{T+1} a(t)V'(\bar{x}(t))u_n(t)\, dt \\ &= \lim_{n\to\infty} (x_n, u_n)_{W^{1,2}([-T-1,T+1])} + (\bar{x}, u_n)_{W^{1,2}([-T-1,T+1])}. \end{aligned} \tag{3.24}$$

Since $\|u_n\|_{W^{1,2}([-T-1,-T])} \to 0$ and $\|u_n\|_{W^{1,2}([T,T+1])} \to 0$ as $n \to \infty$,

$$\begin{aligned} 0 &= \lim_{n\to\infty} (x_n, u_n)_{W^{1,2}([-T,T])} + (\bar{x}, u_n)_{W^{1,2}([-T,T])} \\ &= \lim_{n\to\infty} \int_{-T}^{T} \dot{x}_n(t)(\dot{x}_n(t) - \dot{\bar{x}}(t)) + x_n(t)(x_n(t) - \bar{x}(t)) \\ &\quad + \dot{\bar{x}}(t)(\dot{x}_n(t) - \dot{\bar{x}}(t)) + \bar{x}(t)(x_n(t) - \bar{x}(t))\, dt \\ &= \lim_{n\to\infty} \int_{-T}^{T} \dot{x}_n^2(t) - \dot{\bar{x}}(t)^2 + x_n(t)^2 - \bar{x}(t)^2\, dt \\ &= \lim_{n\to\infty} \int_{-T}^{T} \dot{x}_n^2(t) - \dot{\bar{x}}(t)^2\, dt. \end{aligned} \tag{3.25}$$

Therefore, $\|x_n - \bar{x}\|_{W^{1,2}([-T,T])} \to 0$ as $n \to \infty$.

Suppose $\|x_n - \bar{x}\| \not\to 0$ as $n \to \infty$. Then there exist $\delta > 0$ and a sequence (T_n) with $T_n \to \infty$ and

$$\|x_n - \bar{x}\|^2_{\mathbb{R}\setminus[-T_n,T_n]} \geq 4\delta^2 \tag{3.26}$$

for all n. Along a subsequence, either

$$\|x_n - \bar{x}\|^2_{W^{1,2}((-\infty,-T_n])} \geq 2\delta^2 \quad \text{or} \quad \|x_n - \bar{x}\|^2_{W^{1,2}([T_n,\infty))} \geq 2\delta^2. \tag{3.27}$$

Let us assume the former; the latter case is similar. Since $1 + \bar{x} \in W^{1,2}((-\infty, 0])$,

$$\|x_n + 1\|_{W^{1,2}((-\infty,-T_n])} \geq \delta \tag{3.28}$$

for large n. There are two cases to consider:

Case I: For all $\epsilon > 0$, there exists $M > 0$ such that $|1 + x_n(t)| < \epsilon$ for all n and $t \leq -M$.

Case II: There exists $d \in (0, 1)$ and a sequence $(t_n) \subset \mathbb{R}$ with $t_n \to -\infty$ and $|1 + x_n(t_n)| > d$ for all n.

Case I: let $\xi^* \in (-1, \xi_-)$ and $c \in (0, 1)$ such that

$$V'(x)x \geq c(1 + x)^2 \tag{3.29}$$

for all $x \in [-1-(\xi^*+1), \xi^*]$. This is possible by (V3)-(V5), (3.19), and the definition of ξ_-. Let $M > 0$ be large enough so that

$$|1 + x_n(t)| < \min\big(1 + \xi^*, \frac{c\delta^2}{8(1 + \sqrt{b})}\big) \tag{3.30}$$

for all $n \in \mathbb{N}$, $t \leq -M$. Define $(u_n) \subset W^{1,2}(\mathbb{R})$ by

$$u_n(t) = \begin{cases} 1 + x_n(t) & t \leq -M \\ (1 + x_n(-M))(1 - M - t) & -M \leq t \leq -M + 1 \\ 0 & t \geq -M + 1 \end{cases} \tag{3.31}$$

