Controllability of Open Quantum Optical Systems: Photon Fock States in a Cavity

Byron Henry Lowry

Embry-Riddle Aeronautical University - Daytona Beach

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CONTROLLABILITY OF OPEN QUANTUM OPTICAL SYSTEMS:
PHOTON FOCK STATES IN A CAVITY

BY
BYRON HENRY LOWRY

A Thesis
Submitted to the Department of Physical Sciences
and the Committee on Graduate Studies
In partial fulfillment of the requirements
for the degree of
Master in Science in Engineering Physics

08/2013
Embry-Riddle Aeronautical University
Daytona Beach, Florida
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Byron Henry Lowry

This thesis was prepared under the direction of the candidate’s thesis committee chair, Dr. Bereket Berhane, Department of Physical Sciences, and has been approved by the members of the thesis committee. It was submitted to the Department of Physical Sciences and was accepted in partial fulfillment of the requirements of the Degree of Master of Science in Engineering Physics

THESIS COMMITTEE:

Dr. Bereket Berhane, Chair

Dr. Sergey Drakunov, Member

Dr. Gregory Spradlin, Member

Dr. Peter Erdman, MEEP Graduated Program Coordinator

Dr. Terry Oswalt, Department Chair, Physical Sciences

Dr. Robert Oxley, Associate V.P. for Academics
Abstract

There has always been a significant interest in using optical systems to control quantum phenomena. A major barrier to controllability of quantum optical systems is the fact that the systems are usually infinite dimensional open systems, two cases which have mostly negative controllability results. This thesis develops three new definitions of controllability and reformulates a previous controllability theorem in order to apply the theorem to the system of interest. Then, the controllability of a pumped dissipative quantum optical cavity with engineered decoherence is investigated using previously developed concepts in quantum control theory, as well as the ones developed in this thesis. Positive controllability results were found for Finite Ensemble Population Controllability and Weak Observable Controllability, while Finite Density Matrix Controllability had a negative result. A result for Strong Observable controllability remains elusive, as the normal nonlinear controllability theorems are not applicable to the system.
Acknowledgments

First, I would like to thank Dr. Bereket Berhane, whose invaluable guidance, instruction, and countless office hours over the past few years has made this research possible. I would also like to thank Dr. Sergey Drakunov and Dr. Gregory Spradlin for being members of my thesis committee, and for fielding questions in their fields of expertise enhancing my understanding of the subject matter. I would also like to thank my parents whose endless support for my higher education makes all of this possible. Last but not least I would like to thank my wonderful girlfriend Keri Younger, for being an unshakeable anchor who kept me alive and sane throughout the work.
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Chapter 1

Introduction

With the advent of quantum information and devices quickly approaching length scales on the order of $1 - 10nm$, the need for a complete understanding of quantum control theory is paramount. This opinion is echoed by Dowling and Milburn, who stated that "The development of the general principles of quantum control theory is an essential task for a future quantum technology" [8]. Recent history has seen major advances in the field of quantum control theory, with quantum control methods gaining a stable role in various experimental settings [3], and capstone achievements of the first commercial quantum computer and the realization of continuous feedback quantum quantum controls stabilization of a 4 photon number state in a cavity [16]. Even with all this success, the surface of quantum control theory has barely been cracked. More than 1300 papers are published per year on the subject as more problems and solutions in quantum control theory are found [17].

One active area of modern research in quantum control theory is controllability of open and infinite dimensional quantum systems. Open quantum systems with unitary controls have been declared uncontrollable from multiple sources ([4], [14], [7], [3]). However, it is possible to generate decoherent controls for quantum systems ([9], [18], [17]) which might allow for controllability given additional degrees of freedom. Grigoriu et al. performed controllability analysis on systems with engineered environments, yielding conditions for open quantum system controllability [11]. Grigoriu's
results have the issue that the conditions are prohibitively large even for small systems, therefore some improvement should be made. Infinite quantum systems are similar to open quantum systems in the sense that there have been generally negative results for controllability of quantum systems with a few positive results for specific situations ([13], [5]). Bloch et. al. developed the most promising positive result: he proved a general controllability theorem for arbitrarily sized finite dimensional subspaces of the infinite Hilbert space [5]. However, Bloch’s Finite Dimensional Controllability Theorem does not apply to open quantum systems, making it insufficient for general analysis.

One of the most fundamental and versatile methods of interacting with a quantum system is through electromagnetic radiation: "light". Coherent light has been used with great effectiveness as a control for quantum systems ([19], [5], [3], [21], [16]). Research in decoherent quantum control also shows that incoherent light can also be used as a possible control by tailoring the environment ([9], [17]). A simple system with applications in quantum computing and devices which includes both coherent and decoherent controls is a laser pumped high quality fabry-perot cavity immersed in a tailored environment. Investigating this model will provide improved intuition for general quantum optical systems, as well as investigate the feasibility of such systems for use in quantum engineering.

This thesis investigates a pumped optical cavity within a tailored environment in the presence with constant uncontrollable decoherence. In the analysis of this system, I found that if the initial state of a non-pumped cavity is diagonal, it remains so throughout its evolution. This motivated the development of a new definition of controllability called ensemble population controllability, as well as an accompanying theorem on determining if a system is controllable under these conditions. Also, since the system is infinite, a variant of the finite dimensional controllability theorem which applies to open quantum systems is developed for use in the analysis. Due to the stringent requirements of density matrix controllability and finite state controllability, two new definitions based on the expectation value and variance of an observable are defined, called weak and strong observable controllability. Lastly, the controllability of the system is analyzed using both previously developed methodologies, as well the
new techniques and theorems developed in this thesis.

After presenting a general background on open quantum optical systems and non-linear controllability theory, this thesis discusses and investigates current controllability definitions and conditions for general quantum systems and applies them to quantum optical systems. In addition to the current list of quantum controllability definitions and conditions, two previously undefined categories of controllability called "Ensemble Population Controllability" and "Observable Controllability" are developed. Also, an applicability condition for finite controllability of infinite quantum systems is relaxed so it can be utilized in the analysis of open quantum systems. These theorems are then applied to controllability analysis of a quantum optical system consisting of a laser pumped leaky cavity immersed in an engineered environment.

Throughout this thesis, I adopt a certain number of conventions, the first being the "atomic units" convention i.e. $\hbar = 1$. For the sake of space I also drop the general functional dependence of states after their initial definition i.e. $|\Psi(t)\rangle = |\Psi\rangle$. The same is done for the upper and lower bounds for sums. Also, the words "operator" and "matrix" are used interchangeably.
Chapter 2

Open Quantum Optical Systems

A common assumption in quantum mechanical analysis is that the system is "closed": that the energy and information content of the system is conserved. Dynamics of such systems are described using the Schrödinger equation formalism. However, realistic quantum systems are "open": the system interacts with an external environment where energy and information can enter and leave the system. In open systems, the evolution of the state of the system can no longer be easily described using the Schrödinger equation formalism. Such a description would require knowledge of all the interactions between the system of interest and the environment, as well as tracking all of the states in both systems. Even if a perfect description of the interaction between the environment and the system of interest is known, the overall system would be prohibitively large and essentially unsolvable. This motivates the development of another formalism to handle such systems: the Density Matrix Formalism. Before developing the density matrix formalism and applying it to open quantum systems, it is instructive to first review closed quantum systems and their properties.

2.1 Review of Closed Quantum Systems

In the Schrödinger equation formalism, the state of a closed system is represented by a vector $|\Psi(t)\rangle$ of a complex vector inner product space $\mathcal{H}$. In contrast to classical
control systems, the state vector $|\Psi(t)\rangle$ is not necessarily physical; instead, the probabilistic information about physical observable quantities is encoded in the elements of the state vector. This information is extracted from the state vector using Hermitian operators representing the physical dynamical variable of interest and inner products. One important property is the expectation value or average of a dynamical variable mathematically represented by $\hat{O}$, which is obtained from the inner product

$$\langle \hat{O} \rangle = \langle \Psi | \hat{O} | \Psi \rangle,$$  \hspace{1cm} (2.1.1) where $\langle \Psi \rangle = |\Psi\rangle^\dagger$ is the dual vector of $|\Psi\rangle$.

The evolution of the state vector is given by the time dependent Schrödinger equation

$$i\partial_t |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle \quad |\Psi(t = t_0)\rangle = |\Psi_0\rangle,$$  \hspace{1cm} (2.1.2) where $\hat{H}$ is a Hermitian operator representing the energy of the system called the Hamiltonian, and $|\Psi_0\rangle$ is some initial state. Closed quantum systems have the unique property that their evolution is unitary. Consider an operator $\hat{U}(t)$ called the propagator, which transforms an initial state vector at time $t_0$ to the state vector at time $t$:

$$|\Psi(t)\rangle = \hat{U}(t) |\Psi_0\rangle.$$ Substituting this relation into the Schrödinger equation yields a differential equation describing the evolution of the operator $\hat{U}$, subsequently called the lifted Schrödinger equation:

$$\partial_t \hat{U} = -i \hat{H} \hat{U} \quad \hat{U}(t_0) = \mathbb{I},$$  \hspace{1cm} (2.1.3) where $\mathbb{I}$ is the identity operator. The nature of this operator can be found by taking the Hermitian conjugate of equation (2.1.3):

$$\partial_t \hat{U}^\dagger = i \hat{U}^\dagger \hat{H} \quad \hat{U}^\dagger(t_0) = \mathbb{I}.$$ From the initial conditions of the two above equations, we see that $\hat{U}^\dagger(t_0)\hat{U}(t_0) = \mathbb{I}$.

---

1The Hermitian conjugate, indicated by a superscript dagger, is the complex conjugate transpose of an operator.
The evolution of $\hat{U}(t)^\dagger \hat{U}(t)$ is found by taking the derivative and substituting the lifted Schrödinger equation wherever derivatives of $\hat{U}$ appear:

$$\partial_t \hat{U}^\dagger \hat{U} = \dot{\hat{U}}^\dagger \hat{U} + \hat{U}^\dagger \dot{\hat{U}}$$

$$= i\hat{U}^\dagger \hat{H} \hat{U} - i\hat{U}^\dagger \hat{H} \hat{U}$$

$$= \hat{U}^\dagger (i\hat{H} - i\hat{H}) \hat{U}$$

$$= 0.$$ 

This implies that the propagator retains the property $\hat{U}^\dagger \hat{U} = I$ throughout its evolution; such operators are called unitary.

