An Elliptic Equation With Spike Solutions Concentrating at Local Minima of the Laplacian of the Potential

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An elliptic equation with spike solutions concentrating at local minima of the Laplacian of the potential *

Gregory S. Spradlin

Abstract

We consider the equation $-\epsilon^2 \Delta u + V(z)u = f(u)$ which arises in the study of nonlinear Schrödinger equations. We seek solutions that are positive on $\mathbb{R}^N$ and that vanish at infinity. Under the assumption that $f$ satisfies super-linear and sub-critical growth conditions, we show that for small $\epsilon$ there exist solutions that concentrate near local minima of $V$. The local minima may occur in unbounded components, as long as the Laplacian of $V$ achieves a strict local minimum along such a component. Our proofs employ variational mountain-pass and concentration compactness arguments. A penalization technique developed by Felmer and del Pino is used to handle the lack of compactness and the absence of the Palais-Smale condition in the variational framework.

1 Introduction

This paper concerns the equation

$$-\epsilon^2 \Delta u + V(z)u = f(u)$$

on $\mathbb{R}^N$ with $N \geq 1$, where $f(u)$ is a “superlinear” type function such as $f(u) = u^p$, $p > 1$. Such an equation arises when searching for standing wave solutions of the nonlinear Schrödinger equation (see [3]). For small positive $\epsilon$, we seek “ground states,” that is, positive solutions $u$ with $u(z) \to 0$ as $|z| \to \infty$. Floer and Weinstein ([6]) examined the case $N = 1$, $f(u) = u^3$ and found that for small $\epsilon$, a ground state $u_\epsilon$ exists which concentrates near a non-degenerate critical point of $V$. Similar results for $N > 1$ were obtained by Oh in [10]-[12]. In [3], del Pino and Felmer found that if $V$ has a strict local minimum, then for small $\epsilon$, (1.1) has a ground state concentrating near that minimum. A strict local minimum occurs when there exists a bounded, open set $\Lambda \subset \mathbb{R}^N$.

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with $\inf_{A} V < \inf_{\partial A} V$. They extended their results in [4] to the more general case where $V$ has a “topologically stable” critical point, that is, a critical point obtained via a topological linking argument (see [4] for a precise formulation). Such a critical point persists under small perturbations of $V$. Examples are a strict local extremum and a saddle point. This very strong result is notable because the critical points of $V$ in question need not be non-degenerate or even isolated. Similar results have been obtained by Li [8], and earlier work of Rabinowitz [13] is also interesting. The recent results of [1] and [9] also permit $V$ to have degenerate critical points.

A common feature of all the papers above is that $V$ must have a non-degenerate, or at least topologically stable, set of critical points. Therefore it is natural to try to remove this requirement. That we must assume some conditions on $V$ is shown by Wang’s counterexample [15] - if $V$ is nondecreasing and nonconstant in one variable (e.g. $V(x_1, x_2, x_3) = 2 + \tan^{-1}(x_1)$), then no ground states exist. In [14] the author showed that ground states to (1.1) exist under the assumption that $V$ is almost periodic, together with another mild assumption. Those assumptions did not guarantee that $V$ had a topologically stable critical point.

Aside from periodicity or recurrence properties of $V$, another approach is to impose conditions on the derivatives of $V$. That is the approach taken here. We will assume that $V$ has a (perhaps unbounded) component of local minima, along which $\nabla V$ achieves a strict local minimum. More specifically, assume $f$ satisfies the following:

(F1) $f \in C^1(\mathbb{R}^+, \mathbb{R})$

(F2) $f'(0) = 0 = f(0)$.

(F3) $\lim_{q \to \infty} f(q)/q^s = 0$ for some $s > 1$, with $s < (N + 2)/(N - 2)$ if $N \geq 3$.

(F4) For some $\theta > 2$, $0 < \theta F(q) \leq f(q)q$ for all $q > 0$, where $F(\xi) \equiv \int_0^\xi f(t) \, dt$.

(F5) The function $q \mapsto f(q)/q$ is increasing on $(0, \infty)$.

Assumptions (F1)-(F5) are the same as in [3] and are satisfied by $f(q) = q^s$, for example, if $1 < s < (N + 2)/(N - 2)$. Assume that $V$ satisfies the following:

(V1) $V \in C^2(\mathbb{R}^n, \mathbb{R})$

(V2) $D^\alpha V$ is bounded and Lipschitz continuous for $|\alpha| = 2$.

(V3) $0 < V_- \equiv \inf_{\mathbb{R}^N} V < \sup_{\mathbb{R}^N} V \equiv V^+ < \infty$

(V4) There exists a bounded, nonempty open set $\Lambda \subset \mathbb{R}^N$ and a point $z_0 \in \Lambda$ with $V(z_0) = \inf_{\Lambda} V \equiv V_0$, and

$$\Delta_0 \equiv \inf\{\nabla V(z) \mid z \in \Lambda, \ V(z) = V_0\} < \inf\{\nabla V(z) \mid z \in \partial \Lambda, \ V(z) = V_0\}$$
Note: A special case of (V4) occurs when $\Lambda$ is bounded and $V(z_0) < \inf_{\partial \Lambda} V$; this case is treated, under weaker hypotheses, in [3]. A specific example of (V4) is if $N = 2$ and $V$ satisfies (V1)-(V4), with $V(z_1, z_2) = 1 + (z_1^2 - z_2)^2$ for $z_1^2 + z_2^2 \leq 1$. Then $\Delta V(z_1, z_2) = 8z_1^2 + 2$ for $z_1^2 + z_2^2 \leq 1$, so we may take $\Lambda = B_1(0, 0) \subset \mathbb{R}^2$ and $z_0 = (0, 0)$. Then $V$ has a component of local minima that includes the parabolic arc $\{z_2 = z_1^2\} \cap B_1(0, 0)$, along which $\Delta V$ has a minimum of 2 at $(0, 0)$, with $\Delta V > 2$ at the two endpoints of the arc.

