An Elliptic Equation With Spike Solutions Concentrating at Local Minima of the Laplacian of the Potential

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An elliptic equation with spike solutions concentrating at local minima of the Laplacian of the potential *

Gregory S. Spradlin

Abstract

We consider the equation $-\epsilon^2 \Delta u + V(z)u = f(u)$ which arises in the study of nonlinear Schrödinger equations. We seek solutions that are positive on $\mathbb{R}^N$ and that vanish at infinity. Under the assumption that $f$ satisfies super-linear and sub-critical growth conditions, we show that for small $\epsilon$ there exist solutions that concentrate near local minima of $V$. The local minima may occur in unbounded components, as long as the Laplacian of $V$ achieves a strict local minimum along such a component. Our proofs employ variational mountain-pass and concentration compactness arguments. A penalization technique developed by Felmer and del Pino is used to handle the lack of compactness and the absence of the Palais-Smale condition in the variational framework.

1 Introduction

This paper concerns the equation

$$-\epsilon^2 \Delta u + V(z)u = f(u)$$

(1.1)

on $\mathbb{R}^N$ with $N \geq 1$, where $f(u)$ is a “superlinear” type function such as $f(u) = u^p$, $p > 1$. Such an equation arises when searching for standing wave solutions of the nonlinear Schrödinger equation (see [3]). For small positive $\epsilon$, we seek “ground states,” that is, positive solutions $u$ with $u(z) \to 0$ as $|z| \to \infty$. Floer and Weinstein ([6]) examined the case $N = 1$, $f(u) = u^3$ and found that for small $\epsilon$, a ground state $u_\epsilon$ exists which concentrates near a non-degenerate critical point of $V$. Similar results for $N > 1$ were obtained by Oh in [10]-[12]. In [3], del Pino and Felmer found that if $V$ has a strict local minimum, then for small $\epsilon$, (1.1) has a ground state concentrating near that minimum. A strict local minimum occurs when there exists a bounded, open set $\Lambda \subset \mathbb{R}^N$

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with \( \inf_{A} V < \inf_{\partial A} V \). They extended their results in [4] to the more general case where \( V \) has a “topologically stable” critical point, that is, a critical point obtained via a topological linking argument (see [4] for a precise formulation). Such a critical point persists under small perturbations of \( V \). Examples are a strict local extremum and a saddle point. This very strong result is notable because the critical points of \( V \) in question need not be non-degenerate or even isolated. Similar results have been obtained by Li [8], and earlier work of Rabinowitz [13] is also interesting. The recent results of [1] and [9] also permit \( V \) to have degenerate critical points.

A common feature of all the papers above is that \( V \) must have a non-degenerate, or at least topologically stable, set of critical points. Therefore it is natural to try to remove this requirement. That we must assume some conditions on \( V \) is shown by Wang’s counterexample [15] - if \( V \) is nondecreasing and nonconstant in one variable (e.g. \( V(x_1, x_2, x_3) = 2 + \tan^{-1}(x_1) \)), then no ground states exist. In [14] the author showed that ground states to (1.1) exist under the assumption that \( V \) is almost periodic, together with another mild assumption. Those assumptions did not guarantee that \( V \) had a topologically stable critical point.

Aside from periodicity or recurrence properties of \( V \), another approach is to impose conditions on the derivatives of \( V \). That is the approach taken here. We will assume that \( V \) has a (perhaps unbounded) component of local minima, along which \( \Delta V \) achieves a strict local minimum. More specifically, assume \( f \) satisfies the following:

(F1) \( f \in C^{1}(\mathbb{R}^{+}, \mathbb{R}) \)
(F2) \( f'(0) = 0 = f(0) \).
(F3) \( \lim_{q \to \infty} f(q)/q^s = 0 \) for some \( s > 1 \), with \( s < (N + 2)/(N - 2) \) if \( N \geq 3 \).
(F4) For some \( \theta > 2 \), \( 0 < \theta F(q) \leq f(q)q \) for all \( q > 0 \), where \( F(\xi) \equiv \int_{0}^{\xi} f(t) \, dt \).
(F5) The function \( q \mapsto f(q)/q \) is increasing on \((0, \infty)\).

Assumptions (F1)-(F5) are the same as in [3] and are satisfied by \( f(q) = q^s \), for example, if \( 1 < s < (N + 2)/(N - 2) \). Assume that \( V \) satisfies the following:

(V1) \( V \in C^{2}(\mathbb{R}^{n}, \mathbb{R}) \)
(V2) \( D^{\alpha} V \) is bounded and Lipschitz continuous for \( |\alpha| = 2 \).
(V3) \( 0 < V_{-} \equiv \inf_{\mathbb{R}^{N}} V < \sup_{\mathbb{R}^{N}} V \equiv V^{+} < \infty \)
(V4) There exists a bounded, nonempty open set \( \Lambda \subset \mathbb{R}^{N} \) and a point \( z_{0} \in \Lambda \) with \( V(z_{0}) = \inf_{\Lambda} V \equiv V_{0} \), and

\[
\Delta_{0} \equiv \inf\{\Delta V(z) \mid z \in \Lambda, \ V(z) = V_{0}\} < \inf\{\Delta V(z) \mid z \in \partial \Lambda, \ V(z) = V_{0}\}
\]
Note: A special case of (V4) occurs when \( \Lambda \) is bounded and \( V(z_0) < \inf_{\partial \Lambda} V \); this case is treated, under weaker hypotheses, in [3]. A specific example of (V4) is if \( N = 2 \) and \( V \) satisfies (V1)-(V4), with \( V(z_1, z_2) = 1 + (z_1^2 - z_2^2)^2 \) for \( z_1^2 + z_2^2 \leq 1 \). Then \( \Delta V(z_1, z_2^2) = 8z_1^2 + 1 \) for \( z_1^2 + z_2^2 \leq 1 \), so we may take \( \Lambda = B_1(0,0) \subset \mathbb{R}^2 \) and \( z_0 = (0,0) \). Then \( V \) has a component of local minima that includes the parabolic arc \( \{z_2 = z_1^2\} \cap B_1(0,0) \), along which \( \Delta V \) has a minimum of 2 at \((0,0)\), with \( \Delta V > 2 \) at the two endpoints of the arc.

