A Low-complexity Algorithm to Determine Spacecraft Trajectories

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Joint work with - David Canales, Atahan Kurttisi, and Brian Baker-McEvilly

A Colloquium Series
Department of Mathematics
College of Arts and Sciences

Outline

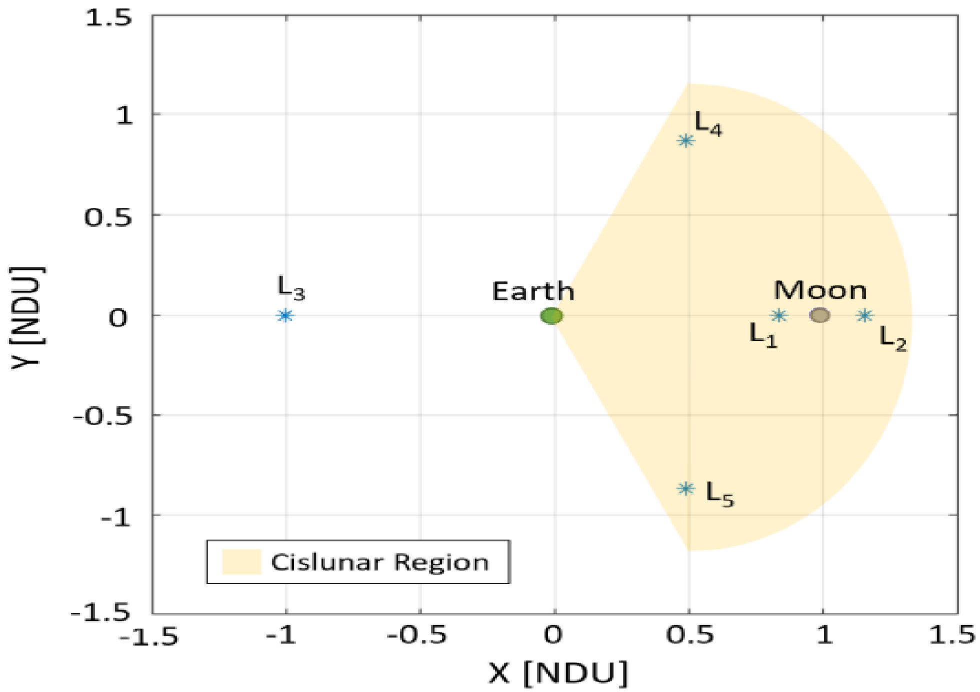


Fig. 1: Cislunar Region.

- **Background**
- **Spacecraft Trajectories for the CR3BP**
- **Algorithm for Spacecraft Trajectories**
- **Numerical Simulations**

Needless to Determine Spacecraft Trajectories

- **The Cislunar region** is gaining increased attention throughout the past few years, **as 90 missions to the Moon are projected by 2030** with additional missions to Mars.
- As **traffic in the Cislunar region** continues to grow, **efficient methods to propagate trajectories**, win the circular restricted three-body problem (CR3BP) are required.
- Many upcoming Cislunar missions are focused on the Lunar South pole, as well as **periodic orbits about L$_1$ and L$_2$** of the **Earth-Moon CR3BP** system.
- Russia's Luna 25 (2022), South Korea's KPLO (2022), Japan and India's joint LUPEX (2023), and India's Chandrayaan-3 (2023) all are **missions** to **observe or land on polar regions of the Moon**.
- **NASA's Artemis program** is a **multi-stage program** to reestablish presence on the **Moon**.
- **The dynamics** that govern the motion in the **CR3BP are highly non-linear and no exact solution** has yet been derived.
- To be able to **design trajectories** in such a model, **different numerical methods:** Gauss-Legendre, Dormand-Prince, and Chebyshev-Picard, Gragg-Bulirsch-Stoer, Adams-Bashforth, or Runge-Kutta integrator, are required to plan space trajectories that satisfy desired behaviors.

A solution is proposed for the well-known CR3BP to determine trajectories via a low-complexity algorithm.

Exploring Structures

- To produce **computationally tractable and inexpensive algorithm**, particularly in the **complex and chaotic three-body dynamics**, it is important to address system structures in relevant equations, development of **novel theories**, and design **low-complexity and reliable algorithm**.
- Many problems in applied sciences and engineering can be reduced to **linear algebra problem**.
- Standard methods may not be practical due to **large dense matrices**.
- **Exploiting the structure lead what???**
- **Speed**

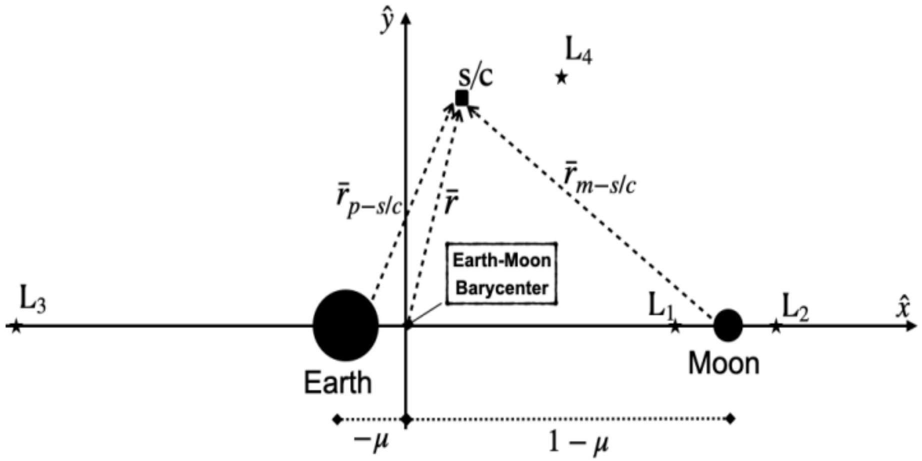
Standard algorithms	$\mathcal{O}(n^3)$ arithmetic operations
Fast algorithms	$\mathcal{O}(n^2)$
Faster algorithms	$\mathcal{O}(n\, log(n))$
Superfast algorithms	$\mathcal{O}(nlog(log(n)))$

Obtain Periodic Orbits in the CR3BP

- The **evolution of a spacecraft (s/c) position** $\bar{r}_{rot} = [x, y, z]^T$ and **velocity** $\dot{\bar{r}}_{rot} = [\dot{x}, \dot{y}, \dot{z}]^T$ is governed by the following equations of motion:

