Robust Nonlinear Estimation and Control Applications using Synthetic Jet Actuators

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by

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Abstract

Limit cycle oscillations (LCO), also known as flutter, cause significant challenges in flight control of small unmanned aerial vehicles (SUAVs), and could potentially lead to structural damage and catastrophic failures. LCO can be described as vibrational motions in the rocking, pitching and plunging displacements of an aircraft wing. To address this, the use of synthetic jet actuators (SJAs) in UAV flight control systems is becoming popular as a practical alternative and to mechanical deflection surfaces.

Synthetic jet actuators are promising tools for LCO suppression systems in small UAVs due to their small size, ease of operation, and low cost. Uncertainties inherent in the dynamics of the synthetic jet actuators present significant challenges in the synthetic jet actuator-based control design. Specifically, the input-output characteristic (voltage-virtual deflection angle relationship) of the synthetic jet actuators is nonlinear and contains parametric uncertainty. Further control design challenges exist in situations where multiple actuators lose effectiveness.

This dissertation focuses on the suppression of limit cycle oscillations on small unmanned air vehicles using synthetic jet actuators. A brief description on how wind gust affects aircraft tracking control is presented. It shows an extension to a paper by adding the wind gust model to the system while also varying the uncertain synthetic jet actuator parameters using a Monte Carlo method. Next, a robust nonlinear control method is presented, which achieves simultaneous aircraft tracking control and limit cycle oscillation suppression using these synthetic jet actuators and a robust controller. Following that, a nonlinear LCO regulation method is presented, which uses a bank of dynamic filters to eliminate the need for pitching and plunging LCO rate measurements. Finally, an alternative method of LCO regulation control is presented, which utilizes a sliding mode observer in lieu of a bank of filters to estimate the pitching and plunging LCO rates.
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Chapter 1

Introduction

1.1 Motivation

Applications involving small unmanned aerial vehicles (SUAV) have become increasingly popular for civilian, military, and surveillance purposes. Due to their smaller size, SUAVs have limited onboard space, and this practical consideration motivates the design of SUAV guidance and control systems that can achieve reliable performance using minimal computational power and mechanical complexity.

One of the main challenges in the design of SUAVs operating in unsteady atmospheric conditions is the aeroelastic phenomenon known as limit cycle oscillations (LCO) (Krishnappa, 2016). LCO can cause flight instabilities and structural damage in SUAV flight; therefore, methods for suppression of LCO are important in these systems. Motivated by these challenges, LCO regulation control methods are presented in this dissertation, which utilize energy-efficient, low-mass synthetic jet actuators (SJA).
1.2 Limit Cycle Oscillations

Limit cycle oscillations (LCO), or flutter, are best described as pitching (rotational) and plunging (vertical) displacements in an airfoil. This happens when two modal frequencies join and the roots of the system are in complex conjugate pairs (Dixit, Kodhanda, Mahesh, & Talole, 2016). In aerospace studies, these oscillations, called aeroelastic phenomena, are the dynamical phenomena resulting from the mutual interaction of the aerodynamic forces and elastic forces on flexible structures (e.g. wings, airplane body, or missile body) (Razak, Rothkegel, & Dimitriadis, 2012). These aeroelastic phenomena are inherently nonlinear and that these nonlinearities lead to phenomena not properly described by linear representations (Strganac, Ko, & Thompson, 2000).

Therefore, understanding the effects of LCO are important because they can cause dynamic instability, which can result in catastrophic damage (Yang, Li, Ren, Tan, & Fan, 2014; Whitmer et al., 2012; Platanitis & Strganac, 2004) and can exceed the limiting safe flight boundaries of any aircraft (Rubillo, Bollt, & Marzocca, 2005). There are three degrees of freedom defining LCO in an aircraft’s wing. One of them

![Figure 1.1: Illustration of the locations of the pitching and plunging displacements on an airfoil.](image-url)
is wing-rock, $\phi$, which is defined as oscillations in the roll axis (Wu, Chen, Gong, & Ye, 2014a). The wing rock phenomenon usually only occurs in higher angles of attack with swept wing platforms and leading edge extensions (Guglieri & Quagliotti, 2001). In this dissertation, only the pitching, $\alpha$, and plunging, $h$, displacements are investigated for the reason that the SUAVs on this dissertation are not reaching higher angles of attack in flight. In Figure 1.1, the $K_h \in \mathbb{R}$ is the spring constant of the wing and $K_\alpha(\alpha) \in \mathbb{R}$ is the highly nonlinear stiffness coefficient that will be defined in the subsequent chapters.

1.3 Synthetic Jet Actuators

The recent surge of interest in applications involving SUAVs has motivated the development of low-mass actuators with reduced power requirements. Based on this, the use of synthetic jet actuators (SJAs) has emerged as a popular tool for UAV control applications. These SJAs can be used in a variety of applications, including trajectory tracking control, limit cycle oscillation suppression, and boundary-layer flow control. They have reduced cost, weight, and mechanical complexity and can be used in conjunction with standard mechanical control surfaces. SJAs are promising tools in SUAV applications, due to the limited onboard power capabilities of SUAVs.

The operation of SJAs is based on an effective combination of electrical, mechanical, and acoustic components. The most common SJA assemblies are piston cylinder, voice-coil magnet, or piezoelectric disk type actuators. These SJA transfer linear momentum to a flow system using a piezoelectric membrane inside a cavity, which creates a train of vortices through the alternating suction and ejection of the air around it
through a small orifice as seen in Figure 1.3. This means that SJAs achieve they have momentum transfer with zero net mass flux across the flow boundary. Figure 1.3 illustrates an example of the installation of these SJAs in an airfoil. They can achieve control effectiveness in high angles of attack, as shown in high fidelity CFD studies (Golubev & Mankbadi, 2012).

Figure 1.2: Schematic layout of a synthetic jet actuator.
1.3. SYNTHETIC JET ACTUATORS

1.3.1 Benefits using Synthetic Jet Actuators

There are some key benefits of using SJAs in SUAVs. For example, SUAVs have limited space for mechanical control surfaces, therefore multiple SJAs (in arrays) can be used as replacements for these control surfaces. Another benefit is deal with flow separation control, which can be achieved using SJAs by virtue of their ability to add or remove momentum to or from the boundary layer (M. G. De Giorgi, De Luca, Ficarella, & Marra, 2015; Tang, Salunkhe, Zheng, Du, & Wu, 2014; Golubev et al., 2015). Moreover, SJAs can improve tracking control maneuverability when used in conjunction with mechanical control surfaces (MacKunis et al., 2013; Deb, Tao, Burkholder, & Smith, 2007; M. De Giorgi, De Luca, Ficarella, & Marra, 2015; Tang et al., 2014). In addition, SJAs do not require space for fuel supply, since they simply utilize the air from the surrounding area.
1.3.2 Challenges using Synthetic Jet Actuators

The challenges in SJA-based control design stem from the fact that the input-output characteristics (voltage-virtual deflection angle relationship) of the SJA are nonlinear as in Figure 1.4 and contain uncertain parameters in its mathematical model. In addition to the challenges involved in control design in the presence of SJA actuator uncertainty, control design for UAV in off-nominal operating conditions (e.g., wind gusts) creates further challenges.

1.3.3 Virtual Surface Deflection using Synthetic Jet Actuators

The dynamics of SJAs are inherently nonlinear. This section discusses an approach to deal with the uncertain nonlinear dynamic model using a robust-inverse control structure.

In standard aircraft flight control applications, the control forces and moments are generated through mechanical deflection surfaces on the aircraft (e.g., ailerons, rudders, elevators, flaps). When using SJA, the control actuation is generated by a given number, $m$, of SJA arrays; and the control input $u(t)$ represents the virtual surface deflection angle resulting from the cumulative effect of these the SJA arrays. The key challenge in SJA-based control design is that the virtual deflection angle due to the $i^{th}$ SJA array is an uncertain nonlinear function of the input voltage applied to the array. Specifically, the dynamics of the virtual surface deflection due to the $i^{th}$ SJA array can be expressed using the empirically determined model (Deb, Tao, Burkholder, & Smith, 2006), (Deb et al., 2007), (Deb, Tao, Burkholder, & Smith,
\begin{equation}
    u_i(t) = \theta^*_2 i - \frac{\theta^{*1}_1}{v_i(t)}, \quad i = 1, ..., m
\end{equation}

where $u_i(t) \in \mathbb{R}$ denotes the virtual deflection angle due to the $i^{th}$ SJA array; $v_i(t) = A^{pp}_{2i}(t) \in \mathbb{R}$ denotes the peak-to-peak voltage applied to the $i^{th}$ SJA array in [Volts]; and $\theta^*_1, \theta^*_2 \in \mathbb{R}$ are uncertain constant physical parameters in [Volt–deg] and [deg], respectively, for the $i^{th}$ SJA array. The parameter $\theta^*_2$ physically represents the maximum surface deflection angle achievable using the $i^{th}$ SJA array. Figure 1.4 shows the different variation of the virtual surface deflection with the voltage provided from the SJA for four different values of the constant physical parameter $\theta^*_1$.

![Figure 1.4: Variation of virtual surface deflection with the voltage from the synthetic jet actuator (Ramos-Pedroza et al., 2017).](image)

To compensate for the SJA actuator nonlinearity and input parametric uncertainty
1.3. SYNTHETIC JET ACTUATORS

in (1.1), a robust-inverse control structure will be utilized (MacKunis et al., 2013), which employs constant, “best-guess” estimates of the uncertain SJA parameters \( \theta_1^*, \theta_2^* \). The robust-inverse control law can be expressed as

\[
v_i (t) = \frac{\hat{\theta}_1 - u_{di} (t)}{\hat{\theta}_2}, \quad i = 1, ..., m
\]

(1.2)

where \( \hat{\theta}_1, \hat{\theta}_2 \in \mathbb{R}^+ \) are constant feedforward estimates of \( \theta_1^* \) and \( \theta_2^* \), respectively; and \( u_{di} (t) \in \mathbb{R} \), for \( i = 1, ..., m \), are the auxiliary control terms. These auxiliary control terms are defined in the subsequent chapters.

**Remark 1 (Avoiding Singularities).** Based on (1.2), the control design \( v_i (t) \) will encounter singularities when \( u_{di} (t) = \hat{\theta}_2 \). To prevent this singularity situation, the control signals \( u_{di} (t) \) for \( i = 1, ..., m \) the following algorithm is incorporated (Mondschein, Tao, & Burkholder, 2011):

\[
u_{di} (t) = \begin{cases} 
\hat{\theta}_2 - \epsilon & \text{if } g (\mu_0 (t), \mu_1 (t)) \geq \hat{\theta}_2 - \epsilon \\
g (\mu_0 (t), \mu_1 (t)) & \text{otherwise}
\end{cases}
\]

(1.3)

where \( \epsilon \in \mathbb{R}^+ \) is a small positive parameter, \( g (\cdot) \) is a subsequently defined control function, and \( \mu_0 (t), \mu_1 (t) \in \mathbb{R}^m \) denote auxiliary control terms. Note that the design parameter, \( \epsilon \), can be selected arbitrarily small, such that system controllability and stability can be proven over a sufficiently wide range of positive control signals \( v_i (t) \).

In addition, the control terms \( u_{di} (t) \) will encounter singularities when \( v_i (t) = 0 \); however, this can be easily avoided by choosing \( \hat{\theta}_1 > 0 \) for \( i = 1, ..., m \).

**Remark 2.** Preliminary results show that the auxiliary control signal \( u_{di} (t) \) in (1.2) can be designed to achieve asymptotic SJA-based tracking control, limit cycle oscil-
1.4 Outline of the Dissertation

This dissertation focuses on the suppression of limit cycle oscillations in small unmanned air vehicles using synthetic jet actuators. Chapter 1 focuses on the motivation and background of LCO and SJAs. Chapter 2 focuses on the various mathematical definitions and control methods used to solve the control problems. Chapter 3 gives a brief description on how wind gust affects aircraft tracking control. It also shows an extension on a paper by adding the wind gust model to the system while also varying the uncertain synthetic jet actuator parameters using a Monte Carlo method. Chapter 4 presents a robust nonlinear control method that achieves simultaneous aircraft tracking control and LCO suppression using these SJAs. Chapter 5 provides a method to eliminate the need for LCO pitching and plunging rate measurements using a bank of dynamic filters in the feedback control law. Finally, in Chapter 6 an alternative method to eliminate LCO rate measurements by utilizing a sliding mode observer to estimate the pitching and plunging rates.
Chapter 2

Mathematical Methods

This chapter describes the key mathematical methods involved in the control development presented in this dissertation. This chapter describes Lyapunov’s first and second stability theorems, basic design methods for linear control and estimation, sliding mode estimation, and details on the basic robust control design methods on which much of the subsequent discussion is focused.

2.1 State-Space Models

The term state-space representation describes the system and the response to any set of inputs (Rowell, 2002). These can come from a fully developed dynamical system to minimum set of state variables. Consider the nonlinear ordinary differential equation

\[ \dot{x} = f(x, u) \quad (2.1) \]

\[ y = h(x) \quad (2.2) \]
2.2. LINEAR STATE CONTROL

where $x(t)$ in $\mathbb{R}^n$ denotes the state vector, $u(t)$ in $\mathbb{R}^n$ is the control input (forcing function), and $y(t)$ in $\mathbb{R}^m$ represents the (sensor) measurement equation (Wie, 2008).

2.2 Linear State Control

A linear state-space controller is the simplest form of a state feedback controller. Linear control and observer methods commonly use a linear time invariant state-space system of the form

\[
\dot{x} = Ax + Bu \tag{2.3}
\]
\[
y = Cx \tag{2.4}
\]

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^m$ is the output vector (i.e., sensor measurement), and $u \in \mathbb{R}^p$ is the input vector. The matrix $A \in \mathbb{R}^{n \times n}$ is the state (or system) matrix, $B \in \mathbb{R}^{n \times p}$ is the input matrix, and $C \in \mathbb{R}^{m \times n}$ is the output matrix. The output $u(t)$ can be designed so the vector $x(t)$ converges to the equilibrium point, $x^*(t)$. As an example, a full state linear feedback control can drive the objective $x(t)$ to 0 and can be achieved by using the control law in the form of

\[
u = -kx \tag{2.5}\]

where $k = \mathbb{R}^{p \times n}$ is a feedback gain matrix (Wie, 2008). Substituting (2.5) into (2.3) we can obtain the following closed-loop system

\[
\dot{x} = (A - Bk)x \tag{2.6}\]
The characteristic equation for (2.6) is in the form

\[ |\lambda I - A + Bk| = 0 \] (2.7)

This matrix, \( k \), can be designed so the closed-loop system utilizing the pole placement method. This method chooses the eigenvalues, \( \lambda \), to have negative real parts and complex conjugate pairs which would make the system stable. However, this type of linear system cannot drive the state to an equilibrium point, \( x(t) \to x^*(t) \), this is where nonlinear control methods in Section 2.4.1 can overcome this challenge.

\[ \text{Desired States} \xrightarrow{\Sigma} \text{Error} \xrightarrow{\text{Controller Input}} \text{Control Input} \xrightarrow{\text{System}} \text{State Output} \]

\[ \nabla \text{Disturbances} \xrightarrow{\text{Estimated States}} \text{Observer} \xrightarrow{\text{Measured States}} \]

Figure 2.1: General block diagram for a control system with system (plant), observer, and controller.

### 2.2.1 Observers

For practical scenarios where not all state variables can be directly measured by the output vector, \( y \), an observer can be utilized to estimate the unmeasurable states (Wie, 2008). Using the form of (2.3), a linear observer that asymptotically estimates the complete state vector \( x(t) \) using only direct measurements of \( y(t) \) can be designed.
2.3. SYSTEM STABILITY

as

\[
\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \nonumber
\]

\[
= (A - LC)\hat{x} + Bu + Ly
\]

where \(\hat{x} \in \mathbb{R}^n\) denotes the estimate of \(x\), and \(L \in \mathbb{R}^{n \times m}\) is an observer gain.

The linear observer in (2.8) can be proven to achieve asymptotic (i.e., zero steady-state error) estimation of the state vector \(x(t)\) for systems in the state-space form given in (2.3). However, the design in (2.8) requires exactly knowledge of the state space model (i.e., the state and input matrices \(A\) and \(B\) in (2.3) must be known exactly). The nonlinear estimation and control methods presented in this thesis are motivated by the desire to achieve reliable control for a more general class of nonlinear systems, under more realistic conditions where the dynamic model is uncertain or ill-defined.

