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Abstract

The problem on classical solutions for the wave equation and the BBM equation is considered. The equations are considered with a forcing term and sufficient conditions of solvability, existence and uniqueness are established.

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0. Introduction

As it is well known the linear wave equation

$$u^{(2,0)} - u^{(0,2)} = 0 \tag{0.1}$$

has far reaching applications. Moreover, it is well known the BBM equation [1]

$$u^{(0,1)} + u^{(1,0)} + u \cdot u^{(1,0)} - u^{(2,1)} = 0 \tag{0.2}$$

is a sufficient model for wave phenomenon under the assumptions of small wave-amplitude and large wavelength, where the notation

$$u^{(m,n)} = \frac{\partial u^{m+n}}{\partial x^m \partial t^n}$$

is being utilized.

In the modeling leading up to either equation (0.1) or (0.2) it is apparent that many of the physical conditions are either overlooked or drastically simplified. This situation is often a reasonable approach and the resulting equations (0.1) or (0.2) can give sufficient approximations to the real world phenomenon under consideration. However, in certain situations it may be required to not overlook or simplify some of the physical conditions in the modeling process.

For example, if one investigates the modeling process of nonlinear dispersive systems of shallow water waves on an inviscid liquid then they will obtain equation (0.2). However, if one considers the same phenomenon but on a viscid liquid then they will obtain the equation

$$u^{(0,1)} + u^{(1,0)} + u \cdot u^{(1,0)} - u^{(2,1)} = \mu u^{(2,0)} \tag{0.3}$$

where the dissipative term $\mu u^{(2,0)}$ is representing the kinematic viscosity of the liquid. In 2007 we presented a lengthy article [2] studying equation (0.3), and we obtained results such as solvability, existence and energy invariance for equation(0.3). In addition, we compared the results for the new viscous equation (0.3) to the inviscid equation (0.2) with the results given in [1].

There are many other examples of how a slight change in the assumptions of the modeling process lead to equations of the form

$$u^{(2,0)} - u^{(0,2)} = f(x, t, u)$$

for classical wave phenomenon or

$$u^{(0,1)} + u^{(1,0)} + u \cdot u^{(1,0)} - u^{(2,1)} = f(x, t, u)$$

for dispersive systems of shallow water waves. In both equations the forcing term, $f(x, t, u)$ is obtained from the modification in the modeling. Thus, in this paper we will investigate the solvability, existence and uniqueness of the later two equations where the forcing term is satisfying a so called Osgood's condition.

In addition, higher order generalizations of these equations are considered, namely

$$u^{(m,n)} = f(x, t, u),$$

where the routine reduction to canonical form of the wave equation has been applied.

1. Statement of results

In the following section we will be refereing to the equation

$$u^{(m,n)} = f(x, t, u) \tag{1.1}$$

and the equation

$$u^{(0,1)} + u^{(1,0)} + u \cdot u^{(1,0)} - u^{(2,1)} = f(x, t, u) \qquad (1.2).$$

Also, we will say function $g \in C([0, \infty])$ satisfies an Osgood's condition if the function $g(z)$ is nondecreasing, $g(z) > 0$ for $z > 0$ and

$$\int_0^1 \frac{1}{g(z)} dz = \infty. \tag{1.3}$$

It is worthy of noting that this Osgood's condition is commonly known in the theory of ordinary differential equations and is often utilized as a way to weaken the famous Lipschitz condition, i.e. Osgood's is often applicable when the equation under consideration does not meet Lipschitz.

Theorem 1.1.
Let $(x_0, t_0) \in R^2$, $a > 0$,$b > 0$, and $D = [x_0, x_0 + a] \times [t_0, t_0 + b]$. Let $f : D \times R \to R$, let g satisfy (1.3), and let f satisfy

$$|f(x,t,u_1) - f(x,t,u_2)| \le g(|u_1 - u_2|) \tag{1.3$'$}$$

for all $(x,t) \in D$ and $u_1, u_2 \in R$. Let $m, n \ge 0$ and let

$$\alpha_i : [t_0, t_0 + b] \to R \text{ for } i = 0, \ldots, m-1,$$

$$\beta_j : [x_0, x_0 + a] \to R \text{ for } j = 0, \ldots, n-1, \text{ and} \tag{1.4}$$

$$\gamma_{i,j} \in R \text{ for } 1 \le i \le m-1,\ 1 \le j \le n-1.$$

Then the initial value problem

$$\frac{\partial^{m+n}}{\partial x^m \partial t^n} = f(x,t,u(x,t)) \text{ on } D \tag{1.5}$$

$$\frac{\partial^i}{\partial x^i} u(x_0, t) = \alpha_i(t) \text{ for all } t_0 \le t \le t_0 + b,\ 0 \le i \le m-1,$$

$$\frac{\partial^j}{\partial t^j} u(x, t_0) = \beta_j(x) \text{ for all } x_0 \le x \le x_0 + a,\ 0 \le j \le n-1,$$

$$\frac{\partial^{i+j}}{\partial x^i \partial t^j} u(x_0, t_0) = \gamma_{i,j} \text{ for all } 1 \le i \le m-1,\ 1 \le j \le n-1$$

has at most one solution.