We prove the following:

**Theorem 1.1** Let \( V \) and \( f \) satisfy (V1)-(V4) and (F1)-(F5). Then there exists \( \epsilon_0 > 0 \) such that if \( \epsilon \leq \epsilon_0 \), then (1.0) has a positive solution \( u_\epsilon \) with \( u_\epsilon(z) \to 0 \) as \( |z| \to \infty \). \( u_\epsilon \) has exactly one local maximum (hence, global maximum) point \( z_\epsilon \in \Lambda \), where \( \Lambda \) is as in (V4). There exist \( \alpha, \beta > 0 \) with \( u_\epsilon(z) \leq \alpha \exp(-\frac{1}{2}|z-z_\epsilon|) \) for \( \epsilon \leq \epsilon_0 \). Furthermore, with \( V_0 \) and \( \Delta_0 \) as in (V4), \( V(z_\epsilon) \to V_0 \) and \( \Delta V(z_\epsilon) \to \Delta_0 \) as \( \epsilon \to 0 \).

For small \( \epsilon \), \( u_\epsilon \) resembles a “spike,” which is sharper for smaller \( \epsilon \). The spike concentrates near a local minimum of \( V \) where \( \Delta V \) has a strict local minimum.

The proof of Theorem 1.1 employs the techniques of [3], with some refinements necessary because \( V \) does not necessarily achieve a strict local minimum. Section 2 introduces the penalization scheme developed by Felmer and del Pino, and continues with the beginning of the proof of Theorem 1.1. These beginning arguments are taken practically verbatim from [3], but are included, since the machinery of the penalization technique is used in the remainder of the proof. The reader is invited to consult [3] for more complete proofs. Section 3 contains the completion of the proof, which is original. This part contains delicate computations involving \( \Delta V \).

## 2 The penalization scheme

Extend \( f \) to the negative reals by defining \( f(q) = 0 \) for \( q < 0 \). Let \( F \) be the primitive of \( f \), that is, \( F(q) = \int_0^q f(t) \, dt \). Define the functional \( I_\epsilon \) on \( W^{1,2}(\mathbb{R}^N) \) by

\[
I_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2} (e^2 |\nabla u|^2 + V(z)u^2) - F(u) \, dz.
\]

\( I_\epsilon \) is a \( C^1 \) functional, and there is a one-to-one correspondence between positive critical points of \( I_\epsilon \) and ground states of (1.1). It is well known that \( I_\epsilon \) and similar functionals in related problems fail the Palais-Smale condition. That is, a “Palais-Smale sequence,” defined as a sequence \( (u_m) \) with \( I_\epsilon(u_m) \) convergent and \( I_\epsilon'(u_m) \to 0 \) as \( m \to \infty \), need not have a convergent subsequence. To get around this difficulty, we formulate a “penalized” problem, with a corresponding “penalized” functional satisfying the Palais-Smale condition, by altering \( f \) outside of \( \Lambda \).
Let $\theta$ be as in (F4). Choose $k$ so $k > \theta/(\theta - 2)$. Let $V_-$ be as in (V3) and $a > 0$ be the value at which $f(a)/a = V_-/k$. Define $\tilde{f}$ by

$$
\tilde{f}(s) = \begin{cases} 
    f(s) & s \leq a; \\
    \frac{sV_-}{k} & s > a,
\end{cases} \tag{2.1}
$$

$g(\cdot, s) = \chi_A f(s) + (1 - \chi_A) \tilde{f}(s)$, and $G(z, \xi) = \int_0^\xi g(z, \tau) \, d\tau$. Although not continuous, $g$ is a Carathéodory function. For $\varepsilon > 0$, define the penalized functional $J_\varepsilon$ on $W^{1,2}(\mathbb{R}^N)$ by

$$
J_\varepsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2}(\varepsilon^2 |\nabla u|^2 + V(z)u^2) - G(z, u) \, dz. \tag{2.2}
$$

A positive critical point of $J_\varepsilon$ is a weak solution of the “penalized equation”

$$
-\varepsilon^2 \Delta u + V(z)u = g(z, u), \tag{2.3}
$$

that is, a $C^1$ function satisfying (2.3) wherever $g$ is continuous. It is proven in [3] that $J_\varepsilon$ satisfies all the hypotheses of the Mountain Pass Theorem of Ambrosetti and Rabinowitz ([2]), including the Palais-Smale condition. Therefore $J_\varepsilon$ has a critical point $u_\varepsilon$, with the mountain pass critical level $c(\varepsilon) = J_\varepsilon(u_\varepsilon)$.

$c(\varepsilon)$ is defined by the following minimax: let the set of paths $\Gamma_{\varepsilon} = \{ \gamma \in C([0, 1], W^{1,2}(\mathbb{R}^N)) \mid \gamma(0) = 0, J_\varepsilon(\gamma(1)) < 0 \}$, and

$$
c(\varepsilon) = \inf_{\gamma \in \Gamma_{\varepsilon}} \max_{\theta \in [0, 1]} J_\varepsilon(\gamma(\theta)).
$$

As shown in ([3]), because of (F4), $c(\varepsilon)$ can be characterized more simply as

$$
c(\varepsilon) = \inf_{u \in W^{1,2}(\mathbb{R}^N) \setminus \{ 0 \}} \sup_{\tau > 0} J_\varepsilon(\tau u). \tag{2.4}
$$

The functions $g(z, q)$ and $f(q)$ agree whenever $z \in \Lambda$ or $q < a$. Therefore if $u$ is a weak solution of (2.3) with $u < a$ on $\Lambda^C \equiv \mathbb{R}^N \setminus \Lambda$, then $u$ solves (1.1). Our plan is to find a positive critical point $u_\varepsilon$ of $J_\varepsilon$, which is a weak solution of (2.3), then show that $u_\varepsilon(z) < a$ for all $z \in \Lambda^C$.