We prove the following:

**Theorem 1.1** Let $V$ and $f$ satisfy (V1)-(V4) and (F1)-(F5). Then there exists $\epsilon_0 > 0$ such that if $\epsilon \leq \epsilon_0$, then (1.0) has a positive solution $u_\epsilon$ with $u_\epsilon(z) \to 0$ as $|z| \to \infty$. $u_\epsilon$ has exactly one local maximum (hence, global maximum) point $z_\epsilon \in \Lambda$, where $\Lambda$ is as in (V4). There exist $\alpha, \beta > 0$ with $u_\epsilon(z) \leq \alpha \exp(-\frac{\beta}{2}|z - z_\epsilon|)$ for $\epsilon \leq \epsilon_0$. Furthermore, with $V_0$ and $\Delta_0$ as in (V4), $V(z_\epsilon) \to V_0$ and $\Delta V(z_\epsilon) \to \Delta_0$ as $\epsilon \to 0$.

For small $\epsilon$, $u_\epsilon$ resembles a “spike,” which is sharper for smaller $\epsilon$. The spike concentrates near a local minimum of $V$ where $\Delta V$ has a strict local minimum. The proof of Theorem 1.1 employs the techniques of [3], with some refinements necessary because $V$ does not necessarily achieve a strict local minimum. Section 2 introduces the penalization scheme developed by Felmer and del Pino, and continues with the beginning of the proof of Theorem 1.1. These beginning arguments are taken practically verbatim from [3], but are included, since the machinery of the penalization technique is used in the remainder of the proof. The reader is invited to consult [3] for more complete proofs. Section 3 contains the completion of the proof, which is original. This part contains delicate computations involving $\Delta V$.

## 2 The penalization scheme

Extend $f$ to the negative reals by defining $f(q) = 0$ for $q < 0$. Let $F$ be the primitive of $f$, that is, $F(q) = \int_0^q f(t) \, dt$. Define the functional $I_\epsilon$ on $W^{1,2}(\mathbb{R}^N)$ by

$$I_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2}(e^2|\nabla u|^2 + V(z)u^2) - F(u) \, dz.$$ 

$I_\epsilon$ is a $C^1$ functional, and there is a one-to-one correspondence between positive critical points of $I_\epsilon$ and ground states of (1.1). It is well known that $I_\epsilon$ and similar functionals in related problems fail the Palais-Smale condition. That is, a “Palais-Smale sequence,” defined as a sequence $(u_m)$ with $I_\epsilon(u_m)$ convergent and $I_\epsilon'(u_m) \to 0$ as $m \to \infty$, need not have a convergent subsequence. To get around this difficulty, we formulate a “penalized” problem, with a corresponding “penalized” functional satisfying the Palais-Smale condition, by altering $f$ outside of $\Lambda$. 

Let $\theta$ be as in (F4). Choose $k$ so $k > \theta/(\theta - 2)$. Let $V_-$ be as in (V3) and $a > 0$ be the value at which $f(a)/a = V_-/k$. Define $\tilde{f}$ by

$$
\tilde{f}(s) = \begin{cases} 
   f(s) & s \leq a; \\
   sV_-/k & s > a,
\end{cases}
$$

(2.1)

g(\cdot, s) = \chi_A f(s) + (1 - \chi_A) \tilde{f}(s)$, and $G(z, \xi) = \int_0^\xi g(z, \tau) \, d\tau$. Although not continuous, $g$ is a Carathéodory function. For $\epsilon > 0$, define the penalized functional $J_\epsilon$ on $W^{1,2}(\mathbb{R}^N)$ by

$$
J_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2} (\epsilon^2 |\nabla u|^2 + V(z) u^2) - G(z, u) \, dz.
$$

(2.2)

A positive critical point of $J_\epsilon$ is a weak solution of the “penalized equation”

$$
-\epsilon^2 \Delta u + V(z) u = g(z, u),
$$

(2.3)

that is, a $C^1$ function satisfying (2.3) wherever $g$ is continuous. It is proven in [3] that $J_\epsilon$ satisfies all the hypotheses of the Mountain Pass Theorem of Ambrosetti and Rabinowitz ([2]), including the Palais-Smale condition. Therefore $J_\epsilon$ has a critical point $u_\epsilon$, with the mountain pass critical level $c(\epsilon) = J_\epsilon(u_\epsilon)$. $c(\epsilon)$ is defined by the following minimax: let the set of paths $\Gamma_\epsilon = \{ \gamma \in \mathcal{C}([0, 1], W^{1,2}(\mathbb{R}^N)) \mid \gamma(0) = 0, \ J_\epsilon(\gamma(1)) < 0 \}$, and

$$
c(\epsilon) = \inf_{\gamma \in \Gamma_\epsilon} \max_{\theta \in [0,1]} J_\epsilon(\gamma(\theta)).
$$

As shown in ([3]), because of (F4), $c(\epsilon)$ can be characterized more simply as

$$
c(\epsilon) = \inf_{u \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}} \sup_{\tau > 0} J_\epsilon(\tau u).
$$

The functions $g(z, q)$ and $f(q)$ agree whenever $z \in \Lambda$ or $q < a$. Therefore if $u$ is a weak solution of (2.3) with $u < a$ on $\Lambda \cup \mathbb{R}^N \setminus \Lambda$, then $u$ solves (1.1). Our plan is to find a positive critical point $u_\epsilon$ of $J_\epsilon$, which is a weak solution of (2.3), then show that $u_\epsilon(z) < a$ for all $z \in \Lambda \cup \mathbb{R}^N \setminus \Lambda$.

