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Numerical Case Study of an Atom-Photon Interaction in a Cavity Exploring Quantum Control

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By

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Abstract

We study the Magnus expansion (ME) approximation scheme for the interaction between an atom and a single quantized cavity mode (Jaynes-Cumming model) in a closed quantum system in resonance or near resonance for a time-dependent coupling coefficient $g(t)$ in both the interaction and rotating picture by implementing a novel numerical method called MG4 and compare our results to the Runge-Kutta 4th (RK4) order solution to demonstrate the conservation of unitary evolution of the ME. A cursory study of open quantum system is given to encourage the study of ME for dissipative systems. Furthermore, we assume that our time-dependent coupling coefficient $g(t)$ can take on two forms, Gaussian and sinusoidal, which are introduced as pulses to study the behavior and response of the cavity. Our results show that ME is a sufficient approximation scheme in our study of closed quantum systems which may have applications in quantum control.

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1

Introduction

As technology continues to evolve towards smaller devices the need for control of such systems becomes evermore pressing as the system moves towards the quantum realm. To begin controlling such systems one must begin with a set of differential equations. Analytical methods with exact solutions are often the preferred and long-sought method in solving differential equations that determine the evolution of physical systems. However, these types of solutions are often impossible to obtain and the physicist must resort to approximations and numerical solutions. In this study we will explore the Magnus Expansion (ME) and use the Jaynes-Cumming Model in a closed quantum system in two different pictures (the interaction picture an the rotating picture) and compare these results to a numerical solution using the 4th Order Rung-Kutta method. We will see how the ME can be used for applications in quantum control theory. We will also implement a method of the ME for linear differential equations called MG4. A brief discussion on open and closed quantum systems will also be given.

A popular approximation scheme in quantum mechanics is perturbation theory which has been used in many applications. In quantum physics, the principle of unitary evolution is an important feature in allowing us to use statistical interpretation of quantum experiments. Wilhelm Magnus in 1954 proposed an alternative to the standard perturbation theory that maintains this principle of unitarity. This new field is often called exponential perturbation theory but we will adopt the name Magnus expansion (ME) in this study. We will show that the Magnus expansion provides an adequate approximation to a complete quantum description of the interaction between a two level system and an electromagnetic field. Hence, we study the resonant or near resonant interaction between a two level atom in a single quantized cavity mode which is commonly known as the Jaynes-Cumming Model (JCM). However, in our model the coupling between the atom and the cavity is time dependent. If the coupling coefficient as a function of time can be control, our work will have applications in quantum control.

The order of this thesis is as follows: we will first give a brief review of quantum mechanics and some important postulates necessary to use quantum mechanics in Chapter 2 [20]. Chapter 2 will also include a discussion on the different "pictures" one can utilize to ease further calculations. We will include a discussion on open quantum systems for real world applications in Chapter 2. A study of open quantum systems is important due the inherent interaction between the atom and the environment[21]. Though we have equations that govern the state of a system deterministic-ally, these equations only hold for ideal cases where we consider the system isolated from its

environment, devoid from any interaction that could cause it to lose energy. In real world applications, however, we must introduce the system to the possibility of energy dissipation and model our system accordingly. Density Matrix formalism and the Master Equation must be introduced to encompass these new fields of study [6, 16, 21]. The study of the density matrix formalism will prove to be a useful tool under the Markovian and Born approximations.

Chapter 3 will delve into the Quantum interaction of light with a two-level atom. Here we will briefly discuss the two level atom and the quantization of the electromagnetic field. With these two concepts at hand we will derive the JCM that we will use in our study of the ME [6, 19, 22].

Chapter 4 will begin our study of ME [5, 15]. We will provide two proofs to the approximations and discuss our applications of ME in the interaction picture. Figures will show a comparison of our approximate solution to a numerical RK4 solution.

Chapter 5 will discuss the application of the ME in the rotating picture for a two-level atom-photon system that is subject to "periodic injections" of atoms. For this study, a method for the Magnus expansion introduced by Iresles et al [12] will be implemented. In contrast to the previous chapter where we introduced the constant coupling coefficient $g(t) \to V_0$, we will introduce *coupling functions* that can take on any form. For our study, we will focus on Gaussian and sinusoidal functions.

Lastly, Chapter 6 will serve as our conclusion and will include any discussion for future work in ME for open quantum systems.

2

Quantum Mechanics Fundamentals

In this chapter we review the fundamental postulates of quantum mechanics necessary for later discussions and introduce two new concepts not covered in undergraduate physics courses (the interaction picture and open quantum systems). The first concept is a combination of two well known pictures or representation of quantum systems, Schrodinger's picture and Heseinberg's picture. The interaction picture is convenient for analyzing quantum systems that includes interactions with external entities, perturbing the system from a well known dynamics. The interaction picture is commonly used in approximation schemes that assumes this type of interaction between a well known system and a perturbing system. The second concept, open quantum systems, is an extension of Schrodinger's equation for isolated quantum systems that interact with an environment.

2.1 Review of Quantum Mechanics of an Isolated System

Quantum mechanics can be summarized by 4 main postulates [20]:

1. The state of the particle living in an n-dimensional Hilbert space is completely specified and represented by a column vector $|\Psi(t)\rangle$ in a Hilbert space in a given eigenbases $(|\Psi_1\rangle, |\Psi_2\rangle, .., |\Psi_n\rangle)$,

2. Independent variable x of classical mechanics (observable) is represented by Hermitian operators X,

3. If a particle is in a state $|\Psi(t)\rangle$, measurement of the variable corresponding to an operator A will yield one of its eigenvalues a with probability $|\langle a|\Psi\rangle|^2$ where $|a\rangle$ is the eigenvector. The system then changes from state $|\Psi(t)\rangle$ to state $|a)\rangle$. Following the Copenhagen iterpretation, the wave function $|\Psi(t)\rangle$ collapses to state $|a\rangle$, and

4. The state vector $|\Psi(t)\rangle$ obeys Schrodinger's equation:

$$i\hbar\frac{d}{dt}|\Psi\rangle = H|\Psi\rangle. \tag{2.1}$$

Recall that a Hilbert space contains vectors that are normalized, allowing one to invoke probabilistic interpretations of quantum mechanics. In addition to these 4 postulate, another important axiom is the expectation value:

$$\langle A\rangle = \langle\Psi(t)|A|\Psi(t)\rangle \tag{2.2}$$

which represents the average value of the observable.

2.2 Schrodinger's Picture, Heisenberg's Picture and Interaction Picture

Schrodinger's picture of quantum mechanics has the state carry the time dependence [21]. In this picture, the state evolves with time while the observable 2.2 remains time-independent[10, 20, 18]. *Heseinberg's picture*, on the other hand, has the operator carry the time dependence while the state of the system remains time-independent. In this picture, the expectation value is:

$$\langle \Psi | \, A_H(t) \, | \Psi \rangle$$

Note that to represent any state or operator in Schrodinger's picture we will adopt the formalism of the absense of a subscript while to represent an operator in Heseinberg's picture we shall use the subscript $_H$. If $U(t)$ is a unitary time operator that transforms the state vector from its initial state to its final state $|\Psi(t)\rangle = U(t) \, |\Psi(0)\rangle$, then a simple transformation law can be derived from the Schrodinger picture to Heseinberg's picture:

$$\langle \Psi(t) | \, A \, | \Psi(t) \rangle$$

$$\langle \Psi(0) | \, U^\dagger(t) A U(t) \, | \Psi(0) \rangle$$

$$A_H(t) = U^\dagger(t) A U(t). \tag{2.3}$$

Since the operator is time-dependent in Heseinberg's picture, we have to construct an equation of motion for Heseinberg operators. If we differentiate equation (2.3), then we get:

$$\begin{aligned}\frac{dA_H(t)}{dt} &= \partial_t \left(U^\dagger(t)AU(t)\right) \\ &= \partial_t \left(U^\dagger(t)\right) AU(t) + U^\dagger(t)A\partial_t \left(U(t)\right).\end{aligned}$$

Now, since $U(t)$ is a unitary operator that transforms $|\Psi(t)\rangle = U(t)\,|\Psi(0)\rangle$, we can derive an differential equation for the evolution operator [21] that obeys:

$$\partial_t U(t,t_0) = -\frac{i}{\hbar} HU(t,t_0). \tag{2.4}$$

Therefore, our equation of motion for Heseinberg operators becomes:

$$\begin{aligned}&= \frac{i}{\hbar}U^\dagger(t)HAU(t) - \frac{i}{\hbar}U^\dagger(t)AHU(t) \\ &= \frac{i}{\hbar}U^\dagger(t)HUU^\dagger(t)AU(t) - \frac{i}{\hbar}U^\dagger(t)AUU^\dagger(t)HU(t).\end{aligned}$$

but because we assume H to be time-independent, $[H, U(t)] = 0$, which leads us to:

$$i\hbar\frac{dA_H(t)}{dt} = [A_H, H]. \tag{2.5}$$

The interaction picture is a hybrid of both pictures. Instead of having the state or the observable carry the time dependence, both carry the time dependence in the interaction picture. The interaction picture a useful representation when dealing with any external systems. In the interaction picture, the hamiltonian is assumed to be of the form

$$H = H_0 + V \tag{2.6}$$

where H_0 is the Hamiltonian with respect to the Heseinberg picture and V is the Hamiltonian with respect to the Schrodinger picture.H_0 is assumed to be the Hamil-

tonian of a system whose dynamics are well known, while the Hamiltonian V is assumed to be due to the external system that perturbs the system slightly. In this picture, it can be interpreted that the state evolves with respect to V while the operators evolve with respect to H_0. To find the state in the interaction picture, we assume that H_0 is time-independent. The transformation from Schrodinger's picture to the interaction picture is then defined as:

$$|\Psi_I(t)\rangle \equiv e^{\frac{iH_0t}{\hbar}} |\Psi(t)\rangle . \tag{2.7}$$

From this definition we can derive a transformation law from Schrodinger's picture to the interaction picture. Starting from the definition of expectation value, we have (for the interaction picture):

$$\langle\Psi_I(t)| A_I(t) |\Psi_I(t)\rangle . \tag{2.8}$$

From eq. (2.7), we have:

$$\langle\Psi(t)| e^{\frac{-iH_0t}{\hbar}} A_I(t) e^{\frac{iH_0t}{\hbar}} |\Psi(t)\rangle \tag{2.9}$$

which is equivalent to:

$$\langle\Psi(t)| A |\Psi(t)\rangle . \tag{2.10}$$

Hence, our transformation for any operator from Schrodinger's picture to the interaction picture is:

$$A_I(t) = e^{\frac{iH_0t}{\hbar}} A e^{\frac{-iH_0t}{\hbar}} \tag{2.11}$$

whose evolution obeys

$$i\hbar\partial_t A_I(t) = [A_I(t), H_0]. \tag{2.12}$$

which is derived from eq. (2.5). It is simple to show that by applying equation eq.