For $\varepsilon > 0$, let $u_\varepsilon$ be a critical point of $J_\varepsilon$ with $J_\varepsilon(u_\varepsilon) = c(\varepsilon)$. Maximum principle arguments show that $u_\varepsilon$ must be positive. Define the “limiting functional” $I_0$ by

$$
I_0(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V_0 u^2) - F(u) \tag{2.5}
$$

and

$$
\mathcal{E} = \inf_{u \in W^{1,2}(\mathbb{R}^N) \setminus \{ 0 \}} \sup_{\tau > 0} I_0(\tau u). \tag{2.6}
$$

The equation corresponding to (2.4) is

$$
-\Delta u + V_0 u = f(u) \tag{2.6}
$$

We will prove Theorem 1.1 by proving the following proposition:
Proposition 2.1 Let $\epsilon > 0$. If $u_{\epsilon}$ is a positive solution of (2.3) satisfying $J_{\epsilon}(u_{\epsilon}) = c(\epsilon)$, then

(i) $\lim_{\epsilon \to 0} \max_{z \in \partial \Lambda} u_{\epsilon} = 0$.

(ii) For all $\epsilon$ sufficiently small, $u_{\epsilon}$ has only one local maximum point in $\Lambda$ (call it $z_{\epsilon}$), with $\lim_{\epsilon \to 0} V(z_{\epsilon}) = V_{0}$

(iii) $\lim_{\epsilon \to 0} \Delta V(z_{\epsilon}) = \Delta_{0}$.

Proof of Theorem 1.1: Assuming Proposition 2.1, there exists $\epsilon_{0} > 0$ such that for $\epsilon < \epsilon_{0}$, $u_{\epsilon} < a$ on $\partial \Lambda$. In [3] it is shown that if we multiply (2.3) by $(u_{\epsilon} - a)_{+}$ and integrate by parts, it follows that $u_{\epsilon} < a$ on $\Lambda^{C}$, so $u_{\epsilon}$ solves (1.1). By the definition of $a$ in (2.1), and the maximum principle, $u_{\epsilon}$ has no local maxima outside of $\Lambda$, so $u_{\epsilon}$ has exactly one local maximum point $z_{\epsilon}$, which occurs in $\Lambda$.

Define $v_{\epsilon}$ by translating $u_{\epsilon}$ from $z_{\epsilon}$ to zero and dilating it by $\epsilon$, that is,

$$v_{\epsilon}(z) = u_{\epsilon}(z_{\epsilon} + \epsilon z).$$

Then $v_{\epsilon}$ is a weak ($C^{1}$) solution of the “translated and dilated” equation

$$-\Delta v_{\epsilon} + V(z_{\epsilon} + \epsilon z)v_{\epsilon} = g(z_{\epsilon} + \epsilon z, v_{\epsilon}).$$

Let $\epsilon_{j} \to 0$. Along a subsequence (called $(z_{\epsilon_{j}})$), $z_{\epsilon_{j}} \to \bar{z} \in \overline{\Lambda}$, with $V(\bar{z}) = V_{0}$ and $\Delta V(\bar{z}) = \Delta_{0}$.

Along a subsequence, $v_{\epsilon_{j}}$ converges locally uniformly to a function $v^{0}$. Pick $R > 0$ so $v^{0} < a$ on $\mathbb{R}^{N} \setminus B_{R}(0)$. For large enough $\epsilon$, $v_{\epsilon} < a$ on $\partial B_{R}(0)$. By the maximum principle arguments of [3], for small $\epsilon$, $v_{\epsilon}$ decays exponentially, uniformly in $\epsilon$.

The proof of Proposition 2.1 will follow if we can prove the following statement.

Proposition 2.2 If $\epsilon_{n} \to 0$ and $(z_{n}) \subset \overline{\Lambda}$ with $u_{\epsilon_{n}}(z_{n}) \geq b > 0$, then

(i) $\lim_{n \to \infty} V(z_{n}) = V_{0}$.

(ii) $\lim_{n \to \infty} \Delta V(z_{n}) = \Delta_{0}$.

It is proven in [3] that $u_{\epsilon}$ has exactly one local maximum point $z_{\epsilon}$ for small $\epsilon$. Since $u_{\epsilon}$ solves (2.3), the maximum principle implies that $u_{\epsilon}(z_{\epsilon})$ is bounded away from zero. Thus Proposition 2.2 and (V4) give Proposition 2.1(ii)-(iii).

To prove Proposition 2.2, let $b$ and $(z_{n})$ be as above. First we repeat the argument of [3] to show that $V(z_{n}) \to V_{0}$: suppose this does not happen. Then, along a subsequence, $z_{n} \to \bar{z} \in \overline{\Lambda}$ with $V(\bar{z}) > V_{0}$. Define $v_{n}$ by translating $u_{\epsilon_{n}}$ from $z_{n}$ to 0 and dilating by $\epsilon_{n}$; that is,

$$v_{n}(z) = u_{\epsilon_{n}}(z_{n} + \epsilon_{n} z).$$

(2.7)
$v_n$ solves the “translated and dilated” penalized equation

$$-\Delta v_n + V(z_n + \epsilon_n z)v_n = g(z_n + \epsilon_n z, v_n)$$

(2.8)
on $\mathbb{R}^N$, with $v_n(z) \to 0$ and $\nabla v_n(z) \to 0$ as $|z| \to \infty$. As shown in [3], $(v_n)$ is bounded in $W^{1,2}(\mathbb{R}^N)$, so by elliptic estimates, $(v_n)$ converges locally along a subsequence (also denoted $(v_n)$) to $v^0 \in W^{1,2}(\mathbb{R}^N)$. Define $\chi_n$ by $\chi_n(z) = \chi(z)$, where $\chi$ is the characteristic function of $\lambda$. $\chi_n$ converges weakly in $L^p$ over compact sets to a function $\chi$, for any $p > 1$, with $0 \leq \chi \leq 1$. Define

$$\bar{g}(z, s) = \chi(z)f(s) + (1 - \chi(z))\bar{f}(s)$$

Then $v^0$ satisfies

$$-\Delta v + V(\bar{z})v = \bar{g}(z, v)$$

(2.9)
on $\mathbb{R}^N$. Define $\bar{G}(z, s) = \int_0^s \bar{g}(z, t)\, dt$. Associated with (2.9) we have the limiting functional $\bar{J}(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V(\bar{z})u^2) - \bar{G}(z, u)\, dz$. $v^0$ is a positive critical point of $\bar{J}$.