2.3 System Stability

In control engineering, stability properties are an important concept used to describe the characteristics of a dynamical system. An equilibrium point, \(x^*\), is considered stable if all solutions starting at nearby points stay nearby. Figure 2.2 illustrates the common representation for a stable system: any small deviations from the equilibrium point \(x^*\) would make the “ball” oscillate at the bottom of this “bowl”. The equilibrium point is asymptotically stable if all solutions starting in the vicinity of the equilibrium point converge to the equilibrium point as time approaches infinity. That means that the “ball” in Figure 2.2 would converge to the bottom of the “bowl”, the equilibrium point \(x^*\), after some time. The equilibrium point would be unstable if it is not stable.
In Figure 2.3 is the common representation for an unstable system where any small deviation would make the “ball” fall off from the “bowl” and would never return to its original position.

![Unstable equilibrium](image)

Figure 2.3: Unstable equilibrium point representation.

2.4 Stability Definitions

Consider the isolated equilibrium point, \( x^* = 0 \), which satisfies

\[
 f(x^*, t) = 0 \quad \forall t \geq 0
\]  

(a) The equilibrium point is stable if for every \( \epsilon > 0 \), there exists a positive, \( \delta = \)

14
δ(ε, t₀) > 0 such that

\[ \|x(t₀)\| < δ \Rightarrow \|x(t)\| < ε, \quad \forall t ≥ t₀ ≥ 0 \]  \hspace{1cm} (2.10)

where \( \| \cdot \| \) denotes the Euclidean norm of a vector, which is defined as

\[ \|x\| \equiv \sqrt{x^T x}. \]  \hspace{1cm} (2.11)

If, in addition, \( δ \) does not depend on \( t₀ \), then the equilibrium point is uniformly stable.

(b) The equilibrium point, \( x^* \), is locally asymptotically stable (LAS) if it is stable (2.10) and

\[ \|x(t₀) - x^*\| < δ \Rightarrow x(t) → x^*, \quad t → ∞ \]  \hspace{1cm} (2.12)

(c) The equilibrium point is globally asymptotically stable (GAS) if it is stable and

\[ x(t) → x^* \quad t → ∞ \quad ∀x(t₀). \]  \hspace{1cm} (2.13)

(d) The equilibrium point is unstable, if it is not stable.

2.4.1 Stability Analysis for Linear and Nonlinear Systems

This section presents some of the most important mathematical tools that are used in the control designs and stability analyses in this dissertation.
2.4.2 Lyapunov Stability

In this dissertation, Lyapunov analyses were used to determine the stability properties of the closed-loop systems. Lyapunov’s second stability theorem was utilized in this dissertation as the primary tool for determining the stability properties of nonlinear differential equations, without explicitly solving the extraneous space.

2.4.2.1 Lyapunov’s First Stability Theorem

Lyapunov’s first stability theorem is based on linearization of a nonlinear system near the equilibrium point, \( x^* \), and then utilizing the resulting linearized system to analyze the local stability properties of the nonlinear system in the neighborhood of that equilibrium point (Wie, 2008).

**Theorem 1.**  
(a) If the origin \( z = 0 \) of the linearized system is asymptotically stable, then the equilibrium point, \( x^* \) of the nonlinear system is locally asymptotically stable. As an example, Figure 2.4 is a representation of a Lyapunov asymptotically stable system.

(b) If the origin \( z = 0 \) of the linearized system is unstable, then the equilibrium point, \( x^* \) of the nonlinear system is unstable. As an example, Figure 2.5 is a representation of an Lyapunov unstable system.

(c) Nothing can be said about the equilibrium point, \( x^* \), of the nonlinear system, if the origin \( z = 0 \) of the linearized system is stable. As an example, Figure 2.6 is a representation of a Lyapunov stable system.
2.4. STABILITY DEFINITIONS

Figure 2.4: Locally asymptotically stable equilibrium point

Figure 2.5: Unstable equilibrium point
2.4. Stability Definitions

2.4.2.2 Lyapunov’s Second Stability Theorem

Lyapunov’s second stability theorem uses a positive definite potential function, called a Lyapunov function, \( V(x) \in \mathbb{R} \) to show that the system is stable if the time derivative of Lyapunov function is negative definite.

**Theorem 2.** Consider a dynamic system in the form (Khalil, 2002)

\[
\dot{x} = f(x,t) \quad f(x^*, t) = 0
\]  

(2.14)

where \( x^* \) is the equilibrium point of the system. In some finite region \( \mathcal{D} \) containing \( x^* \), assume there exists a positive definite continuously differentiable Lyapunov function \( V : \mathcal{D} \rightarrow \mathbb{R} \).
2.4. STABILITY DEFINITIONS

(a) The equilibrium point is stable if

\[ V(x) > 0 \quad \text{in} \quad \mathcal{D} - \{0\} \quad \text{and} \quad V(0) = 0 \quad \forall t \]  \hspace{1cm} (2.15)

and its time derivative along trajectories of the system is negative semi-definite in the sense that

\[ \dot{V}(x) \leq 0. \]  \hspace{1cm} (2.16)

(b) The equilibrium point is locally asymptotically stable if (2.15) is satisfied and \( \dot{V}(x) \) is negative definite in the sense that

\[ \dot{V}(x) < 0 \quad \text{in} \quad \mathcal{D} - \{0\} \quad \text{and} \quad \dot{V}(0) = 0 \quad \forall t \]  \hspace{1cm} (2.17)

(c) The equilibrium point is globally asymptotically stable, if (2.15) is satisfied for any initial state \( x(t_0) \), the time derivative of the Lyapunov candidate function is negative definite, and the function \( V(x) \) is radially unbounded in the sense that

\[ \|x(t)\| \to \infty \Rightarrow V(x) \to \infty \]  \hspace{1cm} (2.18)

Figure 2.7 is a representation of this globally asymptotically stable system.
2.5 Barbalat’s Lemma

In addition to basic stability definitions and methods for determining the stability properties of equilibrium points, here are some mathematical properties that can be used to further analyze the stability of equilibrium points, when Lyapunov’s stability theorems are insufficient. Specifically, Barbalat’s lemma is an important tool that can be used to prove asymptotic stability of an equilibrium point for cases where
Lyapunov’s stability theorems can only be used to prove stability. First, the following definition of *uniform continuity* is an important definition that is utilized in the subsequently defined Barbalat’s lemma (Stewart, 2012).

**Definition 1.** Let $S$ be a subset of $\mathbb{R}$. A function $f : S \rightarrow \mathbb{R}$ is uniformly continuous on $S$ if, for each $\epsilon > 0$, there exists a real number $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in S$ with $|x - y| < \delta$, where $\delta$ depends on $\epsilon$.

**Lemma 1.** Barbalat’s lemma (Khalil, 2002). Let $x : \mathbb{R} \rightarrow \mathbb{R}$ be an uniformly continuous function on $[0, \infty)$. Presume that the following exist and is finite

$$\lim_{t \to \infty} \int_0^t x(\tau) d\tau$$

Then

$$x(t) \to 0 \text{ as } t \to \infty$$

### 2.6 Nonlinear Control Methods

In this section, the robust state control methods used in this dissertation are briefly explained. In the robust control section, the methods of nonlinear damping and sliding mode control will both be described, as well as an example of a sliding mode estimator method.
2.6. NONLINEAR CONTROL METHODS

2.6.1 Robust Control

In robust control methods, the effects of any uncertainty and disturbances in the nonlinear system are assumed to be bounded, and high-frequency or high-gain feedback are utilized to suppress or eliminate their detrimental effects. The following sections describe the nonlinear damping and sliding mode control approaches for reducing these disturbances and stabilizing the closed-loop system.

2.6.1.1 Nonlinear Damping

In nonlinear control, reducing the disturbance effects is an important condition as it helps to eliminate state-state error. The nonlinear damping method provides a way to make sure that the disturbances can be reduced to an arbitrarily small residual set (i.e., an ultimately bounded error). The resulting solution converges to a finite bounded region of the origin, which can be rendered arbitrarily small, but the tracking error cannot be driven to zero using nonlinear damping.

Consider the following scalar system

\[ \dot{x} = f(x, t) + u(t) \]  

(2.23)

where \( x(t) \in \mathbb{R}^n \) is the state space vector, \( u(t) \in \mathbb{R}^n \) is the control input vector, and \( f(x, t) \in \mathbb{R}^n \) is an unknown disturbance that is bounded and sufficiently smooth in the sense that

\[ |f(x, t)| \leq \zeta \quad |\dot{f}(x, t)| \leq \zeta_0 \]  

(2.24)
where $\zeta, \zeta_0 \in \mathbb{R}^+$ are known constants. A control law design, $u(t)$, is utilized to drive the state vector, $x(t)$, to the desire equilibrium point, $x^*$, as

$$u = -(k_s + 1)x \quad (2.25)$$

where $k_s \in \mathbb{R}^+$ is the nonlinear damping gain ($k_s$ could also be defined as a positive definite diagonal gain matrix). The closed loop dynamics are obtained when (2.25) is substituted into (2.23) as

$$\dot{x} = f(x, t) - (k_s + 1)x \quad (2.26)$$

To analyze the stability of (2.26), consider the following positive definite Lyapunov function and its derivative

$$V = \frac{1}{2}x^2 \quad (2.27)$$

$$\dot{V} = x\dot{x} \quad (2.28)$$

Substituting (2.26) into (2.28) results in

$$\dot{V} = xf(x, t) - (k_s + 1)x^2 \quad (2.29)$$

After completing the squares, the Lyapunov derivative can be expressed as

$$\dot{V} \leq -x^2 - k_s \left( |x|^2 - \frac{\zeta}{k_s} |x| \right) \quad (2.30)$$

$$\dot{V} \leq -x^2 + \frac{\zeta^2}{4k_s} \leq -2V + \frac{\zeta^2}{4k_s} \quad (2.31)$$
Based on the expression in (2.31), $x(t)$ is bounded and converges to the compact set described as

$$\mathcal{S} = \left\{ x \mid |x| \leq \frac{\zeta}{2\sqrt{k_s}} \right\}.$$  \hfill (2.32)

Note that the size of the ultimate bound on the tracking error can be made arbitrarily small by increasing the control gain $k_s$.

### 2.6.1.2 Sliding Mode Control

*Sliding mode control* (SMC) is a subset of variable structure control that forces state trajectories to reach a sliding manifold in finite time and to remain on the manifold for all future time (Utkin, 1992; S. V. Drakunov, 1992b). Standard SMC uses a discontinuous control signal that causes the state to asymptotically converge to the desired state or to the origin. They are helpful when handling nonlinear systems with uncertain dynamics and disturbances. Consider a second order system given by

$$\dot{x}_1 = x_2$$  \hfill (2.33)

$$\dot{x}_2 = h(x) + g(x)u$$  \hfill (2.34)

where $h(x)$ and $g(x)$ are unknown nonlinear functions, and $g(x) \geq g_0 \geq 0$ for all $x$. By selecting the sliding manifold as

$$s = a_1 x_1 + x_2 = 0,$$  \hfill (2.35)
then $\dot{x}_1 = -a_1 x_1$, and the control gain $a_1 > 0$ can be selected to yield the desired rate of convergence of the state $x_1(t)$ to zero. The motion on the manifold $s = 0$ is independent of $h$ and $g$. Taking the time derivative of (2.35) and using (2.33) and (2.34), $\dot{s}$ is obtained as

$$\dot{s} = a_1 \dot{x}_1 + \dot{x}_2$$ (2.36)

$$\dot{s} = a_1 x_2 + h(x) + g(x)u$$ (2.37)

It is assumed that $h(x)$ and $g(x)$ satisfy the inequality

$$\left| \frac{a_1 x_2 + h(x)}{g(x)} \right| \leq \chi(x), \quad \forall x \in \mathbb{R}^2$$ (2.38)

for some known function $\chi(x)$. Consider a positive definite Lyapunov candidate function, $V(x)$, as

$$V = \frac{1}{2} s^2$$ (2.39)

After taking the time derivative of (2.39) and using (2.37) and (2.38), the following upper bound is obtained:

$$\dot{V} = s \dot{s} = s [a_1 x_2 + h(x)] + g(x)su \leq g(x)|s|\chi(x) + g(x)su$$ (2.40)

A sliding mode control law can be designed as

$$u = -\beta(x) \text{sgn}(s)$$ (2.41)
where $\beta(x) \geq \chi(x) + \beta_0$, $\beta_0 > 0$, and where the $\text{sgn}(s)$ denotes the discontinuous
signum function, which is defined as

$$\text{sgn}(s) = \begin{cases} 
1 & s > 0 \\
0 & s = 0 \\
-1 & s < 0 
\end{cases}$$ (2.42)

![Figure 2.8: A phase portrait under sliding mode control (Khalil, 2002).](image)

Figure 2.8: A phase portrait under sliding mode control (Khalil, 2002).

Note that this is a simplification of the signum function, which is being used here to simplify the Lyapunov-based stability analysis. The signum function is defined such that the value at zero (i.e., the $\text{sgn}(0)$) is included in the set $(-1, +1)$ (Filippov, 1964)). In Figure 2.8, is a representation of the phase portrait under sliding mode control and how the states tries to reach the sliding surface manifold, $s$, regardless of what trajectory starts with. To analyze the stability properties of the system using the actual definition of the signum function, differential inclusions would be required, and this analysis is not included in this dissertation.
2.6. NONLINEAR CONTROL METHODS

The Lyapunov derivative can be expressed as

\[ \dot{V} = g(x)|s|\chi(x) - g(x) [\chi(x) + \beta_0] \text{sgn}(s) \]  
\[ \quad = -g(x)\beta_0 |s| \leq -g_0\beta_0 |s| \]  

(2.43)

(2.44)

It can be shown that \( W = \sqrt{2V} = |s| \) satisfies the differential inequality

\[ D^+ W \leq g_0\beta_0 \]  

(2.45)

where \( D^+ \) denotes the upper right-hand derivative (also known as the Dini derivative which is a class of generalizations of the derivative).

Remark 3. The upper Dini derivative of a continuous function \( f : \mathbb{R} \to \mathbb{R} \) is denoted as \( f'_+ \) and is defined as

\[ f'_+(t) \triangleq \lim_{h \to 0^+} \sup_{t} \frac{f(t+h) - f(t)}{h}. \]  

(2.46)

The comparison lemma (Khalil, 2002) can then be used to show that

\[ W(s(t)) \leq W(s(0)) - g_0\beta_0 t \]  

(2.47)

This shows that the trajectory reaches the manifold \( s(x) = 0 \) in finite time, where it will remain. Then, \( x_1(t) \to 0 \) as \( t \to \infty \).
2.6. NONLINEAR CONTROL METHODS

2.6.1.3 Sliding Mode Estimator

This section summarizes a sliding mode estimator (or observer) design, which can be utilized to generate state estimates using only available sensor measurements. In contrast to the linear observer method in (2.8), the sliding mode estimator described here can be applied to linear systems or nonlinear systems (S. V. Drakunov, 1992a; S. Drakunov & Utkin, 1995), where the dynamic model is not completely known.