(2.7) to Schrodinger's equation, we obtain the following equation of motion for the interaction picture:

$$i\hbar\partial_t \left|\Psi(t)\right\rangle_I = V \left|\Psi(t)\right\rangle_I . \tag{2.13}$$

It is clear from eq. (2.12) that the evolution of the operator is dependent on H_0 while eq. (2.13) shows that the evolution of the state is dependent on V, which is in agreement with our assumption from eq. (2.6). This is a key feature of the intearction picture.

2.3 Open Quantum Systems

All quantum systems that only deal with system that do not interaction with its environment called closed quantum systems. Quantum systems that includes interactions with its surroundings are called open quantum system. Most practical applications in quantum optics require an understanding of these open quantum systems. To begin with we shall discuss the density operator formalism that will be used in derivation of the Master equation (ME). The ME takes into account any form of damping of the quantum system in terms of spontaneous and stimulated emissions. To derive the ME we must make some necessary assumptions about the behavior of the system. We will discuss the Born-Markov approximations and their corresponding consequences.

2.3.1 Density Matrix Formalism

The density matrix formalism has several applications in quantum mechanics. The density matrix formalism can be used to keep track of several closed quantum systems that are subject to a classical stochastic process. For our purpose the density matrix formalism will be used to keep track of the dissipative energy losses from the quantum system in question to its environment. Since the density matrix represents an ensemble of quantum systems, we call this representation of states *mixed states*. Each state in this ensemble of quantum systems is a *pure state*. Hence, a mixed state is composed of pure states. Given a set of states $|\Psi_0\rangle, |\Psi_1\rangle, ..., |\Psi_n\rangle$, the density matrix is defined to be:

$$\rho \equiv \sum_{i=0}^{n} P_i \left|\Psi_i(t)\right\rangle \left\langle\Psi_i(t)\right| \tag{2.14}$$

where P_i is the probabilty for $|\Psi_i\rangle$. It follows that $\sum_i P_i = 1$. From now on we will utilize Einstein notation for summation. The density matrix obeys the following properties:

1. $Tr(\rho) = 1$
2. $Tr(\rho^2) \leqslant 1$
3. $\rho^\dagger = \rho$ (hermiticity)

For an ensemble of system where $Tr(\rho^2) = 1$, we have a pure state. If $Tr(\rho^2) < 1$, we have a mixed state. The evolution of the density matrix is governed by *Liouville's*

equation:

$$i\hbar\frac{\partial\rho}{\partial t} = [\hat{H}, \rho] \tag{2.15}$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator relation and $\hat{H}$ is the energy operator.

Some important properties of the density matrix is the trace of a tensor product space. We will define the trace to be:

$$Tr(A) \equiv \sum_{\beta} \langle\beta| A |\beta\rangle . \tag{2.16}$$

where $|\beta_i\rangle$ are the basis in the Schrodinger picture in any Hilbert space. It is given that:

$$Tr(A \otimes B) = Tr(A)Tr(B). \tag{2.17}$$

The cycylic property of traces will also play an important role in our derivation of the master equation. It states that operators commute in a cyclic fashion when taken under a trace. In other words:

$$tr(ABC) = Tr(CAB) = Tr(BCA). \tag{2.18}$$

Lastly, another important property is the partial trace which is defined to be a mapping of V and W, which are finite-dimensional vector spaces over a field, to V:

$$T\epsilon L(V \otimes W) \rightarrow Tr_W(T)\epsilon L(V).$$

In other words, if we take the partial trace of $V \otimes W$ over W, we should have $Tr(V)$:

$$Tr_W(V \otimes W) = Tr(V). \tag{2.19}$$

The density matrix also obeys the transformation into the interaction picture:

$$\rho_I = e^{\frac{iH_0t}{\hbar}}\rho_S(t)e^{-\frac{iH_0t}{\hbar}}. \tag{2.20}$$

Note, however, that the density operator in the interaction picture is not necessary time independent. Once the states and operators have been defined in the interaction picture, the evolution of the state in the interaction pictures becomes:

$$i\hbar\partial_t \left|\Psi_I(t)\right\rangle = V_I \left|\Psi_I(t)\right\rangle \tag{2.21}$$

$$i\hbar\partial_t\rho_I = [V_I, \rho_I] \tag{2.22}$$

where $V_I = e^{-\frac{iH_0t}{\hbar}}Ve^{\frac{iH_0t}{\hbar}}$. H_0 is viewed as a Hamiltonian which is well understood, while V is viewed as a more complicated Hamiltonian due to interactions.

If we define $U(t) = e^{-\frac{iH_0t}{\hbar}}$, then the table below summarizes the three different pictures and their respective transformations from the Schrodinger picture:

	Schrodinger's	Heseinberg	Interaction
ket state	$\|\Psi(t)\rangle$	$\|\Psi(0)\rangle$	$\|\Psi_I(t)\rangle = U^\dagger(t)\|\Psi(t)\rangle$
observable	A	$A_H(t)$	$A_I = U^\dagger AU$
density matrix	$\rho_S(t) = Eq.(2.14)$	constant	$\tilde{\rho} = U^\dagger(t)\rho_S(t)U(t)$
expectation value	$\langle\Psi(t)\|A\|\Psi(t)\rangle$	$\langle\Psi(0)\|A_H(t)\|\Psi(0)\rangle$	$\langle\Psi(0)\|A_I(t)\|\Psi(0)\rangle$
Evolution	$i\hbar\partial_t\|\Psi(t)\rangle = H\|\Psi(t)\rangle$	$i\hbar\partial_t A_H(t) = [A_H(t), H]$	Eqs. (2.12) and (2.13)

Table 2.1: Summary of transformations commonly used in quantum mechanics.

2.3.2 Derivation of The Master Equation

By utilizing the interaction picture and the density matrix formalism, we can derive an equation (the Master Equation) for the non-unitary evolution of the density matrix of an open quantum system subject to an external bath. In this formalism, the state

of the total system is ρ_{SB} and the total Hamiltonian of this system is:

$$H = H_S + H_B + H_{SB} \tag{2.23}$$

where H_S, H_R and H_{SB} are the Hamiltonians for the system, bath and system-bath interaction respectively. From equation (2.15), we have:

$$i\hbar\partial_t\rho_{SB} = [H, \rho_{SB}] \tag{2.24}$$

Here, the evolution of the total system (ρ_{SB}) is unitary. We shall introduce the *reduce density matrix* which is defined as:

$$\rho_S \equiv Tr_B[\rho_{SB}]. \tag{2.25}$$

Using our interaction picture, we assign the following relation $H_S + H_B \rightarrow H_0$ and $H_{SB} \rightarrow V$ (note that we used the overhead tilde to indicate in the interaction picture instead of the subsript I). Hence,

$$\tilde{\rho}_{SB} = e^{i(H_S+H_B)t/\hbar}\rho_{SB}(t)e^{-i(H_S+H_B)t/\hbar}$$

$$\tilde{H}_{SB} = e^{i(H_S+H_B)t/\hbar}H_{SB}(t)e^{-i(H_S+H_B)t/\hbar}.$$

According to equation (2.22), we have:

$$i\hbar\partial_t\tilde{\rho}_{SB}(t) = [\tilde{H}_{SB}(t), \tilde{\rho}_{SB}(t)]. \tag{2.26}$$

Integrating this equation from t to $t + \Delta t$,

$$\tilde{\rho}_{SB}(t + \Delta t) - \tilde{\rho}_{SB}(t) = -\frac{i}{\hbar}\int_t^{t+\Delta t} dt'[\tilde{H}_{SB}(t'), \tilde{\rho}_{SB}(t')]$$

iterating once more from t to t',

$$\tilde{\rho}_{SB}(t') = \tilde{\rho}_{SB}(t) - \frac{i}{\hbar}\int_t^{t'} dt''[\tilde{H}_{SB}(t''), \tilde{\rho}_{SB}(t'')]$$

and using this for $\tilde{\rho}_{SB}(t)$ in the previous equation, we get:

$$\tilde{\rho}_{SB}(t+\Delta t) - \tilde{\rho}_{SB}(t) = \\ -\frac{i}{\hbar}\int_t^{t+\Delta t} dt'[\tilde{H}_{SB}(t'), \tilde{\rho}_{SB}(t)] - \frac{1}{\hbar^2}\int_t^{t+\Delta t} dt' \int_t^{t'} dt''[\tilde{H}_{SB}(t'), [\tilde{H}_{SB}(t''), \tilde{\rho}_{SB}(t'')]]. \tag{2.27}$$

