Nonlinear Finite Element Analysis of Laminated Composite Beams Subjected to Harmonic Excitations Using a 20 DOF Beam Element

Mansour Nosrati Kenareh
Embry-Riddle Aeronautical University - Daytona Beach

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by

Mansour Nosrati Kenareh

A Thesis Submitted to Office of Graduate Programs in Partial Fulfillment of the Requirements for the Degrees of Master of Science in Aerospace Engineering

Embry-Riddle Aeronautical University Dayton Beach, Florida May 1994
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Mansour Nosrati Kenareh

This thesis was prepared under direction of the candidate's thesis committee chairman, Dr. Habbib Eslami, Department of Aerospace Engineering, and has been approved by the member of his thesis committee. It was submitted to the Office of Graduate Program and accepted in partial fulfillment of the requirements for the degree of master of Aerospace Engineering.

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ABSTRACT

Author: Mansour Nosrati Kenareh

Title: Nonlinear Finite Element Analysis of Laminated Composite Beams Subjected to Harmonic Excitations Using a 20 DOF Beam Element

Institution: Embry-Riddle Aeronautical University
Degree: Master of Science in Aerospace Engineering
Year: 1994

The purpose of this Thesis is to study the nonlinear analysis of antisymmetrically laminated composite beams including shear deformation subjected to harmonic excitation, using a 20-degree of freedom finite element beam. The beam has 10 degrees of freedom at each node: The axial displacements, the transverse deflection due to bending and transverse shear, the twisting angle, the in-plane shear rotation, and their derivatives along the axial direction. In this study, the effect of different parameters such as damping, shear deformation and different edge conditions on the steady-state frequency-responce will be investigated. The analysis was based on the use of finite element methodology for composite laminated beam structures. The harmonic force matrix represents the externally applied force in matrix form, instead of a vector form. Thus the analysis of nonlinear forced vibration can be performed efficiently to get a converged solution. The analysis was also based on the nonlinear stiffness matrix and both in-plane longitudinal, and transverse deflections are included in the formulation. The amplitude-frequency ratios for different boundary conditions, lamination angles, number of plies and thickness to length ratios are presented. The finite element results are compared with available approximate continuum solutions.
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<th>Symbol</th>
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<tbody>
<tr>
<td>[A]</td>
<td>Extension Stiffness Matrix</td>
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<td>[B]</td>
<td>Coupling Stiffness Matrix</td>
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<td>[D]</td>
<td>Bending Stiffness Matrix</td>
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<td>[k]</td>
<td>Element Stiffness Matrix</td>
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<td>[K]</td>
<td>Global Stiffness Matrix</td>
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<tr>
<td>[M]</td>
<td>Mass Matrix</td>
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<tr>
<td>[n₁]</td>
<td>Element First Order Nonlinear Stiffness Matrix</td>
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<tr>
<td>[n₂]</td>
<td>Element Second Order Nonlinear Stiffness Matrix</td>
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<td>[N₁]</td>
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<td>[q]</td>
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<td>[Bₜ]</td>
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<td>[Bₜₑ]</td>
<td>Nonlinear Strain-Displacement Matrix</td>
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<tr>
<td>[T]</td>
<td>Transformation Matrix</td>
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<tr>
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<td>Strain Matrix</td>
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<td>{κ}</td>
<td>Curvature Matrix</td>
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<tr>
<td>[σ]</td>
<td>Reduced Stiffness Matrix</td>
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<tr>
<td>[σ̃]</td>
<td>Transformed Reduced Stiffness Matrix</td>
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<tr>
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<td>Resultant Force per Unit Length</td>
</tr>
<tr>
<td>Mₓ , Mᵧ , Mₓᵧ</td>
<td>Resultant Moment per Unit Length</td>
</tr>
<tr>
<td>N₁ , N₂ , N₃ , N₄</td>
<td>Hermitian Polynomials</td>
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<tr>
<td>x , y , z</td>
<td>Coordinate System</td>
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<td>u</td>
<td>Axial Deformations</td>
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<tr>
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<td>Transverse Deflection due to Bending</td>
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<tr>
<td>θₑ</td>
<td>Rotation Due to Bending</td>
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<tr>
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<td>Rotation Due to Transverse Shear</td>
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<tr>
<td>τ</td>
<td>Twisting Angle</td>
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<tr>
<td>β</td>
<td>In-Plane Shear Rotation</td>
</tr>
<tr>
<td>λ</td>
<td>Eigenvalue</td>
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<tr>
<td>ωₒ</td>
<td>Natural Frequency of Beam</td>
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<tr>
<td>L</td>
<td>Length of the Beam</td>
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<tr>
<td>b</td>
<td>Width of the Beam</td>
</tr>
<tr>
<td>h</td>
<td>Thickness of the Beam</td>
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<tr>
<td>E</td>
<td>Modulus of Elasticity</td>
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<tr>
<td>G</td>
<td>Shear Modulis</td>
</tr>
<tr>
<td>I</td>
<td>Moment of Inertia</td>
</tr>
<tr>
<td>J</td>
<td>Polar Moment of Inertia</td>
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<tr>
<td>ν</td>
<td>Poissons' Ratio</td>
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<tr>
<td>Symbol</td>
<td>Description</td>
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<td>------------------------------</td>
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<tr>
<td>ρ</td>
<td>Average Mass Density</td>
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<tr>
<td>U</td>
<td>Strain Energy</td>
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<tr>
<td>W</td>
<td>Work Done by Forces</td>
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<tr>
<td>T</td>
<td>Kinetic Energy</td>
</tr>
<tr>
<td>H</td>
<td>Weigh Number</td>
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<tr>
<td>F</td>
<td>Nodal Force and Moment</td>
</tr>
<tr>
<td>P</td>
<td>Potential Energy</td>
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<tr>
<td>Θ</td>
<td>Translational Inertia</td>
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Chapter 1

1.0 Introduction

1.1 Motivation

The aerospace industry has always been a promoter of the advancement of materials, and currently, the increasing use of advanced composites as high performance structural members for aircraft, missiles, launched vehicles, and spacecraft, necessitates prediction methods which reflect their multilayered anisotropy behavior. Vibration problems have been known as one of the most important problems encountered in the design and modifications of most of the engineering problems. As it is shown in fig. (1.1), the problem of vibration of an aircraft wing under aerodynamic loads, introduces an interesting area for analysis of laminated composite plates, in particular the section of a wing skin under transverse excitations due to the lift distribution on the wing. And studies have shown that the combination of transverse and longitudinal excitation will bring about chaotic vibration on the wing skin.

![Diagram of laminated beam under different loading](image)

Fig. (1.1) Illustration of laminated beam under different loading.
The isotropic materials, such as metals and pure polymers, are known to most engineers and material scientists, they have identical mechanical, physical, thermal and electrical properties in every direction. On the other hand, materials exhibiting directional characteristics are called anisotropic materials, for example, composite structures which are composed of two or more materials to form a useful material which exhibits the best qualities of each constituents. Composite materials have been widely used in the aerospace industry, because of their superior strength-to-weight and stiffness-to-weight ratio, as compared to conventional materials. In addition to the above mentioned characteristics, composite structures possess excellent fatigue strength, ease of formability, and negative coefficient of thermal expansion which result in materials with almost unlimited potential. Since the composite panels consist of individual lamina bonded together and since the principal direction may differ from layer to layer, the vibration behavior of such panels may be more complex than conventional panels.

Among different kinds of composite materials, the most widely used are categorized under fibrous, laminated, and particulate composites. The fibrous and laminated composites are the most popular because of their high young's moduli to withstand demanding design requirements.

![Fiber Composite](image1.png) ![Laminar Composite](image2.png)

Figure (1.2) Illustration of Various Types of Composite Material

In fibrous composites, the fibers are bonded together with proper matrix material to distribute the applied loads from fiber to fiber. The laminated composites, consists of a stack of layers to
together, and the orientation of the layer can be different in order to strengthen the structure or give it a property that would best fit the design specification. With the rapid increase of composite materials technology, they are distinguished from some of the more familiar fibers such as glass by having new ultra-high strength and stiffness fibers such as boron, graphite, and kevlar. The matrix material is used to encase the fibers and to transfer loads from fiber to fiber through shear flow in the matrix.

The matrix materials are ductile, of low strength, and have low modulus of elasticity. The most common matrices are polymides and epoxies. They are easy to work with, have excellent creep and fatigue properties, and provide a good bonding between fibers. For a time, the most widely used polymer matrices were epoxy resin. But they proved to deteriorate in humid environments over 350 degrees, therefore new thermoplastic resins, which are less brittle and have a wide range of operating temperatures, are now being considered. Advanced composites can have the same strength and stiffness as high-strength steel, yet are 70% percent lighter.

The work done so far in the area of fiber-reinforced composites is limited, but studies have shown that thin laminated composites subjected to severe periodic transverse loading are likely to encounter flexural oscillations having amplitudes of the order of thickness or even higher, which has a direct relation to the material properties and geometrical configurations [6]. Under these circumstances, the physical behavior of such structures cannot be predicted using linear theory (small deflection theory). Thus, the large deflection analysis (nonlinear theory) is of paramount importance. In this study the von Karman type geometrical nonlinearity was incorporated into the formulation of the problem. The work contained here includes an easy to use finite element program which has a modular programming structure and is also highly capable and adaptable for expansion of research in this area. The programming is based on a FORTRAN 77 code which
consists of subroutines of less than one page length, and the mathematical analyses has been done using the IMSL FORTRAN library.

1.2- Review of Literature

With the growing use of laminated composite structures, there has been a number of research activities dealing with the behavior of such structures and the finite element method has been the predominant tool in accomplishing this task. The existing isotropic and homogeneous finite element programs are capable of analyzing the complicated structures, and they are widely used in industry. However, the analysis of composite structures is more complicated compared that for isotropic materials, because of the anisotropic property which is characterized by bending-stretching, bending-twisting coupling and inplane-shear that takes place under loading. In a linear analysis of isotropic and anisotropic materials, the equilibrium equations, strain-displacement relations and compatibility equations remain the same. However, the stress-strain relations (constitutive relations) which account for anisotropy of composite materials, are different. The increasing demand for more realistic models of structural analysis has encouraged research for solution techniques to deal with nonlinear structural problems.

Most studies have been focused on symmetrically laminated plates which are commonly used and do not exhibit bending-stretching coupling. However, unsymmetrical laminates have proven to be useful for certain practical application where this property can be used to best fit the design criteria; for example, like the skin of aircraft structures for aerodynamic tailoring. Classical beam theory is based on the Bernoulli-Euler assumption that plane sections initially normal to the mid-plane remain plane and normal to the mid-surface after bending. This can lead to a somewhat high percentage of error when analyzing beams or plates composed of anisotropic materials. A
number of research activities considered the effects of shear deformation and rotary inertia on the vibration of isotropic beams. [8], [25], [21]

This thesis focuses on composite beams, but the literature survey covers both isotropic beams and plates. Shames and Dym [34], compared the dimensionless frequency involving only rotatory inertia with the dimensionless frequency of the classical beam theory, they showed that the difference between the two is very small in low frequency modes and for large slenderness ratio. However, for short stubby beams or for higher modes the difference can be significant. For a similar behavior is observed when comparing the dimensionless frequency involving only shear effects to the dimensionless frequency of the classical beam theory. Furthermore, the importance of transverse shear deformation in symmetrically laminated composite beams is discussed by, [10], [11], [31], [19], [24], [28]. The transverse shear deformation significantly affects the lateral displacement, the natural frequency of vibration, and the buckling loads.

A large number of references exist on nonlinear free and forced vibration of isotropic beams and plates, e.g. Mei [18] has done an extensive study in this area. Some work has also been carried out on nonlinear forced vibrations of isotropic and composite plates. Also, there have been number of studies on methods including shear deformation in the analysis of isotropic plates, like, Thomas and Abbas [42] developed a model for the dynamic analysis of Timoshenko's beams which can satisfy all the required force and natural boundary conditions. In this study, they included shear deformations in their analysis by splitting the slope of the deflection curve into the rotation due to direct bending and induced bending due to twisting, and also, the shearing angle of distortion due to the application of a shear force. Their beam element was the first finite element model that could satisfy all of the boundary conditions of a Timoshenko beam. Murty and Shimpi [23] in their studies, included the transverse shear deformations and used an isoparametric rectangular element
using the linearized Reissner-type variational formulation to study large amplitude free vibrations of plates.