Here are some examples of Theorem 1.3:

Example 1: Let $n = 0$. Then the theorem states that the ODE initial value problem

$$u^{(m)}(x) = f(x,u)$$

for $x_0 \le x \le x_0 + a, u^{(i)}(x_0) = \alpha_i \in R$ for $i = 0, 1, \ldots, m-1$, with g and f respectively satisfying (1.3) and (1.3') has at most one solution.

Example 2: Let $m = n = 1$. The the theorem states that the PDE

$$u_{xt} = f(x,t,u)$$

on D with the boundary conditions $u(x_0, t) = \alpha(t)$, $u(x, t_0) = \beta(x)$ for all $x \in [x_0, x_0 + a], t \in [t_0, t_0 + b]$ has at most one solution.

Theorem 1.2. Let $g(x)$ be a continuous function such that

$$\sup_{x\in\Re} |g(x)| \leq b < \infty$$

then there exists a $t_0(b)$ such that the integral equation

$$u(x,y) = g(x) +$$

$$\int_0^t \int_{-\infty}^{+\infty} K(x-\xi)(u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F(\xi,\tau,u(\xi,\tau),)d\xi d\tau,$$

has at most one solution, where $u(x,0) = g(x)$ and F is a function defined by $\frac{dF}{dx} = f$. This solution will be a classical solution of the initial value problem consisting of the partial differential equation (1.2) along with the initial condition $u(x,0) = g(x)$ if the function F satisfies the Osgood's criteria (1.3)

2. Proofs and auxiliary statements

Proof of Theorem 1

Let us assume from now on without lack of generality that $(x_0, t_0) = (0, 0)$. To prove the theorem, we will need several lemmas. First, the following lemma from [3]:

Lemma 2.1 If g satisfies (1.3), ϕ is a nonnegative continuous function on $[0, a]$, and $\phi(x) \leq \int_0^x g(\phi(z))dz$ for all $0 < x \leq a$, then $\phi = 0$ on $[0, a]$.

We will also need a two-dimensional version of Lemma 2.1:

Lemma 2.2 If g satisfies (1.3) and $\phi : [0, a] \times [0, b] \to R^+$ is continuous with

$$\phi(x,t) \leq \int_0^x \int_0^t g(\phi(w,z))\, dw\, dz \tag{2.1}$$

for all $(x, t) \in [0, a] \times [0, b]$, then $\phi = 0$ on $[0, a] \times [0, b]$.

Proof: Define $\Phi(x, t) = \max\{\phi(w, z) \mid 0 \leq w \leq x, 0 \leq z \leq t\}$. Then Φ is non-decreasing in x and in t, $\Phi \geq \phi$, and for any $(x, t) \in [0, a] \times [0, b]$, $\Phi(x, t) = \phi(x_0, t_0)$ for some $0 \leq x_0 \leq x$ and $0 \leq t_0 \leq t$, so

$$\Phi(x,t) = \phi(x_0,t_0) \leq \int_0^{x_0} \int_0^{t_0} g(\phi(w,z))\, dw\, dz \leq\ \leq \int_0^x \int_0^t g(\phi(w,z))\, dw\, dz \leq \int_0^x \int_0^t g(\Phi(z,w))\, dz\, dw.$$

Let $(x, t) \in [0, a] \times [0, b]$. We will prove directly that $\phi(x, t) = 0$. If $x = 0$ or $t = 0$, this is obvious from (2.1). Otherwise, let $m = t/x$ and define $h(w) = \Phi(w, mw)$ for $0 \leq w \leq x$. Let $0 \leq w \leq x$. Then

$$h(w) = \Phi(w, mw) \leq \int_0^w \int_0^{mw} g(\Phi(w', z))\, dz\, dw'$$

$$\leq \int_0^w \int_0^{mw} g(\Phi(w,mw))\,dz\,dw' = = \int_0^w mwg(\Phi(w,mw))\,dw'$$

$$\leq \int_0^w mAg(\Phi(w,mw))\,dw' \equiv \int_0^w mAg(h(w))\,dw'$$

The function mAg has the same required properties as g needed in order to apply Lemma 2.1, so by Lemma 2.1, $0 \leq \phi(x,t) \leq \Phi(x,t) \equiv h(x) = 0$. Lemma 2.2 is proven.