2.3.2.1 Born Approximation

The Born Approximation takes advantage of the assumption that the bath is infinite dimensional and is not affected by the coupling between the system and bath [8, 21, 16]. Hence, *the coupling between the system and the bath are "weak"*. It assumes that the perturbative effects of the system on the bath is negligible and results in a short correlation time between the system and the bath. In other words, any effects done onto the bath due to the system quickly fades away and the inequality $\Delta t >> \tau_c$ must be true. We shall call this correlation time τ_c. This is equivalent to saying that the system-bath state factorizes

$$\tilde{\rho}_{SB}(t) \approx \tilde{\rho}_S(t) \otimes \tilde{\rho}_B$$

throughout its evolution because the system and bath are uncorrelated and hence its tensor product can be factorized. We further assume that

$$Tr_B[\tilde{H}_{SB}(t'), \tilde{\rho_B}] = 0.$$

This follows from the assumption that the initial state of the bath doesn't change with time. This assumption thus implies that

$$Tr_B[\tilde{H}_{SB}(t'), \tilde{\rho}_{SB}(t)] = 0.$$

Thus, after taking the partial trace over the bath of equation (2.27) and together with the cyclic property of traces, the first term cancels out. We are left with:

$$\Delta\tilde{\rho}_S(t) \approx -\frac{1}{\hbar^2}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' Tr_B[\tilde{H}_{SB}(t'), [\tilde{H}_{SB}(t''), \tilde{\rho}_{SB}(t'')]]. \tag{2.28}$$

where $\Delta\rho_S(t) = \rho_S(t+\Delta t) - \rho_S(t)$.

2.3.2.2 Markov Approximation

While *Born's approximation* allows us to assume that the effects of the perturbation on the *bath* are "short-lived", and thus decays very quickly, *Markov's approximation* allows us to assume that the evolution of *system* $\tilde{\rho}(t)$ depends on its current state, not on past history [8, 21, 16]. This amounts to saying that $\rho_{SB}(t'') = \rho_{SB}(t)$ in equation (2.28). This approximation can further be shown by Taylor series expansion of $\rho(t'')$: $\rho(t'') = \rho(t) + O(\Delta t)$ which holds true when $\Delta t \to 0$. However, we still require that $\Delta t >> \tau_c$ due Born's approximation. Hence, from eq. (2.28), we obtain:

$$\Delta\tilde{\rho}_S(t) \approx -\frac{1}{\hbar^2}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' Tr_B[\tilde{H}_{SB}(t'), [\tilde{H}_{SB}(t''), \tilde{\rho}_{SB}(t)]]. \tag{2.29}$$

2.3.2.3 Interaction Hamiltionian (H_{SB})

It can be shown, due to the partial trace over the bath shown in eq. (2.29), that the rapid decay of the *system* is dependent on $\tau = t' - t''$. We shall call τ the relaxation time of the system due to its interation with the bath. To show this we assume $\tilde{H}_{SB}$ to be a sum of products of observable [8, 21]:

$$\tilde{H}_{SB} = \hbar\tilde{S}_\alpha\tilde{B}_\alpha$$

where we assume the Einstein notation for summation. Because the partial trace over the bath "picks-out" $\tilde{B}_\alpha$ and $\tilde{\rho}_B$, leaving $\tilde{S}_\alpha$ and $\tilde{\rho}_S$ unaffected, the resultings terms in eq. (2.29) will result in terms containing $Tr_B\left(\tilde{\rho}_B\tilde{B}_\alpha(t')\tilde{B}_\beta(t'')\right)$ and $Tr_B\left(\tilde{\rho}_B\tilde{B}_\beta(t'')\tilde{B}_\alpha(t')\right)$ due to the cyclic property and the tensor space property of the trace operator. We further make the following change of integration variable to take advantage of the dependence on τ:

$$\begin{aligned}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' &= \int_0^{\Delta t} d\tau \int_{t+\tau}^{t+\Delta t} dt' \\ &\approx \int_0^{\infty} d\tau \int_t^{t+\Delta t} dt'. \end{aligned} \tag{2.30}$$

By switching back to Schrodinger's picture by making the further assumption that:

$$\tilde{S}_\alpha = e^{i(H_S)t} S_\alpha e^{-i(H_S)t} = S_\alpha e^{i\omega_\alpha t}$$

and by making the following substitutions:

$$\begin{aligned}\int_0^{\infty} d\tau e^{-i\omega_\beta \tau} Tr_B\left(\tilde{\rho}\tilde{B}_\alpha(t')\tilde{B}_\alpha(t'')\right) &= \omega^+. \\ \int_0^{\infty} d\tau e^{-i\omega_\beta \tau} Tr_B\left(\tilde{\rho}_B\tilde{B}_\beta(t'')\tilde{B}_\alpha(t')\right) &= \omega^-. \end{aligned}$$

eq. (2.29) reduces to (note that $\rho_s(t) \to \rho(\mathrm{t})$):

$$\begin{aligned}\partial_t\rho(t) = &-\frac{i}{\hbar}[H, \rho(t)] \\ &+ \sum_\alpha \left\{[S_\alpha\rho(t)S_\alpha^\dagger - S_\alpha^\dagger S_\alpha\rho(t)]\omega^+ + [S_\alpha\rho(t)S_\alpha^\dagger - \rho_S(t)S_\alpha^\dagger S_\alpha]\omega^-\right\}. \end{aligned} \tag{2.31}$$

The above equation is called the Master equation and can be further simplified to another form often called the Linblad [6] form which utilizes the superoperator notation:

$$\partial_t\rho(t) = -\frac{i}{\hbar}[H + H_{eff}, \rho(t)] + \sum_\alpha \kappa_\alpha D[S_\alpha]\rho(t) \tag{2.32}$$

where

$$H_{eff} = \hbar \sum_{\alpha} Im[\omega_{\alpha}^{+}] S_{\alpha}^{\dagger} S_{\alpha} \tag{2.33}$$

and

$$D[c]\rho(t) = c\rho(t)c^{\dagger} - \frac{1}{2}[c^{\dagger}c\rho + \rho c^{\dagger}c] \tag{2.34}$$

is our superoperator. Other forms of the master equations can be derive, such as [1].

3

Interactions of Light with a Two-Level Atom

In this chapter, we will look into the behavior of the two-level atom. Due to its simplicity, it is the quintessential example used in quantum physics textbooks to illustrate the behavior of a quantum system in a finite Hilbert space. We will discuss the interaction of a two level atom with light. In the quantum mechanical description the electromagnetic field can be viewed as an infinite set of quantum harmonic oscillators.

Coherent light may also be treated classically when studying its interaction with a two level atom. The resonant interaction of light with a two level atom is characterized by the Rabi frequency g (often called the coupling coefficient in the literature) which describes the strength of the interaction between the atom and the light. By making this coupling coefficient time-dependent $g \rightarrow g(t)$ we see some new behavior and

model this coupling "function" as our control function for future studies. In this study we will focus on approximate solutions and in this chapter we will only be studying the Jaynes-Cumming Model (JCM). Therefore, this chapter will discuss the two-level quantum system and define any necessary operators. Next we will briefly describe the quanitization of light and conclude this chapter with a quick derivation of the Jaynes-Cumming Model.

3.1 Two-Level Atom

The two level atom is an approximation of our atom where only two energy eigenstates are relevant. For a two level atom there are only two basis states. We shall label these states $|e\rangle = (1 \quad 0)^T$ for the excited state and $|g\rangle = (0 \quad 1)^T$ for the ground state. If an atom is in the ground (excited) state, the necessary energy to bring this state to the excited (ground) state is $\hbar\omega_0$. If we assume that the halfway between these two states the energy is zero, the Hamiltonian of this two level atom is:

$$H_{atom} = \frac{1}{2}\hbar\omega_0\sigma_z \tag{3.1}$$

where $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g| = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is the atomic inversion operator, reminiscent of the Pauli spin matrices. It is useful to define the following operators:

$$\sigma = |g\rangle\langle e| = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{3.2}$$

$$\sigma^\dagger = |e\rangle\langle g| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{3.3}$$

where σ is the lowering operator for the two level atom and $\sigma^\dagger$ to be the raising operator. Then, the following commutation relation holds: $[\sigma, \sigma^\dagger] = -\sigma_z$.

3.2 Quantum Description of Light in an Optical Cavity

An optical cavity is an arrangement of mirrors that forms a standing wave cavity resonator for electromagnetic waves. A parameter of optical cavity (called the Q factor) characterizes the reflectivity of the mirrors. A high Q factor indicates a lower rate of energy loss. A diagram of a simple Fabry-Perot cavity is shown below,

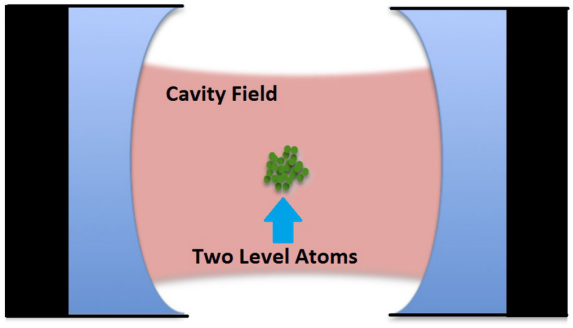

Figure 3.1: Visual schematic of optical cavity [2].