For $\epsilon > 0$, let $u_\epsilon$ be a critical point of $J_\epsilon$ with $J_\epsilon(u_\epsilon) = c(\epsilon)$. Maximum principle arguments show that $u_\epsilon$ must be positive. Define the “limiting functional” $I_0$ by

$$
I_0(u) = \int_{\mathbb{R}^N} \frac{1}{2} (|\nabla u|^2 + V_0 u^2) - F(u)
$$

(2.4)

and

$$
\mathcal{L} = \inf_{u \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}} \sup_{\tau > 0} I_0(\tau u).
$$

(2.5)

The equation corresponding to (2.4) is

$$
-\Delta u + V_0 u = f(u)
$$

(2.6)

We will prove Theorem 1.1 by proving the following proposition:
**Proposition 2.1** Let $\epsilon > 0$. If $u_\epsilon$ is a positive solution of (2.3) satisfying $J_\epsilon(u_\epsilon) = c(\epsilon)$, then

(i) $\lim_{\epsilon \to 0} \max_{z \in \partial \Lambda} u_\epsilon = 0$.

(ii) For all $\epsilon$ sufficiently small, $u_\epsilon$ has only one local maximum point in $\Lambda$ (call it $z_\epsilon$), with $\lim_{\epsilon \to 0} V(z_\epsilon) = V_0$.

(iii) $\lim_{\epsilon \to 0} \Delta V(z_\epsilon) = \Delta_0$.

**Proof of Theorem 1.1:** Assuming Proposition 2.1, there exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, $u_\epsilon < a$ on $\partial \Lambda$. In [3] it is shown that if we multiply (2.3) by $(u_\epsilon - a)_+$ and integrate by parts, it follows that $u_\epsilon < a$ on $\Lambda^c$, so $u_\epsilon$ solves (1.1). By the definition of $a$ in (2.1), and the maximum principle, $u_\epsilon$ has no local maxima outside of $\Lambda$, so $u_\epsilon$ has exactly one local maximum point $z_\epsilon$, which occurs in $\Lambda$.

Define $v_\epsilon$ by translating $u_\epsilon$ from $z_\epsilon$ to zero and dilating it by $\epsilon$, that is,

$$v_\epsilon(z) = u_\epsilon(z_\epsilon + \epsilon z).$$

Then $v_\epsilon$ is a weak ($C^1$) solution of the “translated and dilated” equation

$$-\Delta v_\epsilon + V(z_\epsilon + \epsilon z)v_\epsilon = g(z_\epsilon + \epsilon z, v_\epsilon).$$

Let $\epsilon_j \to 0$. Along a subsequence (called $(z_{\epsilon_j})$), $z_{\epsilon_j} \to \tilde{z} \in \overline{\Lambda}$, with $V(\tilde{z}) = V_0$ and $\Delta V(\tilde{z}) = \Delta_0$.

Along a subsequence, $v_{\epsilon_j}$ converges locally uniformly to a function $v^0$. Pick $R > 0$ so $v^0 < a$ on $\mathbb{R}^N \setminus B_R(0)$. For large enough $\epsilon$, $v_\epsilon < a$ on $\partial B_R(0)$. By the maximum principle arguments of [3], for small $\epsilon$, $v_\epsilon$ decays exponentially, uniformly in $\epsilon$.

The proof of Proposition 2.1 will follow if we can prove the following statement.

**Proposition 2.2** If $\epsilon_n \to 0$ and $(z_n) \subset \overline{\Lambda}$ with $u_{\epsilon_n}(z_n) \geq b > 0$, then

(i) $\lim_{n \to \infty} V(z_n) = V_0$.

(ii) $\lim_{n \to \infty} \Delta V(z_n) = \Delta_0$.

It is proven in [3] that $u_\epsilon$ has exactly one local maximum point $z_\epsilon$ for small $\epsilon$. Since $u_\epsilon$ solves (2.3), the maximum principle implies that $u_\epsilon(z_\epsilon)$ is bounded away from zero. Thus Proposition 2.2 and (V4) give Proposition 2.1(ii)-(iii).

To prove Proposition 2.2, let $b$ and $(z_n)$ be as above. First we repeat the argument of [3] to show that $V(z_n) \to V_0$: suppose this does not happen. Then, along a subsequence, $z_n \to \tilde{z} \in \overline{\Lambda}$ with $V(\tilde{z}) > V_0$. Define $v_n$ by translating $u_{\epsilon_n}$ from $z_n$ to 0 and dilating by $\epsilon_n$; that is,

$$v_n(z) = u_{\epsilon_n}(z_n + \epsilon_n z).$$

(2.7)
\( v_n \) solves the “translated and dilated” penalized equation
\[
-\Delta v_n + V(z_n + \epsilon_n z)v_n = g(z_n + \epsilon_n z, v_n)
\] (2.8)
on \( \mathbb{R}^N \), with \( v_n(z) \to 0 \) and \( \nabla v_n(z) \to 0 \) as \( |z| \to \infty \). As shown in [3], \( v_n \) is bounded in \( W^{1,2}(\mathbb{R}^N) \), so by elliptic estimates, \( (v_n) \) converges locally along a subsequence (also denoted \( (v_n) \)) to \( v^0 \in W^{1,2}(\mathbb{R}^N) \). Define \( \chi_n \) by \( \chi_n(z) = \chi_A(z_n + \epsilon_n z) \), where \( \chi_A \) is the characteristic function of \( A \). \( \chi_n \) converges weakly in \( L^p \) over compact sets to a function \( \chi \), for any \( p > 1 \), with \( 0 < \chi \leq 1 \). Define
\[
\bar{g}(z, s) = \chi(z)f(s) + (1 - \chi(z))\tilde{f}(s)
\]
Then \( v^0 \) satisfies
\[
-\Delta v + V(\bar{z})v = \bar{g}(z, v)
\] (2.9)
on \( \mathbb{R}^N \). Define \( \tilde{G}(z, s) = \int_{\bar{z}}^s \bar{g}(z, t) \, dt \). Associated with (2.9) we have the limiting functional \( \tilde{J}(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V(\bar{z})u^2) - G(z_n + \epsilon_n z, u) \, dz \). \( v^0 \) is a positive critical point of \( \tilde{J} \).