Next, we need

Lemma 2.3 If $\phi^{(n)}(x) = h(x)$ on $[0,a]$, then

$$\phi(x) = \sum_{i=0}^{n-1} \phi^{(i)}(0)\frac{x^i}{i!} + \frac{1}{(n-1)!}\int_0^x (x-z)^{n-1}h(z)\,dz$$

for all $0 \leq x \leq a$.

Proof: the proof is by induction. For $n = 1$, integrating $\phi'(x) = h(x)$ yields $\phi(x) - \phi(0) = \int_0^x h(z)\,dz$. Now suppose the conclusion is true for $n-1$, and $\phi^{(n)}(x) = h(x)$ on $[0,a]$. Integrating from 0 to x yields

$$\phi^{(n-1)}(x) - \phi^{(n-1)}(0) = \int_0^x h(z)\,dt, \phi^{(n-1)}(x) = \phi^{(n-1)}(0) + \int_0^x h(z)\,dt.$$

By the inductive assumption,

$$\phi(x) = \sum_{i=0}^{n-2} \phi^{(i)}(x)\frac{x^i}{i!} + \frac{1}{(n-2)!}\int_0^x (x-z)^{n-2}\left[\phi^{(n-1)}(0) + \int_0^z h(w)\,dw\right]dz$$

$$= \sum_{i=0}^{n-2} \phi^{(i)}(x)\frac{x^i}{i!} + \frac{1}{(n-2)!}\left(\frac{1}{n-1}\phi^{(n-1)}(0)\right)x^{n-1} + \frac{1}{(n-2)!}\int_{w=0}^x \int_{z=w}^x (x-z)^{n-2}h(w)\,dz\,dw$$

$$= \sum_{i=0}^{n-1} \phi^{(i)}(x)\frac{x^i}{i!} + \frac{1}{(n-2)!}\int_{w=0}^x h(w)\left(\frac{1}{n-1}\right)(x-w)^{n-2}\,dw = = \sum_{i=0}^{n-1} \phi^{(i)}(x)\frac{x^i}{i!} + \frac{1}{(n-1)!}\int_0^x (x-z)^{n-1}h(z)\,dz$$

which proves the Lemma.

Now, the last lemma we need is the following:

Lemma 2.4 There exists a function H on D such that if u satisfies the initial value problem (1.5), then

$$u(x,t) = H(x,t) + \frac{1}{(m-1)!(n-1)!}\int_0^t (t-z)^{n-1}\int_0^x (x-w)^{m-1}f(w,z,u(w,z))\,dw\,dz$$

for all $(x,t) \in D$.

Proof: let u solve (1.5). Then by Lemma 2.3, with $\phi = \frac{\partial^m u}{\partial x^m}$,

$$\frac{\partial^m u}{\partial x^m} u(x,t) = \sum_{i=0}^{n-1} \frac{\partial^{m+i}}{\partial x^m \partial t^i} u(x,0) \frac{t^i}{i!} + \frac{1}{(n-1)!} \int_0^t (t-z)^{n-1} f(x,z,u(x,z))\, dz.$$

For the rest of this lemma, let $H(x,t)$ denote a quantity that depends only on x, t, $\{\alpha_i\}$, $\{\beta_j\}$, and $\{\gamma_{i,j}\}$, and which may change from line to line. For convenience define $\gamma_{i,0} = \alpha_i(0)$, $\gamma_{0,j} = \beta_j(0)$. Integrating the above equation from 0 to x yields

$$\frac{\partial^{m-1}}{\partial x^{m-1}} u(x,t) - \frac{\partial^{m-1}}{\partial x^{m-1}} u(0,t)$$

$$= \sum_{i=0}^{n-1} \left(\frac{\partial^{m-1+i}}{\partial x^{m-1} \partial t^i} u(x,0) - \frac{\partial^{m-1+i}}{\partial x^{m-1} \partial t^i} u(0,0) \right) \frac{t^i}{i!} + \frac{1}{(n-1)!} \int_0^t (t-z)^{n-1} \int_0^x f(w,z,u)\, dw\, dz,$$

$$\frac{\partial^{m-1}}{\partial x^{m-1}} u(x,t) = \alpha_{m-1}(t) + \sum_{i=0}^{n-1} \left(\frac{\partial^{m-1+i}}{\partial x^{m-1} \partial t^i} u(x,0) - \gamma_{m-1,i} \right) \frac{t^i}{i!} + \frac{1}{(n-1)!} \int_0^t (t-z)^{n-1} \int_0^x f(w,zu)\, dw\, dz$$

$$= H(x,t) + \sum_{i=0}^{n-1} \frac{\partial^{m-1+i}}{\partial x^{m-1} \partial t^i} u(x,0) + \frac{1}{(n-1)!} \int_0^t (t-z)^{n-1} \int_0^x f(w,zu)\, dw\, dz.$$