We will show (u_n) is uniformly bounded in $W^{1,2}(\mathbb{R})$. Let $K > 0$ so

$$|V'(x)| \leq K \quad \text{and} \quad (x + 1)^2 \leq KV(x) \tag{3.32}$$

for all $x \in [-1-(\xi^*+1), \xi^*]$. This is possible by (V1)-(V5),(3.19), and the definition of ξ_-. For large n,

$$\begin{aligned}\|u_n\|^2 &= \int_{-\infty}^{-M} \dot{x}_n(t)^2 + (1+x_n(t))^2\,dt + (1+x_n(-M))^2 + \frac{1}{2}(1+x_n(-M))\\ &\le (2+\frac{K}{\underline{l}})\int_{-\infty}^{-M} \frac{1}{2}\dot{x}_n(t)^2 + a(t)V(x_n(t))\,dt\\ &\quad + (1+\bar{x}(-M))^2 + \frac{1}{2}(1+\bar{x}(-M)) + 1\\ &\le (2+\frac{K}{\underline{l}})F(x_n) + (1+\bar{x}(-M))^2 + \frac{1}{2}(1+\bar{x}(-M)) + 1\\ &\le (2+\frac{K}{\underline{l}})(2b) + (1+\bar{x}(-M))^2 + \frac{1}{2}(1+\bar{x}(-M)) + 1.\end{aligned} \tag{3.33}$$

Since $F'(x_n) \to 0$, $F'(x_n)u_n \to 0$ as $n \to \infty$. But for large n,

$$\begin{aligned}&F'(x_n)u_n\\ &= \int_{-\infty}^{-M} \dot{x}_n(t)^2 + V'(x_n(t))(1+x_n(t))\,dt + \int_{-M}^{-M+1} (1+x_n(-M))\dot{x}_n(t)\,dt\\ &\quad + \int_{-M}^{-M+1} (1+x_n(-M))(1-M-t)\,dt\\ &\ge \int_{-\infty}^{-M} \dot{x}_n(t)^2 + c(1+x_n(t))^2\,dt\\ &\quad - |1+x_n(-M)|\Big(\int_{-M}^{-M+1} \dot{x}_n(t)^2\,dt\Big)^{1/2} - \frac{1}{2}|1+x_n(-M)|\\ &\ge c\|1+x_n\|^2_{W^{1,2}(-\infty,-M])} - |1+x_n(-M)|(\sqrt{2F(x_n)}+1)\\ &\ge c\delta^2 - (1+2\sqrt{b})|1+x_n(-M)| \ge \frac{1}{2}c\delta^2\end{aligned} \tag{3.34}$$

by (3.30). This is impossible.

Case II: by the arguments of Lemma 3.3,

$$F(x) \ge \sqrt{2\underline{l}}\int_{-1}^{1} \sqrt{V(x)}\,dx \tag{3.35}$$

for all $x \in W$, including $\bar{x}$. Let d and (t_n) be as in Case I. Let $M > 0$ be large enough so that $|1+\bar{x}(t)| < d/2$ for all $t \le -M$, and

$$F_{[-M,M]}(\bar{x}) > \sqrt{2\underline{l}}\int_{-1}^{1} \sqrt{V(x)}\,dx - \frac{1}{10}(2\mathcal{B} + \sqrt{2\underline{l}}\int_{-1}^{1} \sqrt{V(x)}\,dx - b). \tag{3.36}$$

Define $\alpha_n \le \beta_n < -M$ by

$$\beta_n = \max\{t < -M : |1+x_n(t)| = d\}, \quad \alpha_n = \min\{t : |1+x_n(t)| = d\} \tag{3.37}$$