Define \( J_n \) to be the “translated and dilated” penalized functional corresponding to (2.8), that is,
\[
J_n(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V(z_n + \epsilon_n z)u^2) - G(z_n + \epsilon_n z, u) \, dz.
\]
Clearly \( J_n(v_n) = \epsilon_n^{-N} J_n(u_{\epsilon_n}) \). In [3] it is proven that
\[
\lim_{n \to \infty} \inf J_n(v_n) \geq \tilde{J}(v^0).
\] (2.10)
Also, by letting \( w \) be a ground state for (2.6) with \( I_0(w) = c \) (the mountain pass value for \( I_0 \), defined in (2.5) and using \( w \) as a test function for \( J_n \), it is proven that \( c \geq \liminf_{n \to \infty} J_n(v_n) \). Thus \( \tilde{J}(v^0) \leq c \). Therefore, as shown in [3], \( V(\bar{z}) \leq V_0 \). This contradicts our assumption. Thus \( V(z_n) \to V_0 \). All the above is the same as was proven in [3]. Next, we must show that \( \Delta V(z_n) \to \Delta_0 \). That is the focus of the next section.

3 The effect of the Laplacian

Proving \( \Delta V(z_n) \to \Delta_0 \) is a subtle and delicate problem. Making \( \epsilon_n \) approach 0 is equivalent to dilating \( V \), which has the effect of making local minima of \( V \) behave more like global minima. This assists in finding solutions to (1.1). However, making \( \epsilon_n \) small reduces the effect of differences in \( \Delta V \). For this reason, Theorem 1.1 is not only difficult to prove, but is not intuitively compelling, either.

It is known ([7]) that a “least energy solution” of (2.6), that is, a solution \( w \) with \( I_0(w) = c \), must be radially symmetric. We will need to exploit this fact. In order to do this, we will need to work with the maximum points of \( u_{\epsilon_n} \) instead of merely the \( (z_n) \) as given in Proposition 2.2. We need the following concentration-compactness result, which states that the sequence \( (u_{\epsilon_n}) \) of “mountain-pass type solutions” of (2.3) does not “split”:
Lemma 3.1 If \((z_n) \subset \mathbb{A}, (y_n) \subset \mathbb{R}^N\), and \(b > 0\) with \(u_{\epsilon_n}(z_n) > b\) and \(u_{\epsilon_n}(y_n) > b\) for all \(n\), then \(((z_n - y_n)/\epsilon_n)\) is bounded.

**Proof:** define \(v_n(z) = u_{\epsilon_n}(z + \epsilon_n z)\) as in (2.7). Suppose the lemma is false. Then, along a subsequence, \(|y_n - z_n|/\epsilon_n \to \infty\). Let \(x_n = (y_n - z_n)/\epsilon_n\). \(||v_n||\) is bounded in \(W^{1,2}(\mathbb{R}^N)\) and \(|x_n| \to \infty\), so we may pick a sequence \((R_n) \subset \mathbb{N}\) with \(R_n \to \infty\), \(|x_n| - R_n \to \infty\), and \(||v_n||_{W^{1,2}(B_{R_n+1}(0) \setminus B_{R_n-1}(0))} \to 0\) as \(n \to \infty\). Define cutoff functions \(\varphi_{1,2} \in C^\infty(\mathbb{R}^N, [0,1])\) satisfying \(\varphi_1 \equiv 1\) on \(B_{R_n-1}(0)\), \(\varphi_1 \equiv 0\) on \(B_{R_n}(0)\) \(C\), \(\varphi_2 \equiv 1\) on \(B_{R_n+1}(0)\) \(C\), \(\varphi_2 \equiv 0\) on \(B_{R_n}(0)\), and \(||\nabla \varphi_1||_{L^\infty(\mathbb{R}^N)} < 2\), \(||\nabla \varphi_2||_{L^\infty(\mathbb{R}^N)} < 2\). Set \(v_1^n = \varphi_{1}v_n\) and \(v_2^n = \varphi_{2}v_n\), and \(v = v_1^n + v_2^n = (\varphi_{1} + \varphi_{2})v_n\).

Choose \(T_n > 0\) so \(J_n(T_n v) = 0\). We claim that \(T_n\) is well-defined, and bounded in \(n\). Note that the existence of \(T_n\) must be checked for the penalized functional \(J_n\), because of the replacement of \(F\) with \(G\). By elliptic estimates, there exists an open set \(U \subset \mathbb{R}^N\) such that along a subsequence, \(v_1^n > b/2\) on \(U\) and \(U \subset (\Lambda - z_n)/\epsilon_n = \{z \in \mathbb{R}^N | z_n + \epsilon_n z \in \Lambda\}\). Let \(a\) be as in (2.1). For \(t > 2a/b\) and \(z \in U\), \(v_1^n(z) > tb/2 > a\), so \(G(z_n + \epsilon_n z, tv) = F(tv) > F(bt/2)\).

Therefore, for \(t > 2a/b\),

\[
J_n(tv) = t^2 \int_{\mathbb{R}^N} \left( \frac{1}{2}(1 + V^+)||v||^2_{W^{1,2}((\mathbb{R}^N)} \right) - \int_{U} F(tv) \leq \frac{t^2}{2} (1 + V^+)||v||^2_{W^{1,2}((\mathbb{R}^N)} - \lambda(U)F(tb/2),
\]

where \(\lambda\) indicates the Lebesgue measure. By (F4), there exists \(C > 0\) such that for \(t > 2a/b\), \(F(tb/2) > Ct^\theta\). Therefore, for \(t > 2a/b\),

\[
J_n(tv) = \frac{t^2}{2} (1 + V^+)||v||^2_{W^{1,2}((\mathbb{R}^N)} - Ct^\theta. \tag{3.1}
\]

Since \((v_n)\) is bounded in \(W^{1,2}(\mathbb{R}^N)\), this gives the existence and boundedness of \((T_n)\).

