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Lorentz-Violating Gravitoelectromagnetism

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The well-known analogy between a special limit of general relativity and electromagnetism is explored in the context of the Lorentz-violating standard-model extension. An analogy is developed for the minimal standard-model extension that connects a limit of the *CPT*-even component of the electromagnetic sector to the gravitational sector. We show that components of the post-Newtonian metric can be directly obtained from solutions to the electromagnetic sector. The method is illustrated with specific examples including static and rotating sources. Some unconventional effects that arise for Lorentz-violating electrostatics and magnetostatics have an analog in Lorentz-violating post-Newtonian gravity. In particular, we show that even for static sources, gravitomagnetic fields arise in the presence of Lorentz violation.

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I. INTRODUCTION

In its full generality, general relativity (GR) is a highly nonlinear theory that bears little resemblance to classical Maxwell electrodynamics. Nonetheless, it has long been known that when gravitational fields are weak, and matter is slow moving, analogs of the electric and magnetic fields arise for gravity [1]. These fields are sourced by a scalar density and vector current density, just as in electrostatics and magnetostatics. Furthermore, in the geodesic equation for a test body, terms of no more than linear order in the velocity resemble the classical Lorentz-force law arising from effective gravitoelectric and gravitomagnetic fields [2]. Also, a well-known analogy exists between the precession of classical spin in a gravitational field and the precession of the spin of a charged particle in an electromagnetic field [3,4].

In this work we investigate the fate of the standard connection between stationary solutions of the Einstein and Maxwell theories when violations of local Lorentz symmetry are introduced. Recent interest in Lorentz violation has been motivated by the possibility of uncovering experimental signatures from an underlying unified theory at the Planck scale [5–7]. We examine the modified Einstein and Maxwell equations provided by the action-based standard-model extension (SME) framework, which allows for generic Lorentz violation for both gravity and electromagnetism, among other forces [8–10].

In the so-called minimal SME case, the electromagnetic sector contains 23 observable coefficients for Lorentz violation organized into two parts: 4 *CPT*-odd coefficients in $(k_{AF})^{\mu}$ with dimensions of mass and 19 *CPT*-even coefficients in the dimensionless $(k_F)^{\mu\nu\kappa\lambda}$ [9,11]. The former set has been stringently constrained by astrophysical observations at the level of 10^{-42} GeV [12,13]. The latter set has been explored over the last decade using astrophysical observations [14] and sensitive laboratory experiments including resonant-cavity tests [15], among others [16,17]. Currently, constraints on these 19 coefficients are at the level of 10^{-14} to 10^{-32} [18].

In the minimal SME gravitational sector, there are 20 coefficients for Lorentz violation organized into a scalar u, two-tensor $s^{\mu\nu}$, and four-tensor $t^{\mu\nu\kappa\lambda}$ [10]. Within the assumption of spontaneous Lorentz-symmetry breaking, the dominant effects for weak-field gravity are controlled by the subset called $\bar{s}^{\mu\nu}$ [19]. These gravity coefficients have been explored so far in lunar laser-ranging [20] and atom interferometry [21–23], while possibilities exist for other tests such as time-delay and Doppler tests [24].

Since the *CPT*-even portion of the electromagnetic sector of the minimal SME has 19 coefficients and the gravitational sector, apart from an unobservable scaling $\bar{u}$, also has 19 coefficients, one might expect a correspondence between the two sectors—an extension of the conventional analogy. Indeed, as we show in this work, there is a correspondence under certain restrictions. Thus it turns out that, under certain circumstances, Lorentz violation affects classical electromagnetic systems in flat spacetime in a similar manner as gravitational systems are affected by Lorentz violation in the weak-field limit of gravity. As a consequence, some of the unusual effects that occur for Lorentz-violating electromagnetism have an analog in the gravitational case. In addition, from a practical perspective, it is quite useful to be able to translate analytical results in one sector directly into the other, as we illustrate toward the end of this work.

We begin in Sec. II by reviewing the basic field equations for the gravitational and electromagnetic sectors of the SME. Next we explore the solutions to these equations and establish the analogy between the two sectors in both the conventional case and the Lorentz-violating case in Sec. III. In Sec. IV, we explore test-body motion for both sectors and establish the connection in the conventional and Lorentz-violating cases. We conclude this work in Sec. V by illustrating the results with the examples of a pointlike source and a rotating spherical source, and we

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discuss some experimental applications of the results. Finally in Sec. VI, we summarize the main results of the paper. Throughout this work, we take the spacetime metric signature to be $-+++$ and we work in natural units where $c = \epsilon_0 = \mu_0 = 1$.

II. FIELD EQUATIONS

The CPT-even coefficients for Lorentz violation in the photon sector of the minimal SME are denoted $(k_F)^{\mu\nu\kappa\lambda}$, which is assumed totally traceless by convention, and have all of the tensor symmetries of the Riemann tensor and therefore contain 19 independent quantities [9,11]. Following Ref. [13], it is useful to split these 19 coefficients into two independent pieces using the expansion

$$(k_F)^{\mu\nu\kappa\lambda} = C^{\mu\nu\kappa\lambda} + \tfrac{1}{2}[\eta^{\mu\kappa}(c_F)^{\nu\lambda} - \eta^{\mu\lambda}(c_F)^{\nu\kappa} - \eta^{\nu\kappa}(c_F)^{\mu\lambda} + \eta^{\nu\lambda}(c_F)^{\mu\kappa}]. \quad (1)$$

With this decomposition 9 coefficients are contained in the traceless combinations $(c_F)^{\mu\nu} = (k_F)^{\mu\alpha\nu}{}_{\alpha}$ and 10 coefficients are in $C^{\mu\nu\kappa\lambda}$, which is traceless on any two indices. The modified Maxwell equations can then be written in the form

$$\partial_\mu F^{\mu\nu} + C^{\mu\nu\kappa\lambda}\partial_\mu F_{\kappa\lambda} + (c_F)^{\nu\lambda}\partial_\mu F^{\mu}{}_{\lambda} + (c_F)^{\mu\lambda}\partial_\mu F_{\lambda}{}^{\nu} = -j^\nu, \quad (2)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the vector potential. This result follows directly from the electromagnetic action of the minimal SME in Minkowski spacetime, when the electromagnetic field is coupled in the standard way to a conserved four-current $j^\mu = (\rho, \vec{J})$, and when the coefficients are treated as constants in an observer inertial frame.

In the gravitational sector, the coefficients for Lorentz violation are expressed in terms of three independent sets of coefficients: $t^{\mu\nu\kappa\lambda}$, $s^{\mu\nu}$, u. The t coefficients are taken as totally traceless and have the symmetries of the Riemann curvature tensor, implying 10 independent quantities. The s coefficients are traceless and contain 9 independent quantities. With the scalar u, there are, in general, 20 independent coefficients describing Lorentz violation in the gravitational sector.