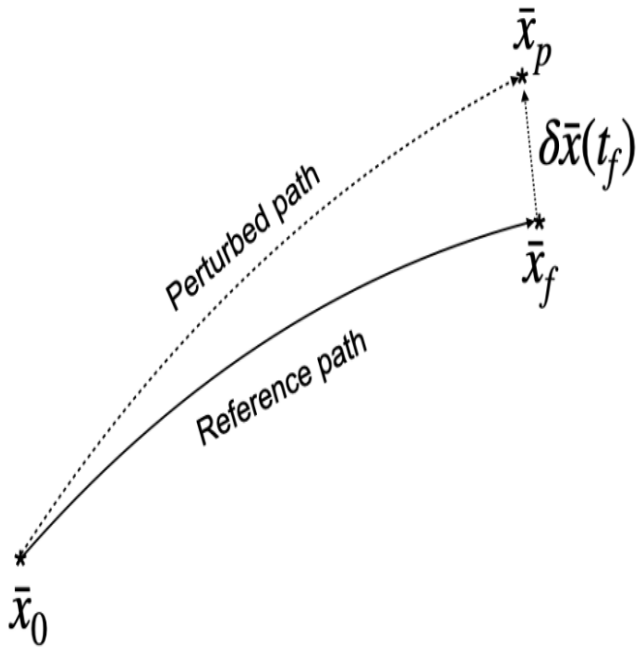
$$\ddot{x} - 2\dot{y} = \frac{\partial U^*}{\partial x}; \; \ddot{y} + 2\dot{x} = \frac{\partial U^*}{\partial y}; \; \ddot{z} = \frac{\partial U^*}{\partial z}$$

where $U^* := \frac{1-\mu}{r_{p-s/c}} + \frac{\mu}{r_{m-s/c}} + \frac{1}{2}(x^2 + y^2)$ represents the pseudo-potential function, mass ratio $\mu = m_M/(m_M + m_E)$ is defined for the system, with m_M and m_E being the masses of the Moon and the Earth, and $r_{p-s/c}$ and $r_{m-s/c}$ are the distances of the s/c to the Earth and the Moon, respectively.



Obtain Periodic Orbits in the CR3BP

- **Spacecraft trajectory design** is mostly based on **numerical strategies: differential corrections** to find trajectories that satisfy specific purposes in different models.
- It is important to find a **correlation** between the variations in the **initial state of a trajectory**, $\delta\bar{x}_0$, with the variations of its **final states**, $\delta\bar{x}_f$, where $\delta\bar{x}(t) = [\delta x, \delta y, \delta z, \delta\dot{x}, \delta\dot{y}, \delta\dot{z}]^T$



- **State Transition Matrix (STM)** is a variable sensitivity matrix and very useful for targeting schemes and stability analysis: $\phi(t_f, t_0) = \frac{\partial\bar{x}(t_f)}{\partial\bar{x}(t_0)}$

Obtain Periodic Orbits in the CR3BP

- The **linear variational equations**, derived from the equations of motion, are provided in the form:

$$\delta\dot{\bar{x}}(t) = \mathbf{A}(t)\delta\bar{x}(t),$$

where $\mathbf{A}(t)$ is the Jacobian matrix comprised of the partials of the equations of motion with respect to the states evaluated at the time t: $\mathbf{A}(t) = \frac{\partial \bar{f}(\bar{x},t)}{\partial \bar{x}(t)}$ and $\dot{\bar{x}} = \bar{f}(\bar{x}, t)$.

- The evolution of $\phi(t, t_0)$ is governed by the following matrix differential equation:

$$\dot{\phi}(t, t_0) = \mathbf{A}(t)\phi(t, t_0)$$

- In the CR3BP,

$$\begin{pmatrix} \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \\ \delta\ddot{x} \\ \delta\ddot{y} \\ \delta\ddot{z} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ U^*_{xx} & U^*_{xy} & U^*_{xz} & 0 & 2 & 0 \\ U^*_{xy} & U^*_{yy} & U^*_{yz} & -2 & 0 & 0 \\ U^*_{xz} & U^*_{yz} & U^*_{zz} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \\ \delta\dot{x} \\ \delta\dot{y} \\ \delta\dot{z} \end{pmatrix}$$

Periodic Orbits via a Multi-variable Newton-Raphson

- Consider that $\bar{\mathcal{Y}}$ contains free variables and the constraints $\bar{\mathcal{F}}(\bar{\mathcal{Y}})$ are defined via $\bar{\mathcal{Y}} = \begin{pmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \\ \vdots \\ \mathcal{Y}_n \end{pmatrix}$ and $\bar{\mathcal{F}}(\bar{\mathcal{Y}}) = \begin{pmatrix} \mathcal{F}_1(\bar{\mathcal{Y}}) \\ \mathcal{F}_2(\bar{\mathcal{Y}}) \\ \vdots \\ \mathcal{F}_n(\bar{\mathcal{Y}}) \end{pmatrix}$, respectively.
- The goal is to find $\bar{\mathcal{Y}}$ that makes the vector of constraints null: $\bar{\mathcal{F}}(\bar{\mathcal{Y}}) = \bar{0}$ with accuracy: $|\bar{\mathcal{F}}| < \epsilon$ where $\epsilon = 10^{-12}$.
- Update the NR algorithm