**Definition 2.1.1.** A **unitary operator** is an operator whose inverse is its Hermitian conjugate: $\hat{U}^{-1} = \hat{U}^\dagger$.

From this background on quantum mechanics of closed systems, the density matrix formalism, and subsequently quantum mechanics of open quantum systems and quantum optical systems can be developed.

### 2.2 Density Matrix Formalism

#### 2.2.1 Density Operators

In the density matrix formalism, the state of a system is represented by an operator $\rho$ called the density operator, which, when represented as a matrix, is called the density matrix. In place of the state of the system being represented by a single state vector $|\Psi\rangle$ like in the Schrödinger equation, the state is represented as an operator $\rho$ representing a statistical mixture or ensemble of individual states. Consider a collection of normalized but not necessarily orthogonal states from a complex vector inner product space $\mathcal{H}$, $\{|\psi_1\rangle, |\psi_2\rangle, \ldots |\psi_n\rangle \in \mathcal{H}$ for all $n$, $\langle \psi_i | \psi_i \rangle = 1 \}$, which have associated with them classical probabilities $\{p_1, p_2, \ldots |\sum_i p_i = 1 \}$.

**Definition 2.2.1.** The **density operator** $\rho$ is defined as the weighted sum of the
outer products of the state vectors with themselves:

\[ \rho := \sum_{m=0}^{N \in \mathbb{N}_0} p_m |\psi_m\rangle\langle \psi_m| . \]

From this definition, the following properties of the density matrix are obtained:

1. \( \rho \) is hermitian (\( \rho^\dagger = \rho \))
2. \( \rho \) has unit trace (\( \text{Tr}(\rho) = 1 \))
3. \( \rho \) is positive semi-definite (\( \langle \phi | \rho | \phi \rangle \geq 0 \) for all \( |\phi\rangle \in \mathcal{H} \))
4. \( \rho \) has a Hilbert-Schmidt norm that is less than or equal to 1 (\( \sqrt{\text{Tr}(\rho^\dagger \rho)} \leq 1 \))

The first three properties are intrinsic to the definition of density operators, with the fourth derived from the other three. For the fourth property, equality only occurs if \( p_i = \delta_{i,k} \) for a single \( k \), which signifies that the density operator consists entirely of a single state \( |\psi\rangle \). Such operators are said to represent \textit{pure} states, where \( \rho = |\Psi\rangle\langle \Psi| \) for some \( |\Psi\rangle \), while the others are said to be \textit{mixed} states. From this, the following rigorous definitions can be stated.

**Definition 2.2.2.**

1. \( \rho \) is in a \textbf{pure state} if and only if \( \text{Tr}(\rho^2) = 1 \).
2. \( \rho \) is in a \textbf{mixed state} if and only if \( \text{Tr}(\rho^2) < 1 \).

Any operator can be expressed as a matrix in an inner product space using any orthonormal basis.\(^2\)

Expressing the density operator as a matrix in the basis \( \mathcal{B} := \{|\phi_1\rangle, |\phi_2\rangle, \cdots |\phi_i\rangle = \delta_{i,j} \} \) gives

\[
\rho_{m,n} = \langle \phi_m | \rho | \phi_n \rangle \\
= \langle \phi_m | \sum_k p_k |\psi_k\rangle \langle \psi_k | \phi_n \rangle \\
= \sum_k p_k c_m^k c_n^k ,
\]

\(^2\)Hence why "operator" and "matrix" are interchangeable.
where $c_n^k = \langle \phi_n | \psi_k \rangle$ is the projection coefficient of the state $|\psi_k\rangle$ into $|\phi_n\rangle$. A diagonal element $\rho_{m,m} = \sum_k p_k c_m^k c_m^k$ is the average of the projections of the states $|\psi_k\rangle$ into $|\phi_m\rangle$. The statistical interpretation of these elements is clear: they represent the classical probability that the system, if measured, is observed to be in the state $|\phi_m\rangle$. From this interpretation, $\rho_{m,m}$ represents a population distribution of the ensemble in the basis $\mathcal{B}$. On the other hand, an off diagonal element $\rho_{m,n}$ represents the coherence between states, an average of interferences induced by $\rho$ between the basis elements $|\phi_m\rangle$ and $|\phi_n\rangle$. Now that the density operator has been defined, how information is extracted from the density operator, as well as how the density operator evolves with time, can be described.

### 2.2.2 Expectation Values of Observables

The equation for finding expectation value of an observable in the density matrix formalism can be derived from equation [2.1.1] of the Schrödinger equation formalism. Recall from section 2.1 that observables are represented by Hermitian operators, and that the expectation value of an observable $\hat{O}$ for a state $|\Psi\rangle$ is $\langle \Psi | \hat{O} | \Psi \rangle$. Expressing $\Psi$ as a vector, $\hat{O}$ as a matrix in the basis $\mathcal{B}$, and recognising that $\text{Tr}(A^\dagger B) = \sum_{m,n} A_{m,n}^* B_{n,m}$, the expectation value of $\hat{O}$ is

$$
\langle \hat{O} \rangle = \sum_{m,n} c_m^* c_n O_{m,n}
= \text{Tr} \left( |\Psi\rangle \langle \Psi | \hat{O} \right).
$$

This relation can be applied to all the members of the ensemble from which $\rho$ is constructed, and the sum over the elements $p_k \text{Tr}(\rho_k \hat{O})$ can be absorbed into the trace by linearity. Therefore, the expectation value of an observable $\hat{O}$ in the density matrix formalism is given by

$$
\langle \hat{O} \rangle = \text{Tr}(\hat{O} \rho).
$$ (2.2.2)
2.2.3 Evolution of the Density Operator

The density operator evolves according to the von Neumann equation, which can be derived from the time derivative of the density operator. Consider a single projection operator $\rho_k = |\psi_m\rangle\langle\psi_m|$. Taking the time derivative of this operator and substituting the Schrödinger equation wherever $|\dot{\psi}_m\rangle$ appears yields the following relation for the evolution of $\rho_k$:

$$\partial_t \rho_k = -i \hat{H} |\psi_m\rangle\langle\psi_m| + i |\psi_m\rangle\langle\psi_m| \hat{H}$$
$$= -i [\hat{H}, \rho_k],$$

where $[A, B] = AB - BA$ is the quantum commutator superoperator. Since the commutator is a linear operator, the evolution equation of the ensemble of states is just the weighted sum of the evolution of the individual projection operators $|\psi_m\rangle\langle\psi_m|$ associated with each state. From this, the von Neumann equation is

$$\partial_t \rho = -i [H, \rho]$$
$$= \text{ad}_{iH} \rho$$
$$= \mathcal{H} \rho,$$

where $\mathcal{H} = \text{ad}_{iH} = -i [H, \cdot]$ are all common notations used to represent the commutators and superoperators. It is important to note that the von Neumann equation as developed is equivalent to the Schrödinger equation; it solely presents a different method of representing the states. Because of this, equation (2.2.6) leads to unitary evolution of $\rho$. However, the propagator pre- and post-multiplies the density operator in order to transition the system to a time $t$,

$$\rho(t) = \hat{U}(t) \rho_0 \hat{U}^\dagger(t),$$

and the dynamics of this operator are given by the lifted von Neumann equation:

$$\partial_t \hat{U} = -i [H, \hat{U}] \quad \hat{U}(0) = \mathbb{I}. $$

(2.2.6)
Even though these evolution equations are just reformulations of the Schrödinger equation, representing the system as a collection of states is a powerful tool which can be used in the analysis of open systems.

2.3 A System Interacting With the Environment

Generally speaking, an open quantum system is a system $S$, with vector space $\mathcal{H}_S$, which is coupled to the environment $R$, with vector space $\mathcal{H}_R$. Qualitatively, the system $S$ is the system of interest, which consists of anything which is measured: an atom, a highly reflective fabry-perot cavity, a Bose Einstein Condensate, etc. The environment then consists of everything else that interacts with and disturbs the system of interest $S$. If the two systems were isolated, the evolution of each system individually would be governed by the von Neumann equation and their respective Hamiltonians $H_S$ and $H_R$. However, since the systems are coupled by definition, the evolution of the individual systems can not be described using the Schrödinger equation or any variant thereof. Therefore, the evolution of a supersystem $SR = S \oplus R$ consisting of the open system and the environment has to be considered. Since the supersystem $SR$ is closed, the dynamics of $SR$ are in fact given by the von Neumann equation

$$\partial_t \rho_{SR} = -i [H_{SR}, \rho_{SR}] \quad \rho_{SR}(t_0) = \rho_{SR0} \rho_{SR} \in \mathcal{H}_{SR}. \quad (2.3.1)$$

The state space of $SR$ is the cartesian product $\mathcal{H}_{SR} = \mathcal{H}_S \otimes \mathcal{H}_R$ of the vector spaces associated with $S$ and $R$. The Hamiltonian $H$ of the encompassing system which describes the evolution of $SR$ is the sum of the Hamiltonians $H_S$ and $H_R$ describing the isolated systems $S$ and $R$, and the interaction Hamiltonian $H_{SR}$:

$$H = H_S \otimes I_R + I_S \otimes H_R + H_{SR}. $$

In this formulation, equation 2.3.1 is essentially unsolvable due to the large number of degrees of freedom and the uncertainty involved with the environment. However, it is possible to average over the effects of the environment, generating a reduced density

---

3 Also referred to in literature as a Reservoir or a Bath
matrix $\rho_S$. An effect of this averaging is that the evolution of the reduced density matrix is no longer given by the von Neumann equation, thus a new equation which governs the reduced system dynamics must be introduced.