Consider a nonlinear system

\[
\dot{x} = f(x) \quad (2.48)
\]
\[
y = h(x) \quad (2.49)
\]

where \(x \in \mathbb{R}^n\), \(y \in \mathbb{R}^m\) and \(f(x) \in \mathbb{R}^n\) is the vector field.

\[
\dot{\hat{x}} = \left( \frac{\partial H(\hat{x})}{\partial x} \right)^{-1} M(\hat{x}) \{ \text{sgn}(V(t) - H(\hat{x})) \}_{eq} \quad (2.50)
\]

where \(\{ \text{sgn}(\cdot) \}_{eq}\) represents a smooth continuous value operator of the discontinuous signum function (S. V. Drakunov, 1992a), \(M(\hat{x}) \in \mathbb{R}^{n \times n}\) is the sliding gain diagonal matrix as previously mentioned of the form

\[
M(\hat{x}, u) = \text{diag} \left[ m_1(\hat{x}) \cdots m_n(\hat{x}) \right], \quad (2.51)
\]

\(H(\hat{x}) \in \mathbb{R}^n\) is a vector of the output derivatives (S. V. Drakunov, 1992a; MacKunis, Drakunov, Reyhanoglu, & Ukeiley, 2011; S. V. Drakunov & Reyhanoglu, 2011) of the
form

\[
H(x) \triangleq \begin{bmatrix}
    h_1(x) & h_2(x) & \cdots & h_n(x)
\end{bmatrix}^T
\]  \hspace{1cm} (2.52)

\[
= \begin{bmatrix}
    h(x) & L_fh(x) & \cdots & L_{n-1}h(x)
\end{bmatrix}^T
\]  \hspace{1cm} (2.53)

where \( L_f = \frac{\partial h}{\partial x} f(x) \) denotes the Lie derivative of the output function, \( h(x) \), along
the direction of the vector field. Lastly, \( V(t) \in \mathbb{R}^n \) is the observer vector in the form
of

\[
V(t) = \begin{bmatrix}
    v_1(t) & \cdots & v_n(t)
\end{bmatrix}^T
\]  \hspace{1cm} (2.54)

\[
= \begin{bmatrix}
    h_1(x) & \cdots & m_i\{\text{sgn}(v_i(t) - h_i(\hat{x}))\}_{eq}
\end{bmatrix}^T
\]  \hspace{1cm} (2.55)

for \( i = 1, \cdots, n \).

### 2.7 Summary of Mathematical Methods

The mathematical definitions and analytical methods presented in this chapter will be utilized throughout this dissertation to design and rigorously analyze SJA-based nonlinear control systems to regulate LCO in SUAV wing sections.
Chapter 3

Wind Gust affecting Aircraft Tracking

In this chapter, an extension of the published paper CDC 2013 paper, \textit{Robust non-linear aircraft tracking control using synthetic jet actuators} by (MacKunis, Subramanian, Mehta, Ton, Curtis and Reyhanoglu, 2013), is presented with a wind gust model and Monte Carlo-type simulation results that demonstrate the capability of a non-linear control system to completely compensate for parametric uncertainty inherent in SJAs. The new extension was then published in Hindawi 2017 with the title \textit{Synthetic Jet Actuator-Based Aircraft Tracking Using a Continuous Robust Nonlinear Control Strategy} by (Ramos Pedroza, Kidambi, MacKunis, and Reyhanoglu, 2016).

In this chapter, only the wind gust model is presented with the results Monte Carlo results.
3.1 Dynamic Model

The dynamic model being considered in this paper incorporates the effects of parametric uncertainty in the aircraft dynamics, along with unmodelled external disturbances, and the inherent SJA actuator nonlinearity and parametric uncertainty. Specifically, the aircraft dynamic model can be expressed as (Golubev et al., 2015; Deb et al., 2007; MacKunis et al., 2013; Deb, Tao, Burkholder, & Smith, 2005; Deb et al., 2008; Mondschein et al., 2011; Singhal, Tao, & Burkholder, 2009)

\[
\dot{x} = Ax + Bu + f(x, t) \tag{3.1}
\]

where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) denote uncertain state and input matrices, respectively; and \( f(x, t) \in \mathbb{R}^n \) represents an unmodelled nonlinear disturbance. The disturbance term \( f(x, t) \) could represent the effects of external disturbances, such as wind gusts, or model inaccuracies resulting from linearization, for example. In (3.1), the control input \( u(t) \triangleq \begin{bmatrix} u_1(t) & \cdots & u_m(t) \end{bmatrix}^T \in \mathbb{R}^m \) represents the virtual surface deflections resulting from \( m \) arrays of SJA discussed in Section 1.3.3. By substituting (1.1) into (3.1), the SJA-based dynamic model can be expressed as

\[
\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i + f(x, t). \tag{3.2}
\]

In (3.2), \( b_i \triangleq \begin{bmatrix} b_{i1} & \cdots & b_{ni} \end{bmatrix}^T \in \mathbb{R}^n \forall i = 1, \ldots, m \), where \( b_{ij} \) represents the \((i, j)^{th}\) element of the uncertain \( B \) matrix.

**Assumption 1.** The nonlinear disturbance \( f(x, t) \) is sufficiently smooth in the sense that the first and second time derivatives \( \dot{f}(x, t) \) and \( \ddot{f}(x, t) \) are bounded, provided
3.2 CONTROL DEVELOPMENT

\( x(t) \) is bounded.

### 3.1.1 Wind Gust Model

This section describes the details of the wind gust model, (i.e., the disturbance term \( f(x,t) \) introduced in (3.1)). It’s important to model this wind gust behavior in the system in order to test the controller’s capability especially during tracking control. The Federal Aviation Regulations (FAR) (Part, 2015) describe a vertical wind gust as a bounded nonlinearity along the longitudinal axis as

\[
f(x,t) = \begin{bmatrix} \frac{1}{V_0} & \frac{U_{ds}}{2} \left[ 1 - \cos \left( \frac{\pi s}{H} \right) \right] \end{bmatrix}.
\] (3.3)

In (3.3), \( H \) denotes the distance in (m) along the airplane’s flight path for the wind gust to reach its peak velocity; \( V_0 \) in (m/s) is the forward velocity of the aircraft when it enters the gust; \( s \in [0 \ 2H] \) denotes the distance penetrated into the wind gust in (m); and \( U_{ds} \) represents the design gust velocity in (m/s). The wind gust model used in the subsequent numerical simulation results is based on the mathematical model in (3.3).

### 3.2 Control Development

The objective is to ensure that the actual aircraft state \( x(t) \) tracks a model reference (desired) state. Based on the mathematical structure of the dynamic model in (3.1),
3.2. CONTROL DEVELOPMENT

the model reference system is designed as

\[
\dot{x}_m = A_m x_m + B_m \delta
\]  

(3.4)

where \(x_m(t) \in \mathbb{R}^n\) is the model reference state (i.e., the desired trajectory), \(A_m \in \mathbb{R}^{n \times n}\) denotes the model reference state matrix, \(B_m \in \mathbb{R}^n\) is the model reference input gain matrix, and \(\delta(t) \in \mathbb{R}\) is the reference input (e.g., a pilot or autopilot command). The parameters of the reference model in (3.4) are selected such that the system achieves favorable flight performance characteristics in terms of convergence time and steady-state error, for example.

**Assumption 2.** The state of the model reference system remains bounded and sufficiently smooth in the sense that \(x_m(t), \dot{x}_m(t), \ddot{x}_m(t), \ldots \in L_\infty \forall t \geq 0\).

3.2.1 Open-loop Error System

To quantify the control objective, a trajectory tracking error \(e(t) \in \mathbb{R}^n\) is defined as

\[
e = x - x_m.
\]  

(3.5)

To facilitate the derivation of the error system dynamics, an auxiliary (filtered) error signal \(r(t)\) is defined as

\[
r = \dot{e} + \gamma e
\]  

(3.6)

where \(\gamma \in \mathbb{R}^+\) is a constant control gain. By calculating the time derivative of (3.6) and substituting (3.1) and (3.5), the open-loop error system dynamics are obtained
3.2. CONTROL DEVELOPMENT

as

\[ \dot{r} = A\dot{e} + A\dot{x}_m + \sum_{i=1}^m b_i \left( \frac{\theta_{1i}^*}{\theta_{1i}} \dot{u}_{di}(t) \right) + \dot{f}(x,t) - \ddot{x}_m + \gamma\dot{e}. \]  

(3.7)

**Remark 4.** Although the constant portion of the SJA actuator model in (1.1) vanishes upon calculating the time derivative to obtain (3.7), the complete SJA model is incorporated in implementation by using (1.1) and (1.2). Thus, the subsequent simulation results incorporate the full SJA actuator model.

The open-loop error system in (3.7) can be rewritten in a more compact form as

\[ \dot{r} = \tilde{N} + N_d + \Omega \dot{u}_d(t) - Se \]  

(3.8)

where \( \Omega \in \mathbb{R}^{n \times m} \) is a constant uncertain matrix, \( S \in \mathbb{R}^{n \times n} \) is a subsequently defined auxiliary matrix, and the auxiliary control vector \( u_d(t) \triangleq \left[ u_{d1}(t) \cdots u_{dm}(t) \right]^T \in \mathbb{R}^m \). In (3.8), the unknown, unmeasurable auxiliary terms \( \tilde{N}(t) \) and \( N_d(t) \) are explicitly defined as

\[ \tilde{N} \triangleq A\dot{e} + \gamma\dot{e} + Se + \left( \dot{f}(x,t) - \dot{f}(x_m,t) \right) \]  

(3.9)

\[ N_d \triangleq A\dot{x}_m - \ddot{x}_m + \dot{f}(x_m,t) \]  

(3.10)

The motivation for the separation of terms as in (3.9) and (3.10) is based on the fact that the following bounding inequalities can be developed:

\[ \|\tilde{N}\| \leq \rho(\|z\|) \|z\|, \quad \|N_d\| \leq \zeta_{N_d}, \quad \|\tilde{N}_d\| \leq \zeta_{\tilde{N}_d} \]  

(3.11)

where \( \rho_0(\cdot) \in \mathbb{R} \) is a positive globally invertible non-decreasing function; \( \zeta_{N_d}, \zeta_{\tilde{N}_d} \in \mathbb{R}^+ \).
3.2. CONTROL DEVELOPMENT

are known bounding constants and \( z(t) \in \mathbb{R}^{2n} \) is an augmented tracking error vector that is defined as

\[
z \triangleq \begin{bmatrix} e^T & r^T \end{bmatrix}^T.
\]  

(3.12)

3.2.2 Closed-loop Error System

Based on the open-loop error dynamics in (3.8) and the subsequent stability analysis, the auxiliary control term \( u_d(t) \) is designed as

\[
u_d(t) = \hat{\Omega}^\# (\mu_0 - \mu_1) \tag{3.13}
\]

where \( \hat{\Omega} \in \mathbb{R}^{n \times m} \) is a constant estimate of \( \Omega \), and \([\cdot]^\#\) denotes the matrix pseudoinverse. In (3.13) \( \mu_0(t), \mu_1(t) \in \mathbb{R}^n \) are subsequently defined feedback control terms. After substituting the time derivative of (3.13) into (3.8), the error dynamics can be expressed as

\[
\dot{r} = \tilde{N} + N_d + \tilde{\Omega} (\dot{\mu}_0 - \dot{\mu}_1) - Se
\]  

(3.14)

where the constant uncertain matrix \( \tilde{\Omega} \in \mathbb{R}^{n \times n} \) is defined as

\[
\tilde{\Omega} = \Omega \hat{\Omega}^\#.
\]  

(3.15)

Lemma 2. (Morse, 1993) Any positive definite matrix \( X \in \mathbb{R}^{n \times n} \) can be decomposed as

\[
X = ST
\]  

(3.16)

where \( S \in \mathbb{R}^{n \times n} \) is a positive definite symmetric matrix and \( T \in \mathbb{R}^{n \times n} \) is an unity upper triangular matrix.
3.2. CONTROL DEVELOPMENT

Proof. Proof of Lemma 2 can be found in (Morse, 1993).

Property 1. Since the matrix $S$ introduced in (3.16) is positive definite and symmetric, its inverse $S^{-1}$ is also positive definite and symmetric. This property will be utilized in the subsequent stability analysis.

Assumption 3. Upper and lower bounds on the elements of the uncertain constant matrix $\Omega \in \mathbb{R}^{n \times n}$ are known such that the constant feed forward estimate $\hat{\Omega} \in \mathbb{R}^{n \times n}$ can be chosen to render the product $\tilde{\Omega} = \Omega \hat{\Omega}^{-1}$ positive definite. Further, the estimate $\hat{\Omega}$ is selected such that

$$\tilde{\Omega} = ST$$

where the unity upper triangular matrix $T$ satisfies the diagonal dominance property

$$\varepsilon \leq |T_{ii}| - \sum_{k=i+1}^{n} |T_{ik}| \leq Q, \quad i = 1, \ldots, n - 1.$$ (3.18)

where $\varepsilon \in (0, 1)$ and $Q \in \mathbb{R}^+$ are known bounding constants, and $T_{ik} \in \mathbb{R}$ denotes the $(i, k)^{th}$ element of the matrix $T$. In (3.17), the matrices $S$ and $T$ are defined in a manner similar to Lemma 2.

Remark 5. The subsequent numerical simulation results demonstrate that Assumption 3 is satisfied over a significant range of uncertainty between the estimated and actual values of the uncertain input-multiplicative matrix (i.e., deviations between $\hat{\Omega}$ and $\Omega$). Specifically, the results show that asymptotic trajectory tracking is achieved when the constant estimates $\hat{\theta}_{1j}$ and $\hat{\theta}_{2j}$ for $j = 1, \ldots, m$ deviate from the actual values by more than 35%.

After using the decomposition technique in (3.17), the open loop error dynamics
in (3.14) can be expressed as

\[
S^{-1} \dot{r} = \tilde{N}_1 + N_{d1} + T(\mu_0 - \dot{\mu}_1) - e
\]

(3.19)

where

\[
\tilde{N}_1 \triangleq S^{-1} \tilde{N}, \quad N_{d1} \triangleq S^{-1} N_d.
\]

(3.20)

Since \(S\) is positive definite, \(\tilde{N}_1(t)\) and \(N_{d1}(t)\) satisfy the inequalities

\[
\left\| \tilde{N}_1 \right\| \leq \rho_1(\|z\|) \|z\|, \quad \|N_{d1}\| \leq \zeta_{N_{d1}}, \quad \left\| \dot{N}_{d1} \right\| \leq \zeta_{\dot{N}_{d1}}
\]

(3.21)

where \(\rho_1(\cdot) \in \mathbb{R}\) is a positive, globally invertible non-decreasing function; \(\zeta_{N_{d1}}, \zeta_{\dot{N}_{d1}} \in \mathbb{R}^+\) are known bounding constants. By using the fact that the uncertain matrix \(T\) is unity upper triangular, the error dynamics in (3.19) can be rewritten as

\[
S^{-1} \dot{r} = \tilde{N}_1 + N_{d1} + \dot{\mu}_0 + \bar{T} \mu_0 - T\dot{\mu}_1 - e
\]

(3.22)

where \(\bar{T} \triangleq T - I_{n \times n}\) is a \textit{strictly upper triangular} matrix, and \(I_{n \times n}\) denotes the \(n \times n\) identity matrix. Based on (3.22) and the subsequent stability analysis, the auxiliary control terms \(\mu_0(t)\) and \(\mu_1(t)\) are designed as

\[
\mu_0 = -(k_s + I_{n \times n}) e(t) - (k_s + I_{n \times n}) e(0) - \int_0^t \gamma (k_s + I_{n \times n}) e(\tau) d\tau.
\]

(3.23)

\[
\mu_1 = \int_0^t \beta \text{sgn}(e(\tau)) d\tau
\]

(3.24)

where \(\beta, k_s \in \mathbb{R}^{n \times n}\) are constant, positive definite, diagonal control gain matrices, and \(\gamma\) is introduced in (3.6). After substituting the time derivative of (3.23) into
(3.2), the closed-loop error system is obtained as

\[ S^{-1}r = \bar{N}_1 + T\dot{\mu}_0 + N_{d1} - (k_s + I_{n\times n}) r - T\dot{\mu}_1 - e. \]  (3.25)

After taking the time derivative of (3.23), the term \( T\dot{\mu}_0 \) can be expressed as

\[ T\dot{\mu}_0 = \begin{bmatrix}
\sum_{j=2}^{n} \bar{T}_{1j}\dot{\mu}_{0j} \\
\sum_{j=3}^{n} \bar{T}_{2j}\dot{\mu}_{0j} \\
\vdots \\
\bar{T}_{(n-1)n}\dot{\mu}_{0n} \\
0
\end{bmatrix} = \begin{bmatrix}
\Lambda_{\rho} \\
0
\end{bmatrix} \]  (3.26)

where the auxiliary signal \( \Lambda_{\rho} \triangleq \left[ \Lambda_{\rho1} \Lambda_{\rho2} \cdots \Lambda_{\rho(n-1)} \right]^T \in \mathbb{R}^{n-1} \), with the individual elements defined as

\[ \Lambda_{\rho i} \triangleq -\sum_{j=i+1}^{n} \bar{T}_{ij} (k_{sj} + 1) r_j \]  (3.27)

for \( i = 1, \ldots, n - 1 \) where the subscript \( j \) indicates the \( j \)th element of the vector.