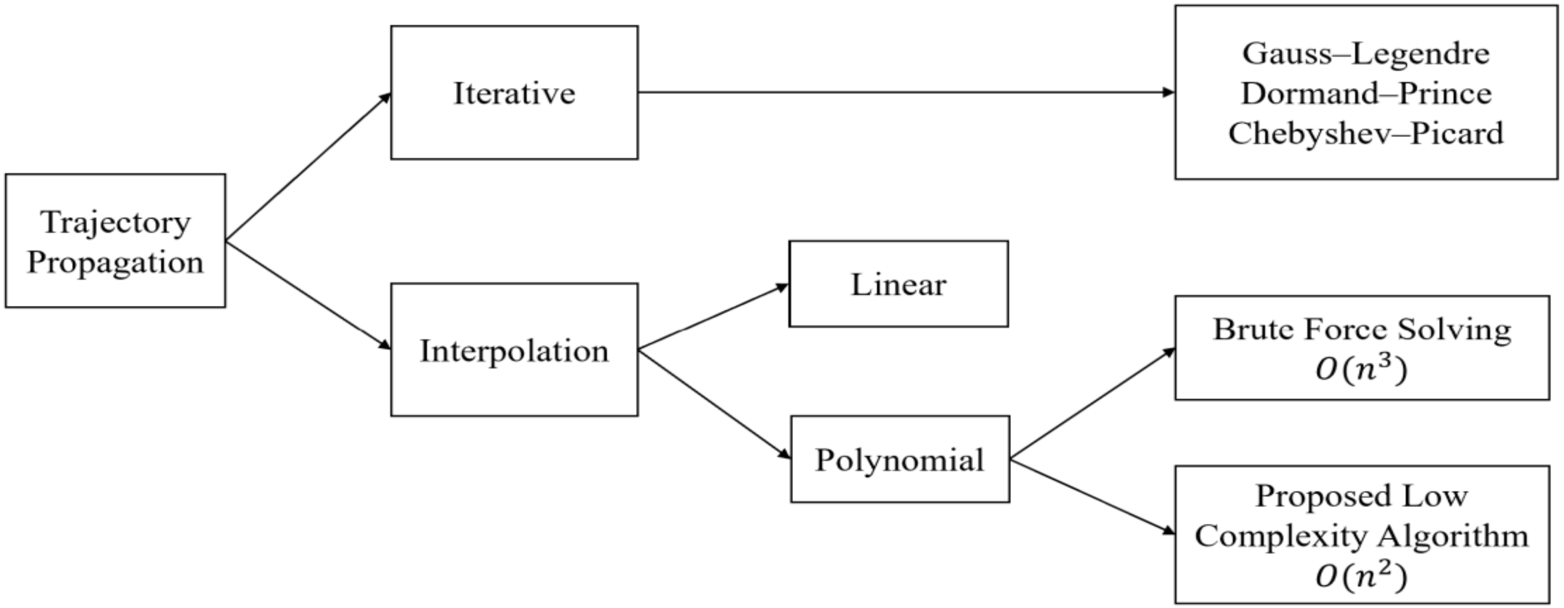
$$\bar{\mathcal{Y}} = \bar{\mathcal{Y}}_0 - D\bar{\mathcal{F}}(\bar{\mathcal{Y}}_0)^T [D\bar{\mathcal{F}}(\bar{\mathcal{Y}}_0) D\bar{\mathcal{F}}(\bar{\mathcal{Y}}_0)^T]^{-1} \bar{\mathcal{F}}(\bar{\mathcal{Y}})$$

where $n \neq j$ and

$$D\bar{\mathcal{F}}(\bar{\mathcal{Y}}_0) = \frac{\partial \bar{\mathcal{F}}(\bar{\mathcal{Y}})_0}{\partial \bar{\mathcal{Y}}_0} = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial \mathcal{Y}_1} & \frac{\partial \mathcal{F}_1}{\partial \mathcal{Y}_2} & \cdots & \frac{\partial \mathcal{F}_1}{\partial \mathcal{Y}_n} \\ \frac{\partial \mathcal{F}_2}{\partial \mathcal{Y}_1} & \frac{\partial \mathcal{F}_2}{\partial \mathcal{Y}_2} & \cdots & \frac{\partial \mathcal{F}_2}{\partial \mathcal{Y}_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{F}_j}{\partial \mathcal{Y}_1} & \frac{\partial \mathcal{F}_j}{\partial \mathcal{Y}_2} & \cdots & \frac{\partial \mathcal{F}_j}{\partial \mathcal{Y}_n} \end{bmatrix}_0 .$$

Comparison between Iterative and Proposed Trajectory Propagation Methods

Interpolation	Iterative
Continuous	Discrete
Best-fit Solution	Approximated Solution
Low-complexity	Computationally expensive



Proposed Spacecraft Trajectories

- Assume that the positions, velocities, and accelerations of the spacecraft at n time intervals are known and given via $(t_i, \bar{x}(t_i))$, $(t_i, \dot{\bar{x}}(t_i))$, and $(t_i, \ddot{\bar{x}}(t_i))$ for $i = 0, 1, \cdots, n$ and $t_0 < t_1 < \cdots < t_n$, where the vectors $\bar{x}(t_i)$, $\dot{\bar{x}}(t_i)$, and $\ddot{\bar{x}}(t_i)$ are in $\mathbb{R}^3$ evaluated at each t_i.
- The equations of motion or **the trajectories of the spacecraft** over n time intervals are described via piecewise-defined functions on the interval $[t_k, t_{k+1}]$ from $\mathbb{R}^3$ to $\mathbb{R}$ such that

$$G_k(x(t)) = g_{0,k} + g_{1,k}t + g_{2,k}t^2 + g_{3,k}t^3 + g_{4,k}t^4 + g_{5,k}t^5,$$

where $t_k \leqslant t \leqslant t_{k+1}$, $k = 0, 1, \cdots, n-1$, and $g_{0,k}, g_{1,k}, \cdots, g_{5,k}$ are constants that depend on the vectors $\bar{x}(t)$, $\dot{\bar{x}}(t)$, and $\ddot{\bar{x}}(t)$.

Arithmetic complexity of the brute-force calculation cost (n^3) operations.

Proposed Spacecraft Trajectories

At the time interval $[t_k, t_{k+1}]$

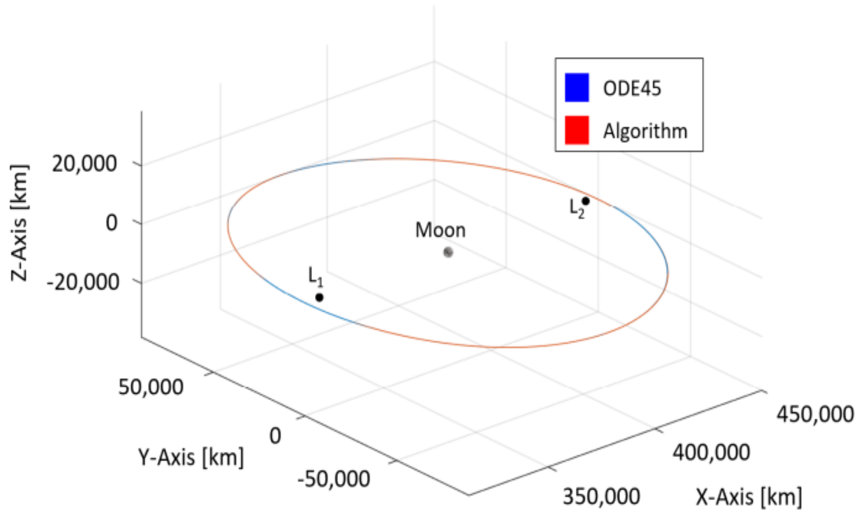
$$\underbrace{\begin{bmatrix} 1 & t_k & t_k^2 & t_k^3 & t_k^4 & t_k^5 \\ 1 & t_{k+1} & t_{k+1}^2 & t_{k+1}^3 & t_{k+1}^4 & t_{k+1}^5 \\ 0 & 1 & 2t_k & 3t_k^2 & 4t_k^3 & 5t_k^4 \\ 0 & 1 & 2t_{k+1} & 3t_{k+1}^2 & 4t_{k+1}^3 & 5t_{k+1}^4 \\ 0 & 0 & 2 & 6t_k & 12t_k^2 & 20t_k^3 \\ 0 & 0 & 2 & 6t_{k+1} & 12t_{k+1}^2 & 20t_{k+1}^3 \end{bmatrix}}_{A_k} \underbrace{\begin{bmatrix} g_{0,k} \\ g_{1,k} \\ g_{2,k} \\ g_{3,k} \\ g_{4,k} \\ g_{5,k} \end{bmatrix}}_{\underline{x}_k} = \underbrace{\begin{bmatrix} x(t_k) \\ x(t_{k+1}) \\ \dot{x}(t_k) \\ \dot{x}(t_{k+1}) \\ \ddot{x}(t_k) \\ \ddot{x}(t_{k+1}) \end{bmatrix}}_{\underline{b}_k}$$