Unlike the SME in Minkowski spacetime, it is not straightforward to proceed directly from the gravitational action to the field equations. This is because introducing externally prescribed coefficients for Lorentz violation into the action can generally conflict with the fundamental Bianchi identities of pseudo-Riemannian geometry [10]. It turns out, however, that spontaneous breaking of Lorentz symmetry evades this difficulty [10,25]. In Ref. [19], the linearized gravitational field equations were derived using a formalism that treats the coefficients for Lorentz violation as dynamical fields inducing spontaneous breaking of Lorentz symmetry, with certain restrictions placed on their dynamics [26]. Similar methods can be adopted for the matter-gravity couplings as well [28]. The linearized equations in this formalism include, as special cases, models of spontaneous Lorentz-symmetry breaking with scalar [29], vector [27], and two-tensor fields [30,31].

In linearized gravity the metric is expanded as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (3)$$

Within the minimal SME approach, the linearized field equations can be written in terms of the vacuum expectation values of the coefficients for Lorentz violation, denoted $\bar{t}^{\mu\nu\kappa\lambda}$, $\bar{s}^{\mu\nu}$, $\bar{u}$, which are taken as constants in a special observer coordinate system [32]. The linearized field equations take the form

$$G_{\mu\nu} = 8\pi G_N(T_M)_{\mu\nu} + \bar{s}^{\kappa\lambda}R_{\kappa\mu\nu\lambda} - \bar{s}^{\kappa}{}_{\mu}R_{\kappa\nu} - \bar{s}^{\kappa}{}_{\nu}R_{\kappa\mu} + \tfrac{1}{2}\bar{s}_{\mu\nu}R + \eta_{\mu\nu}\bar{s}^{\kappa\lambda}R_{\kappa\lambda}, \quad (4)$$

where G_N is Newton's gravitational constant. In this expression $R_{\kappa\mu\nu\lambda}$ is the Riemann curvature tensor, $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, and R is the Ricci scalar. All curvature tensors in (4) are understood as linearized in the fluctuations $h_{\mu\nu}$. Since the $\bar{u}$ coefficient only scales the left-hand side, it is unobservable and is discarded for this work.

Because of a tensor identity [19], the 10 coefficients $\bar{t}^{\mu\nu\kappa\lambda}$ vanish from the linearized equations, thus leaving the 9 coefficients in $\bar{s}^{\mu\nu}$ in this limit. This immediately implies that, should an analogy exist between the photon and gravity sectors of the SME, it involves a subset of the $(k_F)^{\mu\nu\kappa\lambda}$ coefficients. This subset is comprised of the 9 coefficients $(c_F)^{\mu\nu}$.

III. FIELD MATCH

A. Conventional GR case

In GR and Maxwell electrodynamics, the analogy between certain components of the metric fluctuations $h_{\mu\nu}$ and A_μ reveals itself from the field equations in the harmonic gauge:

$$\partial^\mu \bar{h}_{\mu\nu} = 0. \quad (5)$$

Here $\bar{h}_{\mu\nu}$ are the usual trace-reversed metric fluctuations defined by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h^{\alpha}{}_{\alpha}. \quad (6)$$

In the absence of the coefficients for Lorentz violation $(k_F)^{\mu\nu\kappa\lambda}$ and $\bar{s}^{\mu\nu}$, the Einstein equations in this gauge read

$$\Box\bar{h}_{\mu\nu} = -16\pi G_N(T_M)_{\mu\nu}, \quad (7)$$

while the Maxwell equations, in the gauge $\partial^\mu A_\mu = 0$, are

$$\Box A_\mu = -j_\mu. \quad (8)$$

To match the structure of the Maxwell equations one typically makes a slow motion assumption for the matter

source. For example, for perfect fluid matter with ordinary velocity v^j much less than one, and small pressure,

$$\begin{aligned}(T_M)_{00} &\approx \rho,\\ (T_M)_{0j} &\approx -\rho v^j,\\ (T_M)_{jk} &\approx \rho v^j v^k.\end{aligned} \tag{9}$$

Thus, examining Eq. (7), it can be seen that the components $\bar{h}_{jk}$ will be one power of velocity more than $\bar{h}_{0j}$, and hence negligible. To be more precise, if one adopts the standard post-Newtonian expansion and counts terms in powers of mean velocity $\bar{v}$, labeled as $O(1)$, $O(2)$, etc., one finds from Eq. (7) that

$$\begin{aligned}\bar{h}_{00} &\sim O(2),\\ \bar{h}_{0j} &\sim O(3),\\ \bar{h}_{jk} &\sim O(4).\end{aligned} \tag{10}$$

Furthermore, in post-Newtonian counting, partial time derivatives obey the post-Newtonian counting [33,34]

$$\frac{\partial}{\partial t} \sim \frac{\bar{v}}{\bar{r}}, \tag{11}$$

where $\bar{r}$ is the mean distance. A consistent approximation including up to $O(3)$ terms would take $\Box \approx \vec{\nabla}^2$ and Eq. (7) would become

$$\vec{\nabla}^2 \bar{h}_{0\mu} = -16\pi G_N (T_M)_{0\mu}, \tag{12}$$

which can be compared with the stationary equations for electrostatics and magnetostatics

$$\vec{\nabla}^2 A_\mu = -j_\mu. \tag{13}$$

From these two expressions it is clear that, given solutions for A_μ in the stationary limit, the solutions $\bar{h}_{0\mu}$ can be obtained in the manner below.

(1) Replace charge density ρ_q with mass density ρ_m and electric current density J^j with mass-current density ρv^j.
(2) Write down the metric components as $\bar{h}_{0\mu} = -16\pi G_N A_\mu$.

This method agrees with standard results in the literature [35,36].

B. Lorentz-violating case

Equations (7) and (8) lead to a direct correspondence between the solutions for $\bar{h}_{0\mu}$ and A_μ. In the presence of Lorentz violation, this direct analogy involving the trace-reversed metric fluctuations disappears because the coefficients $\bar{s}^{\mu\nu}$ in the modified equations (4) generally mix the components of $\bar{h}_{0\mu}$ with $\bar{h}_{jk}$. As a result of this mixing, $\bar{h}_{jk}$ contains terms of $O(2)$ in post-Newtonian counting, in contrast to the GR case (10), and so there is no particular utility in using the trace-reversed metric fluctuations $\bar{h}_{\mu\nu}$ over the metric fluctuations $h_{\mu\nu}$.

We focus on the stationary limit, where a match between the electromagnetic and gravity sectors can be obtained for the metric components h_{00} and h_{0j}. This gravitoelectromagnetic correspondence is most easily obtained directly from the stationary solutions to Eqs. (2) and (4) for the metric $g_{\mu\nu}$ and the vector potential A_μ. The gravitational solutions were obtained in Ref. [19] while the results in electrodynamics were obtained in Refs. [37,38].