Since $x_n \to \bar{x}$ locally uniformly and $|1+\bar{x}(t)| < d/2 < 1$ for all $t \le -M$, $\beta_n \to -\infty$ as $n \to \infty$. Define $v_n = \tau_{-\beta_n} x_n$. By Fatou's Lemma, the weak lower semicontinuity of $\int_{-\infty}^{\infty} \dot{x}(t)^2\,dt$, and Lemma 3.2, there exists $\bar{v} \in W^{1,2}(\mathbb{R})$ with $F(\bar{v}) < \infty$ and $v_n \to \bar{v}$ locally uniformly and weakly in $W^{1,2}([-T,T])$ for all $T > 0$. By the arguments of (3.21), $F_l'(\bar{v}) = 0$. By the definition of β_n, $\bar{v}(t) \le -1+d < 0$ for all $t > 0$. Therefore, by the arguments of Lemma 3.1 applied to F_l instead of F, $\bar{v}(t) \to -1$ as $t \to \infty$. By the arguments following (1.2), $\dot{\bar{v}}(t) = -\sqrt{2lV(\bar{v}(t))}$ for all $t \in \mathbb{R}$. Let ω_R denote the reversal of ω: $\omega_R(t) = \omega(-t)$ for all t. Clearly $\bar{v} = \tau_\lambda \omega_R$ for some $\lambda \in \mathbb{R}$. By the arguments of (3.22)-(3.25),

$$\|x_n - \tau_{\lambda+\beta_n}\omega_R\|_{W^{1,2}([\beta_n - T, \beta_n + T])} \to 0 \tag{3.38}$$

as $n \to \infty$ for all $T > 0$. This implies $\beta_n - \alpha_n \to \infty$ as $n \to \infty$. For all n and all $t < \alpha_n$, $x_n(t) < -1 + d/2 < 0$. Therefore, arguments similar to those above show that there exists $\lambda_2 \in \mathbb{R}$ with

$$\|x_n - \tau_{\lambda_2+\alpha_n}\omega\|_{W^{1,2}([\alpha_n - T, \alpha_n + T])} \to 0 \tag{3.39}$$

for all $T > 0$ as $n \to \infty$. For $\Omega \subset \mathbb{R}$, define

$$F_{l\Omega}(x) = \int_\Omega \frac{1}{2}\dot{x}(t)^2 + lV(x(t))\,dt \tag{3.40}$$

Still assuming that M is large enough so that (3.36) holds, assume also that M is large enough that

$$\begin{gathered}
F_{l[-M,M]}(\tau_\lambda \omega_R) > \mathcal{B} - \frac{1}{10}(2\mathcal{B} + \sqrt{2\underline{l}}\int_{-1}^{1}\sqrt{V(x)}\,dx - b), \\
F_{l[-M,M]}(\tau_{\lambda_2}\omega) > \mathcal{B} - \frac{1}{10}(2\mathcal{B} + \sqrt{2\underline{l}}\int_{-1}^{1}\sqrt{V(x)}\,dx - b),
\end{gathered} \tag{3.41}$$

Then for large n, (3.36), (3.38)-(3.41), $\alpha_n \to -\infty$, $\beta_n \to -\infty$, and $a(t) \to l$ as $|t| \to \infty$ imply

$$\begin{aligned}
F(x_n) &\ge F_{[\alpha_n - M, \alpha_n + M]}(x_n) + F_{[\beta_n - M, \beta_n + M]}(x_n) + F_{[-M,M]}(x_n) \\
&\ge (\mathcal{B} - \frac{1}{5}(2\mathcal{B} + \sqrt{2\underline{l}}\int_{-1}^{1}\sqrt{V(x)}\,dx - b) \\
&\quad + (\mathcal{B} - \frac{1}{5}(2\mathcal{B} + \sqrt{2\underline{l}}\int_{-1}^{1}\sqrt{V(x)}\,dx - b) \\
&\quad + (\sqrt{2\underline{l}}\int_{-1}^{1}\sqrt{V(x)}\,dx - \frac{1}{5}(2\mathcal{B} + \sqrt{2\underline{l}}\int_{-1}^{1}\sqrt{V(x)}\,dx - b)) \\
&\quad - \frac{1}{5}(2\mathcal{B} + \sqrt{2\underline{l}}\int_{-1}^{1}\sqrt{V(x)}\,dx - b) \\
&= 2\mathcal{B} + \sqrt{2\underline{l}}\int_{-1}^{1}\sqrt{V(x)}\,dx - \frac{4}{5}(2\mathcal{B} + \sqrt{2\underline{l}}\int_{-1}^{1}\sqrt{V(x)}\,dx - b) \\
&\equiv b^+ > b.
\end{aligned} \tag{3.42}$$