$$\left(\prod_{r=1}^{5} L_r\right) U_k \underline{x}_k = \underline{b}_k, \quad \text{where} \quad A_k = \left(\prod_{r=1}^{5} L_r\right) U_k$$

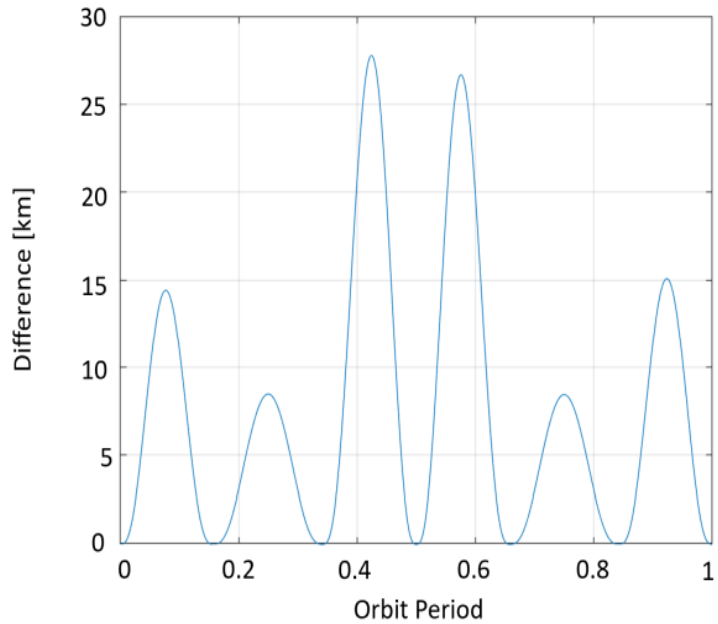
where $L_r \in \mathbb{R}^{6\times 6}$, $r = 1, 2, .., 5$ for bidiagonal lower triangular matrices, and $U_k \in \mathbb{R}^{6\times 6}$ is an upper triangular matrix.

The algorithm cost $\mathcal{O}(n^2)$ as opposed to $\mathcal{O}(n^3)$ complexity.

Distant Retrograde Orbit



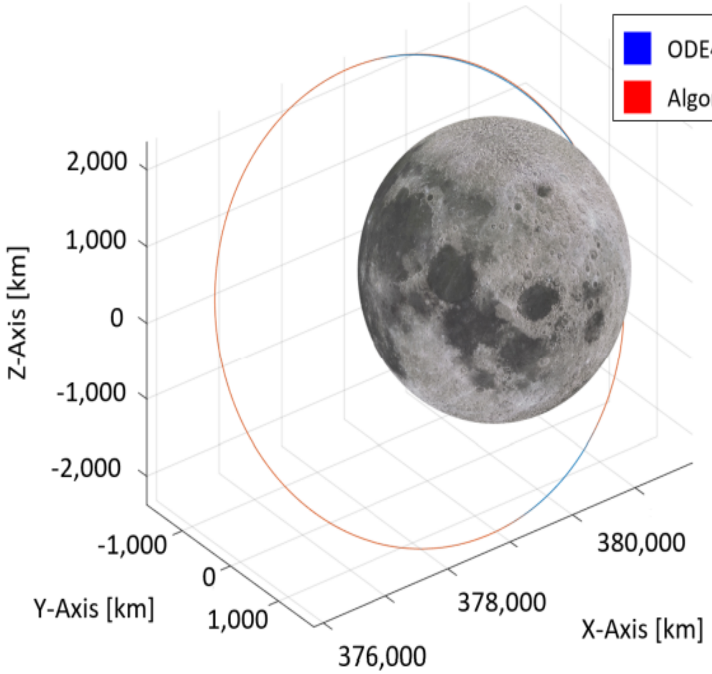
(a) Propagated DRO trajectory.



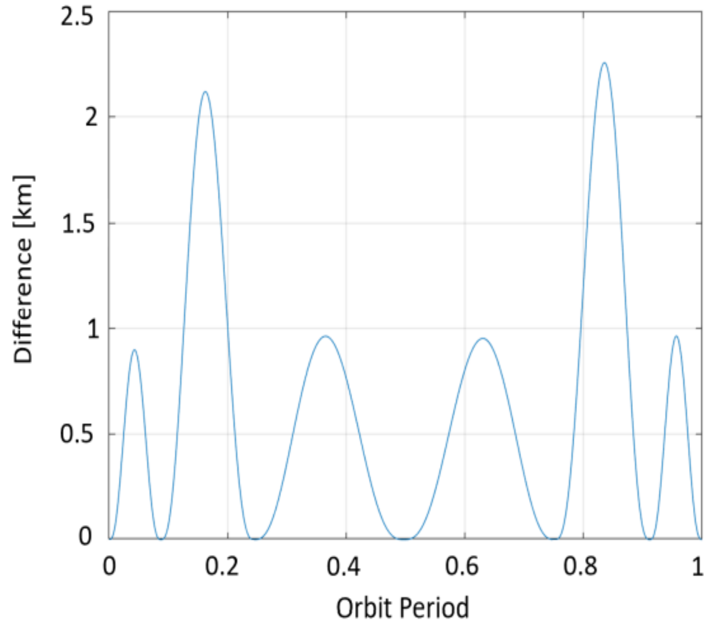
(b) Difference from LCA to ODE45.

The algorithm matches the ODE45 resolved trajectory extremely closely, deviating up to 27 kilometer. A difference of 27 kilometers between ODE45 and the LCA is insignificant in reference to the 100,000 kilometers the DRO stretches across.