Teoh and Huang [44] investigated the free vibration analysis of fiber-reinforced beams including the shear deformation which had a considerable effect on the bending mode, but not on the twisting mode. Hinrichsen and Palazotto [13] developed a quasi-three dimensional finite element using a displacement field expressed with respect to a reference surface plus the displacements through the thickness. The through-the-thickness displacements are modeled using a cubic spline function, and the rotations at interlaminar boundaries are allowed to have degrees of freedom. The nonlinear analysis, which allows large displacements with moderately large rotations and small strains, is formulated using a Lagrangian approach, and solved by the Newton-Raphson technique. Stein and Jegley [40] investigated the effects of transverse shear deformation on cylindrical bending, vibration, and buckling of laminated plates. The displacements for cylindrical bending and stretching of laminated and thick plates are expressed through the thickness by a few algebraic terms and a complete set of trigonometric terms. Only a few terms of this series are needed to get sufficient accurate results for laminated and thick plates. Equation of equilibrium, based on a sufficient number of terms of this series for displacements, are determined using variational theorems from elasticity. Takahashi [45] attempted to minimize the degrading influence of coupling, and he showed that by using a specific thickness distribution of the layers, coupling can be partially or totally eliminated and hence improve the response characteristics of the laminate. Sathyamoorthy [37] has done extensive work and made advances in theoretical and computational procedures for the nonlinear analysis of beams using the finite element method. In that study, the most preferred approach was the Raleigh-Ritz type of finite element approach. However, the Galerkin type of finite element approach has proven to be successful for the geometrical nonlinear
analysis of isotropic beams. Also Prathab and Varadan, T.K., and Bhashyam [26], [5] investigated the study of nonlinear vibration of isotropic beams by using the Galerkin finite element method.

Sarma and Varadam [36] presented a Ritz finite element approach to study the nonlinear vibration of isotropic beams, and later, by using a Lagrange-type formulation in terms of transverse displacement, solved the equation of motion by a Ritz finite element. Chia [8] has done studies on the postbuckling and nonlinear vibration of plates, both isotropic and anisotropic. In this study he showed that odd power derivatives are introduced in the general differential equations which can no longer be solved by the same procedures used for symmetric laminates. Mizuno [22] used the finite element method to analyze thermal buckling, post-buckling, and free vibrations of thermally buckled composite beams including shear deformation. For buckling analysis, the eigenvalue problem was solved for the critical temperature. The scaled first mode shape was used as the trial displacement vector for post-buckling deflection analysis, employing a Newton-Raphson type iterative procedure. In this study, the effect of boundary conditions and temperature on the frequency of the buckled beam were considered. Sivakumaran and Chia [41] studied the large amplitude oscillations of unsymmetrically laminated plates including shear, rotary inertia, and transverse (normal) loads. They showed that odd power derivatives which were used in the general differential equations for symmetric laminates can no longer be used to study the analysis of unsymmetric laminates.

Reddy [31] investigated the forced vibration of laminated plates using a finite element method including transverse shear deformation and rotary inertia using the von Karman type nonlinearity. In his paper, he presents numerical results for the nonlinear analysis of composite plates, and points out the effects of the plate thickness, boundary conditions and loading on the deflection and stresses. He further studied, [32] the effect of coupling on the transient response of laminated
plates, and also developed an isoparametric rectangular element to study nonlinear free vibrations of laminated composite plates. Later he reviewed the application of finite element methods in the analysis of linear and nonlinear anisotropic composite plate problems, the conventional finite element method was based on the total potential energy expression, whereas the mixed method was based on the Reissner-type variational statement and involved the bending moments and deflections as primary dependent variables. From their result, it appeared that, in general, the mixed method yielded better accuracy. Liu and Ready [17] used a refined high-order shear deformable finite element to study nonlinear vibrations of laminated rectangular plates. They showed that as the number of layers is increased the coupling effect dies out for antisymmetric cross-ply laminates, but decreases much slower for unsymmetric laminates. Also Ready and Chao [33] predicted values of natural frequency that are 11% higher than those predicted by the 3-D elasticity theory. This behavior can be seen when transverse shear deformation is neglected. They later, extended the earlier isoparametric rectangular element to include transverse shear in laminated composite materials. Whitney and Leissa [47] formulated the basic governing equations for nonlinear vibration of heterogeneous anisotropic plates in the sense of von Karman. Based on these equations various approximate procedures have been investigated by numerous researchers. The finite element method has proven to be an extremely powerful tool for solving structural problems with complex geometry and loading. Their study agrees with results from a variety of research done on unsymmetrically laminated plates indicating that the bending-stretching coupling effects increase transverse deflection, decrease fundamental frequencies of vibration, and decrease buckling loads. They also showed that the classical plate theory predicts natural frequencies that are 25% higher
than studies which considered a plate shear deformation theory for plates with a width to thickness ratio of 10.

Rao et al.[30] presented a simplified finite element formulation for the large-amplitude free vibration of orthotropic rectangular and circular plates. In that study they proposed a novel scheme of linearizing the nonlinear strain-displacement relations formulating the nonlinear stiffness matrix. Shear deformation and rotary inertia term were also included in the formulation of the problem to study the free flexural vibrations analysis of a simply-supported beams. The results showed that the effects of longitudinal displacement and inertia were to reduce the nonlinearity in the flexural frequency-amplitude relationship. Simply supported and clamped beams were investigated, and comparison of their results with the earlier work confirm the reliability and effectiveness of the linearization of the strain-displacement relations. Kapania and Raciti [16] developed a simple one-dimensional finite element model for a free vibration nonlinear analysis of symmetrically and unsymmetrically laminated composite beams including shear deformation. For unsymmetrically laminated beams, Raciti found out that the nonlinear vibrations had a soft spring behavior for certain boundary conditions as opposed to a hard spring behavior observed in isotropic and symmetrically laminated beams. The in-plane boundary conditions were found to have a significant effects on the nonlinear responses.

Singh, Rao and Iyengar [38] investigated large-amplitude free vibrations of unsymmetrically laminated composite beams using the von Karman large deflection theory. They applied a one dimensional finite element based on the classical lamination theory, first-order shear-deformation theory and higher-order shear-deformation theory having 8, 10, and 12 degrees of freedom per node, respectively. This was done to bring out the effects of transverse shear on the large-amplitude vibrations. Because of the presence of bending-extension coupling, the bending
stiffness of an unsymmetric laminate becomes direction dependent yielding different amplitudes and spatial deformations for the positive and negative deflection half-cycle. Bangera and Chandrashekhara [2] developed a finite-element model to study the large-amplitude free vibrations of generally-layered laminated composite beams. They considered the effects of the Poisson's ratio by including it in the constitutive equation. The direct iteration method was used to solve the nonlinear equation at the point of reversal of motion. The influence of boundary conditions, beam geometry, Poisson effect, and ply orientation on the nonlinear frequencies and mode shapes were demonstrated. Tseng and Dugundji [46] used a multiple-mode expansion in considering the forced response of a clamped beam about its buckled configuration. The buckled beam was excited by the harmonic motion at its supporting base. By using Galerkin's method, the governing partial differential equation was reduced to a modified Duffing equation which was solved by the harmonic balance method. They also conducted the experiment on a straight beam with fixed ends, and the result compared favorably with the corresponding theoretical prediction.

Alturi [1] applied the method of multiple scales to investigate the response of the nonlinear forced vibration of a hinged isotropic beam considering nonlinear inertia terms. He showed that for some cases the non-linearity was of the softening type. Srinivasan [39] solved the free and forced responses of isotropic beams subjected to moderately large-amplitude steady-state oscillations by the Ritz averaging method. The application of this method transforms the governing partial differential equation into a system of nonlinear algebraic equations. He then applied Newton's method to solve those equations. Nayfeh, Mook and Lobitz [25] presented a numerical perturbation method for the nonlinear analysis of forced vibration of isotropic beams. A multiple-mode expansion in terms of the linear mode shapes was considered. The problem was then solved using the method of multiple scales, considering internal resonance.
Some research on nonlinear forced vibrations of composite beams has also been carried out by using various analytical methods such as Galerkin's or Ritz method based on the simplified Berger's hypotheses, Kantorovich averaging method, various perturbation methods and an incremental harmonic balance method. Pai and Nayfeh [28] investigated the forced nonlinear vibration of a symmetrically laminated graphite-epoxy composite beam. The analysis focused on the case of primary resonance of the first flexural-torsional mode. A combination of the fundamental-matrix method, a Galerkin procedure and the method of multiple scales was used to derive four sets of first-order, ordinary differential equations describing the modulations of the amplitudes and phases of the interacting modes with damping, non-linearity and resonance. The results showed that the motion was non-planar although the input was planar. It was further concluded that non-planar responses may be periodic motions, amplitude and phase modulated motions, or chaotically modulated motions.

Chandrashekhara [7] considered the flexural analysis of fiber-reinforced composite beams based on higher-order shear deformation theory. A von Karman type of nonlinearity was incorporated in the formulation of the problem. The finite-element method was used to solve the nonlinear governing equations by direct iteration. Unlike the conventional beam models, Chandrashekhara took into account the y direction strains. The author investigated the differences in the solutions for the cross-ply laminates and the angle-ply laminates. He concluded that the solution obtained from the two approaches differ slightly in the case of the cross-ply laminates, but there exists a considerable difference in the case of angle-ply.

Mei [20] applied the finite element method to large amplitude free vibration of rectangular plate, in that study, the in-plane tensile force induced by the transverse deflection alone was assumed to be constant for each individual plate element. The nonlinear frequencies of rectangular plates with
various boundary conditions agree well with the approximate continuum solution of Chu and Herrmann [55], and Yamaki [56].

Mei and Decha-Umphai [18] were the first to use the finite element methodology to determine the nonlinear harmonic forced response of thin isotropic plates. A force matrix under uniform harmonic excitations is developed for a nonlinear forced vibration analysis. The results obtained were in good agreement with the exact solution using elliptical function, perturbation method, and other approximation solutions. Chiang, Xue and Mei [6] present a finite element formulation for determining the large-amplitude free and steady-state forced vibration response of arbitrarily laminated anisotropic composite thin plates using the Discrete Kirchhoff Theory (DKT) triangular elements. In-plane deformation and inertia was both included in the formulation. The nonlinear stiffness and harmonic force matrices of an arbitrarily laminated composite thin plate element are developed for nonlinear free and forced vibration analyses. The effect of out-of-plane and in-plane boundary conditions, aspect ratio, lamination angle and number of plies on the amplitude-frequency relations are presented. Geometrically nonlinear (GNL) analysis of plates has naturally followed the same developments with the majority of applications being based on the von Karman nonlinear plate equations. The finite element method has been used to analyze large-amplitude forced vibration of rectangular isotropic plates. Mei and Decha-Umphai, [18] Wentz, Mei and Chiang, [57] further extended the finite element method to investigate large-amplitude forced vibration of rectangular composite plates. A finite element formulation was presented for determining the large-amplitude free and steady-state forced vibration of such plates. In this study, the linearized updated-mode method with a nonlinear time function approximation was employed for the solution of the system nonlinear eigenvalue equations, and the results were compared with available approximate continuum solutions. The amplitude-frequency relation for
convergence with gridwork refinement, different boundary conditions, aspect ratios, lamination angles, number of plies, and uniform versus concentrated loads were presented.

Eslami and Kandil [9] used the method of multiple scales in conjunction with Galerkin's method to analyze the nonlinear forced and damped response of a rectangular orthotropic plate subjected to a uniformly distributed transverse loading. The analysis considered simply-supported as well as clamped panels. By using the method of multiple scales, all the possible resonance were investigated; i.e., the primary resonance, subharmonic and superharmonic resonances. Hua [12] studied the geometric nonlinear forced flexural vibration of anisotropic symmetrically laminated composite plates under a harmonic force. He presented the effects of angle of orientation of the symmetrically laminated plates on the amplitude-frequency response.

Huang [14] investigated the nonlinear forced axisymmetric vibrations of an orthotropic composite circular plate with fixed boundary conditions. The governing nonlinear partial differential equations were converted into the corresponding nonlinear ordinary differential equations by elimination of the time variable with a Kantorovitch time-averaging method. The solutions of the eigenvalue problem were obtained by using a Newton-Raphson iteration technique. The results revealed the effects of finite amplitude and anisotropy of materials upon the fundamental responses. Dowell [50] presented a well established equation of motion for aeroelasticity of plates and shell. Results obtained by numerical time integration have been compared to those obtained by topological theories of dynamics and also from experiment. All of these suggest that chaotic self-excited oscillation may occur for this deterministic system.

The purpose of his study is to present a finite element approach for determining the nonlinear forced vibration response of arbitrarily laminated anisotropic composite beams. The formulation,
the solution procedure, and the computer program may be used to solve a variety of examples in
the static response, free linear as well as nonlinear vibrations of isotropic and laminated beams.

1.3- Scope of This Thesis

The purpose of this thesis is to study the nonlinear analysis of symmetrically and
antisymmetrically laminated composite beams including shear deformation subjected to harmonic
and random excitations, using a 20 (DOF) degrees of freedom finite element beam. To the author's
best knowledge, this subject has never been studied in this manner. In previous research the
effects of unsymmetric laminates was not considered. In this thesis, two cases of unsymmetric
laminates will be investigated: unsymmetric angle-ply and unsymmetric cross-ply. For both cases,
the beams are studied either simply-supported or clamped-clamped. The governing equations are
expressed in terms of the lateral displacements and take into account the von Karman geometrical
nonlinearities.
Chapter 2

2.0 Finite Element Method: Linear Analysis

2.1- Basic Concepts

The finite element method is a numerical analysis technique for obtaining approximate solutions to problems by idealizing the continuum model as a finite number of discrete region called elements. These elements are connected at points called nodes where normally the dependent variables such as displacement and mode shapes are determined. Numerical computations for each individual element generate element matrices which are then assembled to form system matrices to represent the entire problem. Generally, the more elements used, the greater the accuracy of the results. In this chapter the finite element formulation of the linear analysis of an arbitrary layered laminated beam is developed. The nonlinear system equations are developed by using the Hermitian Polynomials for interpolation of the displacement function and the Lagrange's equation.