Based on the definitions in (3.23) and (3.26), \( \Lambda_{\rho} \) can be upper bounded as

\[ \|\Lambda_{\rho}\| \leq \rho_{\Lambda 1} \|z\| \]  (3.28)

where \( z(t) \) was previously defined in (3.12), and \( \rho_{\Lambda 1} \in \mathbb{R} \) is a known positive bounding constant.

**Remark 6.** Note that based on (3.26) and (3.27), the bounding constant \( \rho_{\Lambda 1} \) depends
only on elements \( i + 1 \) to \( n \) of the control gain matrix \( k_s \) due to the strictly upper triangular nature of \( \bar{T} \). Thus, the element \( \dot{\mu}_{01}(t) \) of the control vector \( \dot{\mu}_0(t) \) does not appear in the term \( \Lambda_\rho \). This fact will be utilized in the subsequent stability proof (MacKunis et al., 2013).

By utilizing (3.26), the error dynamics in (3.25) can be expressed as

\[
S^{-1} \dot{r} = \tilde{N}_2 + N_{d1} - (k_s + I_{n \times n}) r - T \dot{\mu}_1 - e
\]  

(3.29)

where

\[
\tilde{N}_2 = \tilde{N}_1 + \begin{bmatrix} \Lambda_\rho \\ 0 \end{bmatrix}.
\]  

(3.30)

Based on (3.21), (3.28), and (3.30), \( \tilde{N}_2(t) \) satisfies the inequality

\[
\| \tilde{N}_2 \| \leq \rho_2(\|z\|) \|z\|
\]  

(3.31)

where \( \rho_2(\cdot) \in \mathbb{R} \) is a positive, globally invertible non-decreasing function.

To facilitate the subsequent stability analysis, the control gain \( \beta \) introduced in (3.24) is selected to satisfy

\[
\beta > \frac{1}{\varepsilon} \left( \zeta_{N_{d1}} + \frac{1}{\gamma} \zeta_{\tilde{N}_{d1}} \right)
\]  

(3.32)

where \( \zeta_{N_{d1}} \) and \( \zeta_{\tilde{N}_{d1}} \) are introduced in (3.21), and \( \varepsilon \) is introduced in (3.18).
3.3 Stability Analysis

Let $\mathcal{D} \subset \mathbb{R}^{2n+1}$ be a domain containing $w(t) = 0$, where $w(t) \in \mathbb{R}^{2n+1}$ is defined as

$$w(t) \triangleq \left[ z^T(t) \sqrt{P(t)} \right]^T. \quad (3.33)$$

In (3.33), the auxiliary function $P(t) \in \mathbb{R}$ is defined as the generalized solution to the differential equation

$$\dot{P}(t) = -L(t) \quad (3.34)$$

$$P(0) = \beta Q |e(0)| - e^T(0) N_{d1}(0) \quad (3.35)$$

where the auxiliary function $L(t) \in \mathbb{R}$ is defined as

$$L(t) = r^T (N_{d1}(t) - T\hat{\mu}_1). \quad (3.36)$$

**Lemma 3.** Provided the sufficient condition in (3.32) is satisfied, the following inequality can be obtained:

$$\int_0^t L(\tau) \, d\tau \leq \beta Q |e(0)| - e^T(0) N_{d1}(0). \quad (3.37)$$

Hence, (3.37) can be used to conclude that $P(t) \geq 0$.

**Proof.** Proof of Lemma 3 can be found in the Appendix A. \qed

**Theorem 3.** The robust control law given by (1.2), (1.1), (3.23), and (3.24) achieves
asymptotic trajectory tracking in the sense that

\[ \|e(t)\| \to 0, \quad \text{as} \quad t \to \infty \quad (3.38) \]

provided the control gain matrix \( k_s \) introduced in (3.23) is selected sufficiently large, and \( \beta \) is selected to satisfy the sufficient condition in (3.32).

**Proof.** Let \( V(w,t) : \mathcal{D} \times [0, \infty) \to \mathbb{R} \) be a continuously differentiable, nonnegative function defined as

\[ V = \frac{1}{2} e^T e + \frac{1}{2} r^T S^{-1} r + P. \quad (3.39) \]

which satisfies the inequalities

\[ U_1(w) \leq V(w,t) \leq U_2(w) \quad (3.40) \]

provided the sufficient condition in (3.32) is satisfied. In (3.40), the continuous positive definite functions \( U_1(w), U_2(w) \in \mathbb{R} \) are defined as

\[ U_1(w) \triangleq \eta_1 \|w\|^2, \quad U_2(w) \triangleq \eta_2 \|w\|^2 \quad (3.41) \]

where \( \eta_1, \eta_2 \in \mathbb{R} \) are defined as

\[ \eta_1 \triangleq \frac{1}{2} \min \left\{ 1, \lambda_{\min}(S^{-1}) \right\}, \quad \eta_2 \triangleq \max \left\{ \frac{1}{2} \lambda_{\max}(S^{-1}), 1 \right\} \]

where \( \lambda_{\min}(\cdot), \lambda_{\max}(\cdot) \) denote the minimum and maximum eigenvalues of the arguments, respectively. After taking the time derivative of (3.39), utilizing (3.6), (3.29),
(3.34), and (3.36), and canceling common terms \( \dot{V}(t) \) can be expressed as

\[
\dot{V} = -\gamma \| e \|^2 - \| r \|^2 - r^T \left( \tilde{N}_2 - k_s r \right).
\] (3.42)

After using the upper bound for \( \tilde{N}_2(t) \) given in (3.31) and completing the squares for the parenthetic terms, \( \dot{V} \) can be upper bounded as

\[
\dot{V} \leq -\lambda_0 \| z \|^2 + \frac{\rho_2(\| z \|)^2}{4\lambda_{\min}(k_s)} \| z \|^2
- \lambda_{\min}(k_s) \left( \| r \|^2 - \rho_2(\| z \|) \| r \| \| z \| + \frac{\rho_2(\| z \|)^2}{4\lambda_{\min}^2(k_s)} \| z \|^2 \right)
\] (3.43)

where \( \lambda_0 \triangleq \min \{ \gamma, 1 \} \), and \( \lambda_{\min}(\cdot) \) denotes the minimum eigenvalue of the argument. The upper bound in (3.43) can be rewritten as

\[
\dot{V} \leq - \left( \lambda_0 - \frac{\rho_2(\| z \|)^2}{4\lambda_{\min}(k_s)} \right) \| z \|^2.
\] (3.44)

The following expression can be obtained from (3.44):

\[
\dot{V} \leq -U(w)
\] (3.45)

where \( U(w) = c \| z \|^2 \), for some positive constant \( c \in \mathbb{R} \) is a continuous positive semi-definite function that is defined on the domain

\[
D \triangleq \left\{ w(t) \in \mathbb{R}^{2n+1} \| w \| \leq \rho_2^{-1} \left( 2\sqrt{\lambda_{\min}(k_s)}\lambda_0 \right) \right\}.
\] (3.46)

The expressions (3.40) and (3.44) can be used to prove that \( e(t), r(t) \in L_\infty \) in \( D \).
Given that $e(t), r(t) \in \mathcal{L}_\infty$, (3.6) can be used to show that $\dot{e}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Given that $e(t), \dot{e}(t) \in \mathcal{L}_\infty$, (3.5) can be used along with Assumption 2 to prove that $x(t), \dot{x}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Based on the fact that $x(t) \in \mathcal{L}_\infty$, Assumption 1 can be utilized to show that $f(x,t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $x(t), \dot{x}(t), f(x,t) \in \mathcal{L}_\infty$, (3.1) can be used to show that $u(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $e(t), r(t) \in \mathcal{L}_\infty$, the time derivative of (3.23) and (3.24) can be used to show that $\dot{\mu}_0(t), \dot{\mu}_1(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Given that $e(t), r(t), \dot{\mu}_1(t) \in \mathcal{L}_\infty$, (3.29) can be used along with (3.31) to show that $\dot{r}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $\dot{e}(t), \dot{r}(t) \in \mathcal{L}_\infty$ can be used to show that $e(t),$ and $r(t)$ are uniformly continuous in $\mathcal{D}$. Thus, $z(t)$ is uniformly continuous throughout the closed-loop controller operation. Hence, $U(w)$ and $z(t)$ can be used to prove that $U(w)$ is uniformly continuous in $\mathcal{D}$.

Let $\mathcal{S} \subset \mathcal{D}$ denote a set defined as follows:

$$\mathcal{S} \triangleq \left\{ w(t) \subset \mathcal{D} | U(w(t)) \leq \eta_1 \left( \rho_2^{-1} \left( 2\sqrt{\lambda_{\text{min}}(k_s)\lambda_0} \right) \right)^2 \right\}. \quad (3.47)$$

Theorem 8.4 of (Khalil, 2002) can now be invoked to state that

$$c \| z(t) \|^2 \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \forall \ w(t_0) \in \mathcal{S}. \quad (3.48)$$

Based on the definition of $z(t)$, (3.48) can be used to show that

$$\| e(t) \| \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty \quad \forall \ w(t_0) \in \mathcal{S}. \quad (3.49)$$

Thus, asymptotic regulation of the pitching and plunging displacements can be achieved, provided the initial conditions are within the set $\mathcal{S}$, where $\mathcal{S}$ can be made arbitrarily large by increasing the control gain $k_s$. Hence, this is a semi-global asymptotic
3.4 Simulation Results

A numerical simulation was created to test the performance of the control design in (1.1), (1.2), (3.13), (3.23), and (3.24). The simulation is based on the dynamic model in (3.1) and (1.1), where \( n = 3 \) and \( m = 6 \) (i.e., 3-DOF flight control using 6 SJA arrays). The state vector contains the roll, pitch, and yaw rates, and the tracking error vector can be expressed as

\[
e(t) = \begin{bmatrix} e_1(t) & e_2(t) & e_3(t) \end{bmatrix}^T.
\]

The state and input matrices, \( A \) and \( B \), and reference state and reference input matrices, \( A_m \) and \( B_m \), are defined based on the Barron Associates nonlinear tailless aircraft model (BANTAM) (for further details of the simulation model, see (Deb et al., 2007)). The 3-DOF linearized model for the BANTAM was obtained analytically during trim conditions, where \( M = 0.455 \) is the Mach number, \( \alpha = 2.7 \text{ [deg]} \) is angle of attack, and \( \beta_s = 0 \text{ [deg]} \) denotes the side slip angle. The wind gust model used in the simulation is based on the FAR discrete gust model in (3.3) as described in (Part, 2015) at a velocity of \( U_{ds} = 10.12 \text{ [m/s]} \), \( H = 15.24 \text{ [m]} \), and \( V_0 = 25 \text{ [m/s]} \). The simulation model for the wind gust is based on the expression in (3.3), see Figure 3.1.

The reference state and input matrices used in the simulation are explicitly defined as
3.4. SIMULATION RESULTS

![Simulation Results Diagram]

Figure 3.1: Vertical wind gust velocity using the FAR model.

\[
A_m = \begin{bmatrix}
-61.1446 & 0 & -7.5238 \\
0 & -174.3473 & 0 \\
-7.1579 & 0 & -1.4007 \\
\end{bmatrix}
\quad (3.49)
\]

\[
B_m = \begin{bmatrix}
-1.7517 \\
0 \\
0.3096 \\
\end{bmatrix}
\quad (3.50)
\]

The matrices \( A_m \in \mathbb{R}^{3 \times 3} \) and \( B_m \in \mathbb{R}^{3} \). The model reference (desired) state \( x_m(t) \) in the simulation represents the desired external body axis motion that is generated in response to a reference command of (see (3.4))

\[
\delta(t) = \sin(t).
\]
3.4. SIMULATION RESULTS

The matrices $A$ and $B$ were obtained analytically from the dimensional aerodynamic coefficients of the BANTAM (Deb et al., 2007). These matrices are given by

\[
A = \begin{bmatrix}
-61.1273 & 0 & -7.6409 \\
0 & -174.3472 & 0 \\
-7.2692 & 0 & -0.4543 \\
\end{bmatrix}
\]

(3.51)

\[
B = \begin{bmatrix}
-0.2292 & 0.2292 & -0.2292 & 0.2292 & -0.0306 & 0.0306 \\
0.0599 & 0.0599 & 0.0804 & 0.0804 & -0.0256 & 0.0256 \\
-0.0084 & 0.0084 & -0.535 & 0.0535 & 0.1177 & -0.1177 \\
\end{bmatrix}
\]

(3.52)

Figure 3.2: Closed-loop regulation of the steady state error during closed-loop operation for 20 sets of values of the SJA parameters $\theta_{1i}^\ast$ and $\theta_{2i}^\ast$ with added wind gust.

The results of 20 Monte Carlo-type simulations are shown in Figures 3.2 - 3.6. The results were obtained using control gains selected as $k_s = \text{diag}\{0.10, 0.15, 2.3\}$, $\gamma =$
3.4. SIMULATION RESULTS

Figure 3.3: Virtual deflection angle control commands for the first three SJA arrays (i.e., \( u_1(t) \), \( u_2(t) \), and \( u_3(t) \)) during closed-loop operation for 20 sets of values of the SJA parameters \( \theta^*_{1i} \) and \( \theta^*_{2i} \) with added wind gust.

\[ \begin{array}{c|cccccc} \theta^*_1 \text{ [deg]} & 32.9 & 29.8 & 26.7 & 24.0 & 20.5 & 17.8 \\ \theta^*_2 \text{ [Volt-deg]} & 14.7 & 13.8 & 12.8 & 11.7 & 10.0 & 9.5 \end{array} \]

Remark 7. The capability of the proposed robust nonlinear control method to compensate for SJA parameter deviations of more than 35% demonstrates a significant
3.4. SIMULATION RESULTS

Figure 3.4: Virtual deflection angle control commands for the last three SJA arrays (i.e., \( u_4(t) \), \( u_5(t) \), and \( u_6(t) \)) during closed-loop operation for 20 sets of values of the SJA parameters \( \theta^*_1 \) and \( \theta^*_2 \) with added wind gust.

improvement over standard adaptive control approaches (cf. (Deb et al., 2007, 2008)). Specifically, the results using

Figure 3.2 shows the closed-loop tracking error response and demonstrates rapid convergence of the tracking error to zero in all 20 cases. Figures 3.3 and 3.4 show the virtual surface deflection control commands during closed-loop operation, and Figures 3.5 and 3.6 show the SJA voltage control inputs commanded during closed-loop operation. The results demonstrate that the closed-loop system remains stable in all 20 cases, and asymptotic tracking is achieved throughout the range of uncertainty tested. Figure 3.7 shows the convergence of the actual UAV states to the model reference states during closed-loop operation for the first iteration of our Monte Carlo type simulation. The control commands remain within reasonable limits in all 20 cases.
3.5 Conclusion

A robust nonlinear control method that achieves asymptotic trajectory tracking for a SJA-based aircraft model is presented. The control method is proven to achieve semiglobal asymptotic tracking of a reference trajectory in the presence of SJA actuator parameter uncertainty in addition to external norm-bounded disturbances (i.e., vertical wind gusts). A rigorous stability analysis is carried out to prove that the region of attraction of the closed-loop system can be made arbitrarily large through judicious tuning of a control parameter. The controller is designed to be computationally inexpensive, requiring no function approximators, adaptive laws, or complex computations. By utilizing constant feedforward estimates of the uncertain SJA actuator parameters, a matrix decomposition technique is employed along with a novel
Figure 3.6: Control voltage signals commanded for the last three SJA arrays (i.e., \(v_4(t), v_5(t), \) and \(v_6(t)\)) during closed-loop operation for 20 sets of values of the SJA parameters \(\theta^*_1\) and \(\theta^*_2\) with added wind gust.