This is impossible. Proposition 3.4 is proven. □

Proof of Proposition 2.1. There are two cases: $b < \mathcal{B}$ and $b > \mathcal{B}$. The case $b < \mathcal{B}$ is easier. Let $(x_n) \subset W$ with $F(x_n) \to b < \mathcal{B}$ and $F'(x_n) \to 0$ as $n \to \infty$. By Lemma 3.2, (x_n) converges locally uniformly and weakly in $W^{1,2}([-T,T])$ for all $T > 0$ to some function $\bar{x} \in W^{1,2}_{\rm loc}(\mathbb{R})$. By Fatou's Lemma and the weak lower semicontinuity of $\int_{-\infty}^{\infty}\dot{x}(t)^2\,dt$, $F(\bar{x}) < \infty$. By Proposition 3.4, it suffices to show $\bar{x} \in W$. Suppose $\bar{x} \notin W$. Then by Lemma 3.1, $\bar{x}(t) \to 1$ as $t \to -\infty$ or $\bar{x}(t) \to -1$

as $t \to \infty$. Suppose $\bar{x}(t) \to -1$ as $t \to \infty$ (the proof for $\bar{x}(t) \to 1$ as $t \to -\infty$ is similar). Define

$$\mathcal{B}_\epsilon = \int_{\omega^{-1}(-1+\epsilon)}^{\omega^{-1}(1-\epsilon)} \frac{1}{2}\dot{\omega}^2(t) + lV(\omega(t))\,dt \tag{3.43}$$

for $\epsilon > 0$. Let $\epsilon > 0$ be small enough that

$$\Big(\frac{l-\epsilon}{l}\Big)\mathcal{B}_\epsilon > b. \tag{3.44}$$

Let $T > 0$ be large enough so that $a \ge l - \epsilon$ on $[T, \infty)$ and $\bar{x}(T) < -1 + \epsilon$. Let n be large enough that $x_n(T) < -1 + \epsilon$. Let $T < \alpha < \beta$ with $x_n(\alpha) = -1 + \epsilon$, $x_n(\beta) = 1 - \epsilon$. By arguments similar to those of Lemma 3.3,

$$\begin{aligned} F(x_n) \ge F_{[\alpha,\beta]}(x_n) &= \int_\alpha^\beta \frac{1}{2}\dot{x}_n(t)^2 + a(t)V(x_n(t))\,dt \\ &\ge \int_\alpha^\beta \frac{1}{2}\dot{x}_n(t)^2 + (l-\epsilon)V(x_n(t))\,dt \\ &\ge \frac{l-\epsilon}{l}\int_\alpha^\beta \frac{1}{2}\dot{x}_n(t)^2 + lV(x_n(t))\,dt \\ &\ge \frac{l-\epsilon}{l}\mathcal{B}_\epsilon \equiv b^+ > b. \end{aligned} \tag{3.45}$$

This is a contradiction.

Now suppose $b \in (\mathcal{B}, \mathcal{B} + 2\nu\sqrt{l\underline{l}})$. As before, along a subsequence, (x_n) converges locally uniformly and weakly in $W^{1,2}([-T,T])$ for all $T > 0$ to a function $\bar{x} \in W^{1,2}_{\rm loc}(\mathbb{R})$ with $F(\bar{x}) \le b$. We must show $\bar{x} \in W$; then applying Proposition 3.4 proves Theorem 1.1. Suppose $\bar{x}(t) \not\to 1$ as $t \to \infty$ (the proof for $\bar{x}(t) \not\to -1$ as $t \to -\infty$ is similar). By Lemma 3.1, $\bar{x}(t) \to -1$ as $t \to \infty$. Let $t_n = \max\{t : x_n(t) = 0\}$. Then $t_n \to \infty$ as $n \to \infty$. By the arguments following (3.37) and the arguments of (3.22)-(3.25),