Before displaying the solutions here, it will be convenient to introduce various potential functions that take a similar form for both the electromagnetic and gravitational sectors. The key source quantities appearing in these potentials are the charge (mass) density ρ and the charge (mass) current J^j. The needed potentials are

$$\begin{aligned}U &= \alpha \int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x',\\ U^{jk} &= \alpha \int \frac{\rho(\vec{x}')(x-x')^j(x-x')^k}{|\vec{x}-\vec{x}'|^3}\, d^3x',\\ V^j &= \alpha \int \frac{J^j(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x',\\ X^{jkl} &= \alpha \int \frac{J^j(\vec{x}')(x-x')^k(x-x')^l}{|\vec{x}-\vec{x}'|^3}\, d^3x'.\end{aligned} \tag{14}$$

In the stationary limit, all partial time derivatives of the potentials vanish. The density ρ is time independent and the current is transverse, $\partial_j J^j = 0$. This implies some simplifications of the identities among the potentials listed in Ref. [19], including $\partial_j V^j = 0$ and $\partial_j X^{jkl} = 0$.

The electromagnetic potentials are obtained by interpreting ρ as charge density, J^j as a steady-state current density, and letting the constant $\alpha = 1/4\pi$. For the gravitational sector, the potentials are obtained by interpreting ρ as mass density, $J^j = \rho v^j$ as mass-current density, and letting $\alpha = G_N$.

The components of the metric fluctuations $h_{0\mu}$, relevant for comparison with the electromagnetic sector, can be obtained after an appropriate coordinate gauge choice. We choose coordinates such that

$$\begin{aligned}\partial_j h_{0j} &= 0,\\ \partial_k h_{kj} &= \tfrac{1}{2}\partial_j (h_{kk} - h_{00}),\end{aligned} \tag{15}$$

and the metric fluctuations are time independent. To post-Newtonian $O(3)$, the metric components $h_{0\mu}$ are then given by

$$\begin{aligned}h_{00} &= (2 + 3\bar{s}^{00})U + \bar{s}^{jk}U^{jk} - 4\bar{s}^{0j}V^j,\\ h_{0j} &= -\bar{s}^{0j}U - \bar{s}^{0k}U^{jk} - 4(1 + \tfrac{1}{2}\bar{s}^{00})V^j + 2\bar{s}^{jk}V^k\\ &\quad + 2\bar{s}^{kl}(X^{klj} - X^{jkl}),\end{aligned} \tag{16}$$

where $\alpha = G_N$ is chosen in the expressions (14). Although they are not relevant for the match between the two sectors, for completeness, the remaining components of the metric h_{jk} are given by

$$h_{jk} = [(2 - \bar{s}^{00})U + \bar{s}^{lm}U^{lm}]\delta^{jk} - \bar{s}^{jl}U^{lk} - \bar{s}^{kl}U^{lj} + 2\bar{s}^{00}U^{jk}, \tag{17}$$

which is valid to post-Newtonian $O(2)$.

In the electromagnetic sector, we choose the stationary limit and adopt the $U(1)$ gauge condition $\partial_j A^j = 0$. The modified Maxwell equations have the solutions

$$\begin{aligned} A^0 &= [1 + \tfrac{1}{2}(c_F)^{00}]U_E + \tfrac{1}{2}(c_F)^{jk}U_E^{jk} - (c_F)^{0j}V_E^j \\ &\quad - C^{0j0k}U_E^{jk} - C^{0jkl}X_E^{ljk}, \\ A^j &= \tfrac{1}{2}(c_F)^{0j}U_E + \tfrac{1}{2}(c_F)^{0k}U_E^{jk} + [1 - \tfrac{1}{2}(c_F)^{00}]V_E^j \\ &\quad - \tfrac{1}{2}(c_F)^{jk}V_E^k - \tfrac{1}{2}(c_F)^{kl}[X_E^{klj} - X_E^{jkl}], \\ &\quad - C^{0kjl}U_E^{kl} - C^{0j0k}V_E^k - C^{jklm}X_E^{mkl}, \end{aligned} \tag{18}$$

where the subscript E reminds us to take $\alpha = 1/4\pi$ in the potentials (14).

A glance at Eqs. (16) and (18) reveals that many of the same terms occur in both sectors. However, in the electromagnetic sector the contributions from the 10 independent coefficients $C^{\mu\nu\kappa\lambda}$ do not vanish. To match the two sectors we must first restrict our attention to the special case where

$$C^{\mu\nu\kappa\lambda} = 0. \tag{19}$$

Next we split the terms appearing in A_μ and $h_{0\mu}$ into those involving potentials derived from charge density ρ and those derived from current density J^j. These fields are defined as

$$\begin{aligned} (h_\rho)_{00} &= (2 + 3\bar{s}^{00})U + \bar{s}^{jk}U^{jk}, \\ (h_J)_{00} &= -4\bar{s}^{0j}V^j, \\ (h_\rho)_{0j} &= -\bar{s}^{0j}U - \bar{s}^{0k}U^{jk}, \\ (h_J)_{0j} &= -4(1 + \tfrac{1}{2}\bar{s}^{00})V^j + 2\bar{s}^{jk}V^k + 2\bar{s}^{kl}(X^{klj} - X^{jkl}), \\ (A_\rho)^0 &= [1 + \tfrac{1}{2}(c_F)^{00}]U_E + \tfrac{1}{2}(c_F)^{jk}U_E^{jk}, \\ (A_J)^0 &= -(c_F)^{0j}V_E^j, \\ (A_\rho)^j &= \tfrac{1}{2}(c_F)^{0j}U_E + \tfrac{1}{2}(c_F)^{0k}U_E^{jk}, \\ (A_J)^j &= [1 - \tfrac{1}{2}(c_F)^{00}]V_E^j - \tfrac{1}{2}(c_F)^{jk}V_E^k \\ &\quad - \tfrac{1}{2}(c_F)^{kl}[X_E^{klj} - X_E^{jkl}]. \end{aligned} \tag{20}$$

Note that the split of A_μ and $h_{0\mu}$ corresponds to splitting the terms in the post-Newtonian metric into $O(2)$ and $O(3)$ and splitting the terms in the electromagnetic potentials into "post-Coulombian" terms of $O(2)$ and $O(3)$ [33,39]. The correspondence between the two sectors is summarized in Table I.

TABLE I. The gravitoelectromagnetic correspondence between the electromagnetic and gravitational sectors of the minimal SME.