### 2.3.1 Reduced Density Matrix

The reduced density matrix allows for the environment to be ignored by tracing over the environmental states and averaging out their effects on the state of the system $S$. Consider the bases $\mathcal{B}_S := \{ |s\rangle\langle s'| \}$ and $\mathcal{B}_R := \{ |r\rangle\langle r'| \}$ to the vector spaces $\mathcal{H}_S$ and $\mathcal{H}_R$ respectively. From the definition of $\mathcal{H}_{SR} := \mathcal{H}_S \otimes \mathcal{H}_R$, the basis and density matrix of the combined system $\mathcal{H}_{SR}$ are

$$
\mathcal{B}_{SR} := \{ |s\rangle\langle s'| \otimes |r\rangle\langle r'| \} = \{ |s,r\rangle\langle s',r'| \},
$$

(2.3.2)

$$
\rho_{SR} := \sum_{s,s';r,r'} p_{s,s';r,r'} |s\rangle\langle s'| \otimes |r\rangle\langle r'| = \sum_{s,s';r,r'} p_{s,s';r,r'} |s,r\rangle\langle s',r' |.
$$

(2.3.3)

Recall that in the density matrix formalism, expectation values are found from the trace: the sum of the diagonal elements of the matrix. To reduce the density matrix from $\rho_{SR}$ to $\rho_S$, a partial trace is performed where the trace only applies over the blocks of the density matrix corresponding to the basis elements $|r\rangle\langle r'|$.

**Definition 2.3.1.** The **reduced density matrix** of a system with basis $\{ |s,r\rangle\langle s',r'| \}$ is

$$
\rho_S = \text{Tr}_R(\rho_{SR}) = \sum_r (I_S \otimes |r\rangle\langle r|) \rho_{SS} (I_S \otimes |r\rangle) = \sum_{s,s'} \sum_r p_{s,s';r,r'} |s\rangle\langle s' |.
$$

(2.3.4)

For the most part, the reduced density matrix $\rho_S$ shares the same properties as the general density matrix from definition 2.2.1 with the exception that its evolution is no longer governed by the von Neumann equation.
2.3.2 Reduced System Dynamics

In place of the von Neumann equation, the dynamics of the reduced density matrix \( \rho_S \) is given by the Born-Markov equation, also called the Markovian Master Equation. This equation originates from the von Neumann equation applied to the closed system \( SS \) and is most easily derived in the interaction picture. Defining \( H_0 = H_S \otimes I_R + I_S \otimes H_R \), the transformations to the interaction picture for observables \( \hat{O} \) and the state \( \rho \) are

\[
\tilde{O}(t) = e^{iH_0 t} \hat{O} e^{-iH_0 t}
\]

\[
\tilde{\rho}(t) = e^{iH_0 t} \rho e^{-iH_0 t}. \tag{2.3.6}
\]

Applying these transformations to equation 2.3.1 yields the von Neumann equation for \( SS \) in the interaction picture:

\[
\partial_t \tilde{\rho}_{SS}(t) = -i[H_{SR}(t), \tilde{\rho}_{SS}(t)]. \tag{2.3.7}
\]

Integrating this equation yields an integral equation for the density matrix at time \( t \):

\[
\tilde{\rho}_{SS}(t) = \tilde{\rho}_{SS}(0) - i \int_0^t dt' [H_{SR}(t'), \tilde{\rho}_{SS}(t')]. \tag{2.3.8}
\]

Substituting the integral form (eq 2.3.8) of the von Neumann equation into the differential form (eq 2.3.7) gives a closed form equation:

\[
\dot{\tilde{\rho}}_{SS} = -i[H_{SR}, \tilde{\rho}_{SS}(0)] - \int_0^t dt' [H_{SR}(t), [H_{SR}(t'), \tilde{\rho}(t')]]. \tag{2.3.9}
\]

The objective is to obtain an expression for \( \tilde{\rho}_S = \text{Tr}_R(\tilde{\rho}_{SS}) \). It can generally be assumed that the environmental reservoir has significantly more degrees of freedom and energy content than the system of interest \( S \). From this assumption, it is safe to approximate that the reservoir density operator is static; even though it interacts with the system, it is not changed by this interaction. This is known as the Born approximation. Mathematically, this approximation implies that the density operator can be described as the cartesian product of the density operator of a static environment.
defined at some initial time \( \tilde{\rho}_R(0) \), with the system density operator \( \tilde{\rho}_S(t) \):

\[
\tilde{\rho}_{SS}(t) = \tilde{\rho}_S(t) \otimes \tilde{\rho}_R(0).
\]

After applying the Born approximation, tracing over the reservoir degrees of freedom yields a closed form integro-differential equation for the state of the reduced system:

\[
\partial_t \tilde{\rho}_S(t) = -i \text{Tr}_R \left( [\tilde{H}_{SR}(t), \tilde{\rho}_S(0) \otimes \tilde{\rho}_R(0)] \right) - \int_0^t d\tau \text{Tr}_R \left( [\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t-\tau), \tilde{\rho}_S(t) \otimes \tilde{\rho}_R(0)]] \right). 
\]

(2.3.11)

If the initial commutator does not vanish, then it can be adsorbed into the interaction Hamiltonian. Applying Markovian approximations of no memory and coarse graining yields the Born Markov, or Markov Master equation:

\[
\partial_t \tilde{\rho}_S(t) = -\int_0^\infty d\tau \text{Tr}_R \left( [\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t-\tau), \tilde{\rho}_S(t) \otimes \tilde{\rho}_R(0)]] \right). 
\]

(2.3.12)

For most physical systems of interest in quantum optics, the interaction hamiltonian is of the form \( H_{SR} = \sum_k R_k A_k^\dagger + \tilde{R}_k A_k \). In this case, equation 2.3.12 takes a more succinct form, called the Lindblad form:

\[
\partial_t \rho = -i[H, \rho] + \sum_k \gamma_k \hat{A}_k \rho \hat{A}_k^\dagger - \frac{1}{2} \{ \hat{A}_k^\dagger \hat{A}_k, \rho \},
\]

(2.3.13)

where \( \gamma_k \) are relaxation rates of the decoherence pathways corresponding to the operator \( A_k \). In this thesis, my focus is on systems that can be written in Lindblad form, because it is possible to assume \( \gamma_k \) and \( A_k \) on a phenomenological basis ([2] [3]).

2.4 Quantum Optics of a High-Q Cavity

In general optics, a high quality (high-Q) cavity is a pair of highly reflective parallel mirrors. From the reflectivity, the boundary condition for the electric field at the
surface of each mirror can be assumed to be \( E \approx 0 \). From these boundary conditions, electromagnetic waves which have wavelengths resonant with the cavity spacing can form standing waves within the cavity. Therefore, the cavity isolates individual electromagnetic field modes, the wavelength of which are determined by the cavity spacing. Exciting these field modes generates very non-classical states of light, commonly known as photons and less commonly known as Fock or number states. Due to the quantized nature of the system, a quantum mechanical description must be employed in order to model the system.

### 2.4.1 Quantum Model of a Cavity

Assuming the cavity mirrors are perfectly reflective, the state of the cavity can be modeled using a quantum simple harmonic oscillator. Let \( a \) and \( a^\dagger \) be the photon annihilation and creation operators respectively. The Hamiltonian of the photon number states in the cavity is

\[
H_c = \omega_c \left( a^\dagger a + \frac{1}{2} \right),
\]

(2.4.1)

where \( \omega_c = \frac{2\pi c}{d} \) is the angular frequency of a cavity with separation \( d \). A common simplification is to define the energy of the ground state as 0, which eliminates the \( \frac{1}{2} \omega_c \) offset from the Hamiltonian. The solution to this problem is well known and can be found in any basic textbook on quantum mechanics ([10] [22]).

In the case of a leaky cavity with a pump, the methodology of open quantum systems is used to generate a system model. The system of interest is the cavity as described above, and the reservoir is an equilibrated thermal radiation field with a laser pump. The Hamiltonian associated with the cavity remains unchanged from equation 2.4.1 and the Hamiltonian of the reservoir radiation field is

\[
H_R = \sum_{k,\lambda} \omega_k b_{k,\lambda}^\dagger b_{k,\lambda},
\]

(2.4.2)

where \( \omega_k \) is the angular frequency of a particular reservoir mode, and \( b_{k,\lambda}^\dagger, b_{k,\lambda} \) are
the photon creation and annihilation operators for the reservoir field modes in the direction of $k$. The distribution of photons at a particular angular frequency $\omega_k$ is described by the sum of the Bose-Einstein distribution at a temperature $T$ representing the thermal photons, and by a delta distribution representing a monochromatic laser field:

$$N_{Ph}(\omega_k) = \frac{1}{e^{\frac{\omega_k}{kT}} - 1} + E\delta(\omega_k - \omega_L). \quad (2.4.3)$$

The interaction between the cavity and reservoir is assumed to be linear, and hence take the form

$$H_{SR} = \sum_{k,\lambda} \Omega_{k,\lambda}(a + a^\dagger)(b_{k,\lambda} + b_{k,\lambda}^\dagger), \quad (2.4.4)$$

where $\Omega_{k,\lambda}$ is a coupling constant between the fields. Since the cavity isolates a single mode of the electromagnetic field, it is assumed that $\text{Tr}(H_{SR}\rho_R(t)) = 0$ for all nonresonant field modes. Solving the Born-Markov equation 2.3.12 for this system yields a differential equation in Lindblad form:

$$\partial_t \rho_S = -i[\omega_c a^\dagger a + \beta(e^{-i\omega_L t}a^\dagger + e^{i\omega_L t}a), \rho_S]$$
$$+ \Gamma(a\rho_S a^\dagger - \frac{1}{2}\{a^\dagger a, \rho_S\}) + N_T(\rho^\dagger S a - \frac{1}{2}\{a a^\dagger, \rho_S\})$$
Chapter 3

Accessibility and Controllability

The subjects of accessibility and controllability, hereon referenced under the blanket term "controllability," attempt to answer the following questions:

1. Given some initial state, where can the system be driven, and
2. Can it be driven anywhere?

These are important questions which need to be answered, or at least addressed, before any serious work on developing control algorithms for a system is performed. The mathematical framework for determining controllability lies in the field of differential geometry, specifically in the subfields of Lie groups, Lie algebras, and algebraic topology. In this chapter, the formalism for controllability is developed for control affine systems, with an example on the application of the formalism to a linear control system.