Figure 3.7: Model reference (red) and actual state (blue) during closed-loop controller operation in the presence of 35.75\% SJA parameter uncertainty with added wind gust.
error system derivation to compensate for significant SJA parametric uncertainty (i.e., greater than 35% uncertainty in the SJA parameters). Detailed Monte-Carlo-type numerical simulation results are included to illustrate the effectiveness of the proposed control strategy.
Chapter 4

Nonlinear Tracking Control and Structural Vibration Suppression for Aircraft using Synthetic Jet actuators

In this chapter, synthetic jet actuator-based control method is developed, which is rigorously proven to achieve accurate aircraft trajectory tracking control while simultaneously regulating limit cycle oscillations (LCO) in aircraft wings. This work was published in ICARCV 2016 with the title Nonlinear Tracking Control and Structural Vibration Suppression for Aircraft Using Synthetic Jet Actuators by (Ramos Pedroza, Kidambi, MacKunis, and Reyhanoglu, 2016).

The resulting dynamic model is then utilized to develop a nonlinear control method, which is proven to achieve asymptotic flight trajectory tracking in the pres-
ence of external disturbances in addition to LCO-induced disturbances. A Lyapunov-based stability analysis is utilized to prove semi-global asymptotic trajectory tracking in the presence of LCO disturbances and parametric uncertainty in the SJA actuator model. Numerical simulation results are provided to demonstrate the capability of the proposed SJA-based control method to achieve simultaneous trajectory tracking and LCO regulation.

4.1 Dynamic Model

To incorporate the LCO dynamics into the flight dynamic model, the overall SUAV dynamic model can be expressed as

\[ M(q)\ddot{q} + V_m(q, \dot{q})\dot{q} + G(q) + f(q, \dot{q}) = \tau(t) \]  \hfill (4.1)

\[ y = Cq \] \hfill (4.2)

where \( M(q) \in \mathbb{R}^{n\times n} \) denotes the inertia matrix, \( V_m(q, \dot{q}) \in \mathbb{R}^{n\times n} \) is the centripetal-Coriolis matrix, \( G(q) \in \mathbb{R}^n \) denotes the gravity vector, \( f(q, \dot{q}) \in \mathbb{R}^n \) denotes a general nonlinear disturbance (e.g., elastic forces, unmodeled effects, external disturbances), and \( \tau(t) \in \mathbb{R}^n \) denotes the control input (torque). In (4.2), \( y(t) \in \mathbb{R}^m \) contains the measurable flight states (i.e., roll, pitch, and yaw), and \( C \in \mathbb{R}^{m\times n} \) is a known output matrix. The subsequent control development is based on the assumption that \( M(q), V_m(q, \dot{q}), G(q), \) and \( f(q, \dot{q}) \) are unknown and unmeasurable.

For the combined LCO and tracking control objective addressed in this paper, the
state vector $q(t)$ is composed of three flight and $N$ structural mode variables as

$$q(t) = \begin{bmatrix} y^T(t) & H^T(t) \end{bmatrix}^T$$

(4.3)

where $y(t) \triangleq [\phi(t), \theta(t), \psi(t)]^T \in \mathbb{R}^3$ contains the flight states (i.e., roll, pitch, and yaw), and $H(t) \triangleq [\eta_1(t), \ldots, \eta_N(t)]^T \in \mathbb{R}^N$ contains the LCO (structural) mode displacements. To more clearly describe the coupling between the structural modes and the flight dynamics, the expressions in (4.1) - (4.3) can be utilized to obtain the following expressions:

$$M_a(q) \ddot{y} + C_a(q, \dot{y}) \dot{y} + G_a(q) + f_a(q, \dot{q}) = Bu$$

(4.4)

$$\ddot{\eta}_i - 2\zeta_i \omega_i \dot{\eta}_i - \omega_i^2 \eta_i = N_i(y, \dot{y}), \quad i = 1, \ldots, N.$$  

(4.5)

In (4.4), $M_a(q), C_a(q, \dot{q}) \in \mathbb{R}^{3 \times 3}$ denote inertia and centripetal coriolis matrices, respectively; $G_a(q), f_a(q, \dot{q}) \in \mathbb{R}^3$ denote gravity effects and unmodeled disturbances, respectively; $B \in \mathbb{R}^{3 \times 3}$ represents an uncertain input gain matrix, and $u(t) \in \mathbb{R}^3$ is the control input. In (4.5), $\zeta_i, \omega_i \in \mathbb{R}$, for $i = 1, \ldots, N$, denote the damping factor and natural frequency of the $i^{th}$ structural mode, respectively; and $N_i(q, \dot{q}) \in \mathbb{R}$, for $i = 1, \ldots, N$, denote generalized elastic forces, which depend on the flight states in general.

**Remark 8.** The configuration vector $q(t)$ includes the actuated roll, pitch, and yaw displacements, as well as the $N$ (unactuated) structural modes of vibration. Thus, the control application addressed in this paper is based on the scenario of $n = N + 3$ configuration variables with $m = 3$ control inputs; but the subsequent control development
and stability analysis are presented for the general case of configuration variables with inputs.

**Property 2.** The inertia matrix $M(q)$ is symmetric, positive definite, and satisfies the inequalities $m_1\|\xi\|^2 \leq \xi^TM(q)\xi \leq \bar{m}(q)\|\xi\|^2$, $\forall$ $\xi(t) \in \mathbb{R}^n$, where $m_1 \in \mathbb{R}^+$ is a known bounding constant, $\bar{m}(q) \in \mathbb{R}$ is a known positive function; and $\|\cdot\|$ denotes the standard Euclidean norm.

**Property 3.** The structural mode displacements $\eta_i(t)$, for $i = 1, \ldots, N$, are unmeasurable and are not available for feedback.

**Assumption 4.** If $q(t), \dot{q}(t) \in L_\infty$, then $M(q), V_m(q, \dot{q}), G(q), f(q, \dot{q}) \in L_\infty$. Moreover, if $q(t), \dot{q}(t) \in L_\infty$, then the first and second partial derivatives of $M(q), V_m(q, \dot{q}), G(q), f(q, \dot{q})$ with respect to $q(t)$ exist and are bounded.

**Assumption 5.** If $y(t), \dot{y}(t) \in L_\infty$, then the generalized elastic forces $N_i(y, \dot{y}) \in L_\infty$, for $i = 1, \ldots, N$.

**Assumption 6.** The desired trajectory is designed such that $y_\delta^{(i)}(t) \in L_\infty$, for $i = 1, \ldots, 4$.

### 4.2 Control Development

The tracking control objective is to ensure that the SUAV track a desired time-varying trajectory despite LCO-induced disturbances and uncertainties present in both the SUAV dynamics and in the SJA actuator model. To quantify the control objective, a position tracking error $e_1(t) \in \mathbb{R}^m$ is defined as

$$e_1 = y - y_\delta$$

(4.6)
where \( y_d(t) \in \mathbb{R}^m \) is a desired trajectory, and \( y(t) \) is introduced in (4.2). To facilitate the subsequent control development and stability analysis, auxiliary tracking error variables, denoted \( e_2(t), r(t) \in \mathbb{R}^m \), are also defined as

\[
\begin{align*}
    e_2 &= \dot{e}_1 + \alpha_1 e_1 \\
    r &= \dot{e}_2 + \alpha_2 e_2
\end{align*}
\] (4.7) (4.8)

where \( \alpha_1, \alpha_2 \in \mathbb{R}^+ \) denote constant control gains. Note that the filtered tracking error \( r(t) \) is not a measurable since the expression in (4.8) depends on the acceleration \( \ddot{y}(t) \).

### 4.2.1 Open-loop Error System

By premultiplying (4.8) by \( M_a(q) \) and using (4.1) - (4.4), (4.6), and (4.7), the following is obtained:

\[
M_a(q) r = \chi(t) + N_{d1}(t) + Bu
\] (4.9)

where \( \chi(t), N_{d1}(t) \in \mathbb{R}^m \) are unmeasurable auxiliary functions defined as

\[
\chi(t) \triangleq \begin{align*}
-C_a(q, \dot{q}) \ddot{y} + C_a(q_d, \dot{q}_d) \ddot{y}_d - G_a(q) + G_a(q_d) - f_a(q, \dot{q}) + f_a(q_d, \dot{q}_d) \\
-Ma(q) \ddot{y}_d + Ma(q_a) \ddot{y}_d + Ma(q) \alpha_1 (e_2 - \alpha_1 e_1) + Ma(q) \alpha_2 e_2
\end{align*}
\] (4.10)

\[
N_{d1} \triangleq \begin{align*}
-Ma(q_d) \ddot{y}_d - C_a(q_d, \dot{q}_d) \ddot{y}_d - G_a(q_d) - f_a(q_d, \dot{q}_d)
\end{align*}
\] (4.11)

In (4.10) and (4.11), \( q_d(t) \triangleq q(t)|_{y(t)=y_d(t)} \in \mathbb{R}^n \). After utilizing the SJA actuator model and robust-inverse control definition in (1.1) and (1.2), the expression in (4.9)
can be rewritten as

\[ M_a(q) r = \chi(t) + N_{d1}(t) + \Omega u_d \]  

(4.12)

where \( \Omega \in \mathbb{R}^{m \times m} \) is a constant, uncertain auxiliary matrix, and

\[ u_d(t) \triangleq \begin{bmatrix} u_{d1}(t) & \cdots & u_{dm}(t) \end{bmatrix}^T \in \mathbb{R}^m \]  

(4.13)

where \( u_{di}(t) \), for \( i = 1, \ldots, m \), are introduced in (1.2).

To facilitate the subsequent control design and stability analysis, the time derivative of (4.12) is determined as

\[ M_a(q) \dot{r} = -\frac{1}{2} \dot{M}_a(q) r + \dot{\chi} + N + N_{d2} + \Omega \dot{u}_d - S e_2 \]  

(4.14)

where \( S \in \mathbb{R}^{m \times m} \) denotes a subsequently defined auxiliary matrix; and the unknown, unmeasurable, auxiliary functions \( \tilde{N}(t) \), \( N_{d2}(t) \in \mathbb{R}^m \) are defined as

\[ \tilde{N} \triangleq -\frac{1}{2} \dot{M}_a(q) r + \dot{\chi} + S e_2 \]  

(4.15)

\[ N_{d2} \triangleq \dot{N}_{d1} \]  

(4.16)

The motivation for the selective grouping of terms in (4.15) and (4.16) is based on the fact that the following bounding inequalities can be developed:

\[ \|\tilde{N}\| \leq \rho_0(\|z\|)\|z\|, \quad \|N_{d2}\| \leq \zeta_1, \quad \|\dot{N}_{d2}\| \leq \zeta_2 \]

where \( \rho_0(\cdot) \in \mathbb{R} \) is a positive, globally invertible non-decreasing function; \( \zeta_1, \zeta_2 \in \mathbb{R}^+ \)
4.2. CONTROL DEVELOPMENT

are known bounding constants; and \( z(t) \in \mathbb{R}^{3m} \) is defined as

\[
  z(t) \triangleq \begin{bmatrix}
    e_1^T(t) & e_2^T(t) & r^T(t)
  \end{bmatrix}^T.
\] (4.17)

4.2.2 Closed-loop Error System

Based on the open-loop error dynamics in (4.14) and the subsequent stability analysis, the auxiliary control signal \( u_d(t) \) is designed as

\[
u_d(t) = \hat{\Omega}^{-1} (\mu_0 - \mu_1)
\] (4.18)

where \( \hat{\Omega} \in \mathbb{R}^{m \times m} \) is a constant feedforward estimate of the uncertain matrix \( \Omega \) (i.e., \( \hat{\Omega} \) contains the feedforward estimates \( \hat{\theta}_1, \hat{\theta}_2 \), for \( i = 1, \ldots, m \)). In (4.18) \( \mu_0(t) \), \( \mu_1(t) \in \mathbb{R}^m \) are subsequently defined feedback control terms. After substituting the time derivative of (4.18) into (4.14), the error dynamics can be expressed as

\[
  M_a(q) \dot{r} = -\frac{1}{2} \dot{M}_a(q)r + \dot{N} + N_{d2} + \hat{\Omega} (\dot{\mu}_0 - \dot{\mu}_1) - Se_2
\] (4.19)

where the constant uncertain matrix \( \hat{\Omega} \in \mathbb{R}^{m \times m} \) is defined as

\[
  \hat{\Omega} = \Omega \hat{\Omega}^{-1}.
\] (4.20)

Lemma 4. Any positive definite matrix \( X \in \mathbb{R}^{m \times m} \) can be decomposed as

\[
  X = ST
\] (4.21)
where $S \in \mathbb{R}^{m \times m}$ is a positive definite, symmetric matrix and $T \in \mathbb{R}^{m \times m}$ is a unity upper triangular matrix (Morse, 1993).

**Proof.** Proof of Lemma 4 can be found in (Morse, 1993). \hfill \Box

**Property 4.** Since the matrix $S$ introduced in (4.21) is positive definite and symmetric, its inverse $S^{-1}$ is also positive definite and symmetric. This property will be utilized in the subsequent stability analysis.

**Assumption 7.** Bounds on the uncertain matrix $\Omega \in \mathbb{R}^{m \times m}$ are known such that the constant feedforward estimate $\hat{\Omega} \in \mathbb{R}^{m \times m}$ can be chosen to render the product $\tilde{\Omega} = \Omega \hat{\Omega}^{-1}$ positive definite. Further, the estimate $\tilde{\Omega}$ is selected such that

$$\tilde{\Omega} = ST$$

(4.22)

where the unity upper triangular matrix $T$ is diagonally dominant in the sense that

$$\epsilon \leq |T_{ii}| - \sum_{k=i+1}^{m} |T_{ik}| \leq Q, \quad i = 1, \ldots, m - 1$$

(4.23)

where $\epsilon \in (0, 1)$ and $Q \in \mathbb{R}^+$ are known bounding constants, and $T_{ik} \in \mathbb{R}$ denotes the $(i,k)^{th}$ element of the matrix $T$. In (4.22), the matrices $S$ and $T$ are defined in a similar manner to that in Lemma 4.

After using the decomposition technique in (4.22), the open loop error dynamics in (4.19) can be expressed as

$$M_s(q)\dot{r} = -\frac{1}{2} \dot{M}_s(q)r + \tilde{N}_1 + N_{d3} + T(\dot{\mu}_0 - \dot{\mu}_1) - e_2$$

(4.24)
where \( M_s(q) \triangleq S^{-1} M_a(q) \in \mathbb{R}^{m \times m} \) is positive definite and symmetric. In (4.24), the auxiliary terms \( \tilde{N}_1(t) \) and \( N_{d3}(t) \) are defined as

\[
\tilde{N}_1 \triangleq S^{-1} \tilde{N}, \quad N_{d3} \triangleq S^{-1} N_{d2}.
\] (4.25)

Since \( S \) is positive definite, \( \tilde{N}_1(t) \) and \( N_{d3}(t) \) satisfy the inequalities

\[
\| \tilde{N}_1 \| \leq \rho_1(\| z \|) \| z \|, \quad \| N_{d3} \| \leq \zeta_3, \quad \| \dot{N}_{d3} \| \leq \zeta_4
\]

where \( \rho_1(\cdot) \in \mathbb{R} \) is a positive, globally invertible non-decreasing function; and \( \zeta_3, \zeta_4 \in \mathbb{R}^+ \) are known bounding constants. By using the fact that the uncertain matrix \( T \) is unity upper triangular, the error dynamics in (4.24) can be rewritten as

\[
M_s(q) \dot{r} = -\frac{1}{2} M_s(q) r + \tilde{N}_1 + N_{d3} + \dot{\mu}_0 + \bar{T} \dot{\mu}_0
\] (4.26)

\[
-\bar{T} \dot{\mu}_1 - e_2
\]

where \( \bar{T} \triangleq T - I_{m \times m} \) is a strictly upper triangular matrix, and \( I_{m \times m} \) denotes the \( m \times m \) identity matrix. Based on (4.26) and the subsequent stability analysis, the auxiliary control terms \( \mu_0(t) \) and \( \mu_1(t) \) are designed as

\[
\mu_0 = -(k_s + I_{m \times m}) e_2(t) + (k_s + I_{m \times m}) e_2(0) - \int_0^t \alpha_2(k_s + I_{m \times m}) e_2(\tau) d\tau
\] (4.27)

\[
\mu_1 = \int_0^t \beta \text{sgn}(e_2(\tau)) d\tau
\] (4.28)

where \( \beta \in \mathbb{R}^+ \) is a control gain; \( k_s \in \mathbb{R}^{m \times m} \) is a constant, positive definite, diago-
nal control gain matrix; and $\alpha_2$ is introduced in (4.8). After substituting the time derivative of (4.27) into (4.26), the closed-loop error system is obtained as

$$M_s(q)\dot{r} = -\frac{1}{2} \dot{M}_s(q)r + \bar{N}_1 + \tilde{T}\dot{\mu}_0$$

$$- (k_s + I_{m \times m}) r + N_{d3} - T\dot{\mu}_1 - e_2.$$  

(4.29)

**Property 5.** Note that, based on the strictly upper triangular structure of the matrix $\bar{T}$, the product $\bar{T}\dot{\mu}_0(t)$ contains only the elements $k_{s2}, ..., k_{sm}$ of the control gain matrix $k_s = \text{diag}\{k_{s1}, k_{s2}, ..., k_{sm}\}$ (MacKunis et al., 2013). Thus, the first element $k_{s1}$ of the control gain matrix $k_s$ can be utilized to compensate for the uncertain term $\bar{T}\dot{\mu}_0(t)$. This fact will be leveraged in the subsequent stability proof.