Since \(J_n(T_n v_n) = J_n(T_n v_n^1) + J_n(T_n v_n^2) = 0\) we may pick \(i_n \in \{1, 2\}\) with \(J_n(T_n v_n^{i_n}) \leq 0\). By (F5) and (2.1), the map \(t \to J_n(tv_n^{i_n})\) increases from zero at \(t = 0\), achieves a positive maximum, then decreases to \(-\infty\). We will see more of this in a moment. Thus there exists a unique \(t_n \in (0, T_n)\) with \(J_n(t_n v_n^{i_n}) = \max_{t \geq 0} J_n(tv_n^{i_n})\). We claim that \(t_n\) and \(T_n - t_n\) are both bounded away from zero for large \(n\): by \((f_i) - (f_\leq)\) and (2.1), \(J_n(w) \geq \frac{1}{8} \min(1, V_\leq)||w||^2_{W^{1,2}(\mathbb{R}^N)} - o(||w||^2_{W^{1,2}(\mathbb{R}^N)})\) uniformly in \(n\), so \(\max_{t \geq 0} J_n(tv_n^{i_n})\) is bounded away from zero uniformly in \(n\). It is easy to show that \(J_n\) is Lipschitz on bounded subsets of \(W^{1,2}(\mathbb{R}^N)\), uniformly in \(n\). Since \((T_n)\) is bounded, this implies that \(t_n\) and \(T_n - t_n\) are both bounded away from zero for large \(n\).

By definition of \(v_n\) as a “mountain-pass type critical point” of \(J_n\), we have

\[
\max_{t > 0} J_n(tv_n^{i_n}) = \max_{t > 0} J_n(tv_n).
\]
Using the facts that \( \| v_n - \tilde{v}_n \|_{W^{1,2}(\mathbb{R}^N)} \to 0 \) as \( n \to \infty \), and \( (T_n) \) is bounded, we have

\[
\liminf_{n \to \infty} J_n(t_n v_n^{i_n}) = \liminf_{n \to \infty} \max_{t > 0} J_n(t v_n^{i_n}) \\
\geq \liminf_{n \to \infty} \max_{t > 0} J_n(t v_n) \\
= \liminf_{n \to \infty} \max_{t > 0} J_n(t \tilde{v}_n) \\
= \liminf_{n \to \infty} J_n(t_n \tilde{v}_n) \\
= \liminf_{n \to \infty} (J_n(t_n v_n^{i_n}) + J_n(t_n v_n^{3-i_n})) \\
\geq \liminf_{n \to \infty} J_n(t_n v_n^{i_n}) + \liminf_{n \to \infty} J_n(t_n v_n^{3-i_n}).
\]

Now \( J_n(T_n v_n^{3-i_n}) = -J_n(T_n v_n^{i_n}) \geq 0 \) and \( t_n < T_n \), so \( J_n(t_n v_n^{3-i_n}) \geq 0 \). By (3.2), \( \liminf_{n \to \infty} J_n(t_n v_n^{3-i_n}) \leq 0 \). Therefore \( J_n(t_n v_n^{3-i_n}) \to 0 \) as \( n \to \infty \).

Since \( J_n(w) \geq \frac{1}{2} \min(1, V_-)||w||_{W^{1,2}(\mathbb{R}^N)}^2 - o(||w||_{W^{1,2}(\mathbb{R}^N)}) \) uniformly in \( n \), there exists \( d \in (0, \liminf_{n \to \infty} t_n) \) such that \( \liminf_{n \to \infty} J_n(d v_n^{3-i_n}) > 0 \). Since \( d < t_n \) and \( J_n(d v_n^{3-i_n}) > J_n(t_n v_n^{3-i_n}) \) for large \( n \), the map \( t \mapsto J_n(t v_n^{3-i_n}) \) achieves a maximum at some \( t'_n \in (0, t_n) \), and that maximum is bounded away from zero.

Summarizing the important facts about the mapping \( t \mapsto J_n(t v_n^{3-i_n}) \), we have shown that there exists \( \rho > 0 \) such that for large \( n \),

(i) \( 0 < t'_n < t_n < T_n \)
(ii) \( (T_n) \) is bounded.
(iii) \( (T_n - t_n) \) is bounded away from zero.
(iv) \( J_n(t'_n v_n^{3-i_n}) > \rho > 0 \)
(v) \( J_n(t_n v_n^{3-i_n}) \to 0 \)
(vi) \( J_n(T_n v_n^{3-i_n}) \geq 0 \)

From (i)-(vi) it is apparent that at some \( t^n > t'_n \), the mapping \( t \mapsto J_n(t v_n^{3-i_n}) \) is at once decreasing and concave upward. But this is impossible: let \( n \in \mathbb{N} \) and \( w \in W^{1,2}(\mathbb{R}^N) \setminus \{ 0 \} \). Define \( \psi(t) = J_n(t w) \) for \( t > 0 \). Then

\[
\psi'(t) = \int_{\mathbb{R}^N} |\nabla w|^2 + V(z_n + \epsilon_n z) w^2 \, dz - \int_{\mathbb{R}^N} g(z_n + \epsilon_n z, tw) w \, dz
\]

\[
= t \left[ \int_{\mathbb{R}^N} |\nabla w|^2 + V(z_n + \epsilon_n z) w^2 \, dz - \int_{\{ w \neq 0 \}} \frac{g(z_n + \epsilon_n z, tw)}{tw} w^2 \, dz \right].
\]

By (F5) and (2.1), \( t \mapsto g(z_n + \epsilon_n z, tw)/(tw) \) is nondecreasing, so if \( \psi'(t) \) ever becomes negative, \( \psi'(t) \) is increasing for all time \( t \) after that, and the graph of \( \psi \) is concave down. Therefore the behavior of \( J_n(t v_n^{3-i_n}) \) as described in (i)-(vi) is impossible, and Lemma 3.1 is proven.  