Define $J_n$ to be the “translated and dilated” penalized functional corresponding to (2.8), that is,

$$J_n(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V(z_n + \epsilon_n z)u^2) - G(z_n + \epsilon_n z, u)\, dz.$$ 

Clearly $J_n(v_n) = \epsilon_n^{-N}J_n(u_{\epsilon_n})$. In [3] it is proven that

$$\liminf_{n \to \infty} J_n(v_n) \geq \bar{J}(v^0).$$

(2.10)

Also, by letting $w$ be a ground state for (2.6) with $I_0(w) = c$ (the mountain pass value for $I_0$, defined in (2.5) and using $w$ as a test function for $J_n$, it is proven that $c \geq \liminf_{n \to \infty} J_n(v_n)$). Thus $\bar{J}(v^0) \leq c$. Therefore, as shown in [3], $V(\bar{z}) \leq V_0$. This contradicts our assumption. Thus $V(z_n) \to \Delta_0$. All the above is the same as was proven in [3]. Next, we must show that $\Delta V(z_n) \to \Delta_0$. That is the focus of the next section.

### 3 The effect of the Laplacian

Proving $\Delta V(z_n) \to \Delta_0$ is a subtle and delicate problem. Making $\epsilon_n$ approach 0 is equivalent to dilating $V$, which has the effect of making local minima of $V$ behave more like global minima. This assists in finding solutions to (1.1). However, making $\epsilon_n$ small reduces the effect of differences in $\Delta V$. For this reason, Theorem 1.1 is not only difficult to prove, but is not intuitively compelling, either.

It is known ([7]) that a “least energy solution” of (2.6), that is, a solution $w$ with $I_0(w) = c$ must be radially symmetric. We will need to exploit this fact. In order to do this, we will need to work with the maximum points of $u_{\epsilon_n}$ instead of merely the $(z_n)$ as given in Proposition 2.2. We need the following concentration-compactness result, which states that the sequence $(u_{\epsilon_n})$ of “mountain-pass type solutions” of (2.3) does not “split”:
Lemma 3.1 If \((z_n) \subset \overline{\Lambda}, (y_n) \subset \mathbb{R}^N\), and \(b > 0\) with \(u_{\epsilon_n}(z_n) > b\) and \(u_{\epsilon_n}(y_n) > b\) for all \(n\), then \(((z_n - y_n)/\epsilon_n)\) is bounded.

Proof: define \(v_n(z) = u_{\epsilon_n}(z + \epsilon_n z)\) as in (2.7). Suppose the lemma is false. Then, along a subsequence, \(|y_n - z_n|/\epsilon_n \to \infty\). Let \(x_n = (y_n - z_n)/\epsilon_n\). (\(\|v_n\|\)) is bounded in \(W^{1,2}(\mathbb{R}^N)\) and \(|x_n| \to \infty\), so we may pick a sequence \((R_n) \subset \mathbb{N}\) with \(R_n \to \infty\), \(|x_n| - R_n \to \infty\), and \(\|v_n\|_{W^{1,2}([B_{R_{n+1}}(0) \setminus B_{R_{n-1}}(0)])} \to 0\) as \(n \to \infty\). Define cutoff functions \(\varphi_{1,n}^2 \in C^\infty(\mathbb{R}^N, [0,1])\) satisfying \(\varphi_1 \equiv 1\) on \(B_{R_{n-1}}(0)\), \(\varphi_1 \equiv 0\) on \(B_{R_n}(0)\), \(\varphi_2 \equiv 0\) on \(B_{R_{n+1}}(0)\), \(\varphi_2 \equiv 0\) on \(B_{R_n}(0)\), and \(\|\nabla \varphi_1\|_{L^\infty(\mathbb{R}^N)} < 2\), \(\|\nabla \varphi_2\|_{L^\infty(\mathbb{R}^N)} < 2\). Set \(v_1 = \varphi_1 v_n\) and \(v_2 = \varphi_2 v_n\), and \(\bar{v}_n = v_1 + v_2 = (\varphi_1 + \varphi_2) v_n\).

Choose \(T_n > 0\) so \(J_n(T_n \bar{v}_n) = 0\). We claim that \(T_n\) is well-defined, and bounded in \(n\). Note that the existence of \(T_n\) must be checked for the penalized functional \(J_n\), because of the replacement of \(F\) with \(G\). By elliptic estimates, there exists an open set \(U \subset \mathbb{R}^N\) such that along a subsequence, \(v_1 \not\to 0\) on \(U\) and \(U \subset (\Lambda - z_n)/\epsilon_n = \{z \in \mathbb{R}^N \mid z_n + \epsilon_n z \in \Lambda\}\). Let \(a\) be as in (2.1). For \(t > 2a/b\) and \(z \in U\), \(t \bar{v}_n(z) > tb/2 > a\), so \(G(z_n + \epsilon_n z, t \bar{v}_n) = F(t \bar{v}_n) > F(bt/2)\). Therefore, for \(t > 2a/b\),

\[
J_n(t \bar{v}_n) = \int_{\mathbb{R}^N} \left( \frac{1}{2} |\nabla \bar{v}_n|^2 + V(z_n + \epsilon_n z) \bar{v}_n^2 \right) dx - \int_{\mathbb{R}^N} G(z_n + \epsilon_n z, t \bar{v}_n) dx \\
\leq \int_{\mathbb{R}^N} \left( 1 + V^+ \right) \bar{v}_n^2 dx - \int_U F(t \bar{v}_n) \\
\leq \frac{t^2}{2} \left( 1 + V^+ \right) \int_{\mathbb{R}^N} \bar{v}_n^2 dx - \lambda(U) F(bt/2),
\]

where \(\lambda\) indicates the Lebesgue measure. By (F4), there exists \(C > 0\) such that for \(t > 2a/b\), \(F(bt/2) > Ct^\theta\). Therefore, for \(t > 2a/b\),

\[
J_n(t \bar{v}_n) \leq \frac{t^2}{2} \left( 1 + V^+ \right) \int_{\mathbb{R}^N} \bar{v}_n^2 dx - Ct^\theta. \tag{3.1}
\]

Since \(\bar{v}_n\) is bounded in \(W^{1,2}(\mathbb{R}^N)\), this gives the existence and boundedness of \((T_n)\).