By considering Property 5, the error dynamics in (4.29) can be expressed as

$$M_s(q)\dot{r} = -\frac{1}{2} \dot{M}_s(q)r + \tilde{N}_2 - (k_s + I_{m \times m}) r + N_{d3} - T\dot{\mu}_1 - e_2$$

(4.30)

where

$$\tilde{N}_2 \triangleq \tilde{N}_1 + \tilde{T}\dot{\mu}_0.$$  

(4.31)

Based on (4.25), (4.31), and Property 5, $\tilde{N}_2(t)$ satisfies the inequality

$$\|\tilde{N}_2\| \leq \rho_2(\|z\|)\|z\|.$$  

(4.32)

where $\rho_2(\cdot) \in \mathbb{R}$ is a positive, globally invertible non-decreasing function.

To facilitate the subsequent stability analysis, the control gain $\beta$ introduced in
(4.28) is selected to satisfy
\[ \beta > \frac{1}{\varepsilon} \left( \zeta_3 + \frac{1}{\alpha_2} \zeta_4 \right) \quad (4.33) \]
where \( \zeta_3 \) and \( \zeta_4 \) are introduced in (4.25), and \( \varepsilon \) is introduced in (4.23).

4.3 Stability Analysis

Let \( \mathcal{D} \subset \mathbb{R}^{3m+1} \) be a domain containing \( w(t) = 0 \), where \( w(t) \in \mathbb{R}^{3m+1} \) is defined as
\[ w(t) \triangleq \begin{bmatrix} z^T(t) & \sqrt{P(t)} \end{bmatrix}^T. \quad (4.34) \]

In (4.34), the auxiliary function \( P(t) \in \mathbb{R} \) is defined as the generalized solution to the differential equation
\[ \dot{P}(t) = -L(t) \quad (4.35) \]
\[ P(0) = \beta Q |e_2(0)| - e_2^T(0) N_{d3}(0) \quad (4.36) \]
where \( Q \) is introduced in (4.23), and the auxiliary function \( L(t) \in \mathbb{R} \) is defined as
\[ L(t) = r^T(N_{d3}(t) - T\mu_1). \quad (4.37) \]

**Lemma 5.** Provided the sufficient condition in (4.33) is satisfied, the following inequality can be obtained:
\[ \int_0^t L(\tau) \, d\tau \leq \beta Q |e_2(0)| - e_2^T(0) N_{d3}(0). \quad (4.38) \]
4.3. STABILITY ANALYSIS

Hence, (4.38) can be used to conclude that $P(t) \geq 0$.

**Proof.** Proof of Lemma 5 can be found in Appendix A.

**Theorem 4.** The control law given by (1.2), (4.18), (4.27), and (4.28) achieves asymptotic trajectory tracking in the sense that

$$\|e_1(t)\| \to 0, \quad \text{as} \quad t \to \infty \quad (4.39)$$

provided the control gain matrix $k_s$ introduced in (4.27) is selected sufficiently large, and $\beta$ is selected to satisfy the sufficient condition in (4.33).

**Proof.** Let $V(w, t) : \mathcal{D} \times [0, \infty) \to \mathbb{R}$ be a radially unbounded, positive definite function defined as

$$V(w, t) = \frac{1}{2} e_1^T e_1 + \frac{1}{2} e_2^T e_2 + \frac{1}{2} r^T M_s(q)r + P \quad (4.40)$$

which satisfies the inequalities

$$U_1(w) \leq V(w, t) \leq U_2(w) \quad (4.41)$$

provided the sufficient condition in (4.33) is satisfied. In (4.41), the continuous positive definite functions $U_1(w), U_2(w) \in \mathbb{R}$ are defined as

$$U_1(w) \triangleq \gamma_1\|w\|^2 \quad (4.42)$$

$$U_2(w) \triangleq \gamma_2\|w\|^2$$
where $\gamma_1, \gamma_2 \in \mathbb{R}$ are defined as

$$\gamma_1 \triangleq \frac{1}{2} \min \{1, \lambda_{\min}(M_s(q))\}, \quad \gamma_2 \triangleq \max \left\{ \frac{1}{2} \lambda_{\max}(M_s(q)), 1 \right\}$$

where $\lambda_{\min}(\cdot), \lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues of the arguments, respectively. After taking the time derivative of (4.40), utilizing (4.7), (4.8), (4.30), (4.35), and (4.37), and canceling common terms, the time derivative of (4.40) can be expressed as

$$\dot{V}(w, t) = -\|r\|^2 - \alpha_1 \|e_1\|^2 - \alpha_2 \|e_2\|^2 + e_1^T e_2$$

$$-r^T k_s r + r^T \tilde{N}_2$$

(4.43)

Note that Young’s inequality can be utilized to upper bound the product $e_1^T e_2$ as

$$e_2^T e_1 \leq \frac{1}{2} \|e_1\|^2 + \frac{1}{2} \|e_2\|^2.$$  

(4.44)

By using the upper bounds given in (4.32) and (4.44), (4.43) can be upper bounded as

$$\dot{V}(w, t) \leq -\lambda_0 \|z\|^2 - \lambda_{\min}(k_s)(\|r\|^2 - \frac{\rho_2(\|z\|)}{\lambda_{\min}(k_s)} \|r\| \|z\|)$$

(4.45)

where $\lambda_0 \triangleq \min \{\alpha_1 - \frac{1}{2}, \alpha_2 - \frac{1}{2}, 1\}$. After completing the squares, the upper bound in (4.45) can be expressed as

$$\dot{V}(w, t) \leq -\lambda_0 \|z\|^2 + \frac{\rho_2(\|z\|)}{4 \lambda_{\min}(k_s)} \|z\|^2.$$  

(4.46)
4.3. STABILITY ANALYSIS

The following expression can be obtained from (4.46):

$$\dot{V} \leq -U(w) \quad (4.47)$$

where $U(w) = c\|z\|^2$, for some positive constant $c \in \mathbb{R}$, is a continuous positive semi-definite function that is defined on the domain

$$\mathcal{D} \triangleq \{ w(t) \in \mathbb{R}^{3m+1} \|w\| \leq \rho_2^{-1}\left(2\sqrt{\lambda_{\min}(k_s)\lambda_0}\right) \}. \quad (4.48)$$

The expressions (4.41) and (4.47) can be used to prove that $e_1(t), e_2(t), r(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Given that $e_1(t), e_2(t), r(t) \in \mathcal{L}_\infty$, (4.7) and (4.8) can be used to show that $\dot{e}_1(t), \dot{e}_2(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $e_1(t), \dot{e}_1(t), \dot{e}_2(t), r(t) \in \mathcal{L}_\infty$, (4.6), (4.7), and (4.8) can be used with Assumption 6 to prove that $y(t), \dot{y}(t), \ddot{y}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Given that $y(t), \dot{y}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, (4.3) and (4.5) can be used with Assumption 5 to prove that $\eta_i(t), \dot{\eta}_i(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, for $i = 1, \ldots, 3$, and $q(t), \dot{q}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $e_1(t), \dot{e}_1(t), e_2(t), r(t), q(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$, (4.9) can be used with (4.11) and Assumptions 4 and 6 to prove that $u(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $e_1(t), e_2(t), r(t) \in \mathcal{L}_\infty$, the time derivative of (4.27) and (4.28) can be used to show that $\dot{\mu}_0(t), \dot{\mu}_1(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Given that $e_1(t), e_2(t), r(t), \dot{\mu}_1(t) \in \mathcal{L}_\infty$, (4.30) can be used along with Assumption 4 and the bounds in (4.25) and (4.32) to show that $\dot{r}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $\dot{e}_1(t), \dot{e}_2(t), \dot{r}(t) \in \mathcal{L}_\infty$, $e_1(t), e_2(t)$ and $r(t)$ are uniformly continuous in $\mathcal{D}$. Thus, $z(t)$ is uniformly continuous throughout the closed-loop controller operation. Hence, the definitions of $U(w)$ and $z(t)$ can be used to prove that $U(w)$ is uniformly continuous in $\mathcal{D}$. 

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4.4 Simulation Results

Let \( S \subset \mathcal{D} \) denote a set defined as follows:

\[
S \triangleq \left\{ w(t) \in \mathbb{R}^{3m+1} \mid U_2(w(t)) < \gamma_1 \left( \rho_2^{-1} \left( 2\sqrt{\lambda_{\text{min}}(k_s)\lambda_0} \right) \right)^2 \right\}.
\]

Theorem 8.4 of (Khalil, 2002) can now be invoked to state that

\[
c\|z(t)\|^2 \to 0 \quad \text{as} \quad t \to \infty \quad \forall \ w(t_0) \in S.
\] (4.49)

Based on the definition of \( z(t) \), (4.49) can be used to show that

\[
e_1(t) \to 0 \quad \text{as} \quad t \to \infty \quad \forall \ w(t_0) \in S.
\]

Thus, asymptotic tracking of the desired attitude trajectory can be achieved, provided the initial conditions are within the set \( S \), where \( S \) can be made arbitrarily large by increasing the gain \( k_s \). Hence, this is a semi-global asymptotic result.

4.4 Simulation Results

A numerical simulation was created to test the performance of the proposed control law described in (1.1), (1.2), (4.18), (4.27), and (4.28). To facilitate the simulation design, the dynamic equations in (4.1) are expressed as a set of first-order differential equation, where the simulation configuration vector \( x(t) \in \mathbb{R}^{12} \) includes the 3 flight states, the first 3 structural mode displacements, and their rates (i.e., \( m = 3 \) and \( N = 3 \)) as \( x = [\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi}, \eta_1, \eta_2, \eta_3, \dot{\eta}_1, \dot{\eta}_2, \dot{\eta}_3]^T \). The explicit definitions of the state matrix \( A \in \mathbb{R}^{12 \times 12} \) and input gain matrix \( B \in \mathbb{R}^{12 \times 3} \) were adapted from (MacKunis, Patre, Kaiser, & Dixon, 2010) and are omitted here to adhere to page
4.4. SIMULATION RESULTS

Table 4.1: Constant nominal and estimated values for the SJA parameters used in the simulation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1^*$ [deg]</td>
<td>32.9</td>
<td>29.8</td>
<td>26.7</td>
</tr>
<tr>
<td>$\theta_2^*$ [Volt-deg]</td>
<td>24.0</td>
<td>20.5</td>
<td>17.8</td>
</tr>
<tr>
<td>$\hat{\theta}_1$ [deg]</td>
<td>14.7</td>
<td>13.8</td>
<td>12.8</td>
</tr>
<tr>
<td>$\hat{\theta}_2$ [Volt-deg]</td>
<td>13.7</td>
<td>13.5</td>
<td>12.0</td>
</tr>
</tbody>
</table>

constraints.

The simulation model for the structural modes is described in (4.5), where the damping coefficients and natural frequencies for the first three structural modes in the simulation are $\zeta_i = 0.02$, for $i = 1, 2, 3$ and $\omega_1 = 45$ [rad/s], $\omega_2 = 50$ [rad/s], $\omega_3 = 55$ [rad/s]. In the simulation, the generalized elastic forces $N_i(y, \dot{y})$, for $i = 1, 2, 3$, are assumed to depend linearly on the pitch rate $\dot{\theta}(t)$.

The simulations results were obtained using control gains selected as $\alpha_1 = 7$, $\alpha_2 = 0.09$, $\beta = 0.500$, $k_s = 100.55I_{3\times3}$. An unknown, non-vanishing external disturbance is included in the simulation, which affects the roll, pitch, and yaw moments as $d(t) = [0.1 \sin t, 0.1 \sin t, 0.1 \sin t]^T$.

Figure 4.1 shows the closed-loop trajectory tracking error response and demonstrates rapid convergence of the tracking error to zero. Figure 4.2 also shows the roll, pitch, and yaw rates during closed loop operation. Figure 4.3 show the convergence of the structural mode displacements and Figure 4.4 their rates, Figure 4.5 shows the SJA virtual surface deflection control inputs commanded during closed-loop operation. The results demonstrate that the closed-loop system achieves the proposed control objective, with control inputs remaining within reasonable limits.
4.4. SIMULATION RESULTS

Figure 4.1: Closed-loop regulation of the position states (roll, pitch, yaw) using the proposed control law.

Figure 4.2: Closed-loop regulation of the rate states (roll rate, pitch rate, yaw rate) using the proposed control law.
4.5 Conclusion

A nonlinear SJA-based control method is presented, which is proven to achieve asymptotic SUAV trajectory tracking control while simultaneously regulating LCO in SUAV.
4.5. CONCLUSION

To achieve the result, constant feedforward SJA parameter estimates are utilized to develop a robust-inverse control method, which compensates for the nonlinearity and parametric uncertainty in the SJA actuator model. A matrix decomposition technique is then utilized in the error system development to compensate for the input-multiplicative parametric uncertainty. By combining the LCO dynamics with the flight tracking dynamics in an advantageous form, a nonlinear control method is developed, which is proven to achieve asymptotic trajectory tracking in the presence of external disturbances and structural disturbances due to LCO. A rigorous Lyapunov-based stability analysis is utilized to prove semi-global asymptotic trajectory tracking in the presence of LCO disturbances and parametric uncertainty in the SJA actuator model. Numerical simulation results are provided, which demonstrate the capability of the proposed SJA-based control method to achieve simultaneous trajectory tracking and LCO regulation.
Chapter 5

A Robust Nonlinear Output Feedback Control Method for Limit Cycle Oscillation Suppression using Synthetic Jet Actuators

Generally, limit cycle oscillation suppression systems are usually designed based on the assumption that the full state (i.e., pitching and plunging displacement and velocity measurements) is available for feedback (Dardel & Bakhtiari-Nejad, 2013; Liu, Lara-Rosano, & Chan, 2004; Sun, Haghighat, H.T. Liu, & Bai, 2015; D. Li, Xiang, & Guo, 2011; K. V. Singh, 2015; Wu, Chen, Gong, & Ye, 2014b). Although the availability of velocity measurements is a standard assumption, velocity information can be difficult
5.1 Dynamic Model

In this section, a detailed mathematical model of the pitching and plunging dynamics in an airfoil will be presented, which incorporates nonlinear stiffness effects, unmodeled external disturbances, and SUAV model uncertainty. An additional challenge addressed in the control design is the parametric uncertainty and nonlinearity that is inherent in the SJA dynamic model. A rigorous Lyapunov-based stability analysis is utilized to prove the theoretical result, and numerical simulation results are provided to demonstrate the performance of the proposed control law.
modeled nonlinear external disturbances, and the uncertain nonlinear SJA actuator
dynamics. To facilitate the control design, the LCO dynamics will be expressed in
an advantageous form, which will be utilized to design the LCO suppression control
law.

The equation describing LCO in an UAV wing can be expressed as (Elhami &
Narab, 2012)

\[
M_s \ddot{p} + C_s \dot{p} + F(p)p + d(t) = \begin{bmatrix} -F_L \\ M \end{bmatrix}
\] (5.1)

where the coefficients \(M_s, C_s \in \mathbb{R}^{2 \times 2}\) are the structural mass and damping matrices;
\(F(p(t)) \in \mathbb{R}^{2 \times 2}\) is a nonlinear stiffness matrix; and \(p(t) \triangleq \begin{bmatrix} h(t) & \alpha(t) \end{bmatrix}^T \in \mathbb{R}^2\)
denotes the state vector, where \(h(t), \alpha(t) \in \mathbb{R}\) denote the plunging \([m]\) and pitching
\([\text{rad}]\) displacements, respectively. In (5.1), \(d(t) \in \mathbb{R}^2\) represents a general unknown,
norm-bounded, nonvanishing disturbance.