Quantity	Electromagnetic sector	Gravitational sector
Coefficients	$(c_F)^{\mu\nu}$	$\bar{s}^{\mu\nu}$
Scaling	$1/4\pi$	$G(1 + \bar{s}^{00})$
Density ρ	charge density	mass density
Current J^j	current density J^j	mass current $J^j = \rho v^j$
ρ fields	$(A_\rho)_\mu$	$(h_\rho)_{0\mu}$
J^j fields	$(A_J)_\mu$	$(h_J)_{0\mu}$

Given a stationary solution to the modified Maxwell equations (2) in the Coulomb gauge ($\partial_j A^j = 0$), one can obtain the corresponding metric components by using the following procedure.

(1) Set $C^{\mu\nu\kappa\lambda} = 0$.
(2) Replace $(c_F)^{\mu\nu} \to \bar{s}^{\mu\nu}$.
(3) Separate A^μ into density-sourced and current-sourced terms $(A_\rho)_\mu$ and $(A_J)_\mu$.
(4) Replace charge density ρ_q with mass density ρ_m and electric current density J^j with mass-current density ρv^j.
(5) Write down the metric components

$$\begin{aligned} (h_\rho)_{0\mu} &= -8\pi G_N(1 + \bar{s}^{00})(A_\rho)_\mu, \\ (h_J)_{0\mu} &= -16\pi G_N(1 + \bar{s}^{00})(A_J)_\mu, \end{aligned} \tag{21}$$

and omit any subleading order terms $[O(\bar{s}^2)]$.

The close resemblance of the effects of Lorentz violation on gravity and electromagnetism is remarkable considering the qualitative differences between the theories, particularly in the starting Lagrangians and field equations [10]. On the other hand, since there is a known analogy between A_μ and $h_{0\mu}$ in the conventional case, and both sectors are affected by two-tensor coefficients for Lorentz violation, one might have expected a close correspondence in the appropriate limit. In fact, the map constructed above further justifies the construction of the post-Newtonian metric using the formalism in Ref. [19], which itself relied on several assumptions concerning the dynamics of spontaneous Lorentz-symmetry breaking.

An interesting feature of the solutions for Lorentz-violating electrodynamics is the mixing of electrostatic and magnetostatic effects in the stationary limit. As can be seen from (20), this occurs because a part of the scalar potential A^0 depends on current density and part of the vector potential $\vec{A}$ depends on charge density, a feature absent in the conventional case. This was aptly named electromagnetostatics (EMS) in Ref. [37]. For Lorentz-violating gravity, a similar mixing occurs and h_{00} depends partly on mass current while h_{0j} depends partly on mass density, resulting in what can be called gravitoelectromagnetostatics (GEMS). These features are illustrated with specific examples in Sec. V.

Note that other possibilities are open for exploration concerning the match between the two sectors of the SME. For example, we do not treat here the interesting possibility of whether an analogy persists using gravitational and electromagnetic tidal tensors, as occurs in the Einstein and Maxwell theories [40].

IV. TEST-BODY MOTION

In this section we study another aspect of gravitoelectromagnetism. This concerns the behavior of matter in the presence of the stationary gravitational or electric and magnetic fields. As we show below, if one adopts the appropriate limit, the behavior of test masses in gravitational fields and test charges in electric and magnetic fields is analogous, despite the presence of Lorentz violation. However, differences do arise in the presence of Lorentz violation when comparing the gravitational spin precession to the classical spin precession of a magnetic moment in the presence of electromagnetic fields.

A. Geodesic motion

When Lorentz violation is present in the electromagnetic sector only, test charges e obey

$$\frac{du^\mu}{d\tau} = \frac{e}{m} F^\mu{}_\nu u^\nu, \tag{22}$$

where u^μ is the four-velocity. With the usual identification of the electric and magnetic fields, $E_j = F_{j0}$ and $B_j = (1/2)\epsilon_{jkl}F_{kl}$, we can write the spatial components of (22) as the familiar Lorentz-force law:

$$\frac{du^j}{dt} = \frac{e}{m}[E^j + (\vec{v} \times \vec{B})^j]. \tag{23}$$

For small velocities, $u^j \approx v^j = dx^j/dt$. Thus, with k_F affecting only the electromagnetic sector, the force law for charges is conventional [11].

In the SME, restricted to only the $\bar{s}^{\mu\nu}$ coefficients, freely falling test bodies satisfy the usual geodesic equation

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta. \tag{24}$$

In its full generality, the structure of (24) is quite different from Eq. (22) for charges. Nonetheless, in the weak-field slow motion limit of gravity, there is a correspondence.

Changing variables in (24) to coordinate time, one can solve for the coordinate acceleration $a^j = dv^j/dt$ in terms of the connection coefficients projected into space and time components using standard methods. One obtains the well-known expression [34],

$$a^j = -\Gamma^j{}_{00} - 2\Gamma^j{}_{0l}v^l - \Gamma^j{}_{kl}v^k v^l + (\Gamma^0{}_{00} + 2\Gamma^0{}_{0k}v^k + \Gamma^0{}_{kl}v^k v^l)v^j. \tag{25}$$

So far, Eq. (25) is an exact result, and bears little resemblance to Eq. (23). If one then assumes that the test particle velocity is small and keeps only terms linear in the test particle velocity v^j, the acceleration becomes

$$a^j = -\Gamma^j{}_{00} - 2\Gamma^j{}_{0k}v^k + \Gamma^0{}_{00}v^j. \tag{26}$$

To get a match with Eq. (23) additional assumptions are needed. For example, in the post-Newtonian approximation, the dominant contributions to the connection coefficients are given by the formulas

$$\begin{aligned} \Gamma^j{}_{00} &= \partial_0 g_{0k} - \tfrac{1}{2}g^{jk}\partial_k g_{00}, \\ \Gamma^j{}_{0k} &= \tfrac{1}{2}\partial_0 g_{jk} + \tfrac{1}{2}(\partial_k g_{0j} - \partial_j g_{0k}), \\ \Gamma^0{}_{00} &= -\tfrac{1}{2}\partial_0 g_{00}, \end{aligned} \tag{27}$$

which is valid to post-Newtonian $O(4)$. If the metric is stationary in the chosen coordinate system, ($\partial_0 g_{\mu\nu} = 0$), then the acceleration, in terms of the metric fluctuations $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, is given by

$$a^j = \tfrac{1}{2}\partial_j h_{00} + v^k(\partial_j h_{0k} - \partial_k h_{0j}) - \tfrac{1}{2}h_{jk}\partial_k h_{00}, \tag{28}$$

which neglects terms proportional to the test-mass velocity squared but otherwise is valid to post-Newtonian $O(4)$. This expression now resembles the Lorentz-force law, Eq. (23), except for the last nonlinear term.

To be consistent with the post-Newtonian approximation to $O(4)$, the last term must be included, as well as nonlinear contributions to h_{00} at $O(4)$. This is because the second term in Eq. (28), the so-called gravitomagnetic acceleration term, is an $O(4)$ term in the post-Newtonian expansion.