Since \(J_n(T_n \bar{v}_n) = J_n(T_n \bar{v}_n^1) + J_n(T_n \bar{v}_n^2) = 0\), we may pick \(i_n \in \{1,2\}\) with \(J_n(T_n \bar{v}_n^{i_n}) = 0\). By (F5) and (2.1), the map \(t \to J_n(t \bar{v}_n^{i_n})\) increases from zero at \(t = 0\), achieves a positive maximum, then decreases to \(-\infty\). We will see more of this in a moment. Thus there exists a unique \(t_n \in (0,T_n)\) with \(J_n(t_n \bar{v}_n^{i_n}) = \max_{t > 0} J_n(t \bar{v}_n^{i_n})\). We claim that \(t_n\) and \(T_n - t_n\) are both bounded away from zero for large \(n\): by \((f_1) - (f_4)\) and (2.1), \(J_n(u) \geq \frac{1}{\theta} \min(1,V^-) \int |w|^2_{W^{1,2}(\mathbb{R}^N)} - o(\|w\|_{W^{1,2}(\mathbb{R}^N)})\) uniformly in \(n\), so \(\max_{t > 0} J_n(t \bar{v}_n^{i_n})\) is bounded away from zero, uniformly. It is easy to show that \(J_n\) is Lipschitz on bounded subsets of \(W^{1,2}(\mathbb{R}^N)\), uniformly in \(n\). Since \((T_n)\) is bounded, this implies that \(t_n\) and \(T_n - t_n\) are both bounded away from zero for large \(n\).
Using the facts that \( \|v_n - \bar{v}_n\|_{W^{1,2}(\mathbb{R}^N)} \to 0 \) as \( n \to \infty \), and \( (T_n) \) is bounded, we have

\[
\liminf_{n \to \infty} J_n(t_n v_n^{i_n}) = \liminf_{n \to \infty} \max_{t > 0} J_n(t v_n^{i_n}) \\
\geq \liminf_{n \to \infty} \max_{t > 0} J_n(tv_n) \\
= \liminf_{n \to \infty} \max_{t > 0} J_n(t\bar{v}_n) \\
= \liminf_{n \to \infty} J_n(t_n \bar{v}_n) \\
= \liminf_{n \to \infty} (J_n(t_n v_n^{i_n}) + J_n(t_n v_n^{3-i_n})) \\
\geq \liminf_{n \to \infty} J_n(t_n v_n^{i_n}) + \liminf_{n \to \infty} J_n(t_n v_n^{3-i_n}).
\]

Now \( J_n(T_n v_n^{3-i_n}) = -J_n(T_n v_n^{i_n}) \geq 0 \) and \( t_n < T_n \), so \( J_n(t_n v_n^{3-i_n}) \geq 0 \). By (3.2), \( \liminf_{n \to \infty} J_n(t_n v_n^{3-i_n}) \leq 0 \). Therefore \( J_n(t_n v_n^{3-i_n}) \to 0 \) as \( n \to \infty \).

Since \( J_n(w) \geq \frac{\delta}{2} \min(1, V_-)(\|w\|^2_{W^{1,2}(\mathbb{R}^N)} - \delta(\|w\|^2_{W^{1,2}(\mathbb{R}^N)}) \) uniformly in \( n \), there exists \( d \in (0, \liminf_{n \to \infty} t_n) \) such that \( \liminf_{n \to \infty} J_n(dv_n^{3-i_n}) > 0 \). Since \( d < t_n \) and \( J_n(dv_n^{3-i_n}) > J_n(t_n v_n^{3-i_n}) \) for large \( n \), the map \( t \mapsto J_n(tv_n^{3-i_n}) \) achieves a maximum at some \( t'_n \in (0, t_n) \), and that maximum is bounded away from zero.

Summarizing the important facts about the mapping \( t \mapsto J_n(tv_n^{3-i_n}) \), we have shown that there exists \( \rho > 0 \) such that for large \( n \),

(i) \( 0 < t'_n < t_n < T_n \)

(ii) \( (T_n) \) is bounded.

(iii) \( (T_n - t_n) \) is bounded away from zero.

(iv) \( J_n(t'_n v_n^{3-i_n}) > \rho > 0 \)

(v) \( J_n(t_n v_n^{3-i_n}) \to 0 \)

(vi) \( J_n(T_n v_n^{3-i_n}) \geq 0 \)

From (i)-(vi) it is apparent that at some \( t'_n > t'_n \), the mapping \( t \mapsto J_n(tv_n^{3-i_n}) \) is at once decreasing and concave upward. But this is impossible: let \( n \in \mathbb{N} \) and \( w \in W^{1,2}(\mathbb{R}^N) \setminus \{0\} \). Define \( \psi(t) = J_n(tw) \) for \( t > 0 \). Then

\[
\psi'(t) = t \int_{\mathbb{R}^N} |\nabla w|^2 + V(z_n + \epsilon_n z)w^2 \, dz - \int_{\mathbb{R}^N} g(z_n + \epsilon_n z, tw)w \, dz \\
= t \left[ \int_{\mathbb{R}^N} |\nabla w|^2 + V(z_n + \epsilon_n z)w^2 \, dz - \int_{\{w \neq 0\}} \frac{g(z_n + \epsilon_n z, tw)}{tw}w^2 \, dz \right].
\]

By (F5) and (2.1), \( t \mapsto g(z_n + \epsilon_n z, tw)/(tw) \) is nondecreasing, so if \( \psi'(t) \) ever becomes negative, \( \psi' \) is increasing for all time \( t \) after that, and the graph of \( \psi \) is concave down. Therefore the behavior of \( J_n(tv_n^{3-i_n}) \) as described in (i)-(vi) is impossible, and Lemma 3.1 is proven. \( \diamond \)
As mentioned before, it will be advantageous to work with the maxima of 
\( u_{\epsilon_n} \). Choose \( (y_n) \subset \mathbb{R}^N \) with

\[
u_{\epsilon_n}(y_n) = \max_{\mathbb{R}^N} u_{\epsilon_n}.
\]

We will prove

\[
\Delta V(y_n) \rightarrow \Delta_0.
\]

By Lemma 3.0, \((y_n - z_n)/\epsilon_n\) is bounded, so \( y_n - z_n \rightarrow 0 \). Thus (3.3) gives Proposition 2.2(ii), completing the proof of Theorem 1.1.