Low-Lunar Orbit (LLO) Analysis



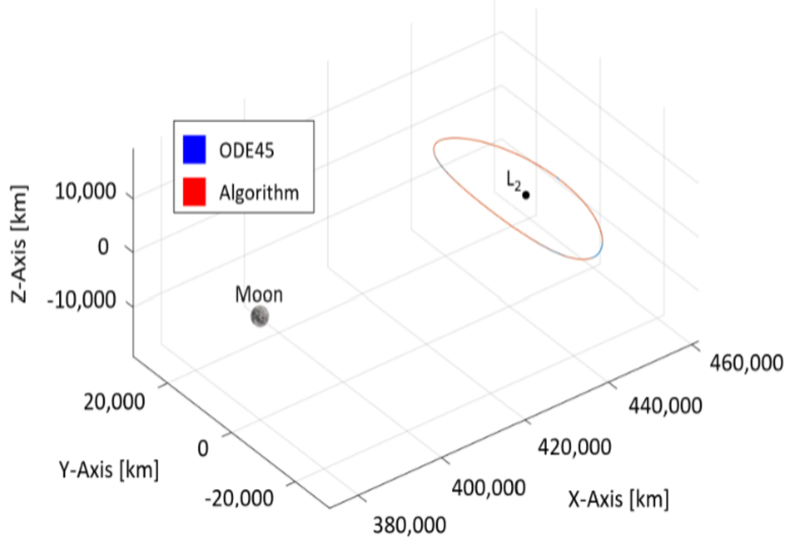
(a) Propagated LLO trajectory.



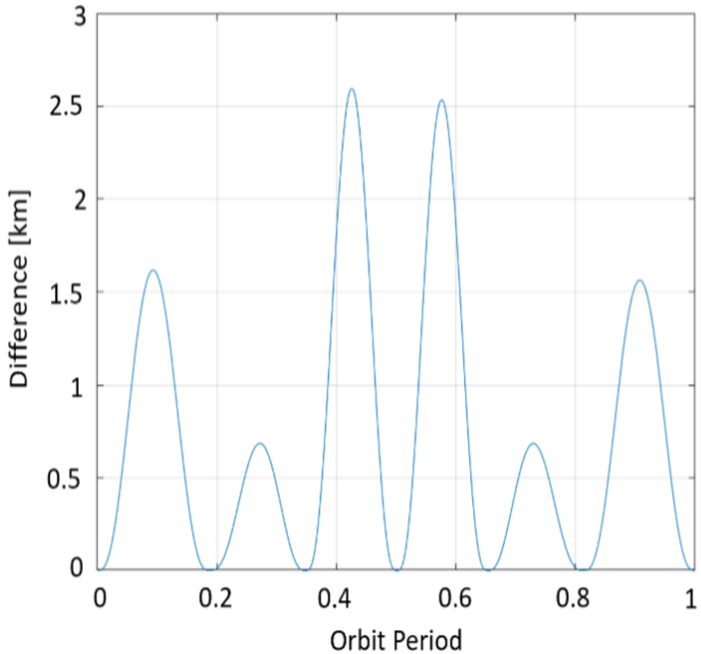
(b) Difference from LCA to ODE45.

The LLO has a peak difference of 2.3 kilometers occurring in the fifth time interval of the algorithm. The 2.3 kilometers difference is also insignificant in reference to the 4,000 kilometers the LLO stretches across.

L_2 Lyapunov Analysis



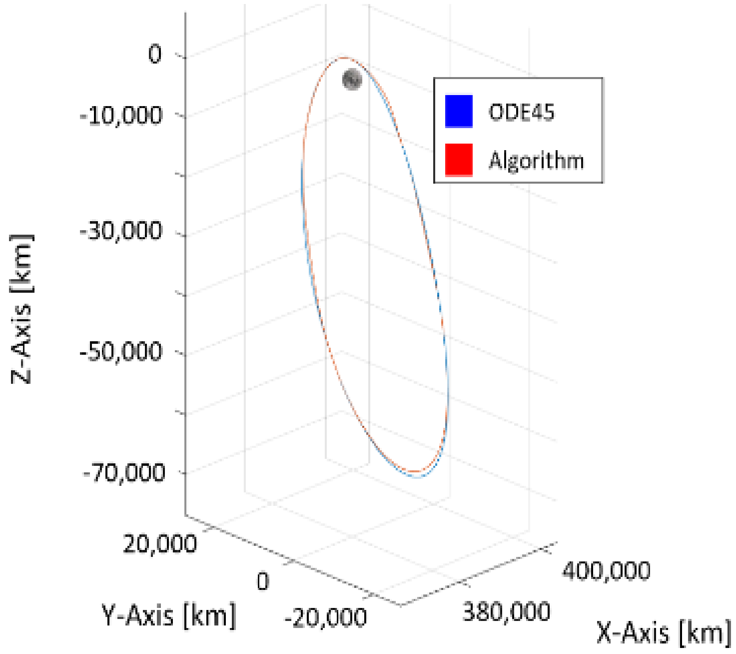
(a) Propagated Lyapunov trajectory.



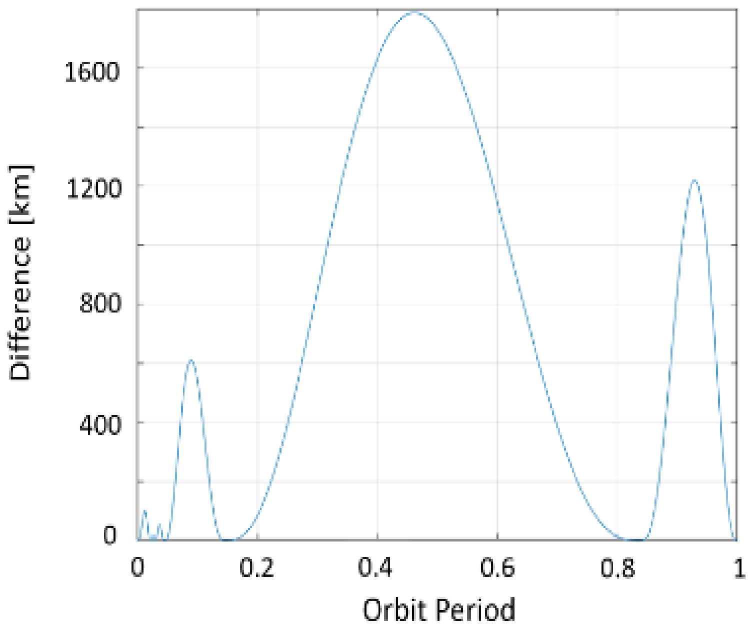
(b) Difference from LCA to ODE45.