In the analysis of a laminated composite beam, the von Karman type geometrical nonlinearity stemming from the nonlinear strain-displacement relationship was used into the formulation of the problem. The shear deformation effects are included into the formulation. A simple one-dimensional beam element with 20 degrees of freedom is used to perform free and forced vibration analyses using a simple, efficient and general approach.
2.2- Shear Deformation

Unlike the Bernouli-Euler beam theory, Timoshenko beam theory includes the effects of shear deformation, which may play a more important role in the analysis of laminated composite beams. In Bernouli-Euler beam theory it is assumed that the cross sections originally perpendicular to the neutral axis of the beam will remain perpendicular after deflection, and the slope is considered to be equal to the derivative of the deflection, \((dw/dx)\). In Timoshenko beam theory, which is known as thick beam theory cross sections will not remain perpendicular to the neutral axis. Timoshenko presented a study of the correction factor for shear of prismatic bars by considering the effect of rotary inertia and shear deformation on a beam and rectangular section.

\[ F_x = \rho A dw \frac{\partial^2 w}{\partial t^2} \quad \text{The Shear Deformation:} \quad (\theta_o) \]

\[ \omega = \rho I dx \frac{\partial^2 \Phi}{\partial t^2} \quad \text{Rotary Inertia of the Beam element} \]

\[ \theta_s = \frac{\partial w}{\partial x} \quad \text{Linear deformation} \]
In Timoshenko beam theory, by considering the element of the beam as it is shown in fig. (2.1), if the effect of shear deformation is disregarded, the tangent to the deflected center line OT coincides with the normal to the face QR (since cross sections normal to the center line would remain normal even after deformation). But due to shear deformation, the tangent to the deformed center line OT will not be perpendicular to the face QR. The tangent angle \( \gamma \) between the tangent of the deformed center line OT and the normal to the face OX' denotes the shear deformation of the element. The positive shear on the right face QR acts downwards. The angle \( \theta_b \) denotes the slope of the deflection curve due to bending deformation alone. Note that because of shear alone, the element undergoes distortion but no rotation. The slope of the deflected beam under load depends on the rotation of the cross section of the beam and the shear deformation. \( \theta_a \) represents the additional rotation of line elements due to shear.

The total rotation of the beam element can be represent by

\[
\theta_b = \frac{\partial w}{\partial x} - \theta_s \tag{2.2.1}
\]

The bending moment \( M \) and shear force \( V \) are related to \( \theta_b \) and \( w \) by the formulas

\[
M = E I \left( \frac{\partial w}{\partial x} \right) \quad \text{and} \quad V = k A G \left( \frac{\partial w}{x} - \theta_b \right) \tag{2.2.2}
\]

Where \( G \) denotes the modules of rigidity of the material of the beam and \( k \) is a constant, also known as Timoshenko's shear coefficient, which depends on the shape of the cross section. For example for a rectangular section the value of \( k \) is 5/6. [43]
2.3- Equation of Motion

The equation of motion for the element shown in Fig. (2.1) can be derived as follows:

For translational inertia of the element in the z direction

\[ [V(x, t) + dV(x, t)] + f(x, t) dx - V(x, t) = \rho A(x) dx \left( \frac{\partial^2 w}{\partial t^2} \right)(x, t) \]  \[2.2.3\]

For the rotation of the element about a line passing through point (D) and parallel to the (y) axis, and the Rotary inertia of the element is:

\[ [M(x, t) + dM(x, t)] - [V(x, t) + dV(x, t)] dx + f(x, t) dx \frac{dx}{2} \frac{dx}{2} - \]

\[ M(x, t) = \rho I(x) dx \left( \frac{\partial^2 \theta_b}{\partial t^2} \right) \]

Using the relations

\[ dV = \left( \frac{\partial V}{\partial x} \right) dx \quad \text{and} \quad dM = \left( \frac{\partial M}{\partial x} \right) dx \]  \[2.2.5\]

and Eq. (2.2.1), Eq. (2.2.2) by disregarding terms involving second powers in (dx), the above equations can be expressed as:

\[ kAG \left( \frac{\partial^2 w}{\partial x^2} - \frac{\partial \theta_b}{\partial x} \right) + f(x, t) = \rho A \left( \frac{\partial^2 w}{\partial t^2} \right) \]  \[2.2.6\]

\[ EI \left( \frac{\partial^2 \theta_b}{\partial x^2} \right) + kAG \left( \frac{\partial w}{\partial x} - \theta_b \right) = \rho I \left( \frac{\partial^2 \theta_b}{\partial t^2} \right) \]  \[2.2.7\]

The equation of motion for the forced vibration of a uniform beam can be determined by solving the eq. (2.2.6) for \( \left( \frac{\partial \theta_b}{\partial x} \right) \) and substituting the result in equation (2.2.7). So, the desired differential equation for the lateral vibration of a uniform beam is

\[ EI \left( \frac{\partial^4 w}{\partial x^4} \right) + \rho A \left( \frac{\partial^2 w}{\partial t^2} \right) - \rho I \left( 1 + \frac{E}{kG} \left( \frac{\partial^4 w}{\partial x^4} \right) + \frac{\rho^2 I}{kG} \left( \frac{\partial^4 w}{\partial t^4} \right) + \frac{EI}{kAG} \left( \frac{\partial^2 f}{\partial x^2} \right) - \frac{\rho I}{kAG} \left( \frac{\partial^2 f}{\partial t^2} \right) - f = 0 \]  \[2.2.8\]
2.4- Displacement Function for Beam Element

An arbitrarily laminated composite beam element of uniform rectangular cross section, with a uniformly distributed load is shown in fig. (2.2).

The element is assumed to have 10 degrees of freedom at each node. The axial displacement $u$, the transverse deflection $w_b$ due to bending, the transverse deflection $w_s$ due to shear, and an angle of rotation or slope $\theta$, the twisting angle $\tau$, the in-plane shear rotation $\beta$ which is equal to $du/dy$, and their derivatives with respect to $x$. Both $\tau$ and $\beta$ are assumed to be constant along the width.
(y-axis) of the beam. And there exists longitudinal force $F$, transverse shearing force $P$, a bending moment $M$, twisting moment $\tau$ and in-plane shearing force $N_{xy}$, acting at each nodal point.

![Free-Body Diagram of Simply Supported Beam With 10 Degrees of Freedom at Each Node](image)

The global and local coordinate systems are represented by $x$ and $\xi$ respectively, and the lateral coordinate system $\xi$ ranges from -1 at node 1 to +1 at node 2, for performing the Gaussian numerical integration and will be used throughout this formulation.

By using the Hermitian polynomials the deflection behavior of the beam element is described by the displacement functions $u(x)$, $\beta(x)$, $w_b(x)$, $w_s(x)$, and $\tau(x)$ in terms of the nodal displacements. These can be written as

\[
\begin{align*}
  u(x) &= N_1u_1 + N_2u_1' + N_3u_2 + N_4u_2' \\
  \beta(x) &= N_1\beta_1 + N_2\beta_1' + N_3\beta_2 + N_4\beta_2' \\
  w_b(x) &= N_1w_{b1} + N_2w_{b1}' + N_3w_{b2} + N_4w_{b2}' \\
\end{align*}
\]
\[ w(x) = N_1 w_{s1} + N_2 w_{s2} + N_3 w_{s3} + N_4 w_{s4} \] \[ \tau(x) = N_1 \tau_1 + N_2 \tau_1' + N_3 \tau_2 + N_4 \tau_2' \] \[2.4.4\]

The prime stands for \( \frac{d}{dx} \) and \( N's \) are the Hermitian polynomials, or shape function which are given as

\[ N_1 = 1 - 3 \left( \frac{x}{L} \right)^2 + 2 \left( \frac{x}{L} \right)^3 \quad N_2 = x - 2 \left( \frac{x^2}{L} \right) + \left( \frac{x^3}{L^2} \right) \] \[2.4.5\]

\[ N_3 = 3 \left( \frac{x}{L} \right)^2 - 2 \left( \frac{x}{L} \right)^3 \quad N_4 = -\left( \frac{x^2}{L} \right) - \left( \frac{x^3}{L^2} \right) \] \[2.4.6\]

Physically, each of the four shape functions represents the deflection curve for the beam element produced by setting the corresponding degree of freedom to be one and setting the rest degrees of freedom to zero [49]. For performing the Gaussian integration, the coordinate has to be change to local coordinate \( \xi \) and the limits of integration has to be from -1 to 1.

The transformed local shape functions in term of \( \xi \) is

\[ N_1 = \frac{1}{4} \left\{ 2 - 3 \xi + \xi^3 \right\} \quad N_2 = \frac{1}{4} \left\{ \xi^3 - \xi^2 - \xi + 1 \right\} \] \[2.4.7\]

\[ N_3 = \frac{1}{4} \left\{ 2 + 3 \xi - \xi^3 \right\} \quad N_4 = \frac{1}{4} \left\{ \xi^3 + \xi^2 - \xi - 1 \right\} \]

where

\[ \xi = 2 \left( \frac{x}{L} \right) - 1 \quad -1 < \xi < 1 \] \[2.4.8\]
2.5- Lamina Constitutive Relations

In chapter 1, a brief discussion of some of the properties and the means of fabricating concepts of composite materials were discussed. However, before understanding the physical behavior of composite beams, plates, and shells, and before being able to quantitatively determine the stresses, strains, deformations, natural frequencies, etc., a clear understanding of anisotropic elasticity is mandatory. As it was mentioned, the superiority of laminated composite materials resides in the fact that it can be tailored to best serve the design requirements. When the fibers are oriented in the direction of the load, the composite material is referred to as unidirectional. On the other hand, if the fibers are oriented in different directions, it is called bi-directional or tridirectional.

Jones [15] has given a detailed presentation of the stress-strain relations for anisotropic materials. The generalized Hooke's law is characterized by six equations leading to 36 non-zero stiffness terms. Then by considering the strain energy, the stiffness constants are reduced to 21 for a conservative process. Since the general anisotropic material is defined by 21 independent stiffness constants, it is referred to as atriclinic material. The stiffness constants can be further reduced by considering planes having symmetric properties. Depending on the number of orthogonal planes of symmetry, the material can be monoclinic if one plane of symmetry exists thus reducing the stiffness constants to 13, orthotropic for two planes of symmetry (symmetry will exist relative to a third mutually orthogonal plane) and 9 independent elastic constants, and isotropic with infinite planes of symmetry and only 2 elastic constant.
2.5.1- Resultant Laminate forces and moments

From the classical plate theory, the resultant forces (N), the transverse shears (Q), and the resultant bending moments (M) acting on the geometric middle surface of the plate regardless of the number and the orientation of the lamina are given as

\[
(N_x, N_y, N_{xy}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) \, dz
\]  

[2.5.1]

\[
(Q_x, Q_y) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\tau_{xz}, \tau_{yz}) \, dz
\]  

[2.5.2]

\[
(M_x, M_y, M_{xy}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{xx}, \sigma_{yy}, \tau_{xy}) \, dz
\]  

[2.5.3]

Where \( N_x, N_y, N_{xy} \) are membrane forces; \( Q_x, Q_y \) are transverse shear forces; and \( M_x, M_y, M_{xy} \) are bending and twisting moments, all per unit length. These forces and moments are shown in fig.(2.6). It represents the positive direction of all the resultant forces, shears and moments, and they are presented in more detail in appendix (A).