The Hamiltonian for an electromagnetic field mode can be derived from the quantization of the electromagnetic field common in the study of Quantum Electrodynamics [3, 21]. The derivation is beyond the scope of this thesis but its results are the necessary components for understanding the JCM. The quantization of EM fields result in a harmonic oscillator Hamiltonian for each mode,

$$H_{EM} = \sum_{i=1}^{\infty} \hbar\omega_i \left(a_i^\dagger a_i + \frac{1}{2} \right) \tag{3.4}$$

which represents a mode of the EM field. These modes are similar to modes generated by the superposition of standing waves on a string with both ends fixed. The frequency ω_i is frequency of oscillation of the quantized electromagnetic fields (EMF). The operators $a_i^\dagger$ and a_i are the creation and annihilation operators for each mode, represented by a harmonic oscillator. The operator $a_i^\dagger a_i$ is the mode number operator whose eigenvalues represent the total number of photons in that mode. The eigenstates of $\sum_{i=1}^{\infty} a^d agger_i a_i$ are called *Fock States* and they represent the number of excitations of each EMF mode with characteristic energy $\hbar\omega_i$. Only one mode resonantly interacts with the atom and will be used to represent the Jaynes-Cumming Model. The system Hamiltonian we wish to observe is the sum of two level Hamiltonian and the quantized EMF Hamiltonian:

$$H_0 = H_{atom} + H_{EM} = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega \left(a^\dagger a + \frac{1}{2} \right). \tag{3.5}$$

Note that because we are treating both atom and field systems as separate quantum

system, we use the tensor product notation

$$|\Psi\rangle = \sum_{i,j} |\Psi_i\rangle_{atom} \otimes |\Psi_j\rangle_{photon} .$$

Similar to how a state of a two level system can be either represented as its ground state or its excited (or a linear combination of both), the state of a quantized light can be represented as a linear superposition of number states (Fock states). For our study, the number of photons produced in the optical cavity due to its interaction with a generic two level atom inside a Fabry-Perot cavity.

3.3 The Janyes-Cumming Model (JCM)

In this section we summarize the derivation of the JCM. For more information about JCM, the reader is encouraged to read [22, 19]. The JCM describes a two-level quantum system interacting with a quantized electromagnetic mode of an optical cavity. JCM is the simplest quantum description of the two-level atom and its interaction with light.

In the dipole approximation, the interaction between the atom and field is given by the potential energy between a dipole and the electric field:

$$U = -\mathbf{d} \cdot \mathbf{E}$$

where the electric field is

$$\mathbf{E}(\mathbf{r}) = -\sqrt{\frac{\hbar\omega}{2\epsilon_0}}\mathbf{f}(\mathbf{r})a + H.c.$$

where $\mathbf{f}(\mathbf{r})$ is the mode profile. Classically, the dipole vector is the separation between

two oppositely charged particles that are bound to each other. For a many-body system, the dipole operator is represented by the following equation:

$$\mathbf{d} = \langle g| \sum_{i=1}^{\infty} e\mathbf{r_i} |e\rangle \left(\sigma + \sigma^\dagger\right)$$

where the operators (σ) and $(\sigma^\dagger)$ are the raising and lowering operators acting on the two-level atom, similar to the creation and annihilation operators acting on the Fock state of the photons. By defining the coupling coefficient:

$$g(\mathbf{r}) = -\sqrt{\frac{\omega}{2\hbar\epsilon_0}} \langle g| \mathbf{d} |e\rangle \cdot \mathbf{f}(\mathbf{r}) = g$$

to be real for a given location $\mathbf{r}$, we end up with the interaction Hamiltonian to be:

$$V = \hbar g \left(\sigma + \sigma^\dagger\right) \left(a + a^\dagger\right). \tag{3.6}$$

In the *rotating wave approximation* we must transform into the interaction picture. The following properties commonly used in operator algebra will be used to make the transformation:

$$e^{iB\lambda} A e^{-iB\lambda} = A + i\lambda[B, A] + \frac{i^2\lambda^2}{2!}[B, [B, A]] + ... \tag{3.7}$$

and

$$[AB, CD] = A[B, CD] + [A, CD]B. \tag{3.8}$$

Together with the commutation relation $[a, a^\dagger] = 1$, we can use the above properties to show that

$$e^{i\omega(a^\dagger a + \frac{1}{2})t} a^\dagger e^{-i\omega(a^\dagger a + \frac{1}{2})t} = a^\dagger e^{i\omega t}$$

$$e^{i\omega(a^\dagger a + \frac{1}{2})t} a e^{-i\omega(a^\dagger a + \frac{1}{2})t} = a e^{-i\omega t}$$

$$e^{i\frac{1}{2}\omega_0\sigma_z t} \sigma^\dagger e^{-\frac{1}{2}i\omega_0\sigma_z t} = \sigma^\dagger e^{i\omega_0 t}$$

$$e^{i\frac{1}{2}\omega_0\sigma_z t} \sigma e^{-\frac{1}{2}i\omega_0\sigma_z t} = \sigma e^{-i\omega_0 t}$$

which leaves us with an interaction Hamiltonian of the form:

$$V(t) = \hbar g \left(\sigma a e^{-i(\omega_0+\omega)t} + \sigma a^\dagger e^{-i(\omega_0-\omega)t} + \sigma^\dagger a e^{i(\omega_0-\omega)t} + \sigma^\dagger a^\dagger e^{i(\omega_0+\omega)t} \right)$$

Under the *rotating wave approximation* $|\omega_0 - \omega| \ll \omega_0 + \omega$, the fast oscillating terms are neglected, and we are thus left with:

$$V(t) = \hbar g \left(\sigma a^\dagger e^{-i(\omega_0-\omega)t} + \sigma^\dagger a e^{i(\omega_0-\omega)t} \right).$$

A transformation back to the Schrodinger picture would yield the us the final form of the JCM for the total Hamiltonian:

$$H = \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) + \hbar g \left(\sigma a^\dagger + \sigma^\dagger a \right). \tag{3.9}$$

4

The Magnus Expansion Approximations

A large part of the study of physics involves solving differential equations of motion that dictate the behavior of the system in question.Some O.D.E can be solved analytically. However an exact solution to the differential equation is often not achievable, leading to numerical schemes invoking numerical methods and to a large extent, approximation methods. A popular approximation scheme used in physics is perturbation theory, which assumes that a system with a well known behavior (an exact solution) experiences a small disturbance (perturbation) caused by it's interaction with another system that we wish to include in our study of the system's differential equation. However, most perturbation theory methods run into a problem with conserving the unitary evolution of the system. The consequence of the statistical interpretation of quantum mechanics necessitates that the normalization of the quantum system remains preserved throughout its evolution in order for the system to make any physical sense, implying that the total probability of the quantum system must be preserved.

This chapter is subdivided into 2 sections. The first introduces a solution that maintains the unitary evolution, proposed by Wilhelm Magnus in his seminal paper of 1954 [15] and as subsequently been known as Magnus expansion approxiation or exponential perturbation theory [4, 5]. We will show that this approximation scheme will preserve unitary evolution of a quantum system by taking advantage of the group structure of quantum systems. Lastly, we will apply the Magnus expansion to the JCM in the interaction picture while taking advantage of the rotating wave approximation (RWA) to eliminate any fast oscillating terms. This method will be compared to a numerical solution using the Runk-Kutta 4th order method.

4.1 The Magnus Expansion

For a first order differential equation of the following form:

$$\frac{dY}{dt} = A(t)Y \tag{4.1}$$

the general solution is

$$Y = e^{\int_{t_0}^{t} A(t)dt}$$

when $A(t)$ is a scalar function. However, for differential equations where A(t) is a matrix function, the above solution doesn't hold because in general, matrices do not commute with itself at a later time. Some approximation schemes have been applied to eq. (4.1), but at the expense of losing some physical feature. An alternative approximation was proposed by Magnus in his theorem [15]:

Theorem 1 *(Magnus 1954). Let $A(t)$ be a known function of t (in general, in an associative ring), and let $Y(t)$ be an unknown function satisfying eq. ?(4.1) with $Y(0) = I$. Then, if certain unspecified conditions of convergence are satisfied, $Y(t)$ can be written in the form*

$$Y(t) = e^{\Omega(t)} Y_0 \tag{4.2}$$

where

$$\frac{d\Omega}{dt} = \sum_{j=0}^{\infty} \frac{B_j}{j!} ad_{\Omega}^{n} A, \tag{4.3}$$

and B_i are the Bernoulli numbers. Integration by iteration leads to an infinite series of $\Omega(t)$:

$$\Omega(t) = \sum_{k=1}^{\infty} \Omega_k(t). \tag{4.4}$$

In addition, we have the following lemma:

Lemma 2 *The derivative of a matrix exponential can be written alternatively as:*

$$\frac{d}{dt} e^{\Omega(t)} = dexp_{\Omega(t)} \Omega'(t) e^{\Omega(t)}, \tag{4.5}$$

where

$$dexp_{\Omega(t)} \Omega'(t) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} ad_{\Omega}^{k}(\Omega'(t)) = \frac{e^{ad_{\Omega}} - I}{ad_{\Omega}}(\Omega'(t)). \tag{4.6}$$

The equations shown in Theorem 1 will be further elaborated in the following section with its proof. For a more comprehensible form of the Magnus expansion we shall explore eqs. (4.3) and (4.4). It must be noted that the matrix function $A(t)$ must be a skew-Hermitian matrix in the context of quantum mechanics due to its consruction from commutators (skew-Hermitian:$A^{\dagger} = -A$). Taken together, these equations

imply that each term in the Magnus expansion can be calculated using the following iterative equation:

$$\Omega_1(t) = \int_0^t A(t_1)dt_1 \tag{4.7}$$

$$\Omega_n(t) = \sum_{j=1}^{n-1} \frac{B_i}{j!} \int_0^t S_n^{(j)}(t_n)dt_n \tag{4.8}$$

where

$$S_n^1 = [\Omega_{n-1}, A] \tag{4.9}$$

$$S_n^j = \sum_{m=1}^{n-j} [\Omega_m, S_{n-m}^{j-1}] \quad 2 \leq j \leq n-1 \tag{4.10}$$