3.1 Control Systems

Simply stated, an affine control system is a differential equation that has a series of adjustable inputs which can be engineered to guide the system from an initial state to a desired final state,

\[ \dot{x}(t) = \hat{f}(x(t)) + \sum_i u_i(t)\hat{g}_i(x(t)). \]  \hspace{1cm} (3.1.1)
Mathematically, an affine control system $\Sigma$ is defined by the triple

$$\Sigma = \left( M, \mathcal{F} = \{ \hat{f}, \hat{g}_1, \hat{g}_2, \cdots, \hat{g}_m \}, U \right), \quad (3.1.2)$$

where $M$ is the state space manifold, $\mathcal{F}$ is a finite set of vector fields on that manifold, and $U$ is the range where the control functions take values

$$U := \left\{ \text{image} \left( [u_1(t), u_2(t), \cdots, u_m(t)]^T \right) \right\} \subset \mathbb{R}^m. \quad (3.1.3)$$

Let $\mathcal{U}$ denote the set of admissible and measurable controls which take values in $U$. The rule of the system is given by the differential equation in equation 3.1.1 where the vector fields and functions have the following definitions and restrictions:

1. $t \in \mathbb{R}^+_0$ (Time is positive)
2. $x : t \rightarrow M$ (The state maps time to the state manifold)
3. $\hat{f}, \hat{g} : M \rightarrow M$ (The vector fields map the state manifold to itself)
4. $u_i : t \rightarrow U \subset \mathbb{R}^m$ (The controls are real scalar maps of time).

### 3.2 Reachable Set

An important aspect of a control system necessary to the subject of controllability is the concept of reachable states, and, by extension, reachable sets. Given an initial state $x(0) = x_i$, a finite amount of time $T$, and a final state $x(T) = x_f$, if equation 3.1.1 has a solution with these boundary conditions for some set of controls $u_i \in \mathcal{U}$, then the state $x_f$ is called **reachable** or **accessible** from the state $x_i$. The set of all reachable states from any initial state $x(0) = x_i$ and time $T$ is called the **reachable set**

$$\mathcal{R}_\Sigma(x_i, T) := \{ x \in M \mid x \text{ is reachable from } x_0 \} \quad (3.2.1)$$
of the system $\Sigma$ from $x_i$. Without a specifically defined final time, the reachable set is defined as the union of all reachable sets from $x_i$ with time greater than zero:

$$R_{\Sigma}(x_i) = \bigcup_{t \geq 0} R_{\Sigma}(x_i, t).$$  \hfill (3.2.2)

Notice that this definition applies only to a particular point $x_i \in M$, not from all possible initial points in $M$. For the question of controllability, the interest is in finding the set of reachable states for all possible initial conditions $x_i \in M$. Such analysis is best performed utilizing methods of Lie group theory. In order to facilitate the use of such methods, if the manifold of the system is not a group, the system $\Sigma$ is lifted from actions on a manifold to actions on a group manifold $G$: a Lie group.

The analysis of lifted systems requires some definitions of properties of vector fields and sets and implies some constraints on the system $\Sigma$.

**Definition 3.2.1** \cite{[20]}. A vector field $\hat{f}$ on a group $G$ with Lie algebra $\mathfrak{g}$ is right-invariant if it takes the form

$$\hat{f}(x) = AX, \quad A \in \mathfrak{g}, \; X \in G.$$  

**Definition 3.2.2.** A subset $U \subset \mathbb{R}^m$ is almost proper if

1. $0 \in \{x | x = \sum_{i=1}^{m} \lambda_i u_i, \; \lambda_i \geq 0, \; \sum_{i=1}^{l} = 1\}$

2. $0 \in \{x | x = \sum_{i=1}^{m} \lambda_i u_i, \; \sum_{i=1}^{l} = 1\}$

The lifted system $\tilde{\Sigma} = (G, \tilde{F} = \{\tilde{K}, \tilde{W}_1, \tilde{W}_2, \cdots, \tilde{W}_m\}, U)$ of $\Sigma$ has a rule which is a group differential equation constructed from the affine combination of right-invariant vector fields $\tilde{K}$ and $\tilde{W}_i$ acting on an element $X \in G$,

$$\dot{X}(t) = \tilde{K}(X(t)) + \sum_i u_i(t)\tilde{W}_i(X(t)),$$  \hfill (3.2.3)

which corresponds to equation $[3.1.1]$. The lifted system, in place of being a control system on a state space, is instead a control system on the manifold of possible transformations called dynamical maps from $M \rightarrow M$. From this interpretation,
any conclusions from dynamical analysis performed on the lifted system apply to the original system. At this point it is important to note a theorem which equates the Lie algebras of the normal and lifted systems.

**Theorem 3.2.3 ([15]).** If $U$ is almost proper, then $\text{Lie}(\mathcal{F}) = \text{Lie}(\tilde{\mathcal{F}})$.

In this theorem, $\text{Lie}(\mathcal{F})$ generates a Lie algebra from the elements of $\mathcal{F}$ with $[A, B] = (\partial_x A)B - (\partial_x B)A$ being the Lie bracket. Also note that in this form, the set of reachable elements from an element $X_i \in G$ has the following significant property:

$$\mathcal{R}_{\tilde{\Sigma}}(X_i) = \mathcal{R}_{\tilde{\Sigma}}(I)X_i.$$  \hspace{1cm} (3.2.4)

From this, it is clear that the reachable set from any element can be classified by the reachable set from the identity. Thankfully, the reachable set from the identity can be determined from previously proven theorems.

Jurdjevic and Sussmann, in their 1972 paper "Controllability on Lie Groups," develop the following theorem and corollary on determining the set of reachable elements from the identity for a lifted system $\tilde{\Sigma}$. Let $\tilde{\mathfrak{f}}$ be the Lie algebra generated by the elements of $\tilde{\mathcal{F}}$, and let $\tilde{F}$ be its associated Lie group.

**Theorem 3.2.4 ([12]).** If there exists a set of constant control functions $u_i$ and a sequence of positive numbers $\{t_n | t_n \geq \delta > 0\}$, for some $\delta$, with the property that $\lim_{n \to \infty} t_n$ exists and belongs to $\tilde{F} = \text{cl}(\tilde{\mathfrak{f}})$ (the closure of $F$), then $\mathcal{R}_{\tilde{\Sigma}}(I) = \tilde{F}$.

**Corollary 3.2.5.** If there exist constant control functions $u_i$ such that the orbit through the identity is periodic, then $\mathcal{R}_{\tilde{\Sigma}}(I) = \tilde{F}$.

If the conditions of this theorem or corollary are satisfied, then the reachable set from any initial element of the group is the Lie group associated with the Lie algebra constructed from $\tilde{\mathcal{F}}$. Now it is possible to define different types of controllability and to present theorems on determining the controllability of a system.
3.3 Controllability Definitions and Theorems

Although there are many types of definitions for controllability for control systems (strong, local, small time, etc.), the most relevant definitions involve the entire state manifold $M$. These definitions are most commonly given the prefix completely or totally to indicate the domain in which they are valid.

**Definition 3.3.1.** Let $\Sigma = (M, \mathcal{F}, U)$ be a control affine system:

1. $\Sigma$ is completely accessible if $\text{int}(\mathcal{R}_\Sigma(x)) \neq \emptyset$ for all $x \in M$.
2. $\Sigma$ is completely controllable if $\mathcal{R}_\Sigma(x) = M$ for all $x \in M$.

From these definitions, accessibility is the ability of a system to drive any state to some open submanifold on $M$, while controllability is the property that any state on the manifold is reachable from any other state. These definitions are dependent on the classification of the reachable set; therefore, from the previous discussion, it is natural to assume that the analysis will be performed on the lifted system $\tilde{\Sigma}$ if $M$ is not already a group.

Let $\Sigma = (M, \mathcal{F}, U)$ be a control affine system and $\tilde{\Sigma} = (G, \tilde{\mathcal{F}}, U)$ its associated lifted system. Let $\tilde{\mathfrak{f}}$ be the Lie algebra generated by $\tilde{\mathcal{F}}$ with an associated Lie group $\tilde{F}$, and let $\mathfrak{g}$ be the associated Lie algebra to $G$.

**Theorem 3.3.2 (Accessibility Rank Condition [20]).** The system $\tilde{\Sigma}$ is totally accessible if and only if $\text{dim}(\tilde{\mathfrak{f}}) = \text{dim}(\mathfrak{g})$.

In contrast to accessibility, which can be determined solely by the dimension of the system algebra, controllability is determined by the structure of the system algebra and group, which leads to a myriad of different conditions representing different structures where controllability is possible. Before the theorems for determining controllability are stated, the following definitions are necessary.

**Definition 3.3.3 (Saturated [20]).** A Lie algebra $\mathfrak{f} \subset \mathfrak{g}$ is saturated if $\mathfrak{f} = \text{LS}(\mathfrak{f}) := \{g \in \mathfrak{g} | gv \in \mathfrak{f}, v \in \mathcal{V}\}$. 
Definition 3.3.4 (Weakly Reversible [23]). A system $\Sigma$ is weakly reversible if $X_f \in R_\Sigma(X_i)$ and $X_i \in R_\Sigma(X_f)$ for all $X_i, X_f \in G$.

Theorem 3.3.5. A system $\tilde{\Sigma}$ is totally controllable if it is totally accessible and one of the following conditions are true:

1. $\tilde{F}$ is a group (necessary)[12]
2. $G$ is connected and compact[12]
3. $\tilde{\Sigma}$ is weakly reversible[23]
4. $\tilde{f}$ is saturated (necessary)[20].

This list is nowhere near exhaustive; for further theorems on controllability, I direct the reader to the theorem references and references therein.