The corresponding loads applied to the beam are as follows

![Fig.(2.6) Resultant Forces and Moments](image)
2.5.2- Lamina Stiffness

To analyze the dynamic response of a laminated beam under dynamic loading, the properties of the lamina can be determined by the classical Kirchhoff's lamination theory. In this method, it is assumed that the lamina is in a state of plane stress, a normal to the middle surface remains straight and stays normal after bending. Unlike the homogeneous plates, in laminated plates a coupling phenomenon can be seen between bending and stretching of the middle-plane. The stress-strain relations in principal material coordinates for orthotropic lamina under plane condition are

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
\]

[2.5.4]

Where \([Q]\) is the reduced stiffness matrix. In the x-y-z coordinate system, the equation can be written as

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

[2.5.5]

Where \([Q]\) is the transformed stiffness matrix, and it depends on the angle of orientation of the laminates, which can be shown as follows:

\[
\begin{bmatrix}
\bar{Q}
\end{bmatrix} = [T]^{-1} [Q] [T]^{-T}
\]

[2.5.6]

and

\[
[T] = \begin{bmatrix}
m^2 & n^2 & 2mn \\
n^2 & m^2 & -2mn \\
-mn & mn & m^2n^2
\end{bmatrix}
\]

[2.5.7]

where \(m=\cos \theta, n=\sin \theta\) and \(\theta\) represents the ply angle.
The basic constitutive relations of classical lamination theory are in the form of

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
k_x \\
k_y \\
k_{xy}
\end{bmatrix}
\]

[2.5.8]

N's are the resultant forces (force per unit length) and the M's are the resultant moments (moments per unit length), the \( \varepsilon \)'s are the strains due to in-plane forces and k's are the bending and twisting curvatures. The \([A_y]\), \([B_y]\) and \([D_y]\) are all different summations of the ply stiffness, and they are called extentional stiffness, coupling stiffness, and bending stiffness, respectively. As it is shown in fig. (2.7) through (2.12), \([A_y]\) is used to evaluate in-plane forces \( N \)'s in relation to the mid-surface strains \( \varepsilon \)'s, \([D_y]\) is used to evaluate the bending moment \( M \)'s in relation to the curvature \( k \)'s. And since \([B_y]\) is used to evaluate both in-plane forces and bending moments \( N \) and \( M \) in relation to \( \varepsilon \)'s and \( k \)'s, it is called the bending-stretching coupling matrix. When \( A_{16} \) and \( A_{26} \) are non-zero, the \textit{stretching-shearing coupling} occurs. When \( B_{12} \) and \( B_{26} \) are non-zero, \textit{twisting-stretching coupling} occurs, and \textit{bending-shearing coupling} appears. And finally \textit{bending-twisting coupling} comes from non-zero values of \( D_{12} \) and \( D_{26} \). And by proper stacking, the (1,6) and (2,6) term can be manipulated so that in some structural applications, these effects could be used to advantage, such as aeroelastic tailoring. So the laminate stiffness matrix values are given by

\[
(A_y, B_y, D_y) = \int_\frac{-h}{2}^{\frac{h}{2}} \bar{Q}_y (1, z, z^2) dz, \quad i, j = 1, 2, 6
\]

[2.5.9]
\[ A_y = \sum_{i=1}^{n} (h_k - h_{k-1}) \bar{Q}^{(k)} \]
\[ B_y = \sum_{i=1}^{n} \frac{1}{2} \left( h_k^2 - h_{k-1}^2 \right) \bar{Q}^{(k)} \]
\[ D_y = \sum_{i=1}^{n} \frac{1}{3} \left( h_k^3 - h_{k-1}^3 \right) \bar{Q}^{(k)} \]

Where \( n \) is the total number of plies, \( k \) designates the individual plys (k-th layer), \( h \) is the laminate thickness, and \( \bar{Q}_y \) is the transformed stiffness coefficient of the k-th layer.

Since this study deals with beams,

\[ N_y = M_y = 0 \quad [2.5.10] \]

that is,

\[ N_y = A_{12} \varepsilon_x + A_{22} \varepsilon_y + A_{26} \gamma_{xy} + B_{12} k_x + B_{22} k_y + B_{26} k_{xy} = 0 \quad [2.5.11] \]
\[ M_y = B_{12} \varepsilon_x + B_{22} \varepsilon_y + B_{26} \gamma_{xy} + D_{12} k_x + D_{22} k_y + D_{26} k_{xy} = 0 \quad [2.5.12] \]

However the in-plane strain \( \varepsilon_y \) and the bending curvature \( k_y \) are assumed to be non-zero, and solving \( \varepsilon_y \) and \( k_y \) in terms of \( \varepsilon_x, \varepsilon_y, k_x \) and \( k_{xy} \), equations (2.5.11) and (2.5.12) become

\[ \varepsilon_y = \frac{1}{\left( \frac{B_{22}}{A_{22}} - A_{22} \right)} \left[ \left( A_{12} - \frac{B_{22} A_{12}}{A_{22}} \right) \varepsilon_x + \left( A_{23} - \frac{B_{26} B_{22}}{D_{22}} \right) \gamma_{xy} + \left( B_{12} - \frac{D_{12} B_{22}}{D_{22}} \right) k_x + \left( B_{26} - \frac{D_{26} B_{22}}{D_{22}} \right) k_{xy} \right] \quad [2.5.13] \]
\[ k_y = \frac{1}{\left( \frac{B_{22}}{A_{22}} - D_{22} \right)} \left[ \left( B_{12} - \frac{B_{22} A_{12}}{A_{22}} \right) \varepsilon_x + \left( B_{26} - \frac{A_{26} B_{22}}{A_{22}} \right) \gamma_{xy} + \left( D_{12} - \frac{B_{12} B_{22}}{A_{22}} \right) k_x + \left( D_{26} - \frac{B_{26} B_{22}}{A_{22}} \right) k_{xy} \right] \quad [2.5.14] \]

or

\[ \varepsilon_y = a_1 \varepsilon_x + a_2 \gamma_{xy} + a_3 k_x + a_4 k_{xy} \quad [2.5.15] \]
\[ k_y = b_1 \varepsilon_x + b_2 \gamma_{xy} + b_3 k_x + b_4 k_{xy} \quad [2.5.16] \]

where
By rearranging the constitutive equation for the beam element, equation (2.5.8) further simplifies as

\[
\begin{bmatrix}
N_x \\
N_{xy} \\
M_x \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{16} & B_{11} & B_{16} \\
A_{16} & A_{66} & B_{16} & B_{66} \\
B_{11} & B_{16} & D_{11} & D_{16} \\
B_{16} & B_{66} & D_{11} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\gamma_{xy} \\
\kappa_x \\
\kappa_{xy}
\end{bmatrix}
+ 
\begin{bmatrix}
A_{11} & B_{12} \\
A_{26} & B_{26} \\
B_{12} & D_{12} \\
B_{26} & D_{26}
\end{bmatrix}
\begin{bmatrix}
\epsilon_y \\
\gamma_{yx}
\end{bmatrix}
\]  

[2.5.18]

Where \( \epsilon_x \) and \( \kappa_x \) are given in equations (2.5.13) and (2.5.14), and finally equation (2.5.18) will reduce to

\[
\begin{bmatrix}
N_x \\
N_{xy} \\
M_x \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
\tilde{D}_{11} & \tilde{D}_{12} & \tilde{D}_{13} & \tilde{D}_{14} \\
\tilde{D}_{21} & \tilde{D}_{22} & \tilde{D}_{23} & \tilde{D}_{24} \\
\tilde{D}_{31} & \tilde{D}_{32} & \tilde{D}_{33} & \tilde{D}_{34} \\
\tilde{D}_{41} & \tilde{D}_{42} & \tilde{D}_{43} & \tilde{D}_{44}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x \\
\gamma_{xy} \\
\kappa_x \\
\kappa_{xy}
\end{bmatrix}
\]  

[2.5.19]

By including the shear deformation, to the bending deformation, the total transverse deformation became:
\[
\omega = \omega_b + \omega_s \quad [2.5.20]
\]

And by including the shear coefficient (k) in the transverse shear force-strain relations, the transverse shear force \( Q_{xz} \) is determined to be:

\[
Q_{xz} = kD_{44} \gamma_{xz} \quad [2.5.21]
\]

\( (D_{44}) \) is the transverse shear stiffness and \( \gamma_{xz} \) is transverse shear strain. By doing so, the constitutive relations are given as:

\[
\begin{bmatrix}
N_x \\
N_{rv} \\
M_x \\
M_{rv} \\
Q_{xz}
\end{bmatrix}
=
\begin{bmatrix}
\bar{D}_{11} & \bar{D}_{12} & \bar{D}_{13} & \bar{D}_{14} & 0 \\
\bar{D}_{21} & \bar{D}_{22} & \bar{D}_{23} & \bar{D}_{24} & 0 \\
\bar{D}_{31} & \bar{D}_{32} & \bar{D}_{33} & \bar{D}_{34} & 0 \\
\bar{D}_{41} & \bar{D}_{42} & \bar{D}_{43} & \bar{D}_{44} & 0 \\
0 & 0 & 0 & 0 & \bar{D}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\gamma_{rv} \\
\kappa_x \\
\kappa_{rv} \\
\gamma_{xz}
\end{bmatrix}
\quad [2.5.22]
\]

where

\[
\bar{D}_{66} = kD_{44}
\]
2.6- Strain-Displacement Relations

In the previous section, the constitutive relations for a laminated composite beam including shear deformation was developed in eq. (2.5.22). This section primarily deals with developing the in-plane strain, bending curvature and nodal-displacements. The strains are written in terms of displacement derivatives and by using eq.(2.4.8) and also by changing the coordinate system from x to local coordinate $\xi$, they can be given as

\[ \varepsilon_x = \frac{du}{dx} = \left( \frac{\partial u}{\partial \xi} \right) \left( \frac{\partial \xi}{\partial x} \right) = \frac{2}{L} \left( \frac{\partial u}{\partial \xi} \right) \]  \[ \text{[2.6.1]} \]

\[ \gamma_{xy} = \beta(x) = \beta(\xi) \]  \[ \text{[2.6.2]} \]

\[ k_x = \left( \frac{\partial^2 w_b}{\partial x^2} \right) = \left( \frac{\partial^2 w_b}{\partial \xi^2} \right) \left( \frac{\partial \xi}{\partial x} \right)^2 = \frac{4}{L^2} \left( \frac{\partial^2 w_b}{\partial \xi^2} \right) \]  \[ \text{[2.6.3]} \]

\[ k_{xy} = 2 \left( \frac{\partial \tau}{\partial x} \right) = 2 \left( \frac{\partial \tau}{\partial \xi} \right) \left( \frac{\partial \xi}{\partial x} \right) = \frac{4}{L} \left( \frac{\partial \tau}{\partial \xi} \right) \]  \[ \text{[2.6.4]} \]

\[ \gamma_{xy} = \left( \frac{\partial w_s}{\partial x} \right) = \left( \frac{\partial w_s}{\partial \xi} \right) \left( \frac{\partial \xi}{\partial x} \right) = \frac{2}{L} \left( \frac{\partial w_s}{\partial \xi} \right) \]  \[ \text{[2.6.5]} \]

Where from eq.(2.4.8), the derivative of the local coordinate axis with respect to the global axis is

\[ \frac{\partial \xi}{\partial x} = \frac{2}{L} \]  \[ \text{[2.6.6]} \]

The mid-plain strain $\varepsilon_x$ is written in terms of derivative of the in-plane displacement u in x direction. $\gamma_{xy}$ is defined in terms of the in-plane shear rotation $\beta$, and it is equal to $du/dy$, and it is assumed to be constant along the y axis. And $\gamma_s$ represents the transverse shear strain and it is in terms of the transverse shear deflection.
Finally by substituting the displacement equations (2.4.1) through (2.4.5) into the strain-displacement eq.(2.6.1-5), the strain-displacement relation may be written as

\[ \{\varepsilon\} = [B]\{q\} \]  

where \( \{q\} \) is the nodal displacement vector (10 degrees of freedom at each node). \([B]\) is the strain-displacement matrix and it is given in the form of

\[ [B] = \frac{2}{L} \begin{bmatrix} N_1' & N_2' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}N_1 \frac{1}{2}N_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

where \( N' \) and \( N'' \) are the shape function derivatives, and the prime denotes the derivative with respect to the local coordinate \( \xi \) which are

\[ N_1' = \left( \frac{3}{4} + \frac{3}{4}\xi^2 \right) \quad N_1'' = \left( \frac{3}{2} \right) \xi \]

\[ N_2' = \left( \frac{3}{4}\xi^2 - \frac{1}{2}\xi - \frac{1}{2} \right) \quad N_2'' = \left( \frac{3}{2}\xi - \frac{1}{2} \right) \]

\[ N_3' = \left( \frac{3}{4} - \frac{3}{4}\xi^2 \right) \quad N_3'' = \left( \frac{3}{2} \right) \xi \]

\[ N_4' = \left( \frac{3}{4}\xi^2 + \frac{1}{2}\xi - \frac{1}{4} \right) \quad N_4'' = \left( \frac{3}{2}\xi + \frac{1}{2} \right) \]
2.7- Element Elastic Stiffness Matrix

The stiffness equations for a beam element can be obtained by using Castigliano's theorem.

\[ F_i = \frac{\partial U}{\partial q_i} \]  \hspace{1cm} [2.7.1]

Where \( F_i \) is the nodal force or moment and \( q_i \) is the nodal displacement or rotation. The subscript \( i \) designates the degree of freedom number, and \( u \) is the strain energy.