$$\|x_n - \tau_{t_n}\omega\|_{W^{1,2}([t_n-M,t_n+M])} \to 0 \tag{3.46}$$

as $n \to \infty$ for all $M > 0$. Let $-1 < e_1 < \xi_i < \xi_-$ with

$$\sqrt{2\underline{l}}\int_{e_1}^{\xi_1} \sqrt{V(x)}\,dx > \nu\sqrt{2\underline{l}} - \frac{1}{5}(\mathcal{B} + 2\nu\sqrt{2\underline{l}} - b). \tag{3.47}$$

Let $c \in (0,1)$ with

$$V'(x)(1+x) \ge cV(x) \tag{3.48}$$

for all $x \in [-1, \xi_1]$. Let $K > 0$ be large enough that

$$|V'(x)| < K \tag{3.49}$$

for all $x \in [-1, 1]$. Let $M > 0$ be large enough that

$$F_{l[-M,M]}(\omega) > \mathcal{B} - \frac{1}{6}(\mathcal{B} + 2\nu\sqrt{2\underline{l}} - b), \tag{3.50}$$

$$1 + \omega(-M) < \min\Big(\frac{c(b-\mathcal{B})}{16(K + 2\sqrt{b})}, 1 + e_1\Big). \tag{3.51}$$

By (3.46) and the fact that $a(t) \to l$ as $t \to \infty$,

$$F_{[t_n-M,t_n+M]}(x_n) < \mathcal{B} + \frac{1}{2}(b - \mathcal{B}) \tag{3.52}$$

for large n, so

$$F_{(-\infty,t_n-M]}(x_n) + F_{[t_n+M,\infty)}(x_n) > \frac{1}{3}(b-\mathcal{B}). \tag{3.53}$$

Assume $F_{(-\infty,t_n-M]} > (b-\mathcal{B})/6$ (the case $F_{[t_n+M,\infty)} > (b-\mathcal{B})/6$ is similar). There are two possible cases: along a subsequence,

Case I: $|1+x_n(\alpha_n)| \geq 1+\xi_1$ for $\alpha_n < t_n - M$,
Case II: $|1+x_n(t)| < 1+\xi_1$ for all $t < t_n - M$.

For Case I, assume $1+x_n(\alpha_n) \geq 1+\xi_1$ (the case $1+x_n(\alpha_n) \leq -(1+\xi_1)$ is similar due to (3.19)). For large n, by Lemma 3.3, (3.47), (3.51), (3.46), (3.50), (A1), and $t_n \to \infty$,

$$\begin{aligned} F(x_n) &\geq F_{(-\infty,\alpha_n]}(x_n) + F_{[\alpha_n,t_n-M]}(x_n) + F_{[t_n-M,t_n+M]}(x_n) \\ &\geq 2(\nu\sqrt{2\underline{l}} - \frac{1}{5}(\mathcal{B}+2\nu\sqrt{2\underline{l}}-b)) + (\mathcal{B} - \frac{1}{5}(\mathcal{B}+2\nu\sqrt{2\underline{l}}-b)) \\ &= \mathcal{B} + 2\nu\sqrt{2\underline{l}} - \frac{3}{5}(\mathcal{B}+2\nu\sqrt{2\underline{l}}-b) \equiv b^+ > b. \end{aligned} \tag{3.54}$$

This is impossible.