1. GR case

Results from GR are contained in (28) and (23) in the limit of vanishing coefficients for Lorentz violation. In the stationary limit of GR, and in the coordinate gauge (15), the acceleration (28) can be written as

$$a^j = \partial_j \phi - 4v^k(\partial_j V^k - \partial_k V^j). \tag{29}$$

Here ϕ is a post-Newtonian potential that includes $O(4)$ terms in GR [33]:

$$\phi = \int \frac{(\rho + \rho\Pi + 3p - 2\rho U)}{|\vec{x} - \vec{x}'|} d^3x' - 2U^2, \tag{30}$$

where p is the perfect fluid pressure and Π is the internal energy per unit mass. Note that ϕ does not satisfy the field equation (12),

$$\vec{\nabla}^2 \bar{h}_{00} = -16\pi G_N \rho. \tag{31}$$

Instead it satisfies

$$\vec{\nabla}^2 \phi = -4\pi G_N(\rho + \rho\Pi + 3p - 2\rho U) - 4(\vec{\nabla}U)^2. \tag{32}$$

Therefore $\phi \neq \bar{h}_{00}$, and it *cannot* be obtained directly from the solutions to A_0 in Eq. (13) using the standard match.

Generally, care is required in discarding the nonlinear terms in ϕ, while keeping the second, gravitomagnetic terms in Eq. (29). A simple estimate for a realistic scenario can establish this. For a rotating spherical body, the solution for V^j is of order $GI\omega/r^2 \sim GMR^2\omega/r^2$, where I is the inertia of the body, R its radius, ω its angular velocity, and r is the coordinate distance from the origin to the location of the test body. The typical test particle velocity v^j is of order $v \sim \sqrt{GM/r}$ or less, where M is the mass of the source body. Thus, the contribution to (29) from the gravitomagnetic force term on a test particle outside of the source body, has an approximate size

$$|\vec{a}_{\rm gm}| \sim \frac{(GM)R^2\omega v}{r^3}. \tag{33}$$

The contribution from the nonlinear terms in ϕ to the test particle acceleration have an approximate size $U\vec{\nabla}U$ or

$$|\vec{a}_{\rm nl}| \sim \frac{GM}{r}\frac{GM}{r^2}. \tag{34}$$

Assuming that the nonlinear contributions are much smaller than the gravitomagnetic contributions, $|\vec{a}_{\rm nl}| \ll |\vec{a}_{\rm gm}|$, amounts to assuming

$$R\omega v \gg \frac{GM}{R}. \tag{35}$$

For example, consider a test-body near the Earth's surface. For this case one finds that condition (35) implies the unrealistic condition that the test particle velocity must be greater than $1/2000$ of the speed of light.

In addition to the above argument, it is important to recall that terms of second and higher order in the test-body velocity v^j were discarded in (26). In terms of post-Newtonian counting, these terms make contributions to the acceleration a^j at the same order $[O(4)]$ as the nonlinear terms. One example is the term $\Gamma^j{}_{kl}v^kv^l$, which can be shown to have an approximate size similar to (34) in the typical post-Newtonian scenario [41]. Furthermore, it is interesting to note that an argument along the lines of the one presented here appeared in the original paper by Lense and Thirring in 1918 [2]. There it was emphasized that nonlinear terms must be included in the equations of motion, in addition to the gravitomagnetic force terms, to properly account, for example, for the precession of the orbital elements of the planets. As an alternative to this reasoning, one can incorporate the nonlinear terms, such as those occuring in Eq. (29), to form "Maxwell-like" equations, as pursued in Ref. [43].

For simplicity here we separate out the gravitomagnetic and gravitoelectric acceleration terms from the nonlinear terms. Thus we write

$$\vec{a} = \vec{a}_{\rm GEM} + \vec{a}_{\rm NL}, \tag{36}$$

where the separate terms are given by

$$\vec{a}_{\rm GEM} = \vec{E}_G + \vec{v}\times\vec{B}_G, \qquad \vec{a}_{\rm NL} \approx \vec{\nabla}(\phi - U). \tag{37}$$

Here we have identified the gravitoelectric and gravitomagnetic fields for GR:

$$\vec{E}_G = \vec{\nabla}U, \qquad \vec{B}_G = -4\vec{\nabla}\times\vec{V}. \tag{38}$$

2. Lorentz-violating case

To see if there is any resemblance for the Lorentz-violating case between the gravitational force law and the electromagnetic force law, we can proceed from Eq. (26). Adopting the stationary limit (28), we restrict attention to the gravitoelectromagnetic portion of the acceleration which we denote $(a')^j_{\rm GEM}$. This acceleration is given by

$$(a')^j_{\rm GEM} = \tfrac{1}{2}\partial_j h_{00} + v^k(\partial_j h_{0k} - \partial_k h_{0j}). \tag{39}$$

For a consistent expansion to first order in the coefficients $\bar{s}^{\mu\nu}$, we take h_{00} to $O(3)$ and h_{0j} to $O(2)$. This produces an acceleration to first order in the coefficients $\bar{s}^{\mu\nu}$ that is at most $O(3)$.

In the presence of the coefficients for Lorentz violation $\bar{s}^{\mu\nu}$, the components of the metric from Sec. III B are needed to this order:

$$h_{00} = (h_\rho)_{00} + (h_J)_{00}, \qquad h_{0j} = (h_\rho)_{0j}. \tag{40}$$

Note that the expansion of h_{0j} is truncated at $O(2)$ since this term is multiplied by a velocity $[O(1)]$ and therefore produces an $O(3)$ term in the acceleration.

With the considerations above, the gravitoelectromagnetic acceleration can be written to $O(3)$ as

$$(\vec{a}')_{\rm GEM} = \vec{E}_G + \vec{v}\times\vec{B}_G, \tag{41}$$

which now resembles the result in Eq. (23). The effective electric and magnetic fields are given by

$$E^j_G = \tfrac{1}{2}\partial_j[(h_\rho)_{00} + (h_J)_{00}], \qquad B^j_G = \epsilon^{jkl}\partial_k(h_\rho)_{0l}. \tag{42}$$

This result demonstrates that in the limit that the gravitoelectromagnetic acceleration terms are considered, the force on a test body takes the same form in the electromagnetic and gravitational sectors of the SME.

To use this result in a manner consistent with the post-Newtonian expansion, additional terms at $O(4)$ but at zeroth order in the coefficients $\bar{s}^{\mu\nu}$ need to be included in the acceleration. Specifically, the total acceleration at $O(4)$ takes the form

$$a^j = (a')^j + a^j_{\rm NL} + v^k[\partial_j(h_J)_{0k} - \partial_k(h_J)_{0j}], \tag{43}$$

where $a_{\rm NL}$ is given by Eq. (37) and the components $(h_J)_{0j}$ are taken to zeroth order in the coefficients $\bar{s}^{\mu\nu}$. In the limit $\bar{s}^{\mu\nu} = 0$, this expression reduces to the standard GR result in (36).