The strain energy expression for the beam element with uniform cross section is given as

\[ U = \frac{b}{2} \int_0^L \{ \varepsilon \}^T \{ N \} \, dx \]  \hspace{1cm} [2.7.2]

Where \( b \) is the width of the beam, \( \{ \varepsilon \} \) is the vector containing the strain and curvatures, and \( \{ N \} \) is the force and moment resultants vector. Expansion of the above equation yields

\[ U = \frac{b}{2} \int_0^L \left[ (N_y \varepsilon_x + N_x \gamma_x) + (M_x k_x + M_y k_y) + (Q_x \gamma_x) \right] \, dx \]  \hspace{1cm} [2.7.3]

The strain energy relation can be determined by substituting eq.(2.5.22) and (2.6.7) into Eq. (2.7.2).

\[ U = \frac{b}{2} \int_0^L \{ q_i \}^T [B] \left[ \tilde{D} \right] [B] \{ q_i \} \, dx \]  \hspace{1cm} [2.7.4]

Since the vector \( \{ q \} \) is not a function of \( x \), it can be taken out of the integral.

\[ U = \frac{b}{2} \{ q_i \}^T \left[ \int_0^L [B] \left[ \tilde{D} \right] [B] \, dx \right] \{ q_i \} \]  \hspace{1cm} [2.7.5]

where

\[ U = \frac{1}{2} \{ q_i \}^T \{ F_i \} \]  \hspace{1cm} [2.7.6]

\( \{ F_i \} \) is the applied load vector, and the load-displacement relation can be given as

\[ \{ F_i \} = [K] \{ q_i \} \]  \hspace{1cm} [2.7.7]
where \([K]\) is element elastic stiffness matrix and is given by:

\[
[K] = b \int_0^L [B]^T [D][B] dx
\]  \[2.7.8\]

To analyze the load matrix, the product of the strain-displacement matrix \([B]\) and the distributed loads \([N]\) must be integrated through interior of the beam for each element. The integration has been done by the Gaussian quadrature numerical method.

\[
\{F\} = b \int_0^L [B]^T [N] dx
\]  \[2.7.9\]

and the eq. (2.7.9) can be solved for the static deflection of a laminated beam by the FORTRAN program \textsc{Compnite} which is developed for this study, Appendix C.
### 2.8- Element Mass Matrix

By using Hamilton's principle, the equations of motion for a beam element are presented as follows:

\[ S = \int_{t_i}^{t_f} L dt \]  \[ \text{[2.8.1]} \]

Where

\[ L = (T - P_E - U) \]

where \( L \) is the Lagrangian, \( T \) is the kinetic energy of the body, \( P_E \) is the potential energy of the applied load, \( U \) is the strain energy, and \( t \) is the time. The above equation leads to the well known Lagrange equations of motion given as

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \left( \frac{\partial L}{\partial q_i} \right) = F_i \]  \[ \text{[2.8.2]} \]

\[ i = 1, 2, 3, \ldots \ldots, 20 \]

The kinetic energy of the beam can be written as

\[ T = \frac{1}{2} \int A \rho \left( \frac{\partial w}{\partial t} \right)^2 dx + \frac{1}{2} \int f \rho I \left( \frac{\partial w}{\partial t} \right)^2 dx \]  \[ \text{[2.8.3]} \]

where \( A \) is the cross sectional area of the beam, \( T \) represents both the kinetic energy due to translatory motion in the lateral direction and the kinetic energy due to rotation of the beam elements which is call rotary inertia. In most cases, for a thin and long beam, the rotary inertia is neglected. However, for a short beam, the rotary inertia is not neglected if the problem is being solved for complex mode shapes at high frequencies. So by neglecting the rotary inertia for this problem, the eq. (2.8.3) becomes

\[ T = \frac{1}{2} \int A \rho \left( \frac{\partial w}{\partial t} \right)^2 dx \]  \[ \text{[2.8.4]} \]

The deflection function of the beam is:

\[ w(x, y) = w_b(x) + w_s(x) - y \tau(x) \]  \[ \text{[2.8.5]} \]
and substituting eq. (2.8.5) into eq. (2.8.4), the kinetic energy expression becomes

\[ T = \frac{1}{2} \int_0^L \rho A \dot{w}^2 (x) dx + \frac{1}{2} \int_0^L J \tau^2 (x) dx \]  

where \( J \) is the polar mass moment of inertia. Substituting the kinetic energy expression eq. (2.8.6) and the strain energy expression eq. (2.7.3) into the Lagrange's equation of motion eq. (2.8.2), the equation of motion for forced vibration will be given as

\[ [K]\{q\} + [m]\{\dot{q}\} = \{F\} \]  

Where \([m]\) is the element mass matrix, \([q]\) is the acceleration vector, and \([K]\) is the element stiffness matrix. The element mass matrix can be written in forms of

\[ (m)_{\text{Tran}} = \int_0^L \rho AN_i(x)N_j(x)dx \]  

\[ (m)_{\text{Tors}} = \int_0^L JN_i(x)N_j(x)dx \]  

These correspond to the transverse and torsional displacements respectively.

By transferring to local coordinates, the eq. (2.8.8) and eq. (2.8.9) become

\[ m_y = \frac{L}{2} \int_{-1}^{1} \rho AN_i(\xi)N_j(\xi)d\xi \]  

\[ m_y = \frac{L}{2} \int_0^L JN_i(\xi)N_j(\xi)d\xi \]  

By using Gaussian integration, the mass matrix can be evaluated.
2.9- Gaussian Quadrature

The numerical integration of all the intervals in this study is done by Gaussian quadrature method

\[ S = \int_{-1}^{+1} g(x) \, dx = \sum_{i=1}^{n} H_i \, g(x_i) \]  \[2.9.1\]

this the general form of the Gaussian quadrature, in this case \( n \) is the number of function evaluations, \( H_i \) is the weight numbers and \( x_i \) are abscissas of the sampling points, and these are presented in table (2.1).

For example, by changing the limits of integration, eq.(2.7.8) becomes

\[ [K] = \frac{L}{2} b \int_{-1}^{+1} [B]^T \tilde{D} [B] d\xi \]  \[2.9.2\]

and the Gaussian integration form using \( n=5 \) is given by:

\[ [K] = \frac{L}{2} \sum_{i=1}^{n} b_i [B]^T \tilde{D} [B] H_i \] \[2.9.3\]

The local and global displacement relations in the Cartesian coordinate system is given by

\[ \{q_L\} = [T] \{q_G\} \] \[2.9.4\]

where the subscripts \( L \) and \( G \) refer to local and global respectively. And the transformation matrix \([T]\) can be determined since the derivatives of the displacements in the local and global coordinates are related by

\[ \left( \frac{\partial}{\partial \xi} \right) = \left( \frac{\partial}{\partial x} \right) \left( \frac{\partial x}{\partial \xi} \right) \] \[2.9.5\]

And since the longitudinal displacements are the same in both local and global coordinates,

\[ u_1(x) = u_1(\xi) \] \[2.9.6\]

Then, from eq. (2.4.8) it can be obtained that
\[ \frac{\partial x}{\partial \xi} = \frac{L}{2} \]

So the main diagonal of the transformation matrix \([T]\) will be equal to (1) corresponding to the displacements, and equal to \(L/2\) corresponding to the derivative of the displacements, and the rest of the elements are equal to zero. And the transformation of the stiffness matrix is as follows:

\[
[K_{G}] = [T][K_{L}][T]^T 
\]

and since the only non-zero elements of the matrix \([T]\) are along the main diagonal, the transformation can be expressed as

\[
[K_{G}] = T_i[K_{L}]_j T_j^T
\]

\[ i, j = 1,2, \ldots \ldots , 20 \]

And by solving the eq. (2.7.9), the displacements in terms of the global coordinate system can be determined.

<table>
<thead>
<tr>
<th>n</th>
<th>Abscissas, ((x_i))</th>
<th>Weight, ((H_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>(\pm \frac{1}{\sqrt{3}})</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>(\pm \sqrt{0.6})</td>
<td>8/9</td>
</tr>
<tr>
<td></td>
<td>0.339981043585</td>
<td>0.65214515486</td>
</tr>
<tr>
<td></td>
<td>0.861136311594</td>
<td>0.34785484514</td>
</tr>
<tr>
<td>4</td>
<td>(\pm 0.538469310106)</td>
<td>0.5688888889</td>
</tr>
<tr>
<td></td>
<td>0.906170845939</td>
<td>0.4786286705</td>
</tr>
<tr>
<td>5</td>
<td>(\pm 0.23861986083)</td>
<td>0.4679139346</td>
</tr>
<tr>
<td></td>
<td>0.661209386466</td>
<td>0.36076157305</td>
</tr>
<tr>
<td></td>
<td>0.932469514203</td>
<td>0.17132449238</td>
</tr>
</tbody>
</table>

Table (2.1) Gaussian Quadrature Abscissas and Weights.
Chapter 3

3.0 Finite Element Method: Nonlinear Analysis

Behavior of most oscillatory systems cannot be predicted or explained by the linear theory. The nonlinear vibration approach leads to a complete new phenomena which are not possible in linear systems; for example, the dependence of frequency, the so called "Jump phenomena", the subharmonic and super harmonic resonance, chaotic vibration and etc. In such cases, nonlinear theory must be used to obtain more accurate results and to explain new phenomena. In linear systems, by doubling the load, the response is doubled and their relation is proportional, but in nonlinear systems this proportionality does not exist. By increasing the magnitude of deformations or the amplitude of oscillations, presence of nonlinear effects can be seen significantly. In general, the source of nonlinearity may be geometric, inertial, or material non-linearity. This study primarily deals with geometric nonlinearity, stems from the nonlinear strain-displacement relations. The nonlinear analysis of beams has gained a considerable attention during the recent years for increasing the efficiency of structural design. More recently, the finite element method has become widely used in the nonlinear analysis of structures as a reliable approach for solving a large variety of problems. As a result, numerous finite element programs, as well as both the material and geometrical nonlinearities for the static and dynamic behavior of the structure, have been developed.
3.1- Geometrically Nonlinear Phenomenon

In general, non-linearities in structural mechanics problems can arise in several ways. When material behavior is nonlinear, a generalized Hook's law is no longer valid. This type of nonlinearity is called material (physical) nonlinearity. Alternatively, material behavior can be assumed to be linear, but structural deformation can become large and cause nonlinear strain-displacement relation. Deformation of a structural member can also reach a magnitude that does not overstrain the material; in such a case, curvature of the deformed median line can no longer be expressed by linear equation. Problems involving large structural deformation are called "geometrically" nonlinear problem. And also, geometric non-linearity may often be combined with material non-linearity such as small strain plasticity. In nonlinear vibration analysis, the conservation energy is used to evaluate the element matrices, and the strain energy relation for a beam element is given by

\[
U = \frac{1}{2} \int_0^L \left( P \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) dx \tag{3.1.1}
\]

where \( P \) is the membrane force and is given by

\[
P = E \cdot A \left\{ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right\} \tag{3.1.2}
\]

And the kinetic energy \( T \) is

\[
T = \frac{1}{2} \int_0^L \rho A (u^2 + \omega^2) \, dx \tag{3.1.3}
\]

Where \( \rho \) is the mass per unit length, and \( L \) is the element length and the rotatory inertia has been neglected. The kinetic energy can be separated into two parts, the kinetic energy due to mid-plane displacement and the kinetic energy due to lateral deflection.
Any large-amplitude deflection of a beam which is restrained at its two end results in some mid-plane stretching. This stretching must be considered in the formulation which can be accomplished by using a nonlinear strain-displacement relationship (geometric nonlinearity). The nonlinear equation of motion describing this situation has been the basis of a number of investigations. Most of these works are based on a single-mode approach.[57]
3.2- Non-Linear Strain-Displacement Relations

In general for all classical problems, it has been implicitly assumed that both displacement and strain developed in the structure are small. In practical terms this means that the geometry of the element basically remains unchange during the loading process. But if more accurate determination of the displacement is needed, geometric non-linearity may have to be considered in some structures. In general the strain-displacement and the curvature-displacement relations in terms of middle surface are given as

\[
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
\frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \\
\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{1}{2} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}
\end{bmatrix} \quad [3.2.2]
\]

\[
\{ k \} =
\begin{bmatrix}
k_x \\
k_y \\
k_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^2 w}{\partial y^2} \\
2 \frac{\partial^2 w}{\partial x \partial y}
\end{bmatrix} \quad [3.2.2]
\]

Where \( \{ \varepsilon^0 \} \) is mid-plane strain matrix and \( \{ k \} \) is the curvature matrix. And \( u_0 \) is the inplane displacement in the x and y directions, are denoted by \( u_0 \) and \( v_0 \) respectively, and \( w \) is the displacement in the z direction. Nonlinear terms appears in eq. (3.2.1) are the so called von-Karman geometrical non-linearity. For this study, the nonlinear strain-displacement relation for a beam element can be simplified as
\[
\{ \varepsilon \} = \left\{ \begin{array}{c}
\frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial x} \\
\frac{\partial w}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
2 \frac{\partial^2 w}{\partial x^2}
\end{array} \right\} + \left\{ \begin{array}{c}
0 \\
0 \\
0 \\
0
\end{array} \right\} = \{ \varepsilon_L \} + \{ \varepsilon_{NL} \}
\]

where \( \{ \varepsilon_L \} \) and \( \{ \varepsilon_{NL} \} \) are the linear and nonlinear strain vectors, respectively. The nonlinear strain vector can be written as

\[
\{ \varepsilon_{NL} \} = \frac{1}{2} \begin{pmatrix}
\frac{\partial w_b}{\partial x} \\
0 \\
0 \\
0
\end{pmatrix}
\left( \frac{\partial w_b}{\partial x} \right) = \frac{1}{2} \{ A \} (\theta)
\]

The derivative (slopes) of \( w_b \) can be related to the nodal displacements as

\[
\frac{\partial w_b}{\partial x} = \frac{2}{L} \left( N'_1 w_{b1} + N'_2 \theta_{b1} + N'_3 w_{b2} + N'_4 \theta_{b2} \right)
\]

or in matrix form as

\[
(\theta_b) = \frac{\partial w_b}{\partial x} = [G] \{ q^b \}
\]

where \([G]\) matrix is defined purely in terms of the coordinates since it contains the shape function's derivatives, and subscript \( b \) refers to the transverse displacement \( w_b \) and rotation due to bending \( \theta_b \).