\( \Diamond \)
As mentioned before, it will be advantageous to work with the maxima of \((u_{\epsilon_n})\). Choose \((y_n) \subset \mathbb{R}^N\) with
\[
 u_{\epsilon_n}(y_n) = \max_{\mathbb{R}^N} u_{\epsilon_n}.
\]
We will prove
\[
 \Delta V(y_n) \to \Delta_0. \tag{3.3}
\]
By Lemma 3.0, \(((y_n - z_n)/\epsilon_n)\) is bounded, so \(y_n - z_n \to 0\). Thus (3.3) gives Proposition 2.2(ii), completing the proof of Theorem 1.1. \(\diamondsuit\)

Along a subsequence, \(y_n \to \tilde{y} \in \mathcal{X}\). By Proposition 2.2(i), \(V(\tilde{y}) = V_0\). Since it is not apparent that \(\tilde{y} \in \Lambda\), we must proceed carefully. We will redefine the \(v_n\)’s like in (2.7), by translating \(u_{\epsilon_n}\) to 0 and dilating it. That is,
\[
 v_n(z) = u_{\epsilon_n}(y_n + \epsilon_n z). \tag{3.4}
\]
Then \(v_n\) is a positive weak solution, vanishing at infinity, of the “penalized, dilated, and translated” PDE
\[
 -\Delta v + V(y_n + \epsilon_n z)v = g(y_n + \epsilon_n z, v).
\]
Like before, \((v_n)\) converges locally uniformly to a function \(v_0\). We claim that \(v_0\) is actually a ground state maximizing at 0 of the autonomous limiting equation (2.6). Proof: As before, define \(\chi_n\) by \(\chi_n(z) = \chi(y_n + \epsilon z)\). As before, along a subsequence, \(\chi_n\) converges weakly in \(L^p\), for any \(p > 1\), on compact subsets of \(\mathbb{R}^N\) to a function \(\chi\) with \(0 \leq \chi \leq 1\). Define \(\tilde{g}\) by
\[
 \tilde{g}(z, s) = \chi(z)f(s) + (1 - \chi(z))\tilde{f}(s).
\]
By the argument of Proposition 2.2, taken from [3], \((v_n)\) converges locally along a subsequence to \(v_0\), a ground state of \(-\Delta v + V_0v = \tilde{g}(z, v)\). The functional corresponding to this equation is \(\tilde{J}(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V_0 u^2) - G(z, u) \, dz\), where 
\[
 G(z, s) = \int_0^s \tilde{g}(z, t) \, dt.
\]
As before, in (2.10), \(\underline{c} \geq \liminf_{n \to \infty} J_n(v_n) \geq \tilde{J}(v_0)\), where \(\underline{c}\) is from (2.5). \(\tilde{J} \geq I_0\), where \(I_0\) is the “autonomous” limiting functional from (2.4), so
\[
 \underline{c} \leq \max_{t > 0} I_0(tv_0) \leq \max_{t > 0} \tilde{J}(tv_0) \leq \underline{c},
\]
and \(v_0\) is actually a ground state of (2.6). \(\diamondsuit\)

Not only does \((v_n)\) converge locally to \(v_0\), but it satisfies the following lemma.

**Lemma 3.2** With \((v_n)\) as in (3.4), for any subsequence of \((v_n)\) there is a radially symmetric ground state \(v_0\) of (2.6) such that \(v_n \to v_0\) uniformly along a subsequence and the \(v_n\)’s decay exponentially, uniformly in \(n\).
Proof: If one establishes uniform convergence, the uniform exponential decay follows readily, using a standard maximum principle argument found in [3]. Suppose the convergence is not uniform. Then there exist a subsequence of \((v_n)\) (denoted \((v_n)\)) and a sequence \((x_n) \in \mathbb{R}^N\) with \(|x_n| \to \infty\) and \(\lim_{n \to \infty} v_n(x_n) > 0\). Let \(d > 0\) with \(d < v_0(0)\) and \(d < \lim_{n \to \infty} v_n(x_n)\). For large \(n\), \(d < v_n(0) = u_{\epsilon_n}(z_n)\) and \(d < v_n(x_n) = u_{\epsilon_n}(z_n + \epsilon_n x_n)\). Letting \(w_n = z_n + \epsilon_n x_n\), we obtain \((w_n - z_n)/\epsilon_n = (x_n)\), which is unbounded, violating Lemma 3.1.

To show \(\Delta V(y_n) \to \Delta_0\), we again argue indirectly. Suppose otherwise. Then, along a subsequence, \(y_n \to \bar{y} \in \overline{\Lambda}\) with
\[
\Delta V(\bar{y}) > \Delta_0. \tag{3.5}
\]
For \(x \in \mathbb{R}^N\), define the translation operator \(\tau_x\) by \(\tau_x u(z) = u(z - x)\), that is, \(\tau_x u\) is \(u\) translated by \(x\). Assume for convenience, and without loss of generality, that
\[
0 \in \Lambda, \; V(0) = V_0, \; \text{and} \; \Delta V(0) = \Delta_0.
\]
We will prove that for large \(n\),
\[
\sup_{t > 0} J_{\epsilon_n}(t\tau_{-y_n} u_{\epsilon_n}) < J_{\epsilon_n}(u_{\epsilon_n}) \tag{3.6}
\]
recalling the definition of \(J_{\epsilon_n}\) in (2.2), and how \(v_n\) is defined from \(u_{\epsilon_n}\) in (3.4). That is, translating \(t u_{\epsilon_n}\), back to the origin reduces the value of \(J_{\epsilon_n}(t v_n)\) because \(V\) has lesser concavity at the origin. This occurs even though shrinking \(\epsilon\) reduces the difference in concavity. (3.6) contradicts the definition of \(u_{\epsilon_n}\).