3.4 Example: Controllability of a Block-Spring System

3.4.1 Linear Systems

In general, the rule of a linear control system has the form

$$\dot{x} = Ax + \sum_i u_i B_i, \quad x, B \in \mathbb{F}^N, A \in \mathbb{F}^{N \times N},$$

for some field $\mathbb{F}$. Corresponding this to the language of controllability theory, the drift and control vector fields $\tilde{f}$ and $\tilde{g}$ are simply $A$ and $B$ respectively. Exploiting the linear and constant nature of the vector fields, the elements of the Lie algebra $\mathfrak{f}$ are $\{B_1, B_2, \cdots, AB_1, \cdots, A^2 B_1 \cdots\}$. The dimension of the algebra is the rank of a matrix $C$ whose columns are the elements of the Lie algebra

$$\dim(\mathfrak{f}) = \text{Rank} \left( [B_1 | \cdots | AB_1 | \cdots | A^{n-1} B_1 | \cdots] \right).$$
It can also be shown that $f$ is saturated [20]. Therefore, the following theorem can be stated:

**Theorem 3.4.1.** If $C$ has full rank, then the linear system is controllable.

### 3.4.2 System Analysis

Let’s investigate the controllability of a linear system corresponding to a block and spring with velocity dependent friction, where the control is a force applied to the block. The differential equation describing the dynamics of the system is

$$\ddot{x} = -\frac{k_s}{m}x - \frac{\mu_k}{m}\dot{x} + u_F,$$

where $k_s$ is the spring constant, $\mu_k$ is the proportionality constant for the friction force, $m$ is the mass of the block, and $u_F$ is the control force. This can be represented in the form $\dot{x} = Ax + uB$ by defining new variables $x_1 = x$ and $x_2 = \dot{x}$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ -\frac{k_s}{m} & -\frac{\mu_k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} + u_F \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Generating the matrix $C$ using the first two elements of the algebra yields

$$C = \begin{bmatrix} 0 & 1 \\ 1 & \frac{\mu_k}{m} \end{bmatrix}.$$  

Clearly, $C$ has full rank, therefore, by theorem 3.4.1 the system is totally controllable.
Chapter 4

Controllability of Quantum Optical Systems

4.1 Notions of Quantum Controllability

When venturing from the realm of classical to quantum controllability, the issue of controllability becomes a more nuanced topic. These nuances arise from two different sources: the different representations of quantum dynamics, and special properties of quantum systems.

First, let’s define the most commonly used definitions of controllability. Recall from Chapter 2 that the dynamics of a quantum system can be modeled in either the density matrix formalism or, if and only if the system is closed, in the Schrödinger equation formalism. This split in modeling methods leads to two types of controllability: density matrix controllability and pure state controllability. Also recall from that same chapter that states are only distinguishable up to a phase factor $e^{i\phi}$, leading to the definition of equivalent state controllability. Furthermore, for any closed quantum system there exists a unitary operator that transforms an initial state to another final state, which leads to the concept of operator controllability. From these descriptions, the following definitions can be stated.

Definition 4.1.1 ([1]). An $N$-dimensional quantum control system $\Sigma = (M, \mathcal{F}, U)$ is
1. Pure State Controllable (PSC): if $M := S_{C}^{N-1}$ and $\Sigma$ is controllable.

2. Equivalent State Controllable (ESC): if $M := S_{C}^{N-1}$ and $\Sigma$ is controllable up to a phase factor $\phi$.

3. Operator Controllable (OC): if $M := \{U(N) \text{ or } SU(N)\}$ and $\Sigma$ is controllable.

4. Density Matrix Controllable (DMC): if $M := \{\rho \in \mathbb{C}^{N \times N} | \rho^{\dagger} = \rho \geq 0, \text{Tr}(\rho) = 1, \text{Tr}(\rho^2) \leq 1\}$ and $\Sigma$ is controllable.

These definitions can be modified to accessibility by changing the controllability requirement to accessibility. For more details on these aspects of controllability, I refer the reader to ([1],[6]). In the case of open quantum systems, the interest is only in density matrix controllability, since evolutionary dynamics are described by the Markovian Master Equation in the density matrix formalism.

In addition to these definitions of controllability, there are three exotic definitions of controllability pertinent to this thesis: ensemble population controllability, observable controllability, and finite controllability. Ensemble population controllability applies to systems where, if given an initial diagonal state, the density matrix remains diagonal throughout the entirety of its evolution.

**Definition 4.1.2.** A $N$-dimensional open quantum control system $\Sigma = (M, \mathcal{F}; U)$ is Ensemble Population Controllable (EPC) if $M := \{x \in \mathbb{R}^{N} \geq 0^{N} ||x||_1 = 1\}$ and $\Sigma$ is controllable.

Observable controllability arises from the desire to only control how a single dynamical variable of the system evolves in place of the entire state of the system. For any observable $\hat{O}$ of a quantum system, there are two statistical values attributed which entirely describe its properties: expectation value and variance. Controlling both of these aspects of a single observable allows for complete control of the measurement outcomes of that observable. Let $x_1 = \langle \hat{O} \rangle$ be the expectation value of $\hat{O}$, $x_2 = \left(\delta \hat{O}\right)^2$ be the variance of $\hat{O}$, and $x = [x_1, x_2]^T$. 
CHAPTER 4. CONTROLLABILITY OF QUANTUM OPTICAL SYSTEMS

Definition 4.1.3. An observable \( \hat{O} \) is **Weakly Observable Controllable (WOC)** if the system \( \Sigma = (M, F, U) \), where \( M \subset \mathbb{R} \), and the rule given by \( \dot{x}_1 = \hat{f}(x_1) + \sum_i u_i(t)\hat{g}(x_1) \) is controllable.

Definition 4.1.4. An observable \( \hat{O} \) is **Strongly Observable Controllable (SOC)** if the system \( \Sigma = (M, F, U) \), where \( M \subset \mathbb{R}^2 \), and the rule given by \( \dot{x} = \hat{f}(x) + \sum_i u_i(t)\hat{g}(x) \) is controllable.

On the other hand, finite controllability applies specifically to the controllability problem on infinite dimensional manifolds on which complete controllability analysis has been mostly negative ([13],[5]). Finite controllability allows us to circumvent this problem by determining the controllability of the system on an arbitrarily large sum manifold of dimension \( \tilde{N}_0 \setminus \{\infty\} \).

Definition 4.1.5 ([5]). An infinite dimensional control system \( \Sigma = (M, F, U) \) is **Finitely "blank" Controllable (F_C)** if there exists an arbitrarily large set \( \mathcal{H} := \{H_1 \subset H_2 \subset H_3 \subset \cdots\} \subset M \) of nested submanifolds of \( M \) where the system is controllable.

Note that this definition presents a prefix to add on to a previous definition of controllability, such as definition 4.1.1 or definition 3.3.1 where the type of controllability is substituted into the blank.

4.2 Density Matrix Controllability of Decoherent Quantum Systems

First, consider the set \( \mathcal{G} \) of \( N \times N \) complex positive semi-definite hermitian matrices:

\[
\mathcal{G} := \left\{ \rho \in \mathbb{C}^{N \times N} \mid \rho = \rho^\dagger, \langle \phi | \rho | \phi \rangle \geq 0 \text{ for all } | \phi \rangle \in \mathbb{C}^N \right\}.
\] (4.2.1)

This condition makes the structure of \( G \) that of a convex cone [3]. This constraint also reduces the real dimension of the set from \( 2N^2 \) to \( N^2 \). Note that the set of possible density matrices \( \mathcal{P} \), a compact convex cone of dimension \( N^2 - 1 \), is a subset of this set \( \mathcal{P} \subset \mathcal{G} \).
Next, consider the Lie group $G$ of automorphisms of $\mathcal{G}$ which preserve the trace:

$$G := \{ \mathcal{L} \in GL^+(\mathcal{G}) | \text{Tr}\{\mathcal{L}(\rho)\} = \text{Tr}(\rho), \forall \rho \in \mathcal{G} \}.$$  \hfill (4.2.2)

Note that by the definition of the General Linear groups on vector spaces, the group $G$ is connected. The underlying structure of this group is linear; therefore, the elements of $\mathcal{G}$ can be "unraveled" into real vectors by applying the affine map

$$\Phi : \mathcal{G} \to \mathbb{R}^{N^2}, \rho \to v, v_i = \text{Tr}(\rho B_i),$$ \hfill (4.2.3)

where $B_i$ is an orthonormal basis element of $\mathcal{G}$: for example, a matrix with a 1 in one element and zeros in the rest. It is clear from this mapping that $\mathcal{G}$ can be identified with $\mathbb{R}^{N^2}$ in what is known as the vector of coherence representation. Therefore, $G$ is isomorphic with

$$\tilde{G} := \left\{ X \in GL(N^2) | \det(X) > 0, \text{Tr}\{X(z)\} = \text{Tr}\{z\} \forall z \in \mathbb{R}^{N^2} \right\}.$$ \hfill (4.2.4)

Let the Lie algebra associated with the Lie group $G$ be represented by $\mathfrak{g} = \text{Lie}(G)$.

Now consider a dynamical system with a rule

$$\dot{\rho} = \mathcal{M}(\rho),$$ \hfill (4.2.5)

where $\mathcal{M}$ is a linear generator (right-invariant vector field) constructed from elements of $\mathfrak{g}$ acting on $\rho \in \mathcal{P} \subset \mathcal{G}$. Since the trace of $\rho$ is invariant by the definition of $\mathcal{M}$, $\partial_t \text{Tr}\{\rho\} = 0$. This yields the following constraint on the general system algebra:

$$\text{Tr}\{\mathcal{M}\rho\} = 0, \forall \rho \in \Gamma_N \mathcal{G}.$$ \hfill (4.2.6)

Representing the state matrix $\rho$ and the generator superoperator $\mathcal{M}$ as a vector length $N$ and an $N^2 \times N^2$ matrix in vector of coherence representation, the Lie algebra of
\( \mathcal{M} \) is isomorphic to a Lie algebra on the set
\[
\left\{ M \in \mathbb{R}^{N^2 \times N^2} \mid \sum_{j=1}^{N^2} M_{j,i} \alpha_j = 0, \forall i = 1, \ldots, N^2 \right\}.
\] (4.2.7)

This set has \( N^2 \) constraints, reducing the overall real dimension of the Lie algebra to
\[
\text{dim}(G) N^4 - N^2 = N^2 (N^2 - 1).
\]

Now define \( \mathcal{H}^G_i \) and \( \mathcal{L}^G_i \) as the elements of \( g \) that are constructed from \( \mathcal{H}_i \) and \( \mathcal{L}_i \) respectively. The evolution equation is associated with the group evolution equation:
\[
\dot{X} = \left( \mathcal{H}^G_0 + \sum_i u_i(t) \mathcal{H}^G_i + \mathcal{L}^G_0 + \sum_j u_j \mathcal{L}^G_j \right) X.
\] (4.2.8)

**Theorem 4.2.1** ([11]). If the Lie algebra generated by \( \mathfrak{f} := \{ \mathcal{H}^G_i, \mathcal{L}^G_i \} \) has dimension \((N^2 - 1)N^2\), then the system is density matrix controllable.