For Case II, define $(u_n) \subset W^{1,2}(\mathbb{R})$ by

$$u_n(t) = \begin{cases} 1+x_n(t) & t \leq t_n - M \\ (1+x_n(t_n-M))(t_n-M+1-t) & t_n-M \leq t \leq t_n-M+1 \\ 0 & t \geq t_n-M+1. \end{cases} \tag{3.55}$$

The sequence (u_n) is uniformly bounded in $W^{1,2}(\mathbb{R})$, as in (3.33). So $F'(x_n)u_n \to 0$. But for large n,

$$\begin{aligned} &F'(x_n)u_n \\ &= \int_{-\infty}^{t_n-M} \dot{x}_n(t)^2 + a(t)V'(x_n(t))(1+x_n(t))\,dt \\ &\quad - (1+x_n(t_n-M)) \int_{t_n-M}^{t_n-M+1} \dot{x}_n(t)\,dt \\ &\quad + (1+x_n(t_n-M)) \int_{t_n-M}^{t_n-M+1} V'(x_n(t))(t_n-M+1-t)\,dt \\ &\geq \int_{-\infty}^{t_n-M} \dot{x}_n(t)^2 + ca(t)V(x_n(t))\,dt \\ &\quad - |1+x_n(t_n-M)|\Big(\int_{t_n-M}^{t_n-M+1} \dot{x}_n(t)^2\,dt\Big)^{1/2} - K|1+x_n(t-M)| \\ &\geq c\int_{-\infty}^{t_n-M} \frac{1}{2}\dot{x}_n(t)^2 + a(t)V(x_n(t))\,dt - (K+2\sqrt{b})|1+x_n(t_n-M)| \\ &= cF_{(-\infty,t_n-M]}(x_n) - (K+2\sqrt{b})|1+x_n(t_n-M)| \\ &\geq \frac{1}{6}c(b-\mathcal{B}) - \frac{1}{12}c(b-\mathcal{B}) = \frac{1}{12}c(b-\mathcal{B}) > 0 \end{aligned} \tag{3.56}$$

by (3.51). This is impossible. Case II is proven. Proposition 2.1 is proven. □

4. Completion of Proof

In this section we tie up some loose ends from Section 2. It was asserted that $c < \mathcal{B} + 2\nu\sqrt{2\underline{l}}$, where c is from (2.8). Define $\gamma_0 \in \Gamma$ by $\gamma_0(t) = \tau_t(\omega)$. We will show $\sup_{t\in\mathbb{R}} F(\omega_0(t)) < \mathcal{B}$. Since $F(\gamma_0(t)) \to \mathcal{B}$ as $|t| \to \infty$, and $F(\gamma_0(t))$ is continuous in t, it suffices to prove that $F(\gamma_0(t)) < \mathcal{B} + 2\nu\sqrt{2\underline{l}}$ for all $t \in \mathbb{R}$. We will prove this for $t = 0$; the proof is similar for other t. After (1.2), it is proven that

$$V(\omega(t)) = \frac{\dot\omega(t)}{\sqrt{2l}}\sqrt{V(\omega(t))} \tag{4.1}$$

for all t. Since $a(t) \to l$ as $|t| \to \infty$, and $\omega(t) \in (-1, 1)$ for all t, (A2) gives us

$$\begin{aligned} F(\gamma_0(0)) = F(\omega) &= \int_{-\infty}^{\infty} \frac{1}{2}\dot\omega(t)^2 + a(t)V(\omega(t))\,dt \\ &< \int_{-\infty}^{\infty} \frac{1}{2}\dot\omega(t)^2 + LV(\omega(t))\,dt \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\dot\omega(t)^2 + lV(\omega(t))\,dt + (L-l)\int_{-\infty}^{\infty} V(\omega(t))\,dt \\ &= \mathcal{B} + \frac{4\nu\sqrt{l\underline{l}}}{\int_{-1}^{1}\sqrt{V(x)}\,dx}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2l}}\dot\omega(t)\sqrt{V(\omega(t))}\,dt = \mathcal{B} + 2\nu\sqrt{2\underline{l}}. \end{aligned} \tag{4.2}$$