Along a subsequence, \( y_n \rightarrow \bar{y} \in \mathbb{x} \). By Proposition 2.2(i), \( V(\bar{y}) = V_0 \). Since it is not apparent that \( \bar{y} \in \Lambda \), we must proceed carefully. We will redefine the \( v_n \)'s like in (2.7), by translating \( u_{\epsilon_n} \) to 0 and dilating it. That is,

\[
v_n(z) = u_{\epsilon_n}(y_n + \epsilon_n z).
\]

Then \( v_n \) is a positive weak solution, vanishing at infinity, of the “penalized, dilated, and translated” PDE

\[-\Delta v + V(y_n + \epsilon_n z)v = g(y_n + \epsilon_n z, v).
\]

Like before, \((v_n)\) converges locally uniformly to a function \( v_0 \). We claim that \( v_0 \) is actually a ground state maximizing at 0 of the autonomous limiting equation (2.6). Proof: As before, define \( \chi_n \) by \( \chi_n(z) = \chi(y_n + \epsilon z) \). As before, along a subsequence, \( \chi_n \) converges weakly in \( L^p \), for any \( p > 1 \), on compact subsets of \( \mathbb{R}^N \) to a function \( \chi \) with \( 0 \leq \chi \leq 1 \). Define \( \tilde{g} \) by

\[
\tilde{g}(z, s) = \chi(z) f(s) + (1 - \chi(z)) \hat{f}(s).
\]

By the argument of Proposition 2.2, taken from [3], \((v_n)\) converges locally along a subsequence to \( v_0 \), a ground state of \(-\Delta v + V_0 v = \tilde{g}(z, v)\). The functional corresponding to this equation is \( J(u) = \frac{1}{2}(|\nabla u|^2 + V_0 u^2) - \frac{1}{2}G(z, u) \), where \( G(z, s) = \int_0^s \tilde{g}(z, t) \, dt \). As before, in (2.10), \( c \geq \lim \inf_{n \rightarrow \infty} J_n(v_n) \geq J(v_0) \), where \( c \) is from (2.5). \( J \geq I_0 \), where \( I_0 \) is the “autonomous” limiting functional from (2.4), so

\[
c \leq \max_{t > 0} I_0(t v_0) \leq \max_{t > 0} \tilde{J}(t v_0) \leq c,
\]

and \( v_0 \) is actually a ground state of (2.6).

Lemma 3.2 With \((v_n)\) as in (3.4), for any subsequence of \((v_n)\) there is a radially symmetric ground state \( v_0 \) of (2.6) such that \( v_n \rightarrow v_0 \) uniformly along a subsequence and the \( v_n \)'s decay exponentially, uniformly in \( n \).
Proof: If one establishes uniform convergence, the uniform exponential decay follows readily, using a standard maximum principle argument found in [3]. Suppose the convergence is not uniform. Then there exist a subsequence of \((v_n)\) (denoted \((v_n)\)) and a sequence \((x_n) \subset \mathbb{R}^N\) with \(|x_n| \to \infty\) and \(\lim_{n \to \infty} v_n(x_n) > 0\). Let \(d > 0\) with \(d < v_0(0)\) and \(d < \lim_{n \to \infty} v_n(x_n)\). For large \(n\), \(d < v_n(0) = u_{\epsilon_n}(z_n)\) and \(d < v_n(x_n) = u_{\epsilon_n}(z_n + \epsilon_n x_n)\). Letting \(w_n = z_n + \epsilon_n x_n\), we obtain \(((w_n - z_n)/\epsilon_n) = (x_n)\), which is unbounded, violating Lemma 3.1.

To show \(\Delta V(y_n) \to \Delta_0\), we again argue indirectly. Suppose otherwise. Then, along a subsequence, \(y_n \to \bar{y} \in \overline{\Lambda}\) with

\[
\Delta V(\bar{y}) > \Delta_0. \tag{3.5}
\]

For \(x \in \mathbb{R}^N\), define the translation operator \(\tau_x\) by \(\tau_x u(z) = u(z - x)\), that is, \(\tau_x u\) is \(u\) translated by \(x\). Assume for convenience, and without loss of generality, that

\[
0 \in \Lambda, \ V(0) = V_0, \ \text{and} \ \Delta V(0) = \Delta_0.
\]

We will prove that for large \(n\),

\[
\sup_{t > 0} J_{\epsilon_n}(t \tau_{-y_n/\epsilon_n} u_{\epsilon_n}) < J_{\epsilon_n}(u_{\epsilon_n}) = \sup_{t > 0} J_{\epsilon_n}(tu_{\epsilon_n}), \tag{3.6}
\]

recalling the definition of \(J\) in (2.2), and how \(v_n\) is defined from \(u_{\epsilon_n}\) in (3.4). That is, translating \(tu_{\epsilon_n}\) back to the origin reduces the value of \(J_{\epsilon_n}(tv_n)\) because \(V\) has lesser concavity at the origin. This occurs even though shrinking \(\epsilon\) reduces the difference in concavity. (3.6) contradicts the definition of \(u_{\epsilon_n}\).