The first four Bernoulli numbers are given as $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$ and $B_4 = -\frac{1}{30}$. Following the above equations, we have the first 3 terms of the Magnus expansion:

$$\begin{aligned}
\Omega_1(t) &= \int_0^t A(t_1)dt_1 \\
\Omega_2(t) &= \frac{1}{2}\int_0^t \int_0^{t_1} [A(t_1), A(t_2)]dt_1dt_2 \\
\Omega_3(t) &= \frac{1}{6}\int_0^t \int_0^{t_1} \int_0^{t_2} [[A(t_1), [A(t_2), A(t_3)]] + [[A(t_1), A(t_2)], A(t_3)]]dt_1dt_2dt_3
\end{aligned} \tag{4.11}$$

Eq.(4.11) coupled with eq. (4.2) gives us a structured set of procedures to follow to obtain an approximation to eq. (4.1) that conserves the unitary behavior necessary for any quantum system. In its simplest form of interpretation, we commute, integrate, and repeat. It is quite common for most systems to truncate the Magnus expansion to 2nd order. For our study we will show that the application of the Magnus expansion to 2nd order will be sufficient. In the following section we will give a cursory proof to the Magnus Expansion.

4.1.1 Proof of Magnus Expansion

The reader is encouraged to peruse [5] for a comprehensive proof of the Magnus Expansion. A cursory proof of the approximation will be given to highlight its key-points. The proof for the Magnus expansion is twofold: the first utilizes the power series expansion of the Bernoulli number that was seen then in Theorem 1, and second from the derivative of a matrix exponential. The Bernoulli numbers is a sequence of rational number whose power series expansion is shown below:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \tag{4.12}$$

whose inverse is also given as

$$\frac{e^x - 1}{x} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} x^n. \tag{4.13}$$

We shall return to eq. (4.12) and (4.13) momentarily. The derivative of the matrix exponential is first shown in eq. (4.5) (Lemma 2). Notice that by taking the time derivative of 4.2, we have:

$$\frac{dY}{dt} = \frac{d}{dt}(e^{\Omega}(t))Y_0 = dexp_{\Omega}\Omega'(t)e^{\Omega(t)}Y_0$$

which follows from Lemma 2. Notice that when compared to eq. (4.1), we obtain

$$A(t) = dexp_{\Omega}\Omega'(t)$$

since $Y = e^{\Omega(t)}Y_0$. Due to the group structure of the system, we are allowed to apply the inverse operator $dexp_{\Omega}^{-1}$. We then obtain

$$\Omega'(t) = dexp_{\Omega}^{-1}(A(t)). \tag{4.14}$$

Notice its similarity with eq. (4.3). The proof, therefore, for Theorem 1 requires us to show that $dexp_{\Omega}^{-1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} ad_{\Omega}^n$. The first step is to assume that $\Omega(t)$ is a

matrix-valued differentiable function and to set

$$Y(\sigma,t)=\frac{\partial}{\partial t}(e^{\sigma\Omega(t)})e^{-\sigma\Omega(t)} \tag{4.15}$$

for σ, $t \exists \mathbb{R}$. Differentiating with respect to σ yield ,

$$\begin{aligned}\frac{\partial Y(\sigma,t)}{\partial\sigma} &= \frac{\partial}{\partial t}(e^{\sigma\Omega(t)}\Omega)e^{-\sigma\Omega(t)}+\frac{\partial}{\partial t}(e^{\sigma\Omega(t)}\Omega)(-\Omega)e^{-\sigma\Omega(t)}\\ &= (e^{\sigma\Omega(t)}\Omega'+\frac{\partial}{\partial t}e^{\sigma\Omega(t)}\Omega)e^{-\sigma\Omega(t)}-\frac{\partial}{\partial t}e^{\sigma\Omega(t)}\Omega e^{-\sigma\Omega(t)}\\ &= e^{\sigma\Omega(t)}\Omega' e^{-\sigma\Omega(t)}\\ &= e^{ad_{\sigma\Omega}}(\Omega')\\ &= \sum_{k=0}^{\infty}\frac{\sigma^k}{k!}ad_{\Omega}^{k}(\Omega'). \end{aligned} \tag{4.16}$$

If we integrate eq. (4.16) from 0 to 1, we obtain:

$$\begin{aligned}\frac{d}{dt}exp(\Omega)exp(-\Omega)=Y(1,t)&=\int_0^1\frac{\partial}{\partial\sigma}Y(\sigma,t)d\sigma\\ &=\int_0^1\sum_{k=0}^{\infty}\frac{\sigma^k}{k!}ad_{\Omega}^{k}(\Omega')d\sigma\\ &=\sum_{k=0}^{\infty}\frac{1}{(k+1)!}ad_{\Omega}^{k}(\Omega')\\ \frac{d}{dt}exp(\Omega)&=\sum_{k=0}^{\infty}\frac{1}{(k+1)!}ad_{\Omega}^{k}(\Omega')exp(\Omega)\end{aligned} \tag{4.17}$$

Notice that eq. (4.17) is equivalent to Lemma 2 (eq. (4.5)). When compared to our power series representation of the Bernoulli number (eqs. (4.12) and eq. (4.13)), it follows that:

$$dexp_{\Omega}^{-1}=\sum_{j=0}^{\infty}\frac{B_j}{j!}ad_{\Omega}^{n}. \tag{4.18}$$

4.1.2 Alternative Proof

A more subtle alternative proof to the Mangus expansion follows by utilizing the Baker-Campbell-Hausdorf formula for the product of two exponentials and the group property of the evolution of operators. The first states that

$$e^X e^Y = exp(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + ...) \tag{4.19}$$

which can be compactly expressed as [14]

$$e^X e^Y = exp(Y + \sum_{j=0}^{\infty} \frac{B_j}{k!} ad_Y^j(X) + \mathcal{O}(X^2)) \tag{4.20}$$

while the second states that any evolution operator obeys the following:

$$U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0). \tag{4.21}$$

We will consider a narrow time interval δt and use the exponential form of the evolution operator:

$$exp(\Omega(t + \delta t, t_0) = exp(\Omega(t + \delta t, t) exp(\Omega(t, t_0))$$

If we assume that during a narrow time interval δt the Hamiltonian of the system is constant, Schrodinger's equation $|\dot{\Psi}\rangle = H(t)|\Psi\rangle$ tells us that $exp(\Omega(t + \delta t, t) \approx exp(H(t)\delta t)$. Hence,

$$exp(\Omega(t + \delta t, t_0) = exp(H(t)\delta t) exp(\Omega(t, t_0)).$$

Invoking eq. (4.20), we obtain:

$$exp(\Omega(t + \delta t, t_0) = exp(\Omega(t, t_0) + \sum_{j=0}^{\infty} \frac{B_j}{k!} ad^j_{\Omega(t,t_0)}(H(t)) + \mathcal{O}(X^2) \tag{4.22}$$

Taking the limit as $\delta t \to 0$, we have

$$\Omega'(t, t_0) = \sum_{j=0}^{\infty} \frac{B_j}{k!} ad^j_{\Omega(t,t_0)}(H(t)). \tag{4.23}$$

4.2 ME in Interaction Picture

For our system, the first order differential equation in question is Schrodinger's equation:

$$\frac{\partial |\Psi\rangle}{\partial t} = \tilde{H}(t) |\Psi\rangle$$

where we define $\tilde{H}(t) = \frac{-iH}{\hbar}$ to represent a skew Hermitian matrix. For the JCM with time dependent coupling coefficient in the interaction picture,

$$\tilde{H}_I(t) = -ig(t) \left(\sigma a^\dagger e^{-i\Delta t} + \sigma^+ a e^{i\Delta t}\right) \tag{4.24}$$

where we introduce the detuning $\Delta = \omega_0 - \omega$. We start by assuming the weak coupling regime $\omega_0 >> g$ and set $\omega_0 = 100g$ and assumed our coupling coefficient is in the order of GHz. From eq. (4.11), the terms in the ME are:

$$\Omega_1(t) = \int_0^t \tilde{H}_I(t) dt_1$$
$$\Omega_2(t) = \frac{1}{2} \int_0^t \int_0^{t_1} [\tilde{H}_I(t_1), \tilde{H}_I(t_2)] dt_1 dt_2$$

Making the following substitutions:

$$f(t) = e^{i\Delta t}$$

$$f^*(t) = e^{-i\Delta t}$$

leads to:

$$\Omega_1(t) = -i \left(\sigma a^\dagger \int_0^t g(t_1) f(t_1) dt_1 + h.c.\right) \tag{4.25}$$

$$\Omega_2(t) = \frac{1}{2} \left(u_1(t) - u_2(t)\right) \left(\sigma^\dagger \sigma + \sigma_z a^\dagger a\right) \tag{4.26}$$

where:

$$u_1(t) = \int_0^t \int_0^{t_1} f(t_1)g(t_1)f^*(t_2)g(t_2)dt_1dt_2 \tag{4.27}$$

$$u_2(t) = \int_0^t \int_0^{t_1} f^*(t_1)g(t_1)f(t_2)g(t_2)dt_1dt_2. \tag{4.28}$$

Furthermore, we initially consider the simplest case where $g(t) = V_0$. Hence,

$$\Omega_1(t) = -\frac{V_0}{\Delta}\left[(e^{i\Delta t} - 1)\sigma a^\dagger - (e^{-i\Delta t} - 1)\sigma^\dagger a\right] \tag{4.29}$$

$$\Omega_2(t) = -\frac{V_0^2}{i\Delta}\left(t - \frac{sin\Delta t}{\Delta}\right)\left(\sigma^\dagger\sigma + \sigma_z a^\dagger a\right). \tag{4.30}$$

An issue with this method is the exponential term containing our control input Δ, which we may vary with time and model as a possible control function. But as our control functions show, we must integrate them first, which could prove difficult.