### 4.3 Ensemble Population Controllability of Decoherent Quantum Systems

The condition for density matrix controllability for open quantum systems can be quite prohibitive, even low dimensional systems require a Lie algebra of large dimension to control \((N = 2 \rightarrow \text{dim}(\mathfrak{f}) = 12)\). Such conditions are difficult to obtain for many quantum systems. However, if the initial state of the system is a completely diagonal mixed or pure state, and the state remains diagonal for all time, then another theorem for the controllability of the system can be proven which requires a much smaller Lie algebra.

#### 4.3.1 Controllability Criterion

Consider the control system \( \Sigma = (M, \mathcal{F}, U) \). Let \( \mathfrak{f} \) designate the Lie algebra generated from \( \mathcal{F} \).
Theorem 4.3.1. If the Lie algebra generated by \( \mathfrak{f} := \{ \mathcal{H}_i^G, \mathcal{L}_i^G \} \) has dimension \( \frac{N(N-1)}{2} \), then the system is ensemble population controllable.

Proof. Let \( \mathcal{V} \) be an \( N \)-dimensional real vector space which has an associated norm

\[
\| \cdot \|_1 : \mathbb{R}^N \to \mathbb{R}^+ \cup \{0\}, \mathbf{x} \to x = \sum_{i=1}^{N} |x_i| \tag{4.3.1}
\]

called the taxicab norm, or the 1-norm.

An important lemma on isometric groups is as follows:

**Lemma 4.3.2.** If \( \mathfrak{g} \) is a Lie algebra associated with Lie group \( G \) of isometric (norm preserving) automorphisms on a vector space \( \mathcal{V} \), then

1. \( G \) is compact, and
2. \( \mathfrak{g} \) is saturated (see def. 3.3.3).

Consider the associated Lie group \( G \) of positive isometric automorphisms of \( \mathcal{V} \)

\[
G := \{ X \in GL^+(\mathcal{V}) | \det(X) > 0, \|Xx\|_1 = \|x\|_1 \forall x \in \mathcal{V} \}. \tag{4.3.2}
\]

Since \( \det(X) > 0 \), \( G \) is connected. By definition, \( G \) preserves the norm; therefore by lemma [4.3.2] \( G \) is compact and \( \mathfrak{g} \) is saturated. A maximal compact subgroup of \( GL^+(\mathcal{V}) \) is \( SO(\mathcal{V}) \); therefore, \( \dim(G) \leq \dim(SO(\mathcal{V})) \). From the saturation property of \( \mathfrak{g} \), \( G \) is also a maximal compact subgroup of \( GL^+(\mathcal{V}) \). All maximal compact subgroups are isomorphic to the Special Orthogonal group \( SO(\mathcal{V}) \), therefore \( G \) has dimension \( \frac{n(n-1)}{2} \). By hypothesis, \( \dim(\mathfrak{f}) = \frac{N(N-1)}{2} = \dim(\mathfrak{g}) \); therefore, \( \mathfrak{f} = \mathfrak{g} \) and \( F = G \). From theorem [3.3.5.2], since \( \mathfrak{f} \) satisfies the rank condition and since \( G \) is compact, the system \( \Sigma \) is ensemble population controllable. \( \square \)

### 4.4 Controllability of Infinite Quantum Systems

First consider a set of finite dimensional nested subspaces of a vector space \( \mathcal{V} \) and a system equation:

\[
\mathcal{V} := \{ \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \} \tag{4.4.1}
\]
\[ \dot{x} = \sum_i u_i(t) \hat{g}_i(x), \ x \in \mathcal{V}. \]  

**Theorem 4.4.1.** Assume without loss of generality that there exists a subset of control operators \( \mathcal{G}_1 \) of \( \mathcal{G} := \{ \hat{g}_1, \hat{g}_2, \ldots, \hat{g}_n \} \) which leaves \( \mathcal{V}_1 \) invariant, and the system is controllable on \( \mathcal{V}_1 \). If for each \( \mathcal{V}_{\alpha \neq 1} \), there exists a subset \( \mathcal{G}_i \) which leaves \( \mathcal{V}_\alpha \) invariant, and for each element in \( \mathcal{V}_\alpha \), the subspace \( \mathcal{V}_{\beta \leq \alpha} \) is accessible, then any initial state in \( \mathcal{V}_i \) can be steered to any final state in \( \mathcal{V}_j \).

Theorem 4.4.1 is a variation on the finite controllability theorem presented in Bloch, Brockett, and Rangan [5]. Their theorem assumes that the control operators are skew-hermitian, leading to overall unitary dynamics of the system. It is surprising that if the skew-hermitian nature of the operators is relaxed, the theorem still holds true. The surprising nature of this result comes from the invocation of time symmetry in order to complete the proof. To illustrate this, the steps of the proof are outlined below.

First, they show that the smallest subspace is controllable, and then that the control directions are able to move the state from a larger subspace to a smaller subspace. Lastly, they prove by induction that one can move from any larger subspace into any smaller subspace, which completes the proof. It is important to note that at this point in the proof they have only shown controllability when moving from larger subspaces to smaller subspaces. The proof finishes by applying a time reversal argument exploiting the assumption that \( \mathcal{G} \) is a set of skew-hermitian operators which causes the evolution of equation 4.4.2 to be unitary.

In the case of decoherent control, it cannot be assumed that the elements of \( \mathcal{G} \) are skew-hermitian; therefore, time reversal is no longer inherently a symmetry of the system. However, it is possible to show that, given controlled evolution between two states \( \rho_i \rightarrow \rho_f \), the control functions driving the system from \( \rho_f \rightarrow \rho_i \) are simply the negative of the original control function.

**Proof.** Consider the evolution of a system under controlled decoherent evolution

\[ \dot{\rho} = \sum_i u_i(t) \mathcal{L}_i \rho. \]  

\[ (4.4.3) \]
Assuming that only a single control function is active at any time, this equation has a well-known solution
\[ \rho(t) = \prod_i e^{\int_{T_i}^{T_{i+1}} u_i(t) \mathcal{L}_i \, dt} \rho(0). \] (4.4.4)

Now assume that the control functions are constant; consider the evolution of the system when the control function \( u_j(t) = 1 \) which drives the initial state \( \rho(0) = \rho_i \) to a final state \( \rho(T_1) = \rho_f \):
\[ \rho_f = e^{\mathcal{L}_j T_j} \rho_i. \] (4.4.5)

Now consider the evolution of the same equation, but under the control function \( u(t) = -1 \), which drives the state from an initial state \( \rho(0) = \rho_f \) to a final state \( \rho(T_2) = \rho'_f \):
\[ \rho'_f = e^{-\mathcal{L}_j T_k} \rho_f = e^{-\mathcal{L}_k T_k} e^{\mathcal{L}_j T_j} \rho_i = e^{\mathcal{L}_j (T_j - T_k)} \rho_i. \] (4.4.6)

From this, it is clear that if \( T_j = T_k \), then \( \rho'_f = \rho_i \). Therefore, the control functions decoherently driving the state from \( \rho_\alpha \rightarrow \rho_\beta \) are simply the negative of the control functions decoherently driving the state from \( \rho_i \rightarrow \rho_f \). This logic can be applied to a sequence of control functions applied in succession, which has a solution
\[ \rho_f = \prod_i e^{\int_{T_i}^{T_{i+1}} u_{i+N}(t) \mathcal{L}_{i+N} \, dt} \prod_i e^{\int_{T_i}^{T_{i+1}} u_i(t) \mathcal{L}_i \, dt} \rho(0). \] (4.4.7)

Akin to the simplified single control scenario, if the successive controls are applied in reverse order with a negative control function value, the state can be driven from any final state back to the initial state.

\[ \square \]

### 4.5 System Model

The model under investigation is a semi-realistic optical cavity in which the electric field boundary conditions produce a standing wave, creating non-classical states of light: photon Fock states. The phrase "semi-realistic" implies that certain simplifying assumptions have been removed, specifically the assumptions that the cavity is at zero kelvin and leakless. This relaxation yields two methods of decoherence where
the system interacts with its environment, allowing photons to leave and enter the system at rates $\Gamma$ and $N_T$, respectively. This system also has three methods of control: a coherent laser field, and two tuneable aspects of the environment which can decoherently add and subtract photons from the system.

Recall that, in general, controlled quantum systems have the form

$$
\dot{\rho} = \frac{1}{i\hbar} \left[ H_0 + \sum_{i>0} u_i(t)H_i, \rho \right] + \mathcal{L}_0\rho + \sum_{j>i} u_j(t)\mathcal{L}_j\rho, \quad (4.5.1)
$$

where $H_0$ and $\mathcal{L}_0$ are the uncontrollable "drift" terms of the system, and $H_i$ and $\mathcal{L}_j$ are the coherent and decoherent control directions with control functions $u_i(t)$ and $u_j(t)$ respectively. This equation can also be represented in the form

$$
\dot{\rho} = \left( H_0 + \sum_{i>0} u_i(t)\mathcal{H}_i + \mathcal{L}_0 + \sum_{j>i} u_j(t)\mathcal{L}_j \right)\rho, \quad (4.5.2)
$$

where $\mathcal{H}_i = \text{ad}_{-iH_i}$ and $\mathcal{L}_j = \frac{1}{2}\{A_j^\dagger A_j, \rho\} - A_j\rho A_j^\dagger$ are the superoperators representing coherent and incoherent evolution of the density matrix.