\[
[G] = \left( \frac{2}{L} \right) \left[ \begin{array}{c}
N'_1, N'_2, N'_3, N'_4
\end{array} \right] \quad \{ q^b \} = \left\{ \begin{array}{c}
w_{b1} \\
\theta_{b1} \\
w_{b2} \\
\theta_{b2}
\end{array} \right\}
\]

Now the total strain-displacement matrix can be represented as a function of the normal displacement as

\[
\left[ \tilde{B} \right] = [B_L] + \left( \frac{1}{2} \right) [B_{NL}(q^b)]
\]

where \([B_L]\) is the linear strain displacement matrix from eq.(2.6.8), and \([B_{NL}]\) is found by taking a variation of \( \{ \varepsilon_{NL} \} \) with respect to \( \{ q^b \} \) as follows: [8]
3.3- Evaluation of Nonlinear Element Stiffness Matrix

The total element stiffness matrix for non-linear analysis can be represented by:

\[ [\mathbf{K}] = [\mathbf{K}_L] + [\mathbf{K}_{NL}] \]  \hspace{1cm} \text{(3.3.1)}

Where \([\mathbf{K}_L]\) and \([\mathbf{K}_{NL}]\) represents the small and large displacement stiffness matrix respectively, and the total strain energy relation can be written by substituting Eq.(3.2.7) into Eq.(2.7.5).

\[ U = \left( \frac{b}{2} \right) \{ q_i \}^T \left[ \int_0^L \left[ B_L + \frac{1}{2} B_{NL}(q_i) \right]^T \left[ \mathbf{D} \right] \left[ B_L + \frac{1}{2} B_{NL}(q_i) \right] \right] \{ q_i \} \]  \hspace{1cm} \text{(3.3.2)}

By performing the following differentiations, the total stiffness matrix can be written as:

\[
\frac{\partial U}{\partial q_i} = b \left[ \int_0^L \left[ [B_L]^T [\mathbf{D}] [B_L] \right] dx + \int_0^L \left[ [B_{NL}(q_i)]^T [\mathbf{D}] [B_L] + \frac{1}{2} [B_L]^T [\mathbf{D}] [B_{NL}(q_i)] \right] dx \right] \{ q_i \} \]  \hspace{1cm} \text{(3.3.3)}

Therefore, the stiffness matrix including nonlinear terms is presented as

\[ [\mathbf{K}] = [\mathbf{K}_L] + [N_1] + [N_2] \]  \hspace{1cm} \text{(3.3.4)}

Where \([\mathbf{K}_L]\), \([N_1]\), and \([N_2]\) are the linear, first order, and second order stiffness matrices respectively, and are given by

\[ [k_L] = b \int_0^L \left[ B_L \right]^T \left[ \mathbf{D} \right] [B_L] \]  \hspace{1cm} \text{(3.3.5)}

\[ [N_1] = \frac{b}{2} \int_0^L \left[ [B_L]^T \left[ \mathbf{D} \right] [B_{NL}(q_i^b)] + [B_{NL}(q_i^b)]^T \left[ \mathbf{D} \right] [B_L] \right] dx \]  \hspace{1cm} \text{(3.3.6)}

\[ [N_2] = \frac{b}{2} \int_0^L \left[ [B_{NL}(q_i^b)]^T \left[ \mathbf{D} \right] \left[ B_{NL}(q_i^b) \right] \right] dx \]  \hspace{1cm} \text{(3.3.7)}

which are obtained by using Gaussian numerical integration.
3.4- Nonlinear Vibrations

The nonlinear equation of motion are obtained by applying the Lagrange's equation. BY using equation (2.7.8), (2.8.10), (2.8.11) and (3.3.4) the equations of motion are expressed as

\[
[K_L] + [N_1] + [N_2] \{q\} + [M] \{\ddot{q}\} = \{F\}
\]  

where \([M]\) is the global system mass matrix, \([K_L]\) is the global linear system stiffness matrix, \([N_1]\) and \([N_2]\) are the global system nonlinear stiffness matrices, and \([F]\) is the global force matrix. By partitioning the displacement vectors \(q\) into the in-plane and transverse displacement vectors \(Q_u\) and \(Q_w\), the following equations can be obtained.

\[
\begin{bmatrix}
K_{uu} & K_{uw} \\
K_{wu} & K_{ww}
\end{bmatrix} + \begin{bmatrix}
0 & N_{uw1} \\
N_{wu1} & N_{ww1}
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & N_{ww2}
\end{bmatrix} \begin{bmatrix}
Q_u \\
Q_w
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
0 & M_{ww}
\end{bmatrix} \begin{bmatrix}
\ddot{Q}_u \\
\ddot{Q}_w
\end{bmatrix} = \begin{bmatrix}
F_u \\
F_w
\end{bmatrix}
\]  

[3.4.2]

For symmetrically laminated beams, there is no coupling between the in-plane and the transverse displacements, so \([K_{uw}], [K_{wu}]\) and \([N_{wu}1]\) equal to zero.

By letting

\[
\{Q_w\} = \{\ddot{Q}_w\} q(t)
\]  

[3.4.3]

In order to have the maximum displacement as unity, the normalized linear mode \(\ddot{Q}_w\) is used, and \(q(t)\) is a time function, and it will be written as \(q\). Since there is no longitudinal (axial) force applied, \(\{F_u\}\) equals to zero, and by substituting Eq. (3.4.3) in to Eq. (3.4.2) and solving it, the in-plane displacement yields

\[
\{F_u\} = 0
\]

\[
\{Q_u\} = -[K_{uu}]^{-1}[K_{uw}]{\ddot{Q}_w} q - [K_{uw}]^{-1}[N_{uw}]{\ddot{Q}_w} q^2
\]  

[3.4.4]
Substituting Eq.(3.4.3) in to Eq.(3.4.2), the equation motion in terms of normal displacements, can be written as:

\[
[K_{ww} - K_{uu} K_{uu}^{-1} K_{wu}] \ddot{Q}_{w} + [N_{ww1} - K_{wu} K_{uu}^{-1} N_{uu1} - N_{ww1} K_{uu}^{-1} K_{uu}] q^2 + \\
+ [N_{ww2} - N_{wu1} K_{uu}^{-1} N_{uu1}] \{ \ddot{Q}_{w} \} q^3 + \{ \ddot{Q}_{w} \} \ddot{q} = \{ F_w \}
\]  

[3.4.5]

In case of symmetrically laminated and isotropic beams, the coefficient of \( \ddot{q} \) is equal to zero. Whereas for the unsymmetrically laminated beams this coefficient may not be zero. By premultiplying eq. (3.4.5) by \( \{ \ddot{Q}_w \}^T \) the following equation can be obtained.

\[
[K_{ww} - K_{uu} K_{uu}^{-1} K_{wu}] q + [N_{ww1} - K_{wu} K_{uu}^{-1} N_{uu1} - N_{ww1} K_{uu}^{-1} K_{uu}] q^2 + \\
+ [N_{ww2} - N_{wu1} K_{uu}^{-1} N_{uu1}] q^3 + [M_{ww}] \ddot{q} = \{ F_w \} \{ \ddot{Q}_w \}^T
\]  

[3.4.6]

and in more simplistic manner:

\[\alpha q + \beta q^2 + \gamma q^3 + \delta \ddot{q} = F_o \cos \Omega t\]  

[3.4.7]

where

\[F_o = \{ F_w \} \{ \ddot{Q}_w \}^T\]  

[3.4.8]

Dividing all terms by \( \delta \), and setting \( \omega_o^2 = \frac{\alpha}{\delta} \), where \( \omega_o \) is the linear frequency of vibration. So eq.(3.4.7) can be written as

\[\left( \omega_o^2 \right) \ddot{q} + \left( \frac{\beta}{\delta} \right) \dot{q}^2 + \left( \frac{\gamma}{\delta} \right) q^3 + \ddot{q} = \frac{F_o}{\delta} \cos \Omega t\]  

[3.4.9]

and for the case of damping, the equation becomes

\[\omega_o^2 \ddot{q} + 2 \xi \omega_o \dot{q} + \mu_o q^3 + 2 \mu_2 \xi \omega_o^2 \ddot{q} + \ddot{q} = F(t)\]  

[3.4.10]
3.5- **Nonlinear Free Vibration**

For the case of free vibration where $F_0 = 0$ in equation (3.4.10), the solution to this equation is obtained using the Lindsted's method [43], which is a perturbation technique used to solve second-order dynamic systems of the form

$$x + \omega^2 x = \varepsilon j(x, x)$$  \[3.5.1\]

where $\varepsilon$ is a small parameter, and $j(x, x)$ is a nonlinear function of $(x)$ and $(x)$.

$$q + \omega^2 q + \varepsilon \omega^2 q^3 = 0$$  \[3.5.2\]

where $q$ is assumed to be:

$$q(t) = A \cos \Omega t$$  \[3.5.3\]

in which $A$ is the amplitude of vibration and $\Omega$ is the nonlinear frequency. The approximate solution to eq.(3.5.2) for a symmetrical laminated beam has the following form for the special case when the coefficient $q^2$ is zero.

$$\Omega \equiv \omega_0 \left[ 1 + \varepsilon \left( \frac{3A^2}{8} - \varepsilon^2 \frac{15A^4}{256} \right) \right]$$  \[3.5.4\]

On the other hand, for the unsymmetrically laminated beam, the solution to eq.(3.4.10) is obtained by using the following relation.[53]

$$\Omega \equiv \omega_0 \left[ 1 + A^2 \left\{ \frac{3}{4} \left( \frac{\mu_1}{\omega_0^2} \right) - \frac{5}{6} \left( \frac{\mu_2}{\omega_0^2} \right)^2 \right\} \right]^{1/2}$$  \[3.5.5\]
3.6- Non-linear Harmonic Vibration

By using the finite element method, the coefficients for the Duffing equation is determined.

\[ \ddot{q} + 2\xi\omega_0 \dot{q} + \omega_0^2 q + \mu_2 q^2 + \mu_3 q^3 = F_0 \cos \Omega t \]  

where \( \xi \) is the damping coefficient.

In the case of symmetrically laminated beams which do not have the quadratic term in the Duffing equation, it can be solved, by using the method of iteration (averaging) [43].

Assuming the first approximation to be

\[ q_0 = A \cos \Omega t \]  

The solution and the relationship between the frequency amplitude and force becomes:

\[ F_0^2 = \left( (\omega_0^2 - \Omega^2) A + \frac{3}{4} \mu_3 A^3 \right)^2 + \left( 2\xi \omega_0 \Omega A \right)^2 \]  

and by solving the following equation in terms of frequency ratio,

\[ A^6 \left[ \frac{9\mu^2}{16\omega_0^4} \right] + A^4 \left[ \frac{3\mu}{2\omega_0^2} \left( 1 - \frac{\Omega^2}{\omega_0^2} \right) \right] + A^2 \left[ 4\xi^2 \left( \frac{\Omega^2}{\omega_0^2} \right) + \left( 1 - \frac{\Omega^2}{\omega_0^2} \right)^2 \right] \frac{F_0^2}{\omega_0^2} = 0 \]  

On the other hand if the quadratic part is present, the solution for the Duffing equation (3.6.1) has to be done by the perturbation method [24]. The damping is taken to be linear, for simplicity and we can assume that the excitation is a linear combination of harmonic functions, that is

\[ F(t) = \sum_{i=1}^{N} F_i \cos (\Omega_i t + \theta_i) \]  

where \( (F_i), (\Omega_i) \), and \( (\theta_i) \) are constants.
In that case, the solution for the steady-state problem yields

\[ A^6 A^4 \left[ \frac{2\sigma}{\beta} \right] + A^2 \left[ \frac{\sigma^2 + 4\omega_0^2 \xi^2}{\beta^2} \right] - \left[ \frac{F_0}{2\omega_0\beta} \right]^2 = 0 \]  

where:

\[ \beta = \left[ \frac{9\mu_3\omega_0^2 - 10\mu_2}{24\omega_0^3} \right] \quad \sigma = \left( \frac{\Omega}{\omega_0} \right) \]

In solving unsymmetrically and cross-ply laminated beams, equation (3.6.6) has to be used to determine the response of the beam.
Chapter 4

4.0- Discussion of Results

In this chapter cases of different problems are solved and the results are tabulated in the form of tables and graphs. The effect of material, and boundary conditions constrains are significant and they are presented in the next section. Using the present finite element formulation, the large-amplitude free and force vibration response of isotropic and arbitrarily laminated composite beams (symmetrically angle-ply, cross-ply, and unsymmetrically angle-ply) are studied. Finite element results are tabulated for both cases of including and neglecting the shear deformations and the difference can be seen in the table [4.4-6]. The boundary conditions for simply supported is defined by letting the longitudinal and transverse deflection at both ends equal to zero, but it is free to rotate at both ends. On the other hand for the clamped-clamped condition, all of the deflections and their slopes equal to zero in both ends. For the case of moveable edges, there is no internal axial force present in the beam, but in immovable condition, the beam bear a longitudinal force which magnifies the nonlinearity effect. The fundamental frequency ratio of non-linear free and forced vibration of isotropic and laminated composite beams under various boundary conditions, fiber orientations, and material properties are studied. The fundamental frequency ratio is defined as the ratio of first nonlinear and linear frequency.