The peak difference occurs in the third time interval set by the boundary conditions, only reaching right over 2.5 kilometers. The LCA's model of the Lyapunov orbit has the closest resemblance to the ODE45 trajectory out of any of all the tested orbits.

Near-rectilinear Halo Orbit



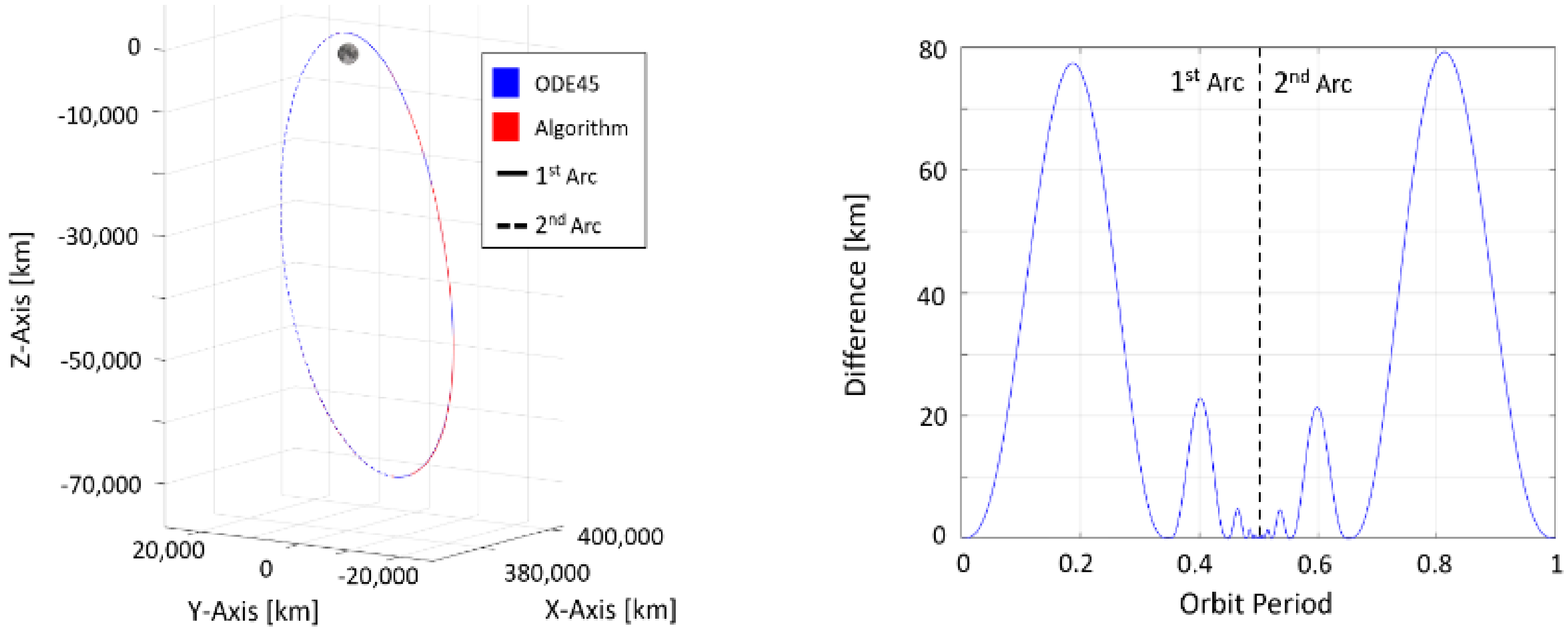
(a) Propagated NRHO trajectory.



(b) Difference from LCA to ODE45.

A challenge occurred in reconstructing NRHO using the LCA.

Near-rectilinear Halo Orbit

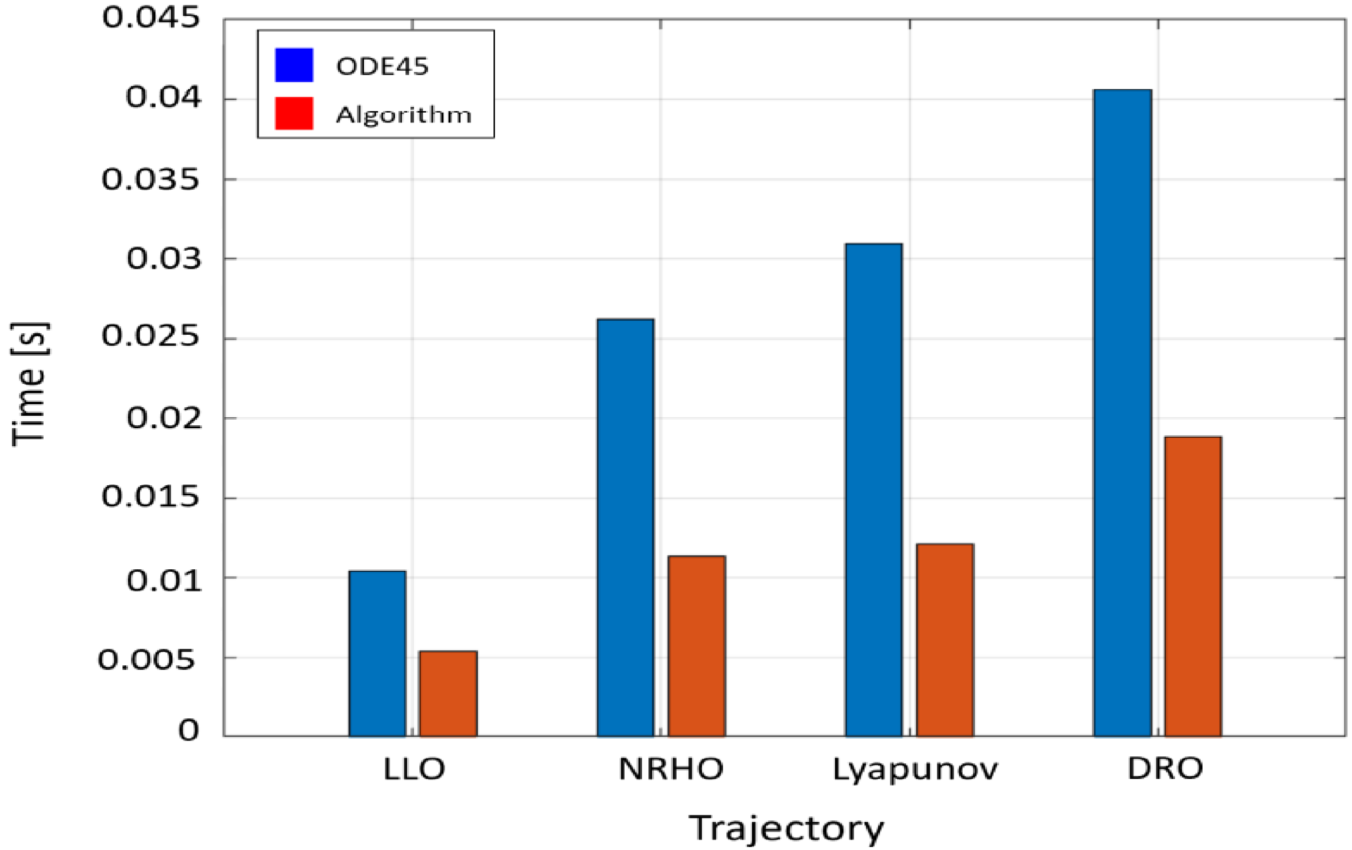


(a) Two arc NRHO trajectory.