**Assumption 8.** The disturbance \(d(t)\) is bounded and sufficiently smooth such that
\(d(t), \dot{d}(t) \in L_\infty\) throughout closed-loop operation.

**Property 6.** The structural mass matrix \(M_s\) is positive definite and symmetric (see
(Elhami & Narab, 2012) and (S. N. Singh & Brenner, 2003)).

In (5.1), the structural linear mass, \(M_s\), structural linear damping, \(C_s\), and the
nonlinear stiffness, \(F(p)\), matrices are described as (Elhami & Narab, 2012)

\[
M_s = \begin{bmatrix} m & mx_ab \\ mx_ab & I_\alpha \end{bmatrix}, \quad C_s = \begin{bmatrix} C_h & 0 \\ 0 & C_\alpha \end{bmatrix}, \quad F(p) = \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha(\alpha) \end{bmatrix}
\] (5.2)
5.1. DYNAMIC MODEL

where \( x_\alpha \in \mathbb{R} \) denotes the non-dimensional distance measured from the elastic axis to the center of mass, \( b \in \mathbb{R} \) is the semi-chord of the wing [m], \( m \in \mathbb{R} \) is the mass of the wing section [kg], and \( I_\alpha \in \mathbb{R} \) is the mass moment of inertia of the wing about the elastic axis [kg\cdot m^2]. The parameter \( C_h \in \mathbb{R} \) denotes the structural damping coefficient in plunge due to viscous damping [kg/s], and \( C_\alpha \in \mathbb{R} \) denotes the structural damping coefficient in pitch due to viscous damping [kg\cdot m^2/s]. The \( K_h \in \mathbb{R} \) is the structural spring constant in plunge [N/m]; and \( K_\alpha (\alpha (t)) \in \mathbb{R} \) is the nonlinear torsion stiffness coefficient [N\cdot m/rad], which is defined via the polynomial

\[
K_\alpha = 2.82\left(1 - 22.1\alpha + 1315.5\alpha^2 - 8580\alpha^3 + 17289.7\alpha^4\right).
\] (5.3)

Remark 9. The number of significant figures in (5.3) is defined by the accepted model in (S. N. Singh & Brenner, 2003).

Also in (5.1), the control force \( F_L (t) \in \mathbb{R} \) and control moment \( M(t) \in \mathbb{R} \) are defined as

\[
F_L = \rho U^2 s_p b c_{l_\alpha} \left[\alpha + \frac{\dot{b}}{b} + \left(\frac{1}{2} - a\right) \frac{b \dot{\alpha}}{U}\right] + \rho U^2 s_p b c_{l_\beta} \beta \tag{5.4}
\]

\[
M = \rho U^2 s_p b^2 c_{m_\alpha} \left[\alpha + \frac{\dot{b}}{b} + \left(\frac{1}{2} - a\right) \frac{b \dot{\alpha}}{U}\right] + \rho U^2 s_p b^2 c_{m_\beta} \beta \tag{5.5}
\]

where \( U \in \mathbb{R} \) denotes forward velocity [m/s], \( s_p \in \mathbb{R} \) is the wing span [m], \( c_{l_\alpha} \in \mathbb{R} \) is the lift coefficient per angle of attack, \( c_{m_\alpha} \in \mathbb{R} \) is the moment coefficient per control surface deflection, \( c_{l_\beta} \in \mathbb{R} \) is the lift coefficient per control surface deflection, \( c_{m_\beta} \in \mathbb{R} \) is the moment coefficient per control surface deflection, and \( a \in \mathbb{R} \) is the non-dimensional distance from the mid-chord to the elastic axis. In (5.4) and (5.5),
the term \( \beta(t) \in \mathbb{R} \) denotes the control surface deflection [deg].

**Property 7.** The control surface deflection \( \beta(t) \) in (5.4) and (5.5) will be generated by means of SJA arrays. In Section 1.3.3, the nonlinear dynamic model for the virtual surface deflection due to arrays of SJAs will be described. In this chapter, it was simplified and the assumption made is that the virtual surface deflection is generated by \( m = 2 \) arrays of SJAs; however, the control design can be easily extended to handle any number \( m \geq 2 \) SJA arrays with little modification (e.g., using the pseudo-inverse of a matrix).

After some rearranging of (5.1), the LCO dynamics can be expressed as

\[
M_s \ddot{p} = \chi(t) - d(t) + Bu
\]

(5.6)

where the unknown, unmeasurable, nonlinear auxiliary signal \( \chi(t) \in \mathbb{R}^2 \) is defined as

\[
\chi(t) \triangleq -C_s \dot{p} - F(p) p
\]

(5.7)

where \( F(p) \) is the nonlinear stiffness. In (5.6), \( u(t) \triangleq \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T \in \mathbb{R}^2 \) denotes the virtual surface deflection angle due to the SJA arrays; and \( B \in \mathbb{R}^{2 \times 2} \) is an uncertain control input gain matrix. To facilitate the following discussion, the constant elements of the control input gain matrix will be denoted as

\[
B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}.
\]

(5.8)

In the definition given in (5.8), the \( i^{th} \) column of \( B \) for \( i = 1, 2 \) contains the aerody-
namic parameters $B_{1,i}$, $B_{2,i}$ corresponding to the $i^{th}$ SJA array. The parameters $B_{1,i}$ and $B_{2,i}$ for $i = 1, 2$ are explicitly defined as (Elhami & Narab, 2012)

$$B_{1,i} = \left( \rho v^2 b^2 c_m \beta s_p + \frac{I_{\alpha}}{m x_a b'} \rho v^2 b c_l \beta s_p \right)_i,$$

$$B_{2,i} = \left( -\rho v^2 b c_l \beta s_p - \frac{1}{x_a b} \rho v^2 b^2 c_m \beta s_p \right)_i.$$

(5.9)

(5.10)

After substituting (1.1) and (1.2) into (5.6), the open-loop dynamics are obtained as

$$M_s \ddot{p} = \chi (t) - d (t) + \Omega u_d$$

(5.11)

where $u_d (t) \triangleq [u_{d1} (t), u_{d2} (t)]^T \in \mathbb{R}^2$, and $\Omega \in \mathbb{R}^{2 \times 2}$ denotes a constant uncertain matrix. Specifically, $\Omega$ contains the uncertain terms from the input gain matrix $B$ in addition to the uncertain SJA parameters $\theta_{1,i}^*$ for $i = 1, 2$.

## 5.2 Control Development

The objective is to design the control signal $u_d (t)$ to regulate the plunging and pitching dynamics (i.e., $h (t)$, $\alpha (t)$) to zero. To quantify the control objective, a LCO regulation error $e (t) \in \mathbb{R}^2$ and auxiliary error signal $r (t) \in \mathbb{R}^2$ are defined as

$$e = p - p_d$$

(5.12)

$$r = \dot{e} + \phi e + e_f$$

(5.13)

where $\phi \in \mathbb{R}^{2 \times 2}$ is a positive definite, diagonal control gain matrix; the desired plunging and pitching states $p_d \triangleq [h, \alpha]^T = [0, 0]^T$ for the LCO suppression objective; and $e_f (t) \in \mathbb{R}^2$ denotes an auxiliary regulation error signal. Thus, the control objec-
tive can be stated as $\|e(t)\| \rightarrow 0$. Note that the auxiliary error signal $r(t)$ is not measurable, since (5.13) depends on $\dot{p}(t)$.

### 5.2.1 Open-loop Error System

In (5.13), the auxiliary regulation error signal, $e_f$, is an output of the dynamic filter (Xian, Dawson, de Queiroz, & Chen, 2004), (Dinh, Bhasin, Kim, & Dixon, 2012) as

\begin{align*}
\dot{q} &= -(k + 2\phi)q - \eta + (k + \phi)^2e + e \\
\dot{\eta} &= q - \phi \eta -(k + \phi)e \\
e_f &= q - (k + \phi)e
\end{align*}

(5.14) \hspace{1cm} (5.15) \hspace{1cm} (5.16)

where $k \in \mathbb{R}^{2 \times 2}$ is a positive definite, diagonal control gain matrix, $\eta(t) \in \mathbb{R}^2$ is another output of the filter, and $q(t) \in \mathbb{R}^2$ is an internal filter variable. Figure 5.1 represents a block diagram how the previous equation synthesize the velocity measurement out of the control law.

![Block Diagram](image)

Figure 5.1: A block diagram representing how the bank of filters synthesize the velocity measurements.
5.2. CONTROL DEVELOPMENT

After using (5.13), (5.14), and (5.16), the time derivative of $e_f(t)$ can be obtained as

$$\dot{e}_f = -\phi e_f - \eta + e - (k + \phi)r.$$  \hspace{1cm} (5.17)

The expression in (5.17) will be utilized in the subsequent Lyapunov-based stability analysis.

After premultiplying (5.13) by $M_s$, taking the time derivative of the result, and utilizing (5.6), the open loop error dynamics are obtained as

$$M_s \dot{r} = \tilde{N}(t) + N_d(t) + \Omega u_d - M_s(k + \phi)r - Se + S\phi e_f$$  \hspace{1cm} (5.18)

where $S \in \mathbb{R}^{n \times n}$ is a subsequently defined uncertain matrix (i.e. see Lemma 6 and Assumption 9), and the unknown, unmeasurable auxiliary signals $\tilde{N}(t), N_d(t) \in \mathbb{R}^2$ are defined as

$$\tilde{N}(t) \triangleq \chi(t) + M_s(\phi r + (1 - \phi^2)e - 2\phi e_f - \eta) + Se - S\phi e_f$$  \hspace{1cm} (5.19)

$$N_d(t) \triangleq -d(t).$$  \hspace{1cm} (5.20)

The motivation for separating the terms as in (5.19) and (5.20) is based on the fact that the following upper bounds can be developed:

$$\|\tilde{N}(t)\| \leq \rho(\|z\|) \|z\|, \quad \|N_d(t)\| \leq \zeta_{d1}, \quad \|\dot{N}_d(t)\| \leq \zeta_{d2}$$  \hspace{1cm} (5.21)

where $\rho(\cdot) \in \mathbb{R}$ is a positive, globally invertible non-decreasing function; $\zeta_{d1}, \zeta_{d2} \in \mathbb{R}^+$
are known bounding constants; and $z(t) \in \mathbb{R}^8$ is defined as

$$z(t) \overset{\Delta}{=} \begin{bmatrix} e^T(t) & e_f^T(t) & r^T(t) & \eta^T(t) \end{bmatrix}^T.$$  \hspace{1cm} (5.22)

A precise definition of $\rho$ is not required in the subsequent stability analysis. It is only required to satisfy the aforementioned mathematical properties.

### 5.2.2 Closed-loop Error System

Based on the open-loop regulation error dynamics in (5.18) and the subsequent stability analysis, the auxiliary control $u_d(t)$ is designed as

$$u_d(t) = \hat{\Omega}^{-1}(\mu_0 - \mu_1)$$  \hspace{1cm} (5.23)

where $\hat{\Omega} \in \mathbb{R}^{2 \times 2}$ is a constant, best-guess estimate of the uncertain matrix $\Omega$. In (5.23), $\mu_0(t), \mu_1(t) \in \mathbb{R}^2$ are subsequently defined feedback control terms.

After substituting (5.23) into (5.18), the closed-loop error dynamics are obtained as

$$M_s \dot{r} = \tilde{N} + N_d + \hat{\Omega}(\mu_0 - \mu_1) - M_s(k + \phi)r - Se + S\phi e_f$$  \hspace{1cm} (5.24)

where the constant uncertain matrix $\tilde{\Omega} \in \mathbb{R}^{2 \times 2}$ is defined as

$$\tilde{\Omega} = \Omega\hat{\Omega}^{-1}.$$  \hspace{1cm} (5.25)

**Lemma 6.** Any positive definite matrix $X \in \mathbb{R}^{n \times n}$ can be decomposed as

$$X = ST,$$  \hspace{1cm} (5.26)
where \( S \in \mathbb{R}^{n \times n} \) is a positive definite symmetric matrix and \( T \in \mathbb{R}^{n \times n} \) is an unity upper triangular matrix (Morse, 1993).

**Proof.** Proof of Lemma 6 can be found in (Morse, 1993).

**Assumption 9.** Bounds on the uncertain matrix \( \Omega \in \mathbb{R}^{n \times n} \) are known such that the constant feed forward estimate \( \hat{\Omega} \in \mathbb{R}^{n \times n} \) can be chosen to render the product, \( \tilde{\Omega} = \Omega \hat{\Omega}^{-1} \) is positive definite. Further, the estimate \( \hat{\Omega} \) is selected such that

\[
\hat{\Omega} = ST,
\]

(5.27)

where the unity upper triangular matrix \( T \) is diagonally dominant in the sense that

\[
\varepsilon \leq |T_{ii}| - \sum_{k=i+1}^{n} |T_{ik}| \leq Q, \quad i = 1, \ldots, n - 1.
\]

(5.28)

where \( \varepsilon \in (0, 1) \) and \( Q \in \mathbb{R}^+ \) are known bounding constants, and \( T_{ik} \in \mathbb{R} \) denotes the \((i,k)\)th element of the matrix \( T \). In (5.27), the matrices \( S \) and \( T \) are defined similarly as in Lemma 6.

After utilizing the decomposition in (5.27), the error dynamics in (5.24) can be rewritten as

\[
Mr = \tilde{N}_1 + N_{d1} + T(\mu_0 - \mu_1) - (k_m + \phi_m)r - e + \phi e_f
\]

(5.29)

where \( M \triangleq S^{-1}M_s \), \( k_m \triangleq S^{-1}M_s k \), and \( \phi_m \triangleq S^{-1}M_s \phi \)

\[
\tilde{N}_1 \triangleq S^{-1}\tilde{N}, \quad N_{d1} \triangleq S^{-1}N_d.
\]

(5.30)
Since $S$ is constant and positive definite, $\tilde{N}_1(t)$ and $N_{d1}(t)$ satisfy the inequalities

$$
\|\tilde{N}_1\| \leq \rho_1(\|z\|), \quad \|N_{d1}\| \leq \zeta_{N_{d1}}, \quad \|\dot{N}_{d1}\| \leq \zeta_{\dot{N}_{d1}}
$$

(5.31)

where $\rho_1(\cdot) \in \mathbb{R}$ is a positive, globally invertible non-decreasing function; $\zeta_{N_{d1}}, \zeta_{\dot{N}_{d1}} \in \mathbb{R}^+$ are known bounding constants. By utilizing the fact that the uncertain matrix $T$ is unity upper triangular, the error dynamics in (5.29) can be rewritten as

$$
\dot{M}r = \tilde{N}_1 + \dot{N}_{d1} - (k_m + \phi_m)r + \mu_0 + \bar{T}\mu_0 - T\mu_1 - e + \phi e_f
$$

(5.32)

where $\bar{T} \triangleq T - I_{2 \times 2}$ is a strictly upper triangular matrix, and $I_{2 \times 2}$ denotes a $2 \times 2$ identity matrix. Based on the open-loop error system in (5.32), the auxiliary control terms $\mu_0(t)$ and $\mu_1(t)$ are designed as

$$
\mu_0 = ke_f
$$

(5.33)

$$
\mu_1 = \beta_f \text{sgn}(\phi e + \phi \eta)
$$

(5.34)

where $k$ is introduced in (5.14)–(5.16); and $\beta_f \in \mathbb{R}^{2 \times 2}$ denotes a positive definite, diagonal control gain matrix. After substituting (5.33) into (5.32), the closed-loop error system is obtained as

$$
\dot{M}r = \tilde{N}_1 + \bar{T}\mu_0 + N_{d1} - (k_m + \phi_m)r - T\mu_1 + (k + \phi) e_f - e.
$$

(5.35)
5.2. CONTROL DEVELOPMENT

The term $\bar{T}_0$ can be expressed as

$$
\begin{align*}
\bar{T}_0 &= \begin{bmatrix} 0, T_{12}; 0, 0 \end{bmatrix} \begin{bmatrix} \mu_0(t) & \mu_0(t) \end{bmatrix}^T = \begin{bmatrix} T_{12} \mu_0(t) & 0 \end{bmatrix}^T = \begin{bmatrix} \Lambda & 0 \end{bmatrix}^T \\
&= \begin{bmatrix} T_{12} \mu_0(t) & 0 \end{bmatrix}^T = \begin{bmatrix} \Lambda \rho \end{bmatrix}^T
\end{align*}
\tag{5.36}
$$

where $\Lambda_\rho$ satisfies the inequality

$$
\|\Lambda_\rho\| \leq \rho_{\Lambda_1} \|z\| \tag{5.37}
$$

where $z(t)$ is defined in (5.22), and $\rho_{\Lambda_1} \in \mathbb{R}^+$ is a known bounding constant.