B. Spin precession

The classical relativistic behavior of a particle with a magnetic moment $\vec{\mu}$ under the influence of external electric and magnetic fields is well known. Consider a particle, such as an electron, with spin $\vec{s}$ defined by

$$\vec{\mu} = \frac{ge}{2m}\vec{s}. \tag{44}$$

Here, e is the charge of the particle, m is the mass, and g is the gyromagnetic ratio for the particle. We can describe the behavior of the spin relativistically using the spin (spacelike) four-vector S^μ which, in an instantaneous comoving rest frame, takes the form ($S^0 = 0$, $S^j = s^j$). The motion of the particle is described with the four velocity u^μ, which satisfies Eq. (22). In addition, we have the identity $S^\mu u_\mu = 0$.

If we ignore field gradient forces and nonelectromagnetic forces, the behavior of the classical spin four-vector S^μ is determined by the dynamical equations [4,44]

$$\frac{dS^\mu}{d\tau} = \frac{e}{m}\left[\frac{g}{2}F^{\mu\nu}S_\nu + \left(\frac{g}{2}-1\right)u^\mu(S_\nu F^{\nu\lambda}u_\lambda)\right]. \tag{45}$$

A formula for the precession of the spin as measured in a locally comoving reference frame can be obtained by projecting S^μ along comoving spatial basis vectors $e^\mu{}_{\hat{j}}$, and making use of Eqs. (45) and (22). With the choice of $g \approx 2$, the lowest order contributions to this precession can be written

$$\frac{dS_{\hat{j}}}{d\tau} = \frac{e}{m}\left[\vec{S}\times\vec{B} - \frac{1}{2}\vec{S}\times(\vec{v}\times\vec{E})\right]^k \delta_{k\hat{j}}. \tag{46}$$

This result holds up to order v^2 in the particle's ordinary velocity. Furthermore, Eqs. (45) and (46) will still hold in the presence of Lorentz violation in the photon sector since the force law takes the conventional form (23).

The behavior of the classical spin four-vector in the presence of gravitational fields is given by the Fermi-Walker transport equation [45]

$$\frac{dS^\mu}{d\tau} = -\Gamma^\mu{}_{\nu\lambda}u^\nu S^\lambda + u^\mu(a^\nu S_\nu), \tag{47}$$

where a^μ is the acceleration of the spinning body. For comparison with the electromagnetic case, we assume that the spin is in free fall ($a^\mu = 0$), and again find the spin precession along the comoving spatial basis $e^\mu{}_{\hat{j}}$, a standard technique [33,45]. The resulting precession was obtained in the post-newtonian limit for an arbitrary metric in Ref. [19] and is given by

$$\frac{dS_{\hat{j}}}{d\tau} = \delta_{k\hat{j}}S^k\left[\frac{1}{4}(v^k\partial_j h_{00} - v^j\partial_k h_{00}) + \frac{1}{2}(\partial_j h_{0k} - \partial_k h_{0j}) + \frac{1}{2}v^l(\partial_j h_{kl} - \partial_k h_{jl})\right], \tag{48}$$

which is valid to post-Newtonian order $O(3)$. Since this result was derived for an arbitrary post-Newtonian metric, it holds for the metric in Eqs. (16) and (17) as well. Note that the expression (48) does not immediately match (46) due to the last terms in (48) dependent on h_{kl} at $O(2)$. However, a judicious choice of coordinate gauge may alleviate the problem, as we show below.

In GR, we can make use of the results of section III A in the harmonic gauge. When expressed in terms of $\bar{h}_{\mu\nu}$ the GR spin precession to $O(3)$ is

$$\frac{dS_{\hat{j}}}{d\tau} = \left[\frac{1}{2}\vec{S}\times(\vec{\nabla}\times\vec{g})^k - \frac{3}{8}\vec{S}\times(\vec{v}\times\vec{\nabla}\bar{h}_{00})^k\right]\delta_{k\hat{j}}, \tag{49}$$

where $g^j = \bar{h}_{0j}$ and we have omitted contributions from $\bar{h}_{jk} \sim O(4)$. The expression (49) now resembles the electromagnetic counterpart, at least up to numerical factors. In fact, one can again define effective electric and magnetic fields for gravity: $\vec{E}_G = (1/4)\vec{\nabla}\bar{h}_{00}$, $\vec{B}_G = \vec{\nabla}\times\vec{g}$.

We next introduce Lorentz violation in the gravitational sector in the form of the post-Newtonian metric (16) and (17). Unlike in GR there are off-diagonal terms in h_{jk} that cannot be eliminated by a choice of coordinate gauge. As a result, we find that the third term in (48) cannot be reduced to a term of the form $\vec{S}\times\vec{\nabla}\Phi$, where Φ is a scalar. Therefore it is not possible to match the form of the spin precession in the gravitational sector to the electromagnetic sector of the SME, the latter of which takes the form (46). Evidently, this is due to the important role of the metric components h_{jk} in the general spin precession expression (48).

V. EXAMPLES AND APPLICATIONS

In this section we illustrate the methods of matching electromagnetic solutions for the fields to gravitational solutions for the metric components. We also demonstrate the match between the two sectors for test-body motion. In our examples we study both a static pointlike source and a rotating sphere. Finally, we comment on the observability of the GEMS mixing effects in specific gravitational tests.

A. Static point source

We consider first a point charge q at rest at the origin in the chosen coordinate system. The potentials in the Coulomb gauge were obtained in Ref. [37] and are given by

$$\begin{aligned} A^0 &= \frac{q}{4\pi r}[1 + (k_F)^{0j0j} - (k_F)^{0j0k}\hat{x}^j\hat{x}^k], \\ A^j &= \frac{q}{4\pi r}[(k_F)^{0kjk} - (k_F)^{jk0l}\hat{x}^k\hat{x}^l], \end{aligned} \tag{50}$$

where $\hat{x} = \vec{x}/r$ and $r = |\vec{x}|$.