We compute the excited population of an atom initially in its excited state with the initial state of the photon in a coherent state and plot with respect to time. We set the number of timesteps to 10,000 with our timescale to be $\Delta t = 1.0001 \times 10^{-14}s$. For figure 4.1 and 4.2, we set our detuning to $2\pi \times 10^9 \frac{1}{s}$ and $2\pi \times 10^7 \frac{1}{s}$, respectively, and compare our approximation to a numerical RK4 solution. Clearly, ME upto 2nd order is more accurate, but as we decrease our detuning ME upto 1st order is sufficient since a smaller detuning represents a stronger resonance between our atom and cavity.

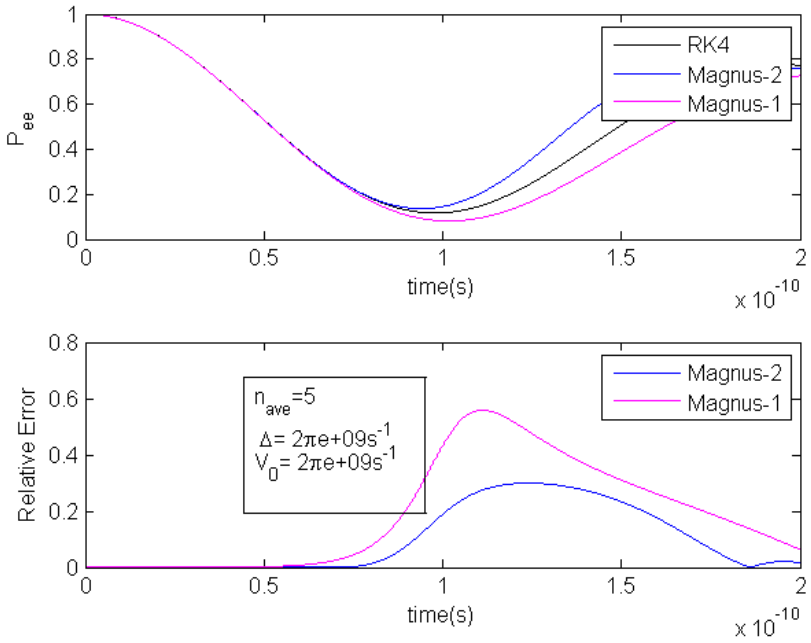

Figure 4.1: RK4 and Magnus Expansion (1st and 2nd order). Initial coherent state. Large detuning ($\Delta = 2\pi \times 10^9 \frac{1}{s}$).

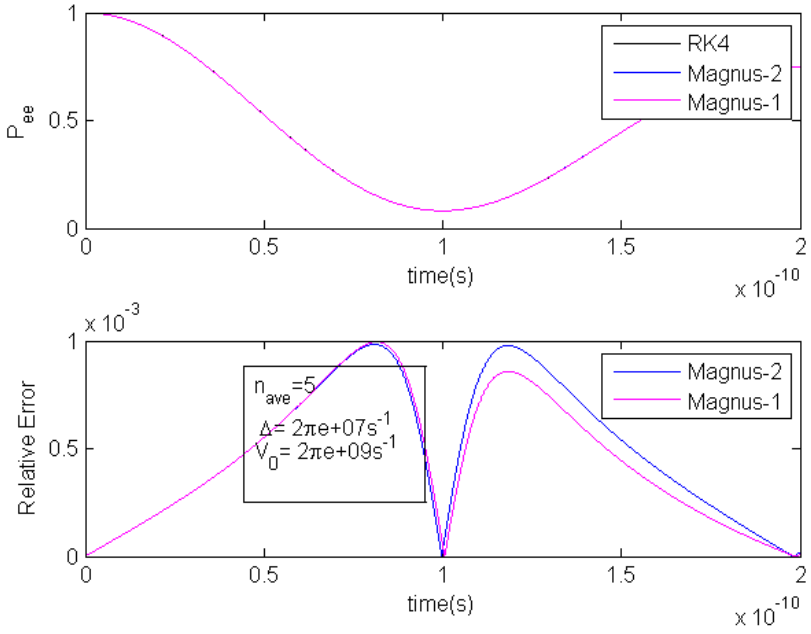

Figure 4.2: RK4 and Magnus Expansion (1st and 2nd order). Initial coherent state. Small detuning ($\Delta = 2\pi \times 10^7 \frac{1}{s}$).

An interesting feature called the collapse and revival occurs when one increases the time from $2 \times 10^{-10}s$ to $100 \times 10^{-10}s$ (figure 4.3). This feature as been observed experimentally [17]. The key observation to note is that ME upto 2nd order is only more accurate at the beginning of the excited population's evolution; ME upto 2nd order is less accurate later in time. However, despite this inaccuracy the ME upto 2nd order still maintains the unitary evolution.

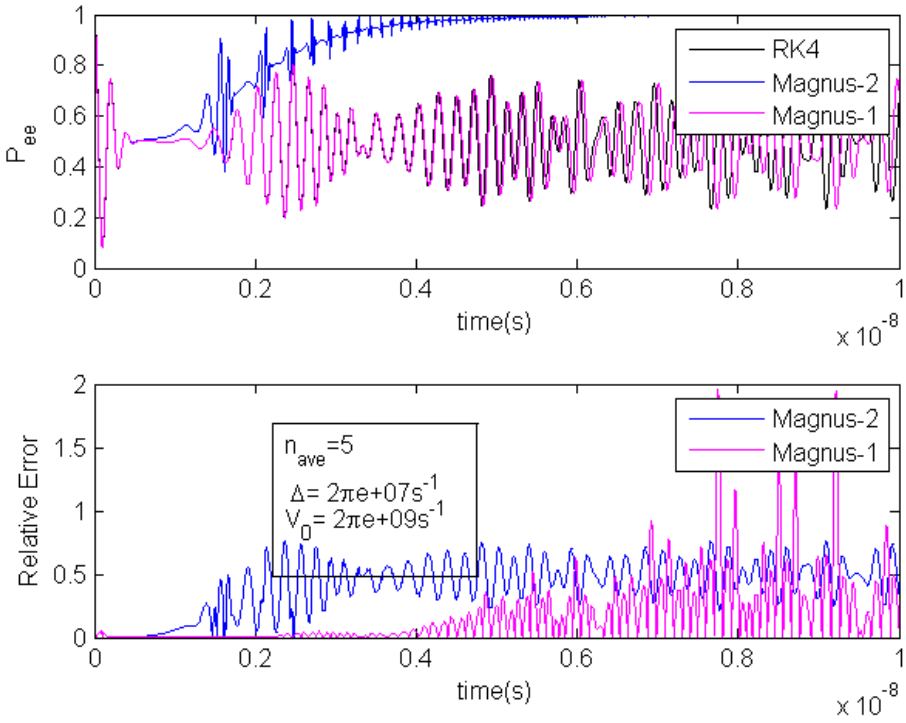

Figure 4.3: RK4 and Magnus Expansion (1st and 2nd order). Initial coherent state. Small detuning ($\Delta = 2\pi \times 10^7 \frac{1}{s}$) from 0 to $100 \times 10^{-10}s$.

5

Pulse Cavity

If we introduce atoms at different time intervals and observe the cavity response, we may model the coupling coefficient as a coupling function and have it take any shape we wish. In this chapter we take the case where $\Delta(t) \to \Delta_0$ is still constant while exploring two different shapes for $g(t)$, a Gaussian function and a sinusoidal function. This will all take place under the rotating picture. Physically, each time an atom is introduced to our atom-cavity system corresponds to a single Gaussian/sinusoidal pulse. A sequence of Gaussian/sinusoidal pulses would therefore represent the periodic introduction of atoms.