Pick \(T > 1\) large enough so that for large \(n\), \(J_n(T v_n) = \epsilon_n^{-N} J_{\epsilon_n}(T u_{\epsilon_n}) < 0\). This is possible by the argument of (3.1). Now (3.6) is equivalent to

\[
\sup_{0 \leq t \leq T} J_{\epsilon_n}(t \tau_{-y_n} u_{\epsilon_n}) < \sup_{0 \leq t \leq T} J_{\epsilon_n}(tu_{\epsilon_n}).
\]

To prove the above, it will suffice to prove the stronger fact that for large \(n\), for all \(t \in (0, T)\),

\[
J_{\epsilon_n}(tu_{\epsilon_n}) > J_{\epsilon_n}(t \tau_{-y_n} u_{\epsilon_n}).
\]

Now, along a subsequence, \(v_n \to v_0\) uniformly, so by the definition of \(v_n\) as a dilation of \(\tau_{-y_n} u_{\epsilon_n}\) (3.4), \(u_{\epsilon_n} \to 0\) uniformly on \(\mathbb{R}^N \setminus \Lambda\) as \(n \to \infty\). Thus for large \(n\) and \(0 \leq t \leq T\), the definition of \(G\) gives \(G(z, t \tau_{-y_n} u_{\epsilon_n}(z)) = F(t \tau_{-y_n} u_{\epsilon_n}(z))\) for all \(z \in \mathbb{R}^N\), so

\[
J_{\epsilon_n}(tu_{\epsilon_n}) - J_{\epsilon_n}(t \tau_{-y_n} u_{\epsilon_n})
\]

\[
= \int_{\mathbb{R}^N} \frac{1}{2} t^2 (|\nabla u_{\epsilon_n}(z)|^2 + V(z) u_{\epsilon_n}(z)^2) - G(z, tu_{\epsilon_n}(z)) \, dz
\]

\[
- \left[ \int_{\mathbb{R}^N} \frac{1}{2} t^2 (|\nabla \tau_{-y_n} u_{\epsilon_n}(z)|^2 + V(z) \tau_{-y_n} u_{\epsilon_n}(z)^2) - F(t \tau_{-y_n} u_{\epsilon_n}(z)) \, dz \right]
\]

\[
\geq \frac{1}{2} t^2 \int_{\mathbb{R}^N} V(z)(u_{\epsilon_n}(z)^2 - u_{\epsilon_n}(z + y_n)^2) \, dz
\]
\[ + \int_{\mathbb{R}^N} F(tu_n(z + y_n) - F(tu_n(z)) \, dz \]
\[ = \frac{1}{2} t^2 \int_{\mathbb{R}^N} (V(z + y_n) - V(z))u_n(z + y_n)^2 \, dz \]
\[ = \frac{1}{2} t^2 \epsilon_n^2 \int_{\mathbb{R}^N} (V(y_n + \epsilon_n z) - V(\epsilon_n z))u_n(\epsilon_n z + y_n)^2 \, dz \]
\[ = \frac{1}{2} t^2 \epsilon_n^2 \int_{\mathbb{R}^N} (V(y_n + \epsilon_n z) - V(\epsilon_n z))v_n(z)^2 \, dz. \]

For \( n = 1, 2, \ldots \), define \( h_n : \mathbb{R} \to \mathbb{R} \) by
\[ h_n(t) = \int_{\mathbb{R}^N} (V(y_n + tz) - V(tz))v_n^2 \, dz. \]
Since \( h_n(\epsilon_n) = \int_{\mathbb{R}^N} (V(y_n + \epsilon_n z) - V(\epsilon_n z))v_n^2 \), we must prove that for large \( n \),
\[ h_n(\epsilon_n) > 0. \] (3.7)

Assume without loss of generality that \( \Lambda \) was chosen so that there exists \( \rho > 0 \) with
\[ \inf_{N_\rho(\Lambda)} V = V_0, \] (3.8)
where \( N_\rho(\Lambda) = \{ x \in \mathbb{R}^N \mid \exists y \in \Lambda \text{ with } |y - x| < \rho \} \). We will prove the following facts about \( h_n \):

**Lemma 3.3** For some \( \beta > 0 \), for large \( n 

(i) \( h_n \in C^2(\mathbb{R}^+, \mathbb{R}) \)
(ii) \( h_n(0) \geq 0 \)
(iii) \( |h_n'(0)|^2 \leq o(1)h_n(0) \)
(iv) \( h_n''(0) > \beta \)
(v) \( h_n'' \) is locally Lipschitz on \( \mathbb{R}^+ \), uniformly in \( n \).

Here \( o(1) \to 0 \) as \( n \to \infty \). Before proving Lemma 3.3, let us prove how it gives (3.7). By (iv)-(v), there exists \( d > 0 \) such that for large \( n \) and \( 0 \leq t \leq d \), \( h_n''(t) > \beta/2 \). For \( t \in [0, d] \), a Taylor’s series expansion shows that for large \( n 
\[ h_n(t) \geq h_n(0) + h_n'(0)t + \frac{\beta}{4} t^2 \equiv l_n(t). \] (3.9)
If \( h_n(0) = 0 \), then by Lemma 3.3(iii), \( h_n'(0) = 0 \), so (3.9) implies that \( h_n(t) > 0 \) for all \( t \in (0, d) \), giving (3.7) if \( n \) is large enough that \( \epsilon_n < d \). If \( h_n(0) > 0 \), then by elementary calculus, \( l_n \) attains a minimum value at \( t = -2h_n'(0)/\beta \), and the minimum value is
\[ \min_{\mathbb{R}} l_n = l_n(-2h_n'(0)/\beta) = h_n(0) - h_n'(0)^2/\beta \geq (1 - o(1))h_n(0), \]
where \( o(1) \to 0 \) as \( n \to \infty \). For large \( n \), if \( h_n(0) > 0 \) then \( l_n(t) > 0 \) for all \( t \in \mathbb{R} \), so \( h_n(t) > 0 \) for all \( t \in (0, d) \) for large \( n \), implying (3.7) if \( n \) is large enough so that \( \epsilon_n < d \).
Proof of Lemma 3.3 Statement (ii) is trivial, since \( h_n(0) = (V(y_n) - V_0) \int_{\mathbb{R}^N} v_n^2 \), and since \( z_n \in \mathbb{K} \) and \( y_n - z_n \to 0 \), (3.8) implies \( V(y_n) \geq V_0 \) for large \( n \). (i) and (v) follow from Leibniz’s Rule, \((V_1) - (V_2)\), and the fact that the \( v_n \)’s decay exponentially, uniformly in \( n \). For \( j = 1, 2 \),

\[
h_n^{(j)}(t) = \int_{\mathbb{R}^N} \sum_{|\alpha|=j} (D^\alpha V(y_n + tz) - D^\alpha V(tz))z^\alpha v_n(z)^2 \, dz.
\]