(b) Difference from LCA to ODE45.

Breaking the trajectory into two arcs enables the maximum difference between the LCA and ODE45 propagation drops from 1800 kilometers to 78 kilometers.

Time Complexity Analysis



Average computational time of 100 propagations of each respective trajectory depicts that the algorithm resolves the trajectory significantly faster than ODE45, clocking in at about half of the time for each orbit.

A Low-complexity Spacecraft Trajectories

- **Spacecraft Trajectories in the Cislunar Region**
- **A Low arithmetic-complexity Algorithm**
- **A Low time-complexity Algorithm**
- **Numerical Simulations for Accuracy**
- **Multiple Spacecraft Trajectories for Future Missions**

The First Female Recognition in the Flight Research Division, NASA

Katherine Johnson (1918 - 2020)

Johnson was an iconic woman of color in STEM. Her mathematical work helped NASA's first crewed spaceflight land on the moon in 1969.

Visiting Scholar Positions

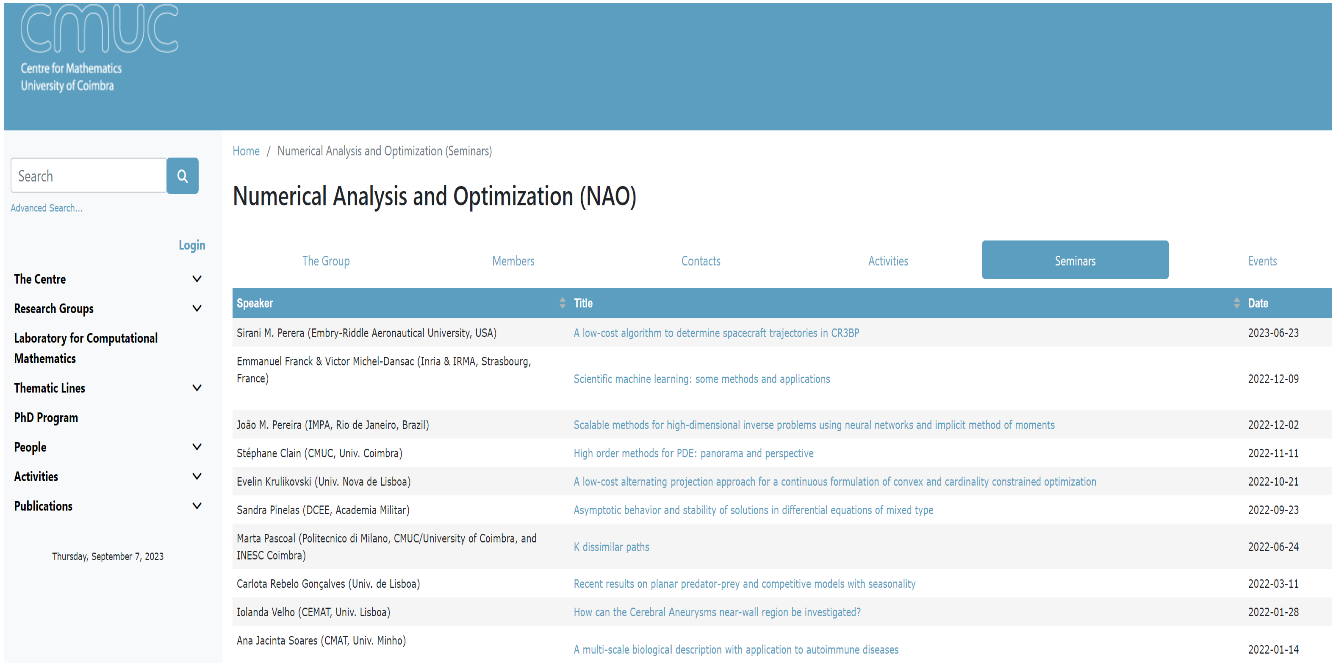
- **Department of Physics and Computer Science, Wilfrid Laurier University, Canada, Oct 2022.**
  - Invited Talk 1: *Utilizing Communication and Sensor Arrays to Navigate Unmanned Aerial Systems*, Teledyne FLIR, Canada
  - Invited Talk 2: *Mathematics is the Centerpiece of Science, Engineering, and Technology*, Laurier Centre for Women in Science (WinS), Canada
  - Invited Talk 3: *A Low-Complexity Algorithm in Phased-array Digital Receivers*, Wilfrid Laurier University, Canada
  - Paper 1: *A Low-complexity Algorithm to Search Legendre Pairs* by Sirani M. Perera and Ilias Kotsireas, submit to Linear Algebra and Its Application, Elsevier, 2023

Visiting Scholar Positions

- **Department of Mathematics, University of Northern Iowa, Iowa, USA, Oct 2022.**
  - Invited Talk 4: *A Low-Complexity Algorithm to Uncouple the Mutual Coupling Effect*
  - Co-PI of the NSF Award 2322922: *Conference: Exchange of Mathematical Ideas Conference 2023*
    Awarded $30,000.00 by the Division of Mathematical Sciences in the NSF, USA, May 2023-May 2024
  - Organizing Committee Member: *Exchange of Mathematical Ideas Conference 2023*
  - Invited Talk 5: *A Low-cost Algorithm to Determine Orbital Trajectories within the Cislunar Region*