The dynamics for the stated system of interest are governed by five superoperators: two for coherent and three for decoherent evolution. The coherent evolution superoperators can be quickly obtained from the Hamiltonians for the free evolution harmonic oscillator $H_0 = \omega_e a^\dagger a$ and from the coherent electric field $H_E = e^{-i\omega_L t}a^\dagger + e^{i\omega_L t}a$. As for the decoherent evolution, the operator $A$ corresponding to cavity leakage and controlled decoherent removal of photons is simply the photon annihilation operator.
a. Similarly, the operator corresponding to thermal and controlled decoherent photon injection is the photon creation operator $a^\dagger$. From these definitions, the set of superoperators which describe evolution of our system completely are

\[
\begin{align*}
\mathcal{H}_0 &= \text{ad}_{iH_0} \\
\mathcal{H}_1 &= \text{ad}_{iH_E} \\
\mathcal{L}_0 &= \alpha \mathcal{L}_1 + \beta \mathcal{L}_2 \\
\mathcal{L}_1\rho &= \frac{1}{2}\{a^\dagger a, \rho\} - a\rho a^\dagger \\
\mathcal{L}_2\rho &= \frac{1}{2}\{aa^\dagger, \rho\} - a^\dagger \rho a.
\end{align*}
\] (4.5.3 - 4.5.7)

4.6 Finite Ensemble Population Controllability

Since this is an infinite quantum system, full controllability cannot be proven. However, finite controllability on invariant subspaces can be shown. Recall that Theorem 4.3.1 states that the requirement for density matrix controllability is that the Lie algebra of the system has dimension \(N^2(N^2 - 1)\), while diagonal state controllability requires dimension \(\frac{N(N-1)}{2}\) from Theorem 4.3.1. The system Lie algebra generated from the operators in Equation 4.1 above. For simplicity, I defined two additional operators which are generated in the algebra: \(\mathcal{K}_1 = \text{ad}_{\gamma a^\dagger - \epsilon a}\) and \(\mathcal{K}_2 = \mathcal{L}_1 + \mathcal{L}_2\).

Note that there are only five linearly independent elements of this algebra, with \(\mathcal{L}_0\) and \(\mathcal{K}_2\) being constructible from \(\mathcal{L}_1\) and \(\mathcal{L}_2\). The smallest physical subspace of the state space \(\mathcal{P}\) is the subspace \(\mathcal{P}_2\), a 2 \times 2 block in the upper left of the density matrix which has dimension 2. The required dimension of the Lie algebra for density matrix controllability is \(2^2(2^2 - 1) = 12\). Since the system Lie algebra has dimension 5 < 12, the system is not density matrix controllable.

It is useful to note that the evolution of the density matrix for this system takes a particularly unique form when there is not an interacting laser field. When there is an interacting laser field, the \(|n\rangle\langle n|\) states are coupled to the \(|n \pm 1\rangle\langle n|\) and \(|n\rangle\langle n \pm 1|\) states, populating the off diagonal "coherence" elements of the density matrix. When
there is not an interacting laser field, as long as the initial state is diagonal, the density matrix remains diagonal for all time, which is shown in appendix ???. This quality allows for diagonal state controllability analysis of the system. Removing the laser interaction term from the algebra also removes $\mathcal{K}_1$, reducing the overall dimension of the algebra to three.

Now consider the set of nested finite dimensional subspaces

$$\mathcal{P}_{\text{sub}} := \{ \mathcal{P}_n | \mathcal{P}_1 \subset \mathcal{P}_2 \subset \cdots, \dim(\mathcal{P}_1) = 2, \rho_0 \in \mathcal{P}_1 \}.$$  \hspace{1cm} (4.6.1)

The subspace $\mathcal{P}_2$ is invariant under the subset of operators $\mathcal{G}_1 := \{ \mathcal{L}_1 \}$. Theorem 4.4.1 requires that the smallest subspace $\mathcal{P}_1$ be controllable using only the subset of operators which leave the subspace invariant. From Theorem 4.3.1 the required dimension of the algebra for diagonal state controllability is $2(2-1)/2 = 1 = \dim(\mathcal{G}_1)$. Therefore, the subspace $\mathcal{P}_1$ is diagonal state controllable. All the other subspaces in $\mathcal{P}_{\text{sub}}$ are also invariant under $\mathcal{L}_1$. Now consider the series of equations

$$\dot{x}_n = u_1(t)(nx_n - (n + 1)x_{n+1}),$$ \hspace{1cm} (4.6.2)

which correspond to the evolution of the system under a controlled $\mathcal{L}_1$ when the initial state is diagonal. It is clear that starting at some initial state in $\mathcal{P}_{k>1}$, the orbit of equation 4.6.2 contains a point in a lower dimensional subspace $\mathcal{P}_{i<k}$. As a result, by theorems ?? and 4.3.1 the system is finitely ensemble population controllable.
4.7 Observable Controllability of Photon Number

For the cavity system, the observable of interest is the photon number \( \hat{N} = a^\dagger a \).

Recall from chapter 2 that the expectation value of any observable \( \hat{O} \) in the density matrix formalism is given by equation 2.2.2:

\[
\langle \hat{O} \rangle(t) = \text{Tr}(\hat{O}\rho(t)).
\]

The time dependence of the expectation value can be found by taking its time derivative and evaluating the generated terms. Since the trace is a linear operator, the derivative applies to the density matrix directly. The von Neumann equation is first order in time; ergo this method will yield a first order linear differential equation governing the evolution of the operator’s expectation value. Substituting the number operator and applying this technique to Equation 4.5.2 yields the following:

\[
\partial_t \langle \hat{N} \rangle = (N_T - \Gamma) \langle \hat{N} \rangle + N_T + u_1(t) \left(-\langle \hat{N} \rangle\right) + u_2(t) \left(\langle \hat{N} \rangle + 1\right).
\]

The same methods applied to deriving the evolution of the expectation value also apply to the evolution of the variance. Recall that variance is defined as

\[
(\delta \hat{O})^2 = \langle (\hat{O} - \langle \hat{O} \rangle)^2 \rangle = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2.
\]

Substituting in the number operator \( \hat{N} \) as the observable, taking the time derivative, and applying similar techniques as in the derivation of the evolution of the expectation value...
value yields the following:

\[
\partial_t (\delta \hat{N})^2 = 2(N_T - \Gamma)(\delta \hat{N})^2 + (N_T + \Gamma)\langle \hat{N} \rangle + N_T \\
+ u_1(t)(\langle \hat{N} \rangle - 2(\delta \hat{N})^2) \\
+ u_2(t) \left(2(\delta \hat{N})^2 + \langle \hat{N} \rangle + 1\right).
\] (4.7.2)

The differential equation governing the dynamics of the variance of the photon number is dependent on both the variance and the expectation value of the photon number; hence, the two equations must be solved simultaneously. Combining this with the equation for the evolution of the expectation value, the following set of differential equations are obtained:

\[
\partial_t \begin{bmatrix} \langle \hat{N} \rangle \\ (\delta \hat{N})^2 \end{bmatrix} = \begin{bmatrix} N_T - \Gamma & 0 \\ N_T + \Gamma & 2(N_T - \Gamma) \end{bmatrix} \begin{bmatrix} \langle \hat{N} \rangle \\ (\delta \hat{N})^2 \end{bmatrix} + \begin{bmatrix} N_T \\ N_T \end{bmatrix} \\
+ u_1(t) \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \langle \hat{N} \rangle \\ (\delta \hat{N})^2 \end{bmatrix} \\
+ u_2(t) \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \langle \hat{N} \rangle \\ (\delta \hat{N})^2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\] (4.7.3)

Throughout the rest of this section, the expectation value and variance are defined by the variables \( x_1 = \langle \hat{N} \rangle \) and \( x_2 = (\delta \hat{N})^2 \) respectively. This notation is used to help better visualize the dynamics of the system in a notation familiar to differential equations and control theory. Also, new control functions \( u'_1(t) = u_1(t) + \Gamma \) and \( u'_2(t) = u_2(t) + N_T \) are defined to simplify the analysis.

### 4.7.1 Weak Observable Controllability

By definition, weak observable controllability concerns itself solely with the expectation value of the operator \( \hat{N} \). Using the new notation, the expectation value evolves
according to equation 4.7.1

\[ \dot{x}_1 = u'_1(t) (-x_1) + u'_2(t) (x_1 + 1). \] (4.7.4)

Clearly, this equation is controllable; however, for the sake of completeness, the solutions to the system are analyzed rigorously. The state manifold for this system is \( M = \mathbb{R}_0^+ \). Due to the inhomogeneity of the equations, group theoretic analysis is not helpful; therefore, a differential equations-based treatment is utilized. Since the equation is a first order linear differential equation, the solutions under various conditions can be found quickly. Equation 4.7.4 can evolve in three different ways: under the control of \( u'_1 \) alone, under the control of \( u'_2 \) alone, and under the control of a combination of \( u'_1 \) and \( u'_2 \). Assuming constant control functions \( u'_1 = k_1 \), and \( u'_2 = k_2 \), and an initial condition \( x_1(0) = x_{1,0} \), the solutions of equation 4.7.4 are

\[
\begin{align*}
    x_1(t) &= x_0 e^{-k_1 t} \quad u'_1 = k_1 \quad (4.7.5) \\
    x_1(t) &= (x_0 + 1) e^{k_2 t} - 1 \quad u'_2 = k_2 \quad (4.7.6) \\
    x_1(t) &= \left(x_0 + \frac{k_2}{k_2 - k_1}\right) e^{-k_1 t} \quad u'_1 = k_1 \text{ and } u'_2 = k_2 \quad (4.7.7) \\
    x_1(t) &= kt + x_0 \quad u'_1 = u'_2 = k, \quad (4.7.8)
\end{align*}
\]

where the fourth solution is the case when solution 3 is singular. Each of these solutions spans the positive real line, the manifold of the system, from any point \( x_0 \) on the real number line. From the basic definition of controllability (def 3.3.1), the expectation value of \( \hat{N} \) is controllable; subsequently, the the Photon number is Weakly Observable Controllable.