After integrating m times, we obtain

$$u(x,t) = H(x,t) + \sum_{i=0}^{n-1} \frac{\partial^i}{\partial t^i} u(x,0) \frac{t^i}{i!} + \frac{1}{(m-1)!(n-1)!} \int_0^t (t-z)^{n-1} \int_0^x (x-w)^{m-1} f(w,z) dw\, dz =$$

$$= H(x,t) + \sum_{i=0}^{n-1} \beta_i(x) \frac{t^i}{i!} + \frac{1}{(m-1)!(n-1)!} \int_0^t (t-z)^{n-1} \int_0^x (x-w)^{m-1} f(w,zu)\, dw\, dz$$

$$= H(x,t) + \frac{1}{(m-1)!(n-1)!} \int_0^t (z-t)^{n-1} \int_0^x (x-w)^{m-1} f(w,z,u)\, dw\, dz.$$

The lemma is proven.

To complete the proof of Theorem (1.1) , suppose that u and $\bar{u}$ are two solutions of (1.5). Then by Lemma 2.4, for $(x,t) \in D$,

$$|u(x,t) - \bar{u}(x,t)| = |\frac{1}{(m-1)!(n-1)!} \int_0^x (x-w)^{m-1} \int_0^t (t-z)^{n-1} (f(w,z,u(w,z) - f(w,z,\bar{u}(w,z)))\, dw\, dz|$$

$$\leq \frac{a^{m-1} b^{n-1}}{(m-1)!(n-1)!} \int_0^x \int_0^t |f(w,z,u(w,z) - f(w,z,\bar{u}(w,z))|\, dw\, dz \leq$$

$$\leq \frac{a^{m-1}b^{n-1}}{(m-1)!(n-1)!}\int_0^x\int_0^t g(|u(w,z)-\bar{u}(w,z)|\,dw\,dz \equiv \; \equiv \int_0^x\int_0^t Kg(|u(w,z)-\bar{u}(w,z)|\,dw\,dz.$$

Kg satisfies (1.3), so by Lemma 2.2, $u(x,t)=\bar{u}(x,t)$ and Theorem (1.1) is proven.

Proof of Theorem 2

To begin we note that (1.2) can be rewritten as

$$u^{(0,1)} - u^{(2,1)} = f - u^{(1,0)} - u^{(1,0)}u.$$

Now, upon some manipulations of the standard derivative operator the above equation can be rewritten as

$$[(1-\partial_x^2)]u^{(0,1)} = -\partial_x[u + \frac{1}{2}u^2 - F].$$

where F is a function such that $\frac{dF}{dx} = f$.

If one views the above equations as a differential equation for $u^{(0,1)}$ they will obtain

$$u^{(0,1)} = \int_{-L}^{L} K(x-\xi)(u(\xi,t) + \frac{1}{2}u^2(\xi,t) - F(\xi,t))d\xi,$$

where L is approaching infinity and the Kernel is defined as $K(x) = \frac{1}{2}sgn(x)e^{-|x|}$. And, this can be rewritten as

$$u(x,y) = g(x) + \int_0^t\int_{-L}^{L} K(x-\xi)(u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F)d\xi d\tau. \tag{1.3}$$

Now, we denote ξ_{t_0} as the set of function which are continuous and uniformly bounded on $[0,R]\times t_0$. In addition, we define the norm $||u|| = sup_{x\in R, 0\leq t\leq t_0}|u^{(i,0)}|$ with $i = 0,1,2,...,5$. And, we define A as the integral operator

$$A[u] = g(x) + \int_0^t\int_{-L}^{L} K(x-\xi)(u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F)d\xi d\tau,$$

hence we can view our partial differential equation (1.2) as

$$u = A[u]$$

Now, if we call v and $\bar{v}$ two solutions to (1.2) and investigate the quantity $|A[v]-A[\bar{v}]|$ we obtain that

$$||A[v]-A[\bar{v}]|| \leq$$

$$\int_0^t\int_{-L}^{L} e^{-|x-\xi|}(|v-\bar{v}| + \frac{1}{2}(|v|+|\bar{v}|)|v-\bar{v}| + F(|v-\bar{v}|)d\xi d\tau +$$

$$\leq \int_0^t\int_{-L}^{L} e^{-|x-\xi|}(|v-\bar{v}| + \frac{1}{2}(|v|+|\bar{v}|)|v-\bar{v}| + g(|v-\bar{v}|)d\xi d\tau +$$

Now, we have obtained

$$||A[v] - A[\bar{v}]|| \leq \int_0^t \int_{-L}^{L} Kg(|v - \bar{v}|)$$

and Kg satisfies (1.3) so by Lemma 2.2 $v = \bar{v}$ which proves Theorem 1.2.

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