We must prove that the gradient vector flow from (2.11) is well-defined on $\mathbb{R}^+ \times W$. Since F is C^2, it suffices to show that for all $A > 0$, there exists $B > 0$ such that if $x \in W$ with $F(x) \le A$, $\|F'(x)\| \le B$: By (V5), it is possible to extend V from $[-1-(\xi_- + 1), 1 + (1-\xi_+)]$ (see (3.19)) to $\mathbb{R}$ such that there exists $K > 0$ with $V'(x)^2 \le KV(x)$ for all real x. Let $x \in W$ with $F(x) \le A$ and $u \in W^{1,2}(\mathbb{R})$ with $\|u\|_{W^{1,2}(\mathbb{R})} = 1$. Then

$$\begin{aligned} F'(x)u &= \int_{-\infty}^{\infty} \dot x(t)\dot u(t) + a(t)V'(x(t))u(t)\,dt \\ &\le \Big(\int_{-\infty}^{\infty}\dot x(t)^2\,dt\Big)^{1/2}\Big(\int_{-\infty}^{\infty}u(t)^2\,dt\Big)^{1/2} \\ &\quad + L\Big(\int_{-\infty}^{\infty}V'(x(t))^2\,dt\Big)^{1/2}\Big(\int_{-\infty}^{\infty}u(t)^2\,dt\Big)^{1/2} \\ &\le \sqrt{2A} + L\Big(\int_{-\infty}^{\infty}KV(x(t))\,dt\Big)^{1/2} \\ &\le \sqrt{2A} + L\sqrt{K/\underline{l}}\Big(\int_{-\infty}^{\infty}a(t)V(x(t))\,dt\Big)1/2 \\ &\le \sqrt{2A} + L\sqrt{KA/\underline{l}}. \end{aligned} \tag{4.3}$$

Here is the "standard deformation argument" alluded to after (2.8): suppose $c > \mathcal{B}$, and suppose there does not exist a Palais-Smale sequence $(x_n) \subset W$ with $F(x_n) \to c$ and $F'(x_n) \to 0$. Then there exist $\epsilon, \delta > 0$ such that $\|F'(x_n)\| > \delta$ for all $x \in W$ with $F(x) \in [c-\epsilon, c+\epsilon]$. Let $\gamma \in \Gamma$ with $\sup_{t\in\mathbb{R}} F(\gamma(t)) < c + \epsilon$. Let $T > 0$ be large enough so that $F(\gamma(t)) < c$ ($> \mathcal{B}$) for $|t| \ge T$. Let $\varphi \in C(\mathbb{R}, [0,1])$ with $\varphi = 0$ on $(-\infty, -T-1] \cup [T+1, \infty)$ and $\varphi = 1$ on $[-T, T]$. Define $\gamma_2 \in \Gamma$ by $\gamma_2(t) = \eta(\frac{2\varphi(t)\epsilon}{\delta^2}, \gamma(t))$, where η is the gradient vector flow from (2.11). Since

$\frac{d}{ds}F(\eta(s,u)) = -\|F'(\eta(s,u))\|^2$ for all $u \in W$, $s \in \mathbb{R}^+$, $F(\gamma_2(t)) < c$ for all $t \in \mathbb{R}$. $F(\gamma_2(t)) \to \mathcal{B}$ as $|t| \to \infty$, so $\sup_{t\in\mathbb{R}} F(\gamma_2(t)) < c$, contradicting the definition of c.

In the $c = \mathcal{B}$ case after (2.8), we have $(x_n) \subset W$ with $F(x_n) \to b \le \mathcal{B}$ as $n \to \infty$ and $x_n(0) = 0$ for all n. Since $F(x_n)$ is bounded, there exists $\bar{x} \in W^{1,2}_{\rm loc}(\mathbb{R})$ and a subsequence of (x_n) (also denoted (x_n)) such that $x_n \to \bar{x}$ locally uniformly and weakly in $W^{1,2}([-T,T])$ for all $T > 0$. As before, $F(\bar{x}) \le b \le \mathcal{B}$. We must prove $\bar{x} \in W$. Suppose otherwise. By Lemma 3.1, $\bar{x}(t) \to 1$ or -1 as $t \to \infty$ and $\bar{x}(t) \to 1$ or -1 as $t \to -\infty$. Suppose $\bar{x}(t) \to -1$ as $t \to \infty$ (the proof for $\bar{x}(t) \to 1$ as $t \to -\infty$ is similar). Let $\mathcal{B}_\epsilon$ be as in (3.43) and let $\epsilon > 0$ be small enough that

$$\frac{l-\epsilon}{l}\mathcal{B}_\epsilon > \mathcal{B} - F_{[-1,1]}(\bar{x})/2. \tag{4.4}$$