The nonlinear free vibration can be simply treated as a limited case of the more general forced vibration problem by setting harmonic force matrix equal to zero. It is obvious that by setting the forcing function to zero, the finite element program would be capable of solving the nonlinear free vibration, and by doing so, the results are compared with the previously established
results. All the results are given for steady-state oscillations. Figures in appendix A shows good agreement between Raciti and Mei’s results [18] [16]. The boundary conditions are simply supported, clamped-clamped, simply supported-clamped, and movable edges. In the case of simply supported, this imposed the condition of a constant end distance instead of the usual theoretical assumption that the axial load in the column remained constant along the beam length.

As it was shown in previous studies, Mei [18] the frequency was dependent upon the amplitude of vibration, and the effect of the amplitude of the vibration becomes more pronounced as the classical linear theory. And also as it was predicted before, the nonlinearity was found to be of the hardening type. The nonlinear effect in this investigation was produced by the axial stretching of the beam.

4.1- Isotropic Beam

To show the validity of this program, various examples of isotropic beams are solved and the results are compared with those of previous work. The result are in good agreement with other works as shown in table (4.2).

The isotropic material that is considered for this problem is an Aluminum beam with the following properties:

<table>
<thead>
<tr>
<th>Isotropic beam Material Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elastic Modules: E=10.0 x 10^6 psi</td>
</tr>
<tr>
<td>Shear Modules: G=3.85 x 10^6 psi</td>
</tr>
<tr>
<td>Poisson's Ratio: v=0.3 in.</td>
</tr>
<tr>
<td>Thickness of the beam: h=0.064 in.</td>
</tr>
<tr>
<td>Width of the Beam: b=1.0 in.</td>
</tr>
<tr>
<td>Length of Beam: L=12 in.</td>
</tr>
<tr>
<td>Number of Element: N=12</td>
</tr>
</tbody>
</table>

Table (4.1)
The effect of shear deformation are tabulated in table (4.2), for three different mode shape.

<table>
<thead>
<tr>
<th>Mode #</th>
<th>Frequency (\omega^2) With Shear</th>
<th>Frequency (\omega^2) Without Shear</th>
<th>Transverse Mode Shape At Node 6 (in.)</th>
<th>Effect of Shear At Node 6 (in.)</th>
<th>% Deference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.292525E+03</td>
<td>0.291542E+03</td>
<td>0.37944E+00</td>
<td>0.27755E-04</td>
<td>% .9999</td>
</tr>
<tr>
<td>2</td>
<td>0.466558E+04</td>
<td>0.466515E+04</td>
<td>-0.17765E+00</td>
<td>-0.51691E-04</td>
<td>% .9932</td>
</tr>
<tr>
<td>3</td>
<td>0.236124E+05</td>
<td>0.236265E+05</td>
<td>-0.22011E+00</td>
<td>-0.14433E-03</td>
<td>% .9993</td>
</tr>
</tbody>
</table>

Table (4.2). Effect of shear deformation on natural frequency for three mode calculation

4.2- Laminated Beams

Numerical results for Graphite/Epoxy laminated composites with the material properties as shown in table (4.3) are presented.

Laminated Beams Material Properties

| Major Elastic Modules: \(E_1=18.5 \times 10^6\) psi |
| Minor Elastic Modules: \(E_2=1.6 \times 10^6\) psi |
| Shear Module: \(G_{12}=.81 \times 10^6\) psi \(G_{13}=.5 \times 10^6\) psi |
| Poisson's Ratio: \(\nu=0.3\) |
| Ply thickness: \(h=0.012\) in./layer |
| Width of beam: \(b=1.0\) in. |
| Length of beam: \(L=12\) in. |
| Number of Element \(N=12\) |

Table (4.3)

The free vibration frequencies and mode position at node 6 and is shown in table (4.4) for beam for laminates with different fiber orientations, like, Symmetrically laminated angle-ply, unsymmetrically angle-ply and cross-ply. And also the influence of boundary condition, number
of layers on the frequency response and nonlinear coefficients is examined. Symmetrically angle-ply laminates were found to be characterized by extensional and bending stiffness, but no bending coupling extension which has considerable effect on nonlinear response.

### Symmetrically Angle-Ply Laminated Beam $[30, -30, 30, -30, -30]$  

<table>
<thead>
<tr>
<th>Mode #</th>
<th>Frequency ($\omega^2_o$) With Shear</th>
<th>Frequency ($\omega^2_o$) Without Shear</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.499258E+03$</td>
<td>$0.504781E+03$</td>
<td>$.0111%</td>
</tr>
<tr>
<td>2</td>
<td>$0.802132E+04$</td>
<td>$0.804500E+04$</td>
<td>$.003%</td>
</tr>
<tr>
<td>3</td>
<td>$0.404729E+05$</td>
<td>$0.407497E+05$</td>
<td>$.006%</td>
</tr>
</tbody>
</table>

Table (4.4). Effect of shear deformation on natural frequency for three mode calculation

The results for moderately large amplitude vibration of laminated beam indicates that the effect of the transverse shear deformation decreases the frequency response and the numerical results for an angle-ply laminated beam is shown in table (4.5) which indicates the effect of shear deformation for first three modes.

### Angle-Ply Laminated Beam $[45, -45, 45, -45, 45]$  

<table>
<thead>
<tr>
<th>Mode #</th>
<th>Frequency ($\omega^2_o$) With Shear</th>
<th>Frequency ($\omega^2_o$) Without Shear</th>
<th>Modal portion Due to Bending At Node 6 (in.)</th>
<th>Modal Portion Due to Shear At Node 6 (in.)</th>
<th>% Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.251830E+03$</td>
<td>$0.252017E+03$</td>
<td>$0.37946E+00$</td>
<td>$0.13253E-03$</td>
<td>$.0007%, $.9997%</td>
</tr>
<tr>
<td>2</td>
<td>$0.402024E+04$</td>
<td>$0.402801E+04$</td>
<td>$-0.17770E+00$</td>
<td>$-0.27640E-03$</td>
<td>$.0019%, $.9984%</td>
</tr>
<tr>
<td>3</td>
<td>$0.203592E+05$</td>
<td>$0.204347E+05$</td>
<td>$-0.22014E+00$</td>
<td>$-0.77121E-03$</td>
<td>$.0037%, $.0065%</td>
</tr>
</tbody>
</table>

Table (4.5). Effect of shear deformation on natural frequency for three mode calculation
Unsymmetrically Angle-Ply Laminated Beam $[45,-45,45,-45,45]$  

<table>
<thead>
<tr>
<th>Mode #</th>
<th>Frequency ($\omega_o^2$)</th>
<th>Frequency ($\omega_o^2$)</th>
<th>Transverse Def. Due to Bending At Node 6 (in.)</th>
<th>Transverse Def. Due to Shear At Node 6 (in.)</th>
<th>% Deference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.523318E+03</td>
<td>0.533447E+03</td>
<td>0.37948E+00</td>
<td>0.30692E-03</td>
<td>%.0194, %.998</td>
</tr>
<tr>
<td>2</td>
<td>0.836134E+04</td>
<td>0.838751E+04</td>
<td>-0.17780E+00</td>
<td>-0.57497E-03</td>
<td>%.0031, %.997</td>
</tr>
<tr>
<td>3</td>
<td>0.421854E+05</td>
<td>0.424910E+05</td>
<td>0.22010E+00</td>
<td>0.16032E-02</td>
<td>%.0072, %.993</td>
</tr>
</tbody>
</table>

Table (4.6). Effect of shear deformation on natural frequency for three mode calculation

4.3- Effect of laminate, fiber Orientation

The fundamental frequency ratio ($\Omega/\omega_o$) of a four-layered angle-ply, Simply supported (L=12 in., b=1 in.) subjected to a uniform harmonic force of $P_o= .2$ are shown in fig. (4.7). The effect of fiber orientation on the natural frequency for various types of laminations are presented in table (4.5), (4.6), (4.7). As the angle of orientation increases the natural frequency decreases and it is the highest at 45 degree.

<table>
<thead>
<tr>
<th>Variation in Angle</th>
<th>Frequency ($\omega_o^2$)</th>
<th>Frequency ($\omega_o^2$)</th>
<th>Transverse Def. Due to Bending At Node 6 (in.)</th>
<th>Transverse Def. Due to Shear At Node 6 (in.)</th>
<th>% Deference</th>
</tr>
</thead>
<tbody>
<tr>
<td>10°</td>
<td>0.637306E+03</td>
<td>0.647207E+03</td>
<td>0.3794E+00</td>
<td>0.3707E-03</td>
<td>%.0156, %.8896</td>
</tr>
<tr>
<td>20°</td>
<td>0.462238E+03</td>
<td>0.461334E+03</td>
<td>0.3794E+00</td>
<td>0.2704E-03</td>
<td>%.0020, %.9993</td>
</tr>
<tr>
<td>30°</td>
<td>0.251830E+03</td>
<td>0.252017E+03</td>
<td>0.3794E+00</td>
<td>0.1477E-03</td>
<td>%.0007, %.9996</td>
</tr>
<tr>
<td>45°</td>
<td>0.971802E+02</td>
<td>0.952104E+02</td>
<td>0.3794E+00</td>
<td>0.5673E-04</td>
<td>%.0203, %.9999</td>
</tr>
</tbody>
</table>

Table (4.7) Effect of angle of orientation on natural frequency
4.4- Effect of number of layers (n) and Thickness to length ratio (h/L)

Numerical results presented for the frequency ratio \( (\Omega/\omega_o) \) cross-ply \((90,0,90,0,90,0,90,0)\) simply supported, laminated beam subjected to a uniform harmonic load with intensity of \( F_o = 0.2 \) lb/in, using 12 elements. Fig (4.8) shows the frequency ratio, vs. Amplitude/Radius of Gyration number of layers ranging from \( N = 2, 4, 6, 8, 10 \) subjected to a uniform harmonic load \( P_o = 0.2 \). As can be seen in table (4.8) for 2 layer laminated beam, the effect of quadratic nonlinearity becomes more significant. The coefficient of cubic nonlinearity converges when the number of layers reach 6.

Cross-Ply Laminated Beam \([90,0,90,0,90,0,90,0]\)

<table>
<thead>
<tr>
<th>Number of Layers</th>
<th>( (h/L) )</th>
<th>( \mu_2 )</th>
<th>( \mu_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2E-3</td>
<td>0.0000E+00</td>
<td>0.1225E+07</td>
</tr>
<tr>
<td>4</td>
<td>4E-3</td>
<td>-0.6586E-06</td>
<td>0.1224E+07</td>
</tr>
<tr>
<td>6</td>
<td>6E-3</td>
<td>0.6287E-03</td>
<td>0.9101E+05</td>
</tr>
<tr>
<td>8</td>
<td>8E-3</td>
<td>0.1094E-02</td>
<td>0.9088E+05</td>
</tr>
<tr>
<td>10</td>
<td>1E-2</td>
<td>-0.2040E-03</td>
<td>0.9081E+05</td>
</tr>
</tbody>
</table>

Table (4.8) Effect of number of layers on coefficient of Duffing equation

4.5- Effect of Boundary Conditions

Numerical results for the frequency ratio \( (\Omega/\omega_o) \) for Isotropic and laminated beam are presented, for six layers cross-ply \((30,-30,30,-30,30,-30)\) simply supported, clamped-clamped, movable edges, laminated beam subjected to uniform harmonic load with intensity of \( F_o = 0.2 \) lb/in, using 12 elements.

Moveable and immovable in-plane boundary conditions as it is shown in appendix A, has significant effect on the non-linearity of the problem. The frequency ratio for a four layered cross-ply laminate subjected to a uniform load The non-linearity is greatly reduced with the
in-plane ends no longer restrained as compared to the case of immovable ends. The effect of three different boundary conditions are examined and by changing the end conditions, the beam shows a softening spring effect. And also the softening spring effect can be seen in the comparison of isotropic and laminated composite beams in appendix A.

<table>
<thead>
<tr>
<th>Boundry Cond.</th>
<th>$\omega^2$</th>
<th>$\mu_2$</th>
<th>$\mu_3$</th>
<th>$P_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS</td>
<td>0.2521E+03</td>
<td>0.6287E-03</td>
<td>0.9101E+05</td>
<td>0.3755E+01</td>
</tr>
<tr>
<td>CC</td>
<td>0.1290E+04</td>
<td>0.6916E-04</td>
<td>0.1465E+06</td>
<td>0.8581E+02</td>
</tr>
<tr>
<td>S-C</td>
<td>0.6148E+03</td>
<td>0.4981E-03</td>
<td>0.1249E+06</td>
<td>0.1493E+02</td>
</tr>
<tr>
<td>Movable Edg.</td>
<td>0.2520E+03</td>
<td>-0.5656E-04</td>
<td>0.1157E+03</td>
<td>0.1071E+05</td>
</tr>
</tbody>
</table>

Table (4.9) Effect of boundary conditions on coefficient of Duffing equation
Chapter 5

5.0- Conclusion

Presented here is the nonlinear analysis of forced vibration and arbitrary laminated fiber-reinforced composite beams by using 20 DOF finite element method. To evaluate the efficiency and validity of the present research, the comparisons are made with a variety of previously published results. Because of the problem's versatile applicability, effect of complex beam configuration, anisotropic material properties, uneven thickness, and various boundary conditions can be studied. The linear and nonlinear stiffness matrix, mass matrix and harmonic force matrix is developed for a beam element subjected to uniform distributed harmonic excitation loading. The amplitude of oscillation for the beam is increased by increasing the loading amplitude. Also as the damping ratio is increased the peak amplitude is decreased. The effect of nonlinearity is more pronounced for immovable edge condition than those of moveable case. One advantage of this type of finite element formulation is that a Duffing type differential equation can be obtained by using the finite element method and the damping effect can be incorporated easily.