Since \((V2)\) holds, \( v_n \) decays exponentially, uniformly in \( n \), \( y_n \to \bar{y} \), and \( v_0 \) is radially symmetric, we have

\[
h_n^{(j)}(0) = \int_{\mathbb{R}^N} \sum_{|\alpha|=2} (D^\alpha V(y_n) - D^\alpha V(0))z^\alpha v_n(z)^2 \, dz
\]

\[
\to \int_{\mathbb{R}^N} \sum_{|\alpha|=2} (D^\alpha V(\bar{y}) - D^\alpha V(0))z^\alpha v_0(z)^2 \, dz
\]

\[
= \int_{\mathbb{R}^N} \sum_{i=1}^N (D^{ii} V(\bar{y}) - D^{ii} V(0))z_2^2 v_0(z)^2 \, dz
\]

\[
= \int_{\mathbb{R}^N} \sum_{i=1}^N (D^{ii} V(\bar{y}) - D^{ii} V(0)) \frac{1}{N} |z|^2 v_0(z)^2 \, dz
\]

\[
= \frac{1}{N} (\Delta V(\bar{y}) - \Delta V(0)) \int_{\mathbb{R}^N} |z|^2 v_0(z)^2 \, dz > 0
\]

by assumption (3.5). Since Lemma 3.3(v) holds, we have Lemma 3.3(iv).

To prove Lemma 3.3(iii), we will need the following calculus lemma:

Lemma 3.4 Let \( U \subset \mathbb{R}^N \) and \( r > 0 \). Let \( V \in C^2(N_r(U), \mathbb{R}) \) with \( \inf_{N_r(U)} V = V_0 > -\infty \), \( |\nabla V| \) bounded on \( N_r(U) \), and \( D^2 V \) Lipschitz on \( N_r(U) \). Then there exists \( C > 0 \) with

\[
|\nabla V(z)|^2 \leq C(V(z) - V_0)
\]

for all \( z \in U \).

Proof: let \( B > 0 \) with \( |D^2 V(z)\xi \cdot \xi| \leq B \) for all \( \xi \in \mathbb{R}^N \) with \( |\xi| = 1 \). Also let \( B \) be big enough so

\[
B > |\nabla V(z)|/r
\]

for all \( z \in U \). Pick \( z \in U \). If \( |\nabla V(z)| = 0 \), then (3.10) is obvious. Otherwise, let \( d = |\nabla V(z)|/B < r \). Define \( \varphi(t) = V(z - t\nabla V(z)/|\nabla V(z)|) \) for \( t \in [0, d] \). \( \varphi \) is \( C^2 \), \( \varphi(0) = V(z) \), and \( \varphi'(0) = -|\nabla V(z)| \). By choice of \( B \) and the fact that \( B_d(z) \subset N_r(U) \), \( |\varphi''(t)| \leq B \) for all \( t \in [0, d] \). Taylor’s theorem gives

\[
\varphi(d) - \varphi(0) = \varphi'(0)d + \varphi''(\xi) \frac{d^2}{2} \leq -|\nabla V(z)|d + Bd^2/2 = -\frac{|\nabla V(z)|^2}{2B}.
\]
Also $\varphi(d) \geq V_0$ because $B_d(z) \subset N_r(U)$. Therefore,

$$\frac{\lvert \nabla V(z) \rvert^2}{2B} \leq \varphi(0) - \varphi(d) \leq V(z) - V_0.$$ 

Lemma 3.4 is proven.

To prove Lemma 3.3(iii), first note that, by the radial symmetry of $v_0$, the uniform exponential decay of $v_n$, and the uniform convergence $v_n \rightarrow v_0$,

$$|h_n'(0)| = \lvert (\nabla V(y_n) - \nabla V(0)) \cdot \int_{\mathbb{R}^N} zv_n^2 \, dz \rvert$$

$$= \lvert \nabla V(y_n) \cdot \int_{\mathbb{R}^N} zv_n^2 \, dz \rvert$$

$$= \lvert \nabla V(y_n) \cdot \int_{\mathbb{R}^N} zv_0^2 \, dz + \nabla V(y_n) \cdot \int_{\mathbb{R}^N} z(v_n^2 - v_0^2) \, dz \rvert$$

$$\leq \lvert \nabla V(y_n) \rvert \int_{\mathbb{R}^N} z(v_n^2 - v_0^2) \, dz$$

$$\leq o(1) \lvert \nabla V(y_n) \rvert,$$

so Lemma 3.4 implies

$$|h_n'(0)|^2 \leq o(1) \lvert \nabla V(y_n) \rvert^2 \leq o(1)(V(y_n) - V_0)$$

$$\leq o(1)(V(y_n) - V_0) \int_{\mathbb{R}^N} v_n^2$$

$$= o(1)h_n(0),$$

since $\int_{\mathbb{R}^N} v_n^2$ is bounded away from zero. Lemma 3.3(iii) is proven. Thence follow (3.7), (3.3), Proposition 2.2, and Theorem 1.1.

**Remarks:** Besides the results cited in the introduction, many important results for equations of type (1.1) have been found recently. For instance, the work in [3]-[5] suggests that Theorem 1.1 could be strengthened by working on a smaller domain than $\mathbb{R}^N$, or by weakening the hypotheses on $V$. It is natural to try to extend Theorem 1.1 to cases where $V$ is not $C^2$, or to the case where the second derivatives of $V$ do not provide a condition like (V4), but higher-order derivatives do.

**References**


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