Let $T > 1$ be large enough so $a \ge l - \epsilon$ on $[t, \infty)$ and $\bar{x}(T) < -1 + \epsilon$. Then, as in (3.45), for large n,

$$F(x_n) \ge F_{[-1,1]}(x_n) + F_{[T,\infty)}(x_n) \ge F_{[-1,1]}(\bar{x})/2 + \frac{l-\epsilon}{l}\mathcal{B}_\epsilon > \mathcal{B}. \tag{4.5}$$

This is impossible.

The final step in the proof is to show that a solution of (1.1) in W takes values in $(-1, 1)$. Suppose $x \in W$ and solves (1.1). If $x(t) > 1$ for some real t, then let $t_{\max} \in \mathbb{R}$ with $x(t_{\max}) = \max_{t\in\mathbb{R}} x(t)$. $\ddot{x}(t_{\max}) \le 0$, but $V(x(t_{\max})) > 0$. This is impossible. Similarly, $x(t) \le 1$ for all real t. Now suppose $x(t^*) = 1$. Then x satisfies the Cauchy problem (1.1), $x(t^*) = 1$, $\dot{x}(t^*) = 0$, so by (V1), $x \equiv 1$. This is a contradiction. Similarly, $x(t) > -1$ for all real t.

References

[1] Ravi P. Aggarwal and Donal O'Regan; Infinite interval problems for differential, difference, and integtral equations, Kluwer A. Publ. Dordrecht, 2001.

[2] H. Brézis; *Analyse Fonctionelle*, Collection mathématiques appliquées pour la maîtrise, Masson, Paris (1983).

[3] D.Bonheure, L. Sanchez ; *Heteroclinic orbits for some classes of second and fourth order differential equations*, Handbook of Differential Equations III, A. Canãda, P. Drabek, A. Fonda (eds), Elsevier, (to appear).

[4] D. Bonheure, L. Sanchez, M. Tarallo, S. Terracini; *Heteroclinic connections between nonconsecutive equilibria of a fourth order differential equation*, Calculus of Variations and Partial Differential Equations **17** (2001), 341-356.

[5] C.-N. Chen, S.-Y. Tzeng; *Existence and multiplicity results for heteroclinic orbits of second order Hamiltonian systems*, J. Differential Equations, **158** (1999), 211-250.

[6] V. Coti Zelati and P. H. Rabinowitz; *Multichain type solutions for Hamiltonian systems, Proceedings of the Conference on Nonlinear Differential Equations*, **5** Southwest Texas State Univ., San Marcos, TX (2000), 223235 (electronic).

[7] V. Coti Zelati and P. H. Rabinowitz; *Heteroclinic solutions between stationary points at different energy levels*, Top. Meth. Nonlinear Analysis, **17** (2001), 1-21.

[8] A. Gavioli; *On the existence of heteroclinic trajectories for asymptotically autonomous equations*, to appear in Top. Meth. Nonlinear Analysis.

[9] A. Gavioli and L. Sanchez; *Heteroclinics for non-autonomous second order differential equations*, to appear in proceedings of NODE, September 2008.

[10] P. Rabinowitz; *Homoclinic and heteroclinic orbits for a class of Hamiltonian systems*, Calc. Var. Partial Differential Equations **1** (1993), 1-36.

Gregory S. Spradlin

Department of Mathematics, Embry-Riddle University, Daytona Beach, Florida 32114-3900, USA

E-mail address: `spradlig@erau.edu`