### 4.7.2 Strong Observabe Controllability

For strong observable controllability, the interest is in controlling both the expectation value and the variance. For the case of the photon number operator, this expands the state space manifold to \( \mathbb{R}_0^+ \otimes \mathbb{R}_0^+ \). In the new notation, the differential equation
representing the evolution of these operators is

\[
\partial_t \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = u_1'(t) \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u_2'(t) \left( \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right). \tag{4.7.9}
\]

Determining strong observable controllability is a non-trivial problem. Lifting the system to the group \( GL(2) \) does not simplify the system, as the differential equation cannot be immediately expressed as a right invariant vector field on \( M \). However, since the system is a second order set of differential equations, it is feasible to analyse the solutions of the differential equation directly. Considering the evolution of the system under the three different possible control permutations \( u_1' = k_1 \), \( u_2' = k_2 \), and \( u_1' = k_1, u_2 = k_2 \), the solutions to this differential equation under those conditions with the initial value \( x(0) = \begin{bmatrix} x_1 & 0 & x_2 & 0 \end{bmatrix}^T \) are

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_{2,0} - x_{1,0})e^{-2k_1t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_{1,0}e^{-k_1t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{4.7.10}
\]

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (x_{1,0} + x_{2,0} + 1)e^{2k_2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (x_{1,0} + 1)e^{k_2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}. \tag{4.7.11}
\]

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \left( x_{2,0} + \frac{k_2^2}{k_1 - k_2} - \frac{(k_1 - k_2)x_{1,0}}{(k_1 - k_2)^2} \right) e^{2(k_2-k_1)t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \left( x_{1,0} - \frac{k_2}{k_1 - k_2} \right) e^{(k_2-k_1)t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{k_2}{k_1 - k_2} \\ \frac{k_1 - k_2}{(k_1 - k_2)^2} \end{bmatrix}. \tag{4.7.12}
\]

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kt + x_{1,0} \\ k^2t^2 + 2kx_{1,0} + k) + x_{2,0} \end{bmatrix}. \tag{4.7.13}
\]

For strong observable controllability, one of the equations \(4.7.10\) to \(4.7.13\) should have a solution for an arbitrary final condition \( x = [x_{1,f}, x_{2,f}]^T \). The only equation which has the possibility of having a solution for any \( k_1 \) and \( k_2 \) is equation \(4.7.13\). Neither by hand analysis nor the computer algebra capabilities of MATLAB could determine solutions for this equation yielded any positive or negative results.
\[ i \rightarrow i \]
Chapter 5

Conclusions and Further Work

This thesis develops three new definitions of quantum controllability applicable to open quantum systems and redevelops another previously constructed definition and theorem to a more general form. These new definitions of controllability, as well as standard definitions from the literature, are applied to an open quantum optical system consisting of a leaky cavity in a tuneable environment pumped by a laser. The described system was found to not be Density Matrix Controllable. However, without a pumping laser, the state of the system was found to be diagonal through its evolution. In addition, the Lie algebra of the system satisfied the rank condition for accessibility, and the newly developed controllability theorem was successfully used to prove Ensemble Population Controllability for the system on a small subspace. The modified finite controllability theorem was also successfully used to show that the system is controllable on an arbitrarily large subspace of the infinite dimensional vector space. Determining observable controllability proved difficult. Weak Observable Controllability was shown using methods of differential equations, while Strong Observable Controllability remains evasive, but without definite negative results.

Further work in this field includes obtaining either a positive definite or negative definite result on strong observable controllability for the quantum optical system. Also, a cavity which is coupled to an atom in addition to the other conditions presented in this thesis should be investigated using both the previous and newly developed techniques of control theory. Effects of restrictions on the values of the control
functions need to be investigated as well, in both a classical and quantum sense. It is possible that the results of this thesis, as well as the body of present literature, are misleading, as this aspect of quantum controllability has not been fully developed. A possible outcome of controllability analysis is that the controllability conditions might allow for control functions which generate nonphysical states.
Appendix A

Lie Theory

Two of the most commonly used mathematical tools in both control theory and physics are the concepts of Lie groups and Lie Algebras. The mathematics of Lie groups and algebras deals with the concepts of topologies and manifolds. Some preliminary concepts must be defined.

A.1 Lie Groups

A.1.1 Definitions

Definition A.1.1 (Lie Group). A Lie Group is a smooth manifold $G$ with an operation $\cdot$ which follows the following axioms:

1. $a \cdot b \in G$ for all $a, b \in G$
2. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b \in G$
3. There exists an identity element $\mathbb{I} \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$
4. There exists an inverse element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$ for all $a \in G$
5. $a \cdot b^{-1}$ is smooth for all $a, b \in G$
This definition is sufficient in order to continue with the work. There are also two important properties of Lie groups, connectedness and compactness, which appear in later theorems and proofs and hence need to be defined.

**Definition A.1.2 (Compact Group).** A compact group is a topological group where every open cover has a finite subcover.

**Definition A.1.3 (Connected Group).** A connected group is a group which cannot be represented as the union of two or more disjoint nonempty open sets.

### A.1.2 Examples

The set of $N \times N$ invertible matrices forms over the field $\mathbb{F}$ forms a disconnected noncompact Lie group called the *General Linear Group*:

$$GL(N, \mathbb{F}) := \{X \in \mathbb{F}^{N \times N} | \det(X) \neq 0\}. \quad (A.1.1)$$

The two disconnected subgroups are the matrices with positive determinant and negative determinant. $GL^+(N, \mathbb{F})$ is the subgroup that contains the identity element. In place of defining the general linear group specifically as a set of invertible matrices over a field, it is possible to define it as the set of invertible transformations over a vector space $\mathcal{V}$.

The subgroup of the general linear group which has unit determinant is called the special linear group $SL(N, \mathbb{F})$

$$SL(N, \mathbb{F}) := \{X \in \mathbb{F}^{N \times N} | \det(X) = 1\}. \quad (A.1.2)$$

The set of complex $N \times N$ matrices whose inverse is their conjugate transpose forms a Lie group called the *Unitary Group* $U(N)$. If, in addition, the matrices are of determinant 1, then the group is the Special Unitary Group $SU(N)$

$$U(N) := \{X \in \mathbb{C}^{N \times N} | X^\dagger X = XX^\dagger = I\}. \quad (A.1.3)$$
\[ SU(N) := \{ X \in \mathbb{C}^{N \times N} | X^\dagger X = XX^\dagger = I, \det(X) = 1 \}. \quad (A.1.4) \]

The set of real \( N \times N \) matrices whose inverse is their transpose forms a Lie group called the Orthogonal Group \( O(N) \). If, in addition, the matrices are of determinant 1, then the group is the Special Orthogonal Group \( SO(N) \):

\[ O(N) := \{ X \in \mathbb{R}^{N \times N} | X^T X = XX^T = I \} \quad (A.1.5) \]

\[ SO(N) := \{ X \in \mathbb{R}^{N \times N} | X^T X = XX^T = I, \det(X) = 1 \}. \quad (A.1.6) \]

\section*{A.2 Lie Algebras}

\subsection*{A.2.1 Definition}

A Lie algebra is a vector space \( \mathfrak{g} \) over a field \( \mathbb{F} \) with an operation \([\cdot, \cdot]\) called the Lie Bracket which satisfies the following axioms:

1. \([ax + by, z] = a[x, z] + b[y, z]\) for all \( x, y, z \in \mathfrak{g} \) and \( a, b \in \mathbb{F} \)

2. \([x, x] = 0\) for all \( x \in \mathfrak{g} \)

3. \([a, [y, z]] + [z, [x, y]] + [y, [z, x]]\]

\subsection*{A.2.2 Examples}

Any vector space \( V \) endowed with a Lie bracket is a Lie algebra.

If the vector space is the set of \( N \times N \) matrices, the algebra is the General Linear Lie algebra \( \mathfrak{gl}(N) \).

If the vector space is the set of traceless \( N \times N \) matrices, the algebra is the Special Linear Lie algebra \( \mathfrak{sl}(N) \).

If the vector space is the set of skew-symmetric \( N \times N \) matrices, the algebra is the Orthogonal Lie algebra \( \mathfrak{o}(N) \).
If the vector space is the set of traceless skew-symmetric $N \times N$ matrices, the algebra is the Unitary Lie algebra $u(N)$.

A.3 Group Lift

One technique in analysis of dynamical systems is to "lift" the system from actions on a submanifold of a field such as $\mathbb{R}$ or $\mathbb{C}$ to actions on a group. For example, consider the Schrödinger equation in atomic units

$$\dot{\psi} = -i\hat{H}\psi, \quad \psi \in \mathbb{C}^N \quad \psi(0) = \psi_0$$

which has a trivial solution

$$\psi(t) = e^{-i\hat{H}t}\psi_0.$$ 

Note that $-i\hat{H}$ is a skew hermitian operator and hence is an element of the Lie algebra $u$. The exponential map, which is defined for all $t$, then maps the element of the Lie Algebra $u$ to the Lie group $SU$. From this, the Schrödinger equation can be written as

$$\dot{X} = -i\hat{H}X, \quad X \in U, \quad X(0) = I,$$

which is known as the Lifted Schrödinger equation.
Appendix B

Diagonal State Evolution of the Lindblad Equation

Proof. For an imperfect quantum optical cavity within an engineered environment, the equation governing the system dynamics is

\[ \dot{\rho}(\rho, t) = (u_1(t) + \Gamma)\mathcal{L}_1 \rho + (u_2(t) + N_{\text{therm}})\mathcal{L}_2 \rho, \]  

(B.0.1)

where \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) are the Lindblad superoperators:

\[ \mathcal{L}_1 \rho = a^\dagger a \rho + \rho a^\dagger a - 2a \rho a^\dagger \]  

(B.0.2)

\[ \mathcal{L}_2 \rho = aa^\dagger \rho + \rho aa^\dagger - 2a^\dagger \rho a. \]  

(B.0.3)

A first order approximation to the evolution at time \( t_0 + \delta t \) using Euler’s method is

\[ \rho_{i+1} = \rho_i + \dot{\rho}(\rho_i, t_i) \delta t. \]

Assuming the initial state is a general diagonal state

\[ \rho_0 = \sum_N P_n |n\rangle \langle n|, \]
the density matrix at time $t_0 + \delta t$ is
\[
\rho_1 = \rho_0 + [(u_1(t_0) + \Gamma)\mathcal{L}_1 \rho_0 + (u_2(t_0) + N_{\text{therm}})\mathcal{L}_2 \rho_0].
\]
Evaluating the action of the superoperators $\mathcal{L}_1$ and $\mathcal{L}_2$ on the state $\rho_0$ yields
\[
\rho_1 = \rho_0 + \left[(u_1(t_0) + \Gamma) \sum_n 2n P_n (|n\rangle\langle n| - |n - 1\rangle\langle n - 1|) + (u_2(t_0) + N_{\text{therm}}) \sum_n 2(n + 1) P_n (|n\rangle\langle n| - |n + 1\rangle\langle n + 1|)\right] \delta t.
\]
Since $\rho_0$ is diagonal, and the term in square brackets is also diagonal, the evolution of $\rho$ remains diagonal for all time. \qed
Bibliography


[15] Andrew D. Lewis. A brief on controllability of nonlinear systems. Technical report, Queen’s University, Department of Mathematics and Statistics, Queen’s University, Kingston, ON K7L 3N6, Canada, Nov. 2001.