Visiting Scholar Positions

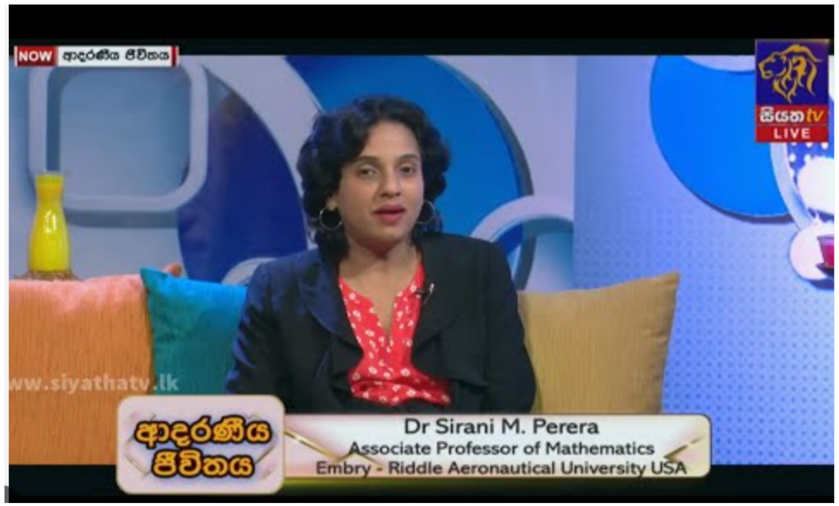
- **Department of Mathematics, University of Coimbra, Portugal, June 2023.**
  - Invited Talk 5: *A low-cost algorithm to determine spacecraft trajectories in CR3BP*,
  - Paper 2: *A Low-cost and Numerically Stable Algorithm to Solve Tridiagonal Systems via Quasiseparable Matrices*, Sirani M. Perera and Natalia Bebianos, submitted to Numerical Algorithms, Springer Nature, 2023



Speaker	Title	Date
Sirani M. Perera (Embry-Riddle Aeronautical University, USA)	A low-cost algorithm to determine spacecraft trajectories in CR3BP	2023-06-23
Emmanuel Franck & Victor Michel-Dansac (Inria & IRMA, Strasbourg, France)	Scientific machine learning: some methods and applications	2022-12-09
João M. Pereira (IMPA, Rio de Janeiro, Brazil)	Scalable methods for high-dimensional inverse problems using neural networks and implicit method of moments	2022-12-02
Stéphane Clain (CMUC, Univ. Coimbra)	High order methods for PDE: panorama and perspective	2022-11-11
Evelin Krulikovski (Univ. Nova de Lisboa)	A low-cost alternating projection approach for a continuous formulation of convex and cardinality constrained optimization	2022-10-21
Sandra Pinelas (DCEE, Academia Militar)	Asymptotic behavior and stability of solutions in differential equations of mixed type	2022-09-23
Marta Pascoal (Politecnico di Milano, CMUC/University of Coimbra, and INESC Coimbra)	K dissimilar paths	2022-06-24
Carlota Rebelo Gonçalves (Univ. de Lisboa)	Recent results on planar predator-prey and competitive models with seasonality	2022-03-11
Iolanda Velho (CEMAT, Univ. Lisboa)	How can the Cerebral Aneurysms near-wall region be investigated?	2022-01-28
Ana Jacinta Soares (CMAT, Univ. Minho)	A multi-scale biological description with application to autoimmune diseases	2022-01-14

Junior Consultant

- **Faculty of Engineering, University of Sri Jayewardenepura, Sri Lanka, Jan 2023 - May 2023.**
  - TV Broadcast: *Importance of Convergence in Research and Addressing the Objectives of the "Symposium on Interlacing Engineering Research"*
  - Academia-based Symposium: *Interlacing Engineering Research* funded by the Asian Development Bank (ADB) through award number R1/SJ/02 with the amount Rs. 315,000.00.
  - Industry-based Symposium: *Mezclair:* Bridging the Gap Between the University System and Industries sponsored by 14 Industries, including, SYNOPSYS, Hayleys, Unilever, gap HQ, MAS, OREL, SULECO, TRONIC.LK with the amount $\approx$ Rs. 1,000,000.00.
  - Invited Talk 6: *Utilizing Unmanned Aerial Systems to Tackle Climate-Induced Challenges*



International Linear Algebra Society

- **Minisymposium:** *Numerical linear algebra applications in data science* by James Nagy and Sirani M. Perera
- **Minisymposium:** *State-of-the-art in algorithms and applications* by Sirani M. Perera and Natalia Bebiano
- **Talk 1:** *A Low-complexity Algorithm in Navigating Unmanned Aerial Systems* by Sirani M. Perera
- **Talk 7:** *Structured Matrices Approach for Legendre Pairs* by Ilias Kotsireas
- **Talk 8:** *A Vandermonde Neural Operator: Extending the Fourier Neural Operator to Nonequispaced Distributions* by Levi Lingsch

Other NSF Work

- **PI** of the NSF award entitled Collaborative Research: SWIFT: AI-based Sensing for Improved Resiliency via Spectral Adaptation with Lifelong Learning - NSF award number 2229473
  - Paper 3: *A Low-complexity Algorithm to Digitally Uncouple the Mutual Coupling Effect in Antenna Arrays* by Sirani M. Perera, Levi Lingsch, Arjuna Madanayake, and Leonid Belostotski, submitted to Journal of Computational and Applied Mathematics, 2023
  - Paper 4: *Vandermonde Neural Operator* by Levi Lingsch, Mike Michelis, Sirani M. Perera, Sirani M. Perera, Robert K. Katzschmann, Siddartha Mishra, 2023
  - Poster 1: *AI-based Sensing for Improved Resiliency* by Arjuna Madanayake, Sirani M. Perera, Houbing Song, and Francesco Restuccia, NSF Spectrum Week, NSF, 2023
- **Co-PI** of the NSF award entitled Distributed Learning for Undergraduate Programs in Data Science at Diverse Universities - NSF award number 2142514
  - Talk 9: *Learn Linear Algebra to Familiarize with Deep Learning*, 2023
  - Videos: 30 conceptual glass door filming for mathematical modeling and simulations
- **Mentor** and **Senior Personnel** of the award entitled REU Site: Swarms of Unmanned Aircraft Systems in the Age of AI/Machine Learning - NSF award number 2150213
  - Paper 5: *Multi-beam Beamforming-based ML Algorithm to Optimize the Routing of Drone Swarms* by Rodman J. Myers, Sirani M. Perera, Grace McLewee, David Huang, and Houbing Song, submitted to ACM Transactions on Autonomous and Adaptive Systems, 2023
- **The Convergence Research (CORE) Fellow** in Tackling Climate-Induced Challenges with AI, NSF Convergence Accelerator, USA

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Paper 6: David Canales, Sirani M. Perera, Atahan Kurttisi, and Brian Baker-McEvilly, A Low-Complexity Algorithm to Determine Trajectories within the Circular Restricted Three-Body Problem, in the review of the Journal of the Astronautical Sciences, Springer Nature, (2023)