Pick \(T > 1\) large enough so that for large \(n\), \(J_n(T v_n) = \epsilon_n^{-N} J_{\epsilon_n}(T u_{\epsilon_n}) < 0\). This is possible by the argument of (3.1). Now (3.6) is equivalent to
\[
\sup_{0 \leq t \leq T} J_{\epsilon_n}(t\tau_{-y_n} u_{\epsilon_n}) < \sup_{0 \leq t \leq T} J_{\epsilon_n}(t u_{\epsilon_n}).
\]

To prove the above, it will suffice to prove the stronger fact that for large \(n\), for all \(t \in (0, T)\),
\[
J_{\epsilon_n}(t u_{\epsilon_n}) > J_{\epsilon_n}(t \tau_{-y_n} u_{\epsilon_n}).
\]
Now, along a subsequence, \(v_n \to v_0\) uniformly, so by the definition of \(v_n\) as a dilation of \(\tau_{-y_n} u_{\epsilon_n}\) (3.4), \(u_{\epsilon_n} \to 0\) uniformly on \(\mathbb{R}^N \setminus \Lambda\) as \(n \to \infty\). Thus for large \(n\) and \(0 \leq t \leq T\), the definition of \(G\) gives \(G(z, t \tau_{-y_n} u_{\epsilon_n}(z)) = F(t \tau_{-y_n} u_{\epsilon_n}(z))\) for all \(z \in \mathbb{R}^N\), so
\[
J_{\epsilon_n}(t u_{\epsilon_n}) - J_{\epsilon_n}(t \tau_{-y_n} u_{\epsilon_n})
\]
\[
= \int_{\mathbb{R}^N} \frac{1}{2} t^2 (|\nabla u_{\epsilon_n}(z)|^2 + V(z) u_{\epsilon_n}(z)^2) - G(z, t u_{\epsilon_n}(z)) \; dz
\]
\[
- \left[ \int_{\mathbb{R}^N} \frac{1}{2} t^2 (|\nabla \tau_{-y_n} u_{\epsilon_n}(z)|^2 + V(z) \tau_{-y_n} u_{\epsilon_n}(z)^2) - F(t \tau_{-y_n} u_{\epsilon_n}(z)) \; dz \right]
\]
\[
\geq \frac{1}{2} t^2 \int_{\mathbb{R}^N} V(z)(u_{\epsilon_n}(z)^2 - u_{\epsilon_n}(z + y_n)^2) \; dz
\]
If $0 with facts about $t$ for all $n$, then by elementary calculus, $l_n(t) = 1 + \int_{\mathbb{R}^N} F(tu_n(z) + y_n) - F(tu_n(z))\,dz = \frac{1}{2} t^2 \int_{\mathbb{R}^N} (V(z + y_n) - V(z))u_{\epsilon_n}(z + y_n)^2\,dz = \frac{1}{2} t^2 \epsilon_n^2 \int_{\mathbb{R}^N} (V(y_n + \epsilon_n z) - V(\epsilon_n z))u_{\epsilon_n}(\epsilon_n z + y_n)^2\,dz = \frac{1}{2} t^2 \epsilon_n^2 \int_{\mathbb{R}^N} (V(y_n + \epsilon_n z) - V(\epsilon_n z))u_n(z)^2\,dz.$

For $n = 1, 2, \ldots$, define $h_n: \mathbb{R} \to \mathbb{R}$ by

$$h_n(t) = \int_{\mathbb{R}^N} (V(y_n + tz) - V(tz))u_n^2\,dz.$$ 

Since $h_n(\epsilon_n) = \int_{\mathbb{R}^N} (V(y_n + \epsilon_n z) - V(\epsilon_n z))u_n^2\,dz$, we must prove that for large $n$, $h_n(\epsilon_n) > 0. \tag{3.7}$

Assume without loss of generality that $\Lambda$ was chosen so that there exists $\rho > 0$ with

$$\inf_{N_\rho(\Lambda)} V = V_0, \tag{3.8}$$

where $N_\rho(\Lambda) = \{x \in \mathbb{R}^N \mid \exists y \in \Lambda \text{ with } |y - x| < \rho\}$. We will prove the following facts about $h_n$:

**Lemma 3.3** For some $\beta > 0$, for large $n$,

(i) $h_n \in C^2(\mathbb{R}^+, \mathbb{R})$

(ii) $h_n(0) \geq 0$

(iii) $|h_n'(0)|^2 \leq o(1)h_n(0)$

(iv) $h_n''(0) > \beta$

(v) $h_n''$ is locally Lipschitz on $\mathbb{R}^+$, uniformly in $n$.

Here $o(1) \to 0$ as $n \to \infty$. Before proving Lemma 3.3, let us prove how it gives (3.7). By (iv)-(v), there exists $d > 0$ such that for large $n$ and $0 \leq t \leq d$, $h_n(t) > \beta/2$. For $t \in [0, d]$, a Taylor’s series expansion shows that for large $n$,

$$h_n(t) \geq h_n(0) + h_n'(0)t + \frac{\beta}{4} t^2 \equiv l_n(t). \tag{3.9}$$

If $h_n(0) = 0$, then by Lemma 3.3(iii), $h_n'(0) = 0$, so (3.9) implies that $h_n(t) > 0$ for all $t \in (0, d)$, giving (3.7) if $n$ is large enough that $\epsilon_n < d$. If $h_n(0) > 0$, then by elementary calculus, $l_n$ attains a minimum value at $t = -2h_n'(0)/\beta$, and the minimum value is

$$\min_{\mathbb{R}} l_n = l_n(-2h_n'(0)/\beta) = h_n(0) - h_n'(0)^2/\beta \geq (1 - o(1))h_n(0),$$

where $o(1) \to 0$ as $n \to \infty$. For large $n$, if $h_n(0) > 0$ then $l_n(t) > 0$ for all $t \in \mathbb{R}$, so $h_n(t) > 0$ for all $t \in (0, d)$ for large $n$, implying (3.7) if $n$ is large enough so that $\epsilon_n < d$. 