Using the method outlined in Sec. III B, we can obtain the corresponding metric components $h_{0\mu}$ in the fixed coordinate gauge (15). First we expand the coefficients

$(k_F)^{\kappa\lambda\mu\nu}$ into C and c_F terms using (1). Next, we set all of the coefficients $C = 0$, according to step 1. Then we make the replacement in the remaining coefficients $c_F \to \bar{s}$. At this intermediate stage the potentials are given by

$$A^0 = \frac{q}{4\pi r}\left[1 + \frac{1}{2}\bar{s}^{00} + \frac{1}{2}\bar{s}^{jk}\hat{x}^j\hat{x}^k\right],$$
$$A^j = \frac{q}{8\pi r}[\bar{s}^{0j} + \bar{s}^{0k}\hat{x}^k\hat{x}^j]. \tag{51}$$

Since there is no dependence of the potentials on any current density, for step 3 we simply note that in Eq. (51) $A^0 = (A_\rho)^0$ and $A^j = (A_\rho)^j$. We make the replacement $q \to m$ and multiply the potentials by a factor of $-8\pi G_N(1 + \bar{s}^{00})$ and cancel subleading order terms $[O(\bar{s}^2)]$. This yields

$$h_{00} = \frac{2G_N m}{r}\left[1 + \frac{3}{2}\bar{s}^{00} + \frac{1}{2}\bar{s}^{jk}\hat{x}^j\hat{x}^k\right],$$
$$h_{0j} = -\frac{G_N m}{r}[\bar{s}^{0j} + \bar{s}^{0k}\hat{x}^k\hat{x}^j]. \tag{52}$$

In a similar manner, we can also obtain effective gravitoelectric and gravitomagnetic fields using (42):

$$E^j_G = -\frac{G_N m}{r^2}\left[\hat{x}^j\left(1 + \frac{3}{2}\bar{s}^{00} + \frac{3}{2}\bar{s}^{kl}\hat{x}^k\hat{x}^l\right) - \bar{s}^{jk}\hat{x}^k\right],$$
$$B^j_G = -\frac{2G_N m}{r^2}\epsilon^{jkl}\bar{s}^{0k}\hat{x}^l. \tag{53}$$

Using these expressions the acceleration of a test mass can be written in the Lorentz-force law form (41).

An interesting feature arises from this simple solution. In Lorentz-violating electromagnetism, even a static source will generate a magnetic field. For gravity, the analog of this effect occurs. For example, consider the scenario in which the coefficients $\bar{s}^{jk} = 0$. Apart from a scaling, the gravitoelectric force appears conventional. However, even when the source body is static, a test body with some initial velocity $\vec{v}_0$ will experience a gravitomagnetic force.

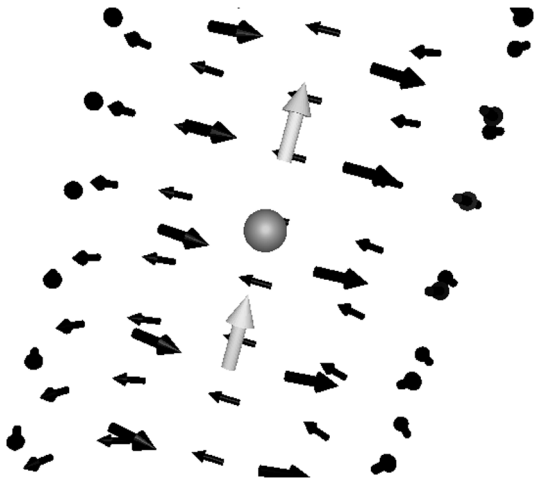

FIG. 1. The gravitomagnetic field $\vec{B}_G$ (dark arrows) from a static point mass m (center). The field curls around the direction of $\vec{s}$ (light gray arrows) and falls off as the inverse square of the distance. An approaching test body is deflected opposite $\vec{s}$.

The nature of this gravitomagnetic force is illustrated in Fig. 1. The gravitomagnetic field itself falls off as the inverse square of the distance from the point mass, and curls around the direction of the vector denoted $\vec{s}$, where $s^j = -\bar{s}^{0j}$. A test mass approaching the pointlike source will be deflected in the opposite direction of $\vec{s}$, as illustrated in the figure.

B. Rotating sphere

We next turn our attention to a more involved example, a spherical distribution of charge or mass that is rotating. In Ref. [37], a scenario was considered that involved a magnetized sphere with radius a and uniform magnetization $\vec{M}$. In conventional magnetostatics, an idealized scenario would allow for the sphere to have zero charge density and no electrostatic field surrounding it, thus it would only produce a dipole magnetic field. In the presence of Lorentz violation, however, a dipole electric field persists, with an effective dipole moment controlled by the parity-odd coefficients for Lorentz violation $(k_F)^{0jkl}$.

Since we aim to find the gravitational analog of this solution, we cannot consider an object with zero charge density. Instead we study a closely related example: a charged rotating sphere, which produces an effective magnetic dipole moment $\vec{m}$ in the conventional case. For this example, the current-induced portion of the electric scalar potential, $(A_J)^0$, can be obtained directly from Eq. (31) in Ref. [37]:

$$(A_J)^0 = \frac{\epsilon^{jkl}(c_F)^{0j}\hat{x}^k m^l}{4\pi r^2}, \tag{54}$$

which holds for the region outside the sphere. For a rotating charged sphere

$$m^j = \tfrac{1}{3}I_E\omega^j, \tag{55}$$

where $\vec{\omega}$ is the angular velocity of the sphere. The quantity I_E is the charge analog of the spherical moment of inertia for massive body,

$$I_E = \int d^3x\rho|\vec{x}|^2. \tag{56}$$

Comparing (54) with the standard dipole potential, the effective dipole moment is

$$p^j = \epsilon^{jkl}m^k(c_F)^{0l}. \tag{57}$$

The effective electric field therefore takes the standard form

$$\vec{E} = \frac{3\vec{p}\cdot\hat{x}\,\hat{x} - \vec{p}}{4\pi r^3}. \tag{58}$$

The gravitational analog for the solutions (54) and (58) can be obtained using the methods in Sec. III B. Since the $C^{\kappa\lambda\mu\nu}$ coefficients do not appear, step 1 is redundant. We next make the replacement $(c_F)^{0j} \to \bar{s}^{0j}$. All that remains is to change $\rho_q \to \rho_m$ and multiply (54) by $16\pi G_N$ which yields

$$(h_J)_{00} = \frac{4G_N I \epsilon^{jkl}\hat{x}^j \omega^k \bar{s}^{0l}}{3r^2}, \tag{59}$$

where now I is the spherical moment of inertia of the massive body, given by Eq. (56) using mass density. Note that this produces an extra component of the gravitoelectric field $\vec{E}_G = (1/2)\vec{\nabla}(h_J)_{00}$.

In the electromagnetic case, part of the electrostatic field arises from the effective current of the rotating charged sphere, a feature absent in the standard Maxwell theory. This unconventional mixing of electrostatics and magnetostatics has an analogy for stationary gravitational fields produced by a rotating mass, in the presence of Lorentz violation. Thus, a uniformly rotating sphere of mass produces a gravitoelectric field whose strength depends on the rotation rate, a feature absent in standard GR.

As in the point-mass example, the vector $\vec{s}$ is responsible for the effect. In Fig. 2, the effective dipole moment of a rotating spherical mass is depicted. The dipole moment is obtained from the cross product of $\vec{s}$ with $4I\vec{\omega}/3$.