5.1- Further studies

The most simplest and the widespread practical application of laminated composites are in the form of plates, but beams are, of course, simple to analyze because of their one dimensional nature. The finite element program presented here can be modified to deal with plates, shells or more complex structures, such as those used in aircraft structures.
Appendix A

Free and Forced Vibration Plots
Free Vibration (Amplitude vs. Frequency Ratio)

Simply Supported Isotropic Beam

- Present Study
- Raciti
- Mei

(Frequency Ratio)**2

Figure: A1
Free Vibration of a Symmetric Cross-Ply

Comparison of Results For Simply Supported

Figure: A2
Free Vibration (Amplitude vs. Frequency Ratio)

Isotropic & Laminated Beam with Different Fiber Orientations

Figure: A3
Forced Vibration of Isotropic Beam

Comparison of Results (A/R vs. FR)

Figure: A4
Harmonic Vibration of Isotropic Beam

(Amplitude vs. Frequency Ratio)

+ Galerkin Method
• Finite Element Method

Po = 0.2

Simply Supported

Figure: 25
Free & Forced Vibration of Isotropic Material

(Amplitude vs. Frequency Ratio)
Harmonic Excitation (Amplitude vs. Frequency Ratio)

Isotropic & Laminated Composite Beam with Different Fiber Orientations

Simply Supported

P₀ = .2

Figure: A7
Forced Vibration of Cross-Ply Laminated Beam

Different Boundary Conditions (A/RG vs. Fr)

- Clamped-Clamped
- Movable Ends

[30,-30,30,-30,30,-30]

Figure: A8
Forced Vibration of Cross-Ply Laminated Beam

For Linear & Nonlinear Analysis (A/RG vs. Fr.)

Figure: A9
Amplitude vs. Frequency

For Simply Supported, Cross-Ply Laminated Beam with Different Layers

- 2 Layers
- 4 Layers
- 6 Layers
- 8 Layers
- 10 Layers

Figure: A10
Harmonic Vibration of Isotropic Beam with Different Loading

(Amplitude vs. Frequency Ratio)

Simply Supported

- Po=0.0
- Po=0.05
- Po=0.1
- Po=0.2
- Po=0.4
- Po=0.6

Figure: All
Harmonic Excitation (Amplitude vs. Frequency Ratio)

Symmetric Angle_ply with Different Damping Coefficient

Simply Supported

P_0 = 0.2

- Frequency Ratio
- Amplitude/Radius of Gyration

Damping Coefficients:
- ◦ Damping Coefficient = 0.0
- + Damping Coefficient = 0.2
- • Damping Coefficient = 0.1
- • Damping Coefficient = 0.05

Figure: A12
Forced Vibration of Cross-Ply Laminated Beam

With & Without Shear Effect (A/RG vs. Fr)

![Graph showing forced vibration of cross-ply laminated beam with and without shear effect. The graph illustrates the relationship between amplitude/radius of gyration and frequency ratio. The x-axis represents frequency ratio ranging from 0.0 to 4.0, and the y-axis represents amplitude/radius of gyration ranging from 0.0 to 10.0. Two curves are shown: one for without shear effect and another for with shear effect.](Figure: A13)
Appendix B

Out-put Results
KENAREHM@EPSILON (10)% !7
comp3.out

ENTER THE TYPE OF PROBLEM TO BE SOLVED (1,2,3,4,5)

******************************************************************************
1) STATIC ANALYSIS *
2) LINEAR FREE VIBRATION *
3) NONLINEAR FREE VIBRATION *
4) NONLINEAR FORCED VIBRATION *
5) NONLINEAR RANDOM VIBRATION *
******************************************************************************

4

WOULD YOU LIKE TO USE INPUTS IN FILE ? y

ENTER (NEL, NON, NMOD, ALPHA, TRANSLOAD)
10, 11, 3, 0.83333333, .2

ENTER NATURAL BOUNDARY CONDITIONS.
U, UP, WB, THEB, WS, THEs, T, TP, BET, BETP
0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1
0, 1, 0, 1, 0, 1, 1, 1, 1, 1

ENTER (LENGTH, WIDTH, DENSITY) FOR EACH ELEMENT
1, 1, .055
1, 1, .055
1, 1, .055
1, 1, .055
1, 1, .055
1, 1, .055
1, 1, .055
1, 1, .055
1, 1, .055

ENTER (NL, E1, E2, V12, G12, G13)
6, 18.7e6, 1.36e6, .3, .7479e6, .624e6

ENTER (ANGLE OF ORIENTATION)(LAYER THICKNESS) FOR EACH ELEMENT
45, .012
45, .012
45, .012
45, .012
45, .012

-----------------------------------------------
A MATRIX

0.4321E+06  0.3244E+06  -0.1562E-01
0.3244E+06  0.4321E+06  -0.1953E-01
-0.1562E-01 -0.1953E-01  0.3487E+06
**B MATRIX**

\[
\begin{bmatrix}
0.2441E-03 & -0.1221E-03 & -0.1885E+04 \\
-0.1221E-03 & 0.2441E-03 & -0.1885E+04 \\
-0.1885E+04 & -0.1885E+04 & 0.0000E+00
\end{bmatrix}
\]

**D MATRIX**

\[
\begin{bmatrix}
0.1867E+03 & 0.1401E+03 & -0.5086E-05 \\
0.1401E+03 & 0.1867E+03 & -0.5086E-05 \\
-0.5086E-05 & -0.5086E-05 & 0.1506E+03
\end{bmatrix}
\]

**C MATRIX**

\[
\begin{bmatrix}
0.1886E+06 & -0.4046E-02 & 0.5650E-03 & -0.4698E+03 & 0.0000E+00 \\
-0.4046E-02 & 0.3296E+06 & -0.4698E+03 & -0.1258E-03 & 0.0000E+00 \\
0.5650E-03 & -0.4698E+03 & 0.8145E+02 & -0.2600E-05 & 0.0000E+00 \\
-0.4698E+03 & -0.1258E-03 & -0.18515E-02 & 0.1424E+03 & 0.0000E+00 \\
0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.0000E+00 & 0.3744E+05
\end{bmatrix}
\]

**DYNAMIC ANALYSIS:**

**MODE NO. 1**

**EIGENVECTORS AT EIGENVALUE= 0.199896E+03**

**FREQUENCY (RAD/S)= 0.141385E+02**

**NODAL DISPLACEMENTS:**

**NODE NO. 1**

- **U:** 0.00000E+00
- **DER.U:** 0.42220E-06
- **W-B:** 0.00000E+00
- **THETA-B:** 0.13285E+00
- **W-S:** 0.00000E+00
- **THETA-S:** 0.28689E-04
- **TAU:** -0.18515E-02
- **DER.TAU:** 0.48879E-05
- **BETA:** -0.69527E-07
- **DER.BETA:** -0.17915E-04

**NODE NO. 2**

- **U:** 0.38388E-07
- **DER.U:** 0.22288E-06
- **W-B:** 0.13068E+00
- **THETA-B:** 0.12635E+00
- **W-S:** 0.27902E-04
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<th>THETA-B</th>
<th>W-S</th>
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EIGENVECTORS AT EIGENVALUE=0.144780E+04
FREQUENCY (RAD/S)=0.380500E+02

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### Nodal Displacements:

#### MODE NO. 3

**Eigenvectors at Eigenvalue:** 0.144840E+04  
**Frequency (Rad/S):** 0.380579E+02

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**NONLINEAR COEFFICIENT:**

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<td></td>
<td><strong>X1= 0.7037E+00</strong></td>
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<td><strong>X2= -0.3882E-05</strong></td>
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<td><strong>X3= 0.2955E+03</strong></td>
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<td><strong>X4= 0.3542E-02</strong></td>
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**NONLINEAR COEFFICIENT IN GENERAL FORM**

\[ 0.1987E+03 \quad q + -0.1096E-02 \quad q^2 + 0.8342E+05 \quad q^3 + q'' = F(T) \]

**NONLINEAR COEFFICIENTS FOR HARMONIC VIBRATION:**

\[ 0.1987E+03 \quad q + -0.1096E-02 \quad q^2 + 0.8342E+05 \quad q^3 + q'' = -0.4564E+01 \quad F_0 \cos(\omega t) \]
Appendix C

Program Listing
MAIN PROGRAM

LINEAR AND NONLINEAR ANALYSIS OF ANTISYMMETRICALLY LAMINATED COMPOSITE BEAM SUBJECTED TO HARMONIC AND RANDOM EXCITATION USING 20 (DOF) FINITE ELEMENT METHOD

PROGRAM COMPITE

REAL A(3,3), B(3,3), D(3,3), D44, C(5,5), EX(5), H(5), XMODE(300),
+ L(40), W(40), GSTIFF(120,120), GINCRE(120,120), GMASS(120,120),
+ GLOAD(120,1), GNON1(120,120), GNON2(120,120), GNON1(120,120), C,
+ GNON2(120,120), DEN(40), GNN1(120,120), GNN2(120,120), TQW(1,50),
+ GTT(120,120), VL(120), BETA(120), XN1(20,20), ALPHA, XN2(20,20),
+ GSTIFF1(120,120), GMASS1(120,120), GINCR1(120,120), GLOAD1(120),
+ X1, X2, X3, X4, X11, X22, X33, PO(1,1), TRANSLOAD
INTEGER NEL, NON, NMOD, SELECT, ND, GDOF(40,10), BCN(40,10),
+ TV(120), IW
COMPLEX ALFA(120)
CHARACTER ANSI

CALL MENU(SELECT)

CALL INPUT(NEL, NON, NMOD, ALPHA, BCN, L, W, EX, H, DEN, VL, SELECT, ANSI,
+ TRANSLOAD)

CALL ABD(A, B, D, D44, T, ANSI)

CALL CMAT(A, B, D, C, D44, ALPHA)

CALL INITIAL(NON, BCN, GDOF, GSTIFF, GMASS, GINCRE, GNON1, GNON2,
+ GNN1, GNN2, GTT, GNON1, GNON2, GLOAD, ND, GMASS1, GSTIFF1)

CALL GLOBAL(C, EX, GDOF, H, SELECT, L, W, GLOAD, T, ND, GINCRE, GSTIFF1, GMASS1, GINCR1, GLOAD1, TRANSLOAD)

IF (SELECT.EQ.1) THEN
  CALL STATIC(NON, BCN, GSTIFF, ND, VL)
ELSEIF (SELECT.EQ.2) THEN
  CALL LFEVIB(ND, NMOD, GMASS, GSTIFF, ALFA, BETA, BCN, NON, XMODE, VL)
ELSEIF (SELECT.EQ.3) THEN
  CALL LFEVIB(ND, NMOD, GMASS, GSTIFF, ALFA, BETA, BCN, NON, XMODE, VL)
  CALL NONLINANL(C, EX, L, XMODE, W, H, XN1, XN2, GDOF, GSTIFF1, GMASS1,
+ BCN, TV, NON, ND, VL, NEL, IW, TQW, X1, X2, X3, X4, X11, X22, X33)
  CALL GRAPHFREE(X11, X22, X33, T)
ELSEIF (SELECT.EQ.4) THEN
  CALL LFEVIB(ND, NMOD, GMASS, GSTIFF, ALFA, BETA, BCN, NON, XMODE, VL)
  CALL NONLINANL(C, EX, L, XMODE, W, H, XN1, XN2, GDOF, GSTIFF1, GMASS1,
+ BCN, TV, NON, ND, VL, NEL, IW, TQW, X1, X2, X3, X4, X11, X22, X33)
  CALL NLFCDVDIB(IW, TQW, L, NEL, X11, X22, X33, X4, GLOAD1, ND)
  CALL GRAPHFCD(X11, X22, X33, T, PO)
ELSEIF (SELECT.EQ.5) THEN
  CALL LFEVIB(ND, NMOD, GMASS, GSTIFF, ALFA, BETA, BCN, NON, XMODE, VL)
  CALL NONLINANL(C, EX, L, XMODE, W, H, XN1, XN2, GDOF, GSTIFF1, GMASS1,
+ BCN, TV, NON, ND, VL, NEL, IW, TQW, X1, X2, X3, X4, X11, X22, X33)
  CALL NLIRANVIB(IW, TQW, L, NEL, X11, X22, X33)
  CALL GRAPHRAND(X11, X22, X33, T)
ENDIF
END