Proof of Lemma 3.3 Statement (ii) is trivial, since $h_n(0) = (V(y_n) - V_0)\int_{\mathbb{R}^N} v_n^2$, and since $z_n \in \mathcal{X}$ and $y_n - z_n \to 0$, (3.8) implies $V(y_n) \geq V_0$ for large $n$. (i) and (v) follow from Leibniz’s Rule, $(V_1)-(V_2)$, and the fact that the $v_n$’s decay exponentially, uniformly in $n$. For $j = 1, 2$,

$$h_n^{(j)}(t) = \int_{\mathbb{R}^N} \sum_{|\alpha|=j} (D^\alpha V(y_n + tz) - D^\alpha V(tz))z^\alpha v_n(z)^2 dz.$$ 

Since (V2) holds, $v_n$ decays exponentially, uniformly in $n$, $y_n \to \bar{y}$, and $v_0$ is radially symmetric, we have

$$h_n''(0) = \int_{\mathbb{R}^N} \sum_{|\alpha|=2} (D^\alpha V(y_n) - D^\alpha V(0))z^\alpha v_0(z)^2 dz$$

$$= \int_{\mathbb{R}^N} \sum_{i=1}^{N} (D^{ii} V(\bar{y}) - D^{ii} V(0))z^2 v_0(z)^2 dz$$

$$= \int_{\mathbb{R}^N} \sum_{i=1}^{N} (D^{ii} V(\bar{y}) - D^{ii} V(0)) \frac{1}{N} |z|^2 v_0(z)^2 dz$$

$$= \frac{1}{N} (\Delta V(\bar{y}) - \Delta V(0)) \int_{\mathbb{R}^N} |z|^2 v_0(z)^2 dz > 0$$

by assumption (3.5). Since Lemma 3.3(v) holds, we have Lemma 3.3(iv).

To prove Lemma 3.3(iii), we will need the following calculus lemma:

**Lemma 3.4** Let $U \subset \mathbb{R}^N$ and $r > 0$. Let $V \in C^2(N_r(U), \mathbb{R})$ with $\inf_{N_r(U)} V = V_0 > -\infty$, $|\nabla V|$ bounded on $N_r(U)$, and $D^2 V$ Lipschitz on $N_r(U)$. Then there exists $C > 0$ with

$$|\nabla V(z)|^2 \leq C(V(z) - V_0) \quad (3.10)$$

for all $z \in U$.

**Proof:** let $B > 0$ with $|D^2 V(z)\xi \cdot \xi| \leq B$ for all $\xi \in \mathbb{R}^N$ with $|\xi| = 1$. Also let $B$ be big enough so

$$B > |\nabla V(z)|/r$$

for all $z \in U$. Pick $z \in U$. If $|\nabla V(z)| = 0$, then (3.10) is obvious. Otherwise, let $d = |\nabla V(z)|/B < r$. Define $\varphi(t) = V(z - t\nabla V(z)/|\nabla V(z)|)$ for $t \in [0, d]$. $\varphi$ is $C^2$, $\varphi(0) = V(z)$, and $\varphi'(0) = -|\nabla V(z)|$. By choice of $B$ and the fact that $B_d(z) \subset N_r(U)$, $|\varphi''(t)| \leq B$ for all $t \in [0, d]$. Taylor’s theorem gives

$$\varphi(d) - \varphi(0) = \varphi'(0)d + \varphi''(\xi)d^2/2 \leq -|\nabla V(z)|d + Bd^2/2 = -\frac{|\nabla V(z)|^2}{2B}.$$
Also \( \varphi (d) \geq V_0 \) because \( B_d(z) \subset N_r(U) \). Therefore,
\[
\frac{|\nabla V(z)|^2}{2B} \leq \varphi (0) - \varphi (d) \leq V(z) - V_0.
\]
Lemma 3.4 is proven.

To prove Lemma 3.3(iii), first note that, by the radial symmetry of \( v_0 \), the uniform exponential decay of \( v_n \), and the uniform convergence \( v_n \to v_0 \),
\[
|h_n'(0)| = |(\nabla V(y_n) - \nabla V(0)) \cdot \int_{\mathbb{R}^N} zv_n^2 \, dz| \\
= |\nabla V(y_n) \cdot \int_{\mathbb{R}^N} zv_n^2 \, dz| \\
= |\nabla V(y_n) \cdot \int_{\mathbb{R}^N} zv_n^2 \, dz + \nabla V(y_n) \cdot \int_{\mathbb{R}^N} z(v_n^2 - v_0^2) \, dz| \\
= |\nabla V(y_n) | \int_{\mathbb{R}^N} z(v_n^2 - v_0^2) \, dz| \\
\leq |\nabla V(y_n) | \int_{\mathbb{R}^N} (v_n^2 - v_0^2) \, dz| \\
\leq o(1)|\nabla V(y_n)|,
\]
so Lemma 3.4 implies
\[
|h_n'(0)|^2 \leq o(1)|\nabla V(y_n)|^2 \leq o(1)(V(y_n) - V_0) \\
\leq o(1)(V(y_n) - V_0) \int_{\mathbb{R}^N} v_n^2 \\
= o(1)h_n(0),
\]
since \( \int_{\mathbb{R}^N} v_n^2 \) is bounded away from zero. Lemma 3.3(iii) is proven. Thence follow (3.7), (3.3), Proposition 2.2, and Theorem 1.1.

Remarks: Besides the results cited in the introduction, many important results for equations of type (1.1) have been found recently. For instance, the work in [3]-[5] suggests that Theorem 1.1 could be strengthened by working on a smaller domain than \( \mathbb{R}^N \), or by weakening the hypotheses on \( V \). It is natural to try to extend Theorem 1.1 to cases where \( V \) is not \( C^2 \), or to the case where the second derivatives of \( V \) do not provide a condition like (V4), but higher-order derivatives do.

References


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