5.1 ME in Rotating Picture

The rotating picture is another transformation commonly used for time-dependent unitary transformation. Remember that we've decomposed our Hamiltonian into

two parts: $H = H_0 + V$, both of which were assumed to be time-independent. If, however, we assume V to be time-dependent $V \to V(t)$, then we may apply a form of perturbation theory to obtain a differential equation that may be amendable to control. We begin with JCM Hamiltonian and and make the following substitution $\omega(t) = \omega_0 + \Delta(t)$ where we maintain the following interpretation that $\Delta(t)$ is the detuning between the atom and light natural frequencies. We assume that we are able to control the frequency of light by modifying the cavity, thus converting our detuning to a time-dependent frequency. Our modified JCM in Schrodinger's picture becomes:

$$\partial_t \left|\Psi(t)\right\rangle = -i\left(\frac{1}{2}\omega_0\sigma_z + \omega_0 a^\dagger a + \Delta(t)a^\dagger a + g(t)(\sigma a^\dagger + \sigma^\dagger a)\right)\left|\Psi(t)\right\rangle \tag{5.1}$$

where we have neglected the ground photon level $a^\dagger a + \frac{1}{2} \to a^\dagger a$. We may now make the following transformation, substituting:

$$\left|\Psi(t)\right\rangle = e^{-iH_0 t}\left|\Psi_R(t)\right\rangle \tag{5.2}$$

where we have separated the Hamiltonian of eq. (5.1) into:

$$H_0 = \frac{1}{2}\omega_0\sigma_z + \omega_0 a^\dagger a \tag{5.3}$$

and

$$H_1(t) = \Delta(t)a^\dagger a + g(t)(\sigma a^\dagger + \sigma^\dagger a). \tag{5.4}$$

We have also adopted the notational subscript R to indicate the rotating frame. By taking the time derivative of eq. (5.2) and substituting eq. (5.1) and eq. (5.2) while

using our notation for the Hamiltonian, we obtain:

$$-i[H_0 + H_1(t)]\,|\Psi(t)\rangle = e^{-iH_0t}\partial_t\,|\Psi_R(t)\rangle - iH_0\,|\Psi(t)\rangle$$

$$-iH_1(t)\,|\Psi(t)\rangle = e^{-1H_0t}\partial_t\,|\Psi_R(t)\rangle$$

$$\partial_t\,|\Psi_R(t)\rangle = -ie^{iH_0t}\left[\Delta(t)a^\dagger a + g(t)(\sigma a^\dagger + \sigma^\dagger a)\right]e^{-iH_0t} \tag{5.5}$$

By invoking eq. (3.7) we obtain our final form for the JCM under the rotating picture:

$$\frac{\partial\,|\Psi_R(t)\rangle}{\partial t} = -i\left[\Delta(t)a^\dagger a + g(t)(\sigma a^\dagger + \sigma^\dagger a)\right]|\Psi_R(t)\rangle\,. \tag{5.6}$$

In this section we shall be including terms of the Magnus expansion upto 4th order. Making the following substitutions:

$$N = a^\dagger a$$

$$R = \sigma^+ a + \sigma a^\dagger$$

$$S = \sigma^+ a - \sigma a^\dagger$$

$$M = a^\dagger a + \sigma^+\sigma$$

$$Q = \sigma_z M \tag{5.7}$$

the following commutation relations hold:

$$[R, N] = S$$

$$[S, N] = R$$

$$[R, S] = -2Q$$

$$[Q, N] = 0$$

$$[R, Q] = -2MS$$

$$[S, Q] = -2MR. \tag{5.8}$$

Hence, our goal now is to solve the following:

$$\Omega(t) = u_1(t)N + u_2(t)R + u_3(t)S + u_4(t)Q + u_5(t)MS \tag{5.9}$$

where

$$\begin{aligned}
u_1(t) &= -i\int_0^t \Delta_1 dt_1 \\
u_2(t) &= -i\int_0^t g_1 dt_1 + i\int_0^t\int_0^{t_1}\int_0^{t_2}(\Delta_1\Delta_2 g_3 + g_1\Delta_2\Delta_3 - 2\Delta_1 g_2\Delta_3)dt_1dt_2dt_3 \\
u_3(t) &= \int_0^t\int_0^{t_1}(\Delta_1 g_2 - g_1\Delta_2)dt_1dt_2 + \int_0^t\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}(\Delta_1\Delta_2 g_3\Delta_4 - \Delta_1 g_2\Delta_3\Delta_4)dt_1dt_2dt_3dt_4 \\
u_4(t) &= 2i\int_0^t\int_0^{t_1}\int_0^{t_2}(2g_1\Delta_2 g_3 - g_1 g_2\Delta_3 - \Delta_1 g_2 g_3)dt_1dt_2dt_3 \\
u_5(t) &= 8\int_0^t\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}(g_1\Delta_2 g_3 g_4 - g_1 g_2\Delta_3 g_4)dt_1dt_2dt_3dt_4
\end{aligned} \tag{5.10}$$

where we adopt the notation $A(t_1) = A_1$. The set of eq. (5.10) proves quite difficult to implement computationally due to the exponential matrix function used in Matlab using a time-dependent coupling coefficient. Fortunately, a method introduced by Iserles in his paper [12] provides a more computationally friendly method in implementing the ME. We shall apply this method in the rotating picture in the following section.

5.2 MG4: Implementing ME for Linear Differential Equations

The method introduced by Iserles et al. [12] is fairly straightforward if one is familiar with numerical techniques in solving differential equations. Its implementation is

similar to how one implements RK4 in solving differential equations. We shall adopt the same name that Iserles et al gives their new method, MG4. The method states that the solution to the linear differential equation is:

$$y_{n+1} = e^{\Omega^n} y_n \tag{5.11}$$

where

$$\begin{aligned} a_1(t) &= a(t_n + (\frac{1}{2} - \frac{\sqrt{3}}{6}h) \\ a_2(t) &= a(t_n + (\frac{1}{2} + \frac{\sqrt{3}}{6}h) \\ \Omega^n &= \frac{1}{2}h(a_1(t) + a_2(t)) + \frac{\sqrt{3}}{12}h^2[a_2(t), a_1(t)] \end{aligned} \tag{5.12}$$

and h is the timestep of our numerical scheme. Using eq. (5.11) proves to be computationally efficient especially when one deals with coupling functions. For our JCM in the rotating picture (eq. (3.7)), $a(t) \to H(t)$.

5.3 Pulsed Gaussian Coupling Function

We shall model $g(t)$ as a Gaussian function in this section. We begin by substituting the following equation for $g(t)$:

$$g(t) = V_0 \left(A + Be^{-\frac{(b-t)^2}{2c^2}} \right) \tag{5.13}$$

where A is a constant that determines the lower limit of the Gaussian function, B is a constant that determines its peak and V_0 is the constant coupling coefficient. From the usual definition of a Gaussian function, the coefficients b and c determines the location and standard deviation of the Gaussian function respectively. If we introduce

the Gaussian functions periodically as a sequence of pulses, we have to split our use of the ME into several pieces corresponding to the number of pulses we wish to introduce to our system.

For the figures shown below we used 5 and 15 Gaussian pulses to show the behavior of the average number of photons in the cavity as well as its uncertainty. In fig. 5.1 and fig. 5.2, we notice two points in the uncertainty that dips, but not to the desired value. The main goal of quantum control is to drive the uncertainty of the sought out expectation value to zero. Hence, it's of scientific interest to study the effects of changing the coupling functions at these points to observe how the uncertainty behaves.

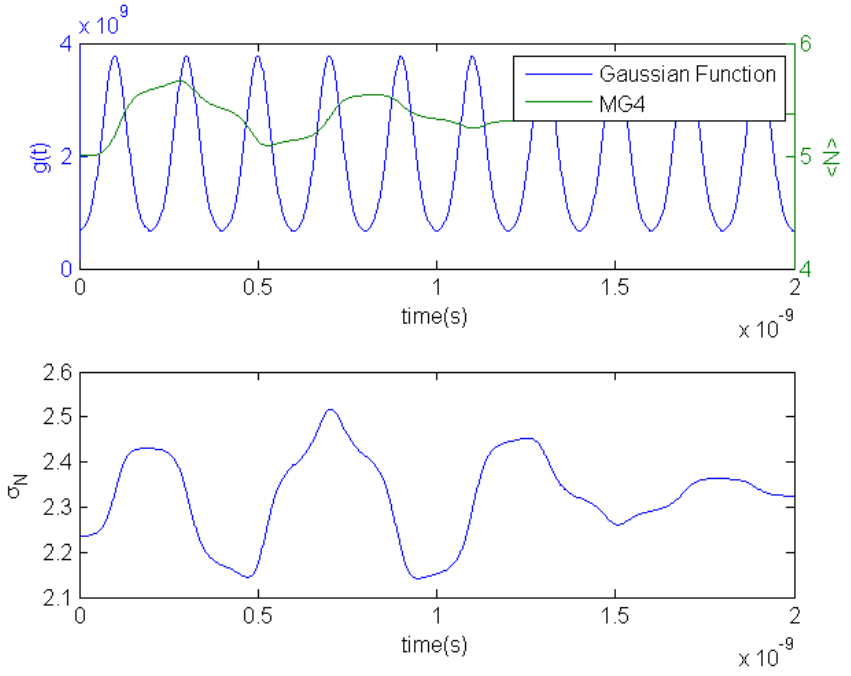

Figure 5.1: Gaussian Coupling (5 pulses). Average photon number vs. time

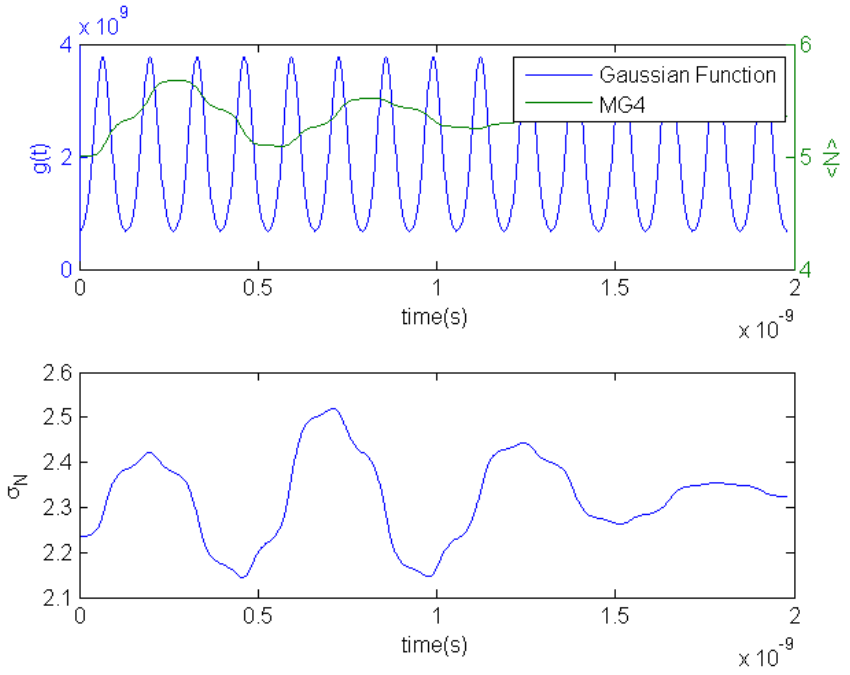

Figure 5.2: Gaussian Coupling (15 pulses). Average photon number vs. time

5.4 Pulsed Sinusoidal Coupling Function

We begin by substituting the following equation for $g(t)$:

$$g(t) = V_0 \left(sin(\omega t) + 1\right) \tag{5.14}$$

where ω is the frequency of pulses for our time interval. The results of our test-bed using the sinusoidal coupling function for 5 and 15 pulses are shown below:

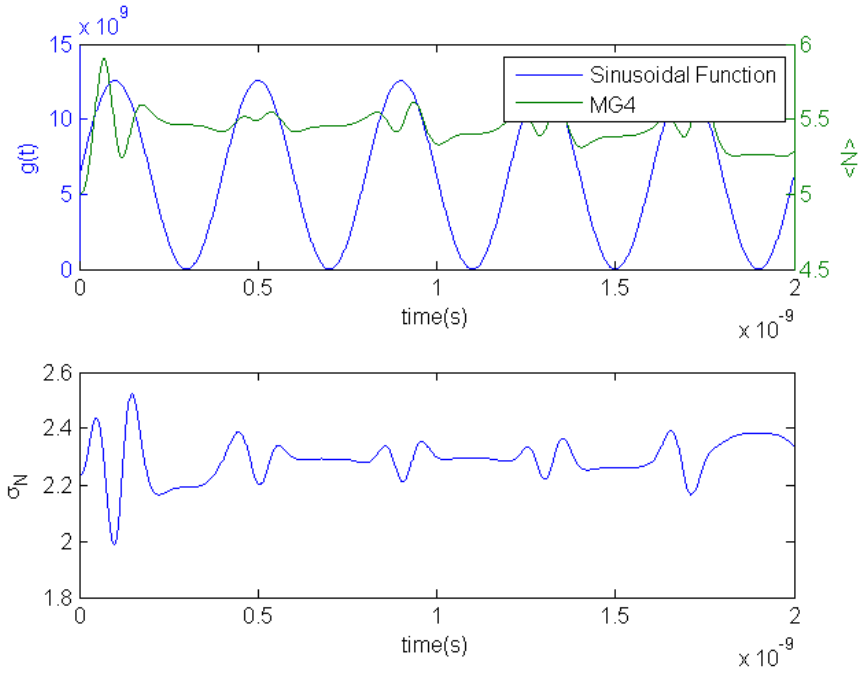

Figure 5.3: Sinusoidal Coupling (5 pulses). Average photon number vs. time.

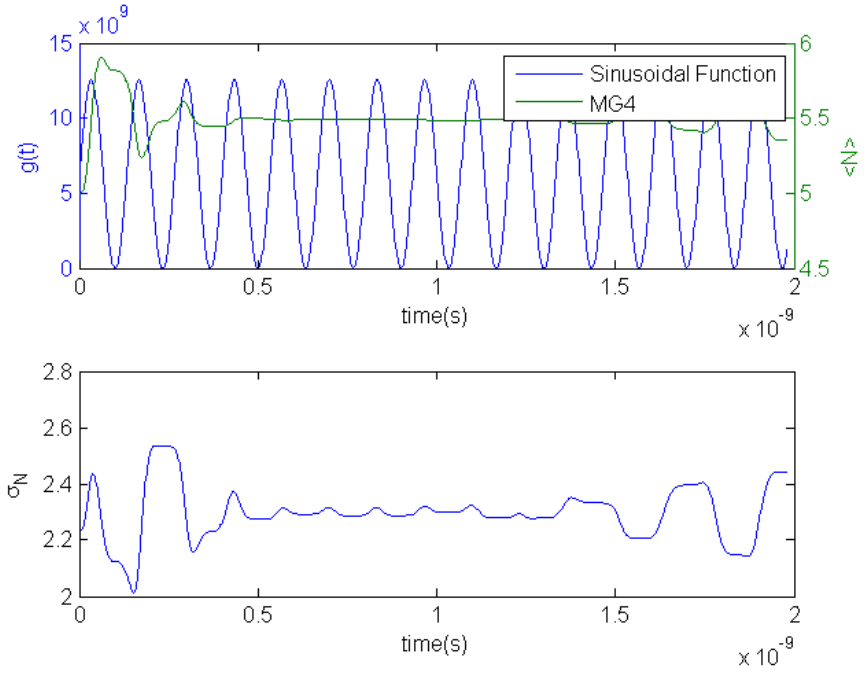

Figure 5.4: Sinusoidal Coupling (15 pulses). Average photon number vs. time.

It is evident that different coupling functions exhibit different behaviors in uncer-

tainty. If one compares figure 5.4 to figure 5.2, it would appear that large number of pulses encourages early dips in the uncertainty for the sinusoidal function as compared to the Gaussian function. An interesting feature of figure 5.4 shows a recurrence of the collapse and revival of the JCM when one continues to increase in pulse number.

6

Conclusions and Future Work

We have shown that the Magnus expansion is a useful tool for studying the Janyes-Cumming Model in both the interaction picture and the rotating picture when compared to the RK4 numerical solution. Furthermore, we have shown that Schrodinger's equation in the rotating picture is a more appealing picture than the interaction picture due to the absence of the exponential term which would be useful for future studies in quantum control. Our study has also shown that the ME is only useful for the case when $1/\omega_0 << t$, that is to say when the interaction time between the atom and cavity is much smaller than the interaction time t. This is evident in our plots where the approximations increases in error over time. We have also shown that characteristic features are maintained, such as the collapse and revival for initial coherent states of photons and the unitary evolution of the system.

It should be noted we have addressed the possible applications of the Magnus expansion to control problems but haven't executed them. Control theorists prefer

the following form for first order differential equations:

$$\dot{X}(t) = \sum_k u_k(t)\hat{\mathscr{L}_k}X(t)$$

where $X\epsilon\mathbb{C}^n$ and $\hat{\mathscr{L}}\epsilon\mathbb{C}^{nxn}$. In chapter 4 and 5 we have shown that the ME provides such forms for Schrodinger's equation.

Some Matlab codes have been create to incorporate any effects on the system given different shapes of our coupling function $g(t)$ (such as a Gaussian and sinusoidal shapes). However, due to time constraints of writing a Master's thesis, we haven't been able to include any effects of the detuning $\Delta(t)$. Future work could address our control functions discussed in Chapter 4 and 5 and study which functions could drive the system to a desired state. The field of quantum control is steadily increasing and the reader is encouraged to read the following articles [7, 9, 11, 13].

Another path for future study of the ME is in its use of open quantum system. This thesis introduced and discussed the necessary tools to begin the study of ME for open quantum system but was restricted due to time constraint. Application on how to apply the ME to superoperators of the Master Equation in Lindblad form could prove useful for real world applications of quantum systems. However, an initial glance of the study of ME in terms of the Lindblad form and superoperators would require large computation time due to the nature of open quantum systems in using the density matrix formalism. None-the-less, it is an attractive field for active research.

Appendices

A

Matlab Codes

A.1 MG4 and RK4 codes

```
function [Y_PRIME] = MG4 (H,psi_0,t0,N,dt,atom_dim,photon_dim)

% initialize output arrays

S_p_time = zeros(photon_dim*atom_dim, N); % for S_p as a function of time

t = zeros(1,N);

t (1,1) = t0;

% assign initial conditions

S_p_time (:,1) = psi_0;   % total space in time.

S_p = psi_0;

for ii =1:N-1

    % updates counter for rung kutta 4 to keep track of previous solution

    S_p_counter  = S_p;
```

```
    mg4_1 = H(t(1,ii)+(0.5-(sqrt(3)/6))*dt);

    mg4_2 = H(t(1,ii)+(0.5+(sqrt(3)/6))*dt);

    mg4 = 0.5*dt*(mg4_1 + mg4_2)+(sqrt(3)/12)*dt^2*(mg4_2*mg4_1 - mg4_1*mg4_2);

    % updates State

    S_p = expm(mg4)*S_p_counter;

    % update S_p in time

    S_p_time (:,ii+1) = S_p_counter;

    t(ii+1) = t0 + ii*dt;

end

Y_PRIME = S_p_time;

end


function [P_PRIME,t] = RK4_coupled_gps (H,psi_0,t0,N,dt,atom_dim,photon_dim)

% initialize output arrays

S_p_time = zeros(photon_dim*atom_dim, N); % for S_p as a function of time

t = zeros(1,N);

t (1,1) = t0;

% assign initial conditions

S_p_time (:,1) = psi_0;   % total space in time.

S_p = psi_0;

for ii=1:N-1

    % updates counter for rung kutta 4 to keep track of previous solution

    S_p_counter  = S_p;
```

```
    k1 = H(t(1,ii))*S_p_counter;

    k2 = H(t(1,ii)+dt/2)*(S_p_counter+0.5*dt*k1);

    k3 = H(t(1,ii)+dt/2)*(S_p_counter+0.5*dt*k2);

    k4 = H(t(1,ii)+dt)*(S_p_counter+dt*k3);

    % updates State

    S_p = S_p_counter + dt*(k1+2*k2+2*k3+k4)/6;

    % update S_p in time

    S_p_time (:,ii+1) = S_p_counter;

    t(ii+1) = t0 + ii*dt;

end

P_PRIME = S_p_time;

end
```

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