The full solution for the case of a rotating massive or charged sphere can be constructed using the potentials U, U^{jk}, V^j, and X^{jkl} in Eqs. (14). For the electromagnetic case ($\alpha = 1/4\pi$) we obtain, for the region outside the sphere $r > R$,

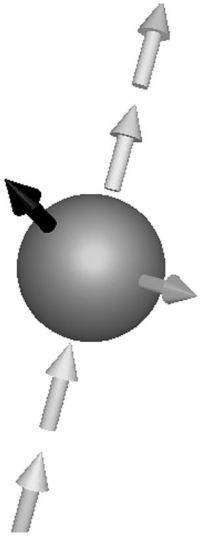

FIG. 2. A depiction of the effective dipole moment that develops for a rotating sphere in the presence of the coefficients for Lorentz violation $\vec{s}$. The dipole moment (medium gray arrow) is proportional to the cross product of $\vec{s}$ (light gray arrows) with the angular momentum of the sphere (dark arrow).

$$\begin{aligned}
U_E &= \frac{Q}{4\pi r^2}, \\
U_E^{jk} &= \frac{Q\hat{x}^j\hat{x}^k}{4\pi r^2} + \frac{I_E}{12\pi r^3}(\delta^{jk} - 3\hat{x}^j\hat{x}^k), \\
V_E^j &= \frac{I_E \epsilon^{jkl}\omega^k\hat{x}^l}{12\pi r^2}, \\
X_E^{jkl} &= 3V_E^j\left[\hat{x}^k\hat{x}^l\left(1 - \frac{I_E'}{I_E r^2}\right) + \frac{I_E'\delta^{kl}}{5I_E r^2}\right] \\
&\quad + \frac{I_E(\epsilon^{jkm}\hat{x}^l\omega^m + \epsilon^{jlm}\hat{x}^k\omega^m)}{12\pi r^2}\left(1 - \frac{3I_E'}{5I_E r^2}\right),
\end{aligned} \tag{60}$$

where I_E' is a spherical moment given by the integral in Eq. (56) with $|\vec{x}|^4$ instead of $|\vec{x}|^2$. Using these expressions it is straightforward to calculate the associated electric and magnetic fields as well as the gravitoelectric and magnetic fields. The expressions are lengthy and omitted here.

C. Applications

A full analysis of the dominant observable effects in gravitational experiments and observations has been performed in Ref. [19]. However, the coefficients were analyzed collectively and the separation of various distinct Lorentz-violating effects was not fully studied. Here we focus specifically on the observability of the novel gravitomagnetic force shown to arise in the point-mass example in Sec. V A and illuminate its role in a key test.

Lunar laser ranging and atom interferometry have measured 8 of the 9 coefficients in $\bar{s}^{\mu\nu}$ and the combined results are tabulated in Ref. [22]. These results are reported in the standard Sun-centered celestial-equatorial frame, where coordinates are denoted with capital letters for clarity. In this frame, the current constraints on $\bar{s}^{JK}$ are at the 10^{-9} level. For $\bar{s}^{TJ}$, the constraints are at the weaker level of $10^{-6} - 10^{-7}$. The gravitomagnetic force due to the effective gravitomagnetic field in the second of Eqs. (53) is controlled by the $\bar{s}^{TJ}$ coefficients. This force has been measured by both lunar laser-ranging and, effectively, atom interferometry. However, its specific effects are most easily discernable in orbital tests such as the lunar laser-ranging scenario, so we focus on this case.

The principle effects from the $\bar{s}^{TJ}$ coefficients for lunar laser ranging are modifications to the relative acceleration of the Earth and Moon. This acceleration includes such terms as the gravitomagnetic terms considered in Eqs. (53). In fact, from the results in Ref. [19], one can read off the portion of the Earth-Moon acceleration δa^J responsible for the effective force that is described in Fig. 1. In the Sun-centered celestial-equatorial frame coordinates, it reads

$$\delta a^J = \frac{2G\delta m}{r^3} v^K(\bar{s}^{TK} r^J - \bar{s}^{TJ} r^K), \tag{61}$$

where δm is the mass difference between the Earth and Moon, r^J is the coordinate difference between the Earth

and Moon center of mass positions, and v^J is their relative coordinate velocity.

The dominant observable effects from Eq. (61) are oscillations in the lunar range at the mean lunar orbital freqeuncy ω. In the lunar laser-ranging scenario, these oscillations are controlled by two linear combinations of the $\bar{s}^{TJ}$ coefficients called $\bar{s}^{01}$ and $\bar{s}^{02}$, which are expressed in the mean orbital plane of the lunar orbit. These two quantities control the size of the Lorentz-violating gravitomagnetic force for this case. Using over three decades of lunar laser-ranging data, analysis reveals that $\bar{s}^{01} = (-0.8 \pm 1.1) \times 10^{-6}$ and $\bar{s}^{02} = (-5.2 \pm 4.8) \times 10^{-7}$ [20]. Therefore there is no compelling evidence for the gravitomagnetic force controlled by $\bar{s}^{TJ}$ coefficients. However, ongoing tests such as the Apache Point Observatory Lunar Laser-Ranging Operation have already improved on lunar ranging capability and could significantly improve sensitivity to this effect [46].

VI. SUMMARY

In this work we have shown that an analogy exists between the gravitational sector and the electromagnetic sector of the SME at two levels. First we showed that in the stationary limit and for a particular coordinate choice, part of the post-Newtonian metric $h_{0\mu}$ in the gravity sector can be obtained from the vector potential A_μ in the electromagnetic sector by essentially making a series of substitutions, most notably the exchange of the coefficients $C^{\mu\nu} \to \bar{s}^{\mu\nu}$, as outlined in Sec. III B. For the equations of motion of a test body, the gravitational case was shown to resemble the electromagnetic Lorentz-force law, so long as nonlinear terms in the geodesic equation are disregarded.

In Sec. V, we provided two examples of how the mixing of electrostatics and magnetostatics in Lorentz-violating electrodynamics has an analog in the gravitational case. In the same manner as a point charge produces a magnetic field in the presence of the electromagnetic coefficients C^{0j}, we showed that a point mass will produce a gravitomagnetic field controlled by the coefficients $\bar{s}^{0j}$. Similarly, we also explored the converse of this example, demonstrating that a moving mass produces an additional gravitoelectric field. We also discussed the observability of the gravitomagnetic force controlled by the $\bar{s}^{0j}$ in lunar laser-ranging tests.

Several areas are open for future investigation. One possibility is to systematically isolate the GEMS mixing effects from others in the various predicted signals for Lorentz violation in gravitational experiments [19], along the lines of the discussion in Sec. V C. It also would be interesting to investigate whether any analogy is possible in the presence of the matter sector coefficients that play a role in gravitational experiments [28]. Furthermore, using a method similar to the one developed in this paper, it may be possible to extend the class of signals for Lorentz violation by looking for gravitational analogs of the nonminimal electromagnetic sector of the SME, which goes beyond the minimal $(k_F)^{\kappa\lambda\mu\nu}$ coefficients [13].

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