By utilizing (5.36), the error dynamics in (5.35) can be expressed as

$$
M \dot{r} = \tilde{N}_2 + N_{d1} - (k_m + \phi_m) r - T_0 + (k + \phi) e_f - e 
\tag{5.38}
$$

where

$$
\tilde{N}_2 = \tilde{N}_1 + \bar{T}_0 = \tilde{N}_1 + \begin{bmatrix} \Lambda \rho & 0 \end{bmatrix}^T. \tag{5.39}
$$

Based on (5.31), (5.37), and (5.39), $\tilde{N}_2(t)$ satisfies the inequality

$$
\|\tilde{N}_2\| \leq \rho_2 (\|z\|) \|z\| \tag{5.40}
$$

where $\rho_2 (\cdot) \in \mathbb{R}$ is positive, globally invertible nondecreasing function.

To facilitate the following stability analysis, the control gain $\beta_f$ introduced in (5.34) is selected to satisfy the sufficient condition

$$
\beta_f > \frac{1}{\varepsilon} \left( \zeta_{N_{d1}} + \frac{1}{\phi} \zeta_{\tilde{N}_{d1}} \right) \tag{5.41}
$$
where $\zeta_{N_d1}$ and $\zeta_{N_d1}$ are introduced in (5.31), $\varepsilon$ is introduced in (5.28), and $\phi$ is introduced in (5.13).

### 5.3 Stability Analysis

To facilitate the following stability analysis, let $D \subset \mathbb{R}^9$ be a domain containing $y(t) = 0$, where $y(t) \in \mathbb{R}^9$ is defined as

$$y(t) \triangleq \begin{bmatrix} z^T(t) & \sqrt{P(t)} \end{bmatrix}^T.$$  \hspace{1cm} (5.42)

In (5.42), the non-negative auxiliary function $P(t) \in \mathbb{R}$ is the generalized solution to the differential equation

$$\dot{P}(t) = -L(t)$$  \hspace{1cm} (5.43)

$$P(0) = \beta_f Q |e(0) + \eta(0)| - (e(0) + \eta(0))^T N_{d1}(0)$$  \hspace{1cm} (5.44)

where the auxiliary function $L(t) \in \mathbb{R}$ is defined as

$$L(t) \triangleq r^T(t) (N_{d1}(t) - T\mu_1(t)).$$  \hspace{1cm} (5.45)

**Lemma 7.** Provided the sufficient condition in (5.41) is satisfied, the following inequality can be obtained:

$$\int_0^t L(\tau) d\tau \leq \beta_f Q |e(0) + \eta(0)| - (e(0) + \eta(0))^T N_{d1}(0).$$  \hspace{1cm} (5.46)

Hence, (5.43), (5.44), and (5.46) can be used to conclude that $P(t) \geq 0$.  

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5.3. STABILITY ANALYSIS

Proof of Lemma 7 can be found in the appendix section.

Theorem 5. The robust output feedback control law given in (5.12), (5.14), (5.15), (5.16), and (5.23) ensures asymptotic regulation of LCO in the sense that

\[ \| e(t) \| \to 0 \quad \text{as} \quad t \to \infty \]  \hspace{1cm} (5.47)

provided the control gain, \( k \), introduced in (5.33) is selected sufficiently large (see the subsequent proof), and \( \beta_f \) is selected to satisfy the sufficient condition in (5.41).

Proof. Let \( V(y, t) : \mathcal{D} \times [0, \infty) \to \mathbb{R} \) be a radially unbounded, positive definite function defined as

\[
V = \frac{1}{2} e^T e + \frac{1}{2} e_f^T e_f + \frac{1}{2} r^T Mr + \frac{1}{2} \eta^T \eta + P \]

which satisfies the inequalities

\[
U_1(y) \leq V(y, t) \leq U_2(y) \] \hspace{1cm} (5.49)

provided the sufficient condition in (5.41) is satisfied. In (5.49), the continuous positive definite functions \( U_1(y), U_2(y) \in \mathbb{R} \) are defined as

\[
U_1(y) \triangleq \eta_1 \| y \|^2, \quad U_2(y) \triangleq \eta_2 \| y \|^2 \]

where \( \eta_1, \eta_2 \in \mathbb{R} \) are defined as

\[
\eta_1 \triangleq \frac{1}{2} \min \{ 1, \lambda_{\min}(M) \}, \quad \eta_2 \triangleq \max \left\{ \frac{1}{2} \lambda_{\max}(M), 1 \right\} \]

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where $\lambda_{\min} (\cdot)$, $\lambda_{\max} (\cdot)$ denote the minimum and maximum eigenvalues of the arguments, respectively. After taking the time derivative of (5.48) and utilizing (5.13), (5.14), (5.15), (5.16), (5.17), (5.24), (5.43), and (5.45), $\dot{V} (t)$ can be expressed as

\[
\dot{V} = e^T (r - \phi e - ef) + e_f^T (e - \phi ef - \eta - (k + \phi)r) \\
+ r^T \left( \tilde{N}_2 + N_{d1} - T\mu_1 - (k_m + \phi_m) r + (k + \phi) ef - e \right) \\
+ \eta^T (ef - \phi \eta) - r^T (N_{d1} - T\mu_1) 
\] (5.51)

where the symmetry of $M = S^{-1} M_s$ was utilized. After canceling common terms in (5.51), $\dot{V} (y, t)$ can be simplified as

\[
\dot{V} = -\phi \|e\|^2 - \phi \|ef\|^2 - \phi \|\eta\|^2 - \phi_m \|r\|^2 + r^T \left( \tilde{N}_2 - k_m r \right). 
\] (5.52)

After using the bounding inequality in (5.40), the expression in (5.52) can be upper bounded as

\[
\dot{V} \leq -\lambda_0 \|z\|^2 - \left( k_m \|r\|^2 - \rho_2 (\|z\|) \|r\| \|z\| \right) 
\] (5.53)

where $\lambda_0 \triangleq \min \{ \lambda_{\min} (\phi), \lambda_{\min} (\phi_m) \}$, where $\lambda_{\min} (\cdot)$ denotes the minimum eigenvalue of the argument. By completing the squares for the last two terms in (5.53), the upper
5.3. STABILITY ANALYSIS

Bound on $\dot{V}(t)$ can be expressed as

$$
\dot{V} \leq -\lambda_0 \|z\|^2 - k_m \left( \|r\|^2 - \frac{\rho_2(\|z\|)}{k_m} \|z\| + \frac{\rho_2^2(\|z\|)}{4k_m^2} \|z\|^2 \right) + \frac{\rho_2^2(\|z\|)}{4k_m^2} \|z\|^2
$$

(5.54)

$$
\dot{V} \leq -\lambda_0 \|z\|^2 - k_m \left( \|r\| - \frac{\rho_2(\|z\|)}{2k_m} \|z\| \right)^2 + \frac{\rho_2^2(\|z\|)}{4k_m} \|z\|^2
$$

(5.55)

$$
\dot{V} \leq -\left( \lambda_0 - \frac{\rho_2^2(\|z\|)}{4k_m} \right) \|z\|^2.
$$

(5.56)

The following expression can be obtained from (5.56):

$$
\dot{V} \leq -U(y)
$$

(5.57)

where $U(y) = c \|z\|^2$, for some positive constant $c \in \mathbb{R}$ is a continuous positive semi-definite function that is defined on the domain

$$
\mathcal{D} \triangleq \left\{ y(t) \in \mathbb{R}^9 | \|y\| \leq \rho^{-1} \left( 2\sqrt{k_m \lambda_0} \right) \right\}.
$$

(5.58)

The expressions in (5.49) and (5.57) can be used to prove that $e(t), e_f(t), r(t), \eta(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Given that $e(t), e_f(t), r(t), \eta(t) \in \mathcal{L}_\infty$, (5.13) can be used to show that $\dot{e}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$; (5.17) can be used to show that $\dot{e}_f(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$; and (5.14) can be used to show that $q(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. It then follows that $\dot{\eta}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$ from (5.15). Given that $e_f(t) \in \mathcal{L}_\infty$, the expressions in (5.23), (5.33), and (5.34) can be used along with (1.1) and (1.2) to prove that the control signals $u_{di}(t), v_i(t), u_i(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Since $e(t), e_f(t), r(t), \eta(t) \in \mathcal{L}_\infty$, (5.31), (5.38), and (5.40) can be used to prove that $\dot{r}(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Given that $\dot{e}(t), \dot{e}_f(t), \dot{r}(t), \dot{\eta}(t) \in \mathcal{L}_\infty$, it follows that $e(t), e_f(t), r(t)$, and $\eta(t)$ are uniformly continuous in
5.4 Simulation Results

A numerical simulation was created to demonstrate the performance of the proposed control law. In order to develop a realistic stepping stone to high-fidelity numerical simulation results using detailed CFD models, the following simulation results are based on detailed dynamic parameters and specifications. The simulation is based

\[ D; \text{ thus, } z(t) \text{ is uniformly continuous throughout closed-loop controller operation.} \]

Hence, the definitions of \( U(y) \) and \( z(t) \) can be used to prove that \( U(y) \) is uniformly continuous in \( D \).

Let \( S \subset D \) denote a set defined as follows:

\[ S \triangleq \{ y(t) \subset D | U(y(t)) \leq \eta_1 \left( \rho^{-1} \left( 2\sqrt{k_m \lambda_0} \right) \right)^2 \}. \quad (5.59) \]

Theorem 8.4 of (Khalil, 2002) can now be invoked to state that

\[ c \| z(t) \|^2 \to 0 \quad \text{as} \quad t \to \infty \quad \forall \ y(t_0) \in S. \]

Based on the definition of \( z(t) \), (5.59) can be used to show that

\[ \| e(t) \| \to 0 \quad \text{as} \quad t \to \infty \quad \forall \ y(t_0) \in S. \]

Thus, asymptotic regulation of the pitching and plunging displacements can be achieved, provided the initial conditions are within the set \( S \), where \( S \) can be made arbitrarily large by increasing the control gain \( k \) (recall that \( k_m = S^{-1} M_s k \)). Hence, this is a semi-global asymptotic result.
5.4. SIMULATION RESULTS

on the dynamic model given in (5.1)–(5.5). The dynamic parameters utilized in the simulation are summarized in Table 5.1 and were obtained from (Elhami & Narab, 2012). The actual parameters \( \theta_1^* \) and \( \theta_2^* \) and their estimates \( \hat{\theta}_1,i \) and \( \hat{\theta}_2,i \) are described in Table 5.2 were obtained from (MacKunis et al., 2013).

Table 5.1: Dynamic parameters and geometric dimensions of the wing section

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>1.225 kg/m(^3)</td>
</tr>
<tr>
<td>( m )</td>
<td>12.387 kg</td>
</tr>
<tr>
<td>( C_\alpha )</td>
<td>0.036 kg·m(^2)/s</td>
</tr>
<tr>
<td>( I_a )</td>
<td>0.065 kg·m</td>
</tr>
<tr>
<td>( K_h )</td>
<td>2844.4 N/m</td>
</tr>
<tr>
<td>( a )</td>
<td>-0.6</td>
</tr>
<tr>
<td>( b )</td>
<td>0.125 m</td>
</tr>
<tr>
<td>( c_{l,\alpha} )</td>
<td>3.358</td>
</tr>
<tr>
<td>( s_p )</td>
<td>0.6 m</td>
</tr>
<tr>
<td>( v )</td>
<td>13 m/s</td>
</tr>
<tr>
<td>( c_{m,\alpha} )</td>
<td>-0.635</td>
</tr>
<tr>
<td>( c_{m,\beta} )</td>
<td>-0.635</td>
</tr>
</tbody>
</table>

Table 5.2: SJA parameters and their estimates

<table>
<thead>
<tr>
<th>( \theta_{1,i}^* )</th>
<th>( \theta_{1,i}^* )</th>
<th>( \theta_{2,i}^* )</th>
<th>( \theta_{2,i}^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>32.9 Volt-deg</td>
<td>32.7 Volt-deg</td>
<td>16.5 deg</td>
<td>16.4 deg</td>
</tr>
<tr>
<td>29.8 Volt-deg</td>
<td>29.7 Volt-deg</td>
<td>15.9 deg</td>
<td>15.8 deg</td>
</tr>
</tbody>
</table>

The following simulation results were obtained using control gains selected as

\[
\beta_f = \text{diag}(0.68, 0.178), \quad k = \text{diag}(1, 3.55), \quad \phi = \text{diag}(0.05, 5). \quad (5.60)
\]

The control gains given in (5.60) were selected based on achieving a desirable response in terms of settling time and required control effort. To test the case where the input gain matrix \( B \) is uncertain, it is assumed in the simulation that the actual value of \( B \) is the 2 × 2 identity matrix, but the constant feedforward estimate \( \hat{B} \) used in the control law is given by

\[
\hat{B} = \begin{bmatrix}
0.9 & 0.1 \\
-0.1 & 1.1
\end{bmatrix} \quad (5.61)
\]
The initial conditions used in the simulation are

\[ p(0) = \begin{bmatrix} 0.02 & 0.2 \end{bmatrix}^T. \]

The external disturbance used in the simulation is

\[ d = 0.01 \begin{bmatrix} 0.003\sin(0.5t) & 0.003\sin(0.5t) + 0.001\sin(0.5t) \end{bmatrix}^T. \]

Figure 5.2: Open loop plunging and pitching displacements of the LCO response in the simulated system.

Fig. 5.2 shows that the open-loop system exhibits pitching and plunging displacements of the LCO. Fig. 5.3 shows the close-loop system and how the proposed control law affects the LCO driving it to zero. Figs. 5.4 - 5.6 show the performance of the proposed control law to regulate the pitching and plunging displacements of the LCO. The figures show the performance of the control law for 10 different sets of off-nominal values of the uncertain SJA parameters \( \theta_1^* \) and \( \theta_2^* \) using a Monte Carlo-
5.4. SIMULATION RESULTS

Figure 5.3: Closed-loop regulation of pitching and plunging LCO using the proposed control law.

Figure 5.4: Closed-loop regulation of pitching and plunging LCO using the proposed control law for 10 different off-nominal sets of values for $\theta_1^*$ and $\theta_2^*$.

type simulation. In each of the 10 trials, the values of the uncertain parameters were generated randomly to test a range of uncertainty of $\pm 1\%$ off nominal. Fig. 5.5
5.4. SIMULATION RESULTS

Figure 5.5: Virtual deflection angle $\beta(t)$ control commands during closed-loop operation for 10 sets of values of the SJA parameters $\theta_1^*$ and $\theta_2^*$.

Figure 5.6: SJA control voltage input signals commanded during closed-loop operation for the 10 sets of values of the SJA parameters $\theta_1^*$ and $\theta_2^*$.

shows the virtual surface deflection angle control inputs during closed-loop controller operation, and Fig. 5.6 shows the corresponding SJA control voltage inputs in the
closed-loop for the 10 trials. The commanded control inputs remain within reasonable limits throughout the duration of the simulation for all 10 trials.

5.5 Conclusion

A SJA-based output feedback control method is presented, which achieves asymptotic LCO regulation in small unmanned aerial vehicle (SUAV) wings in the presence of uncertain SUAV dynamics and unmodeled external disturbances. In addition, the proposed control method compensates for the parametric uncertainty and nonlinearity inherent in the SJA actuator dynamics. Motivated by the limitations characteristic of SUAV applications, the control method is designed to be computationally inexpensive, eliminating the need for time-varying parameter update laws, function approximators, or heavy computations. To achieve the result, a robust-inverse control method is utilized, which is proven to compensate for the SUAV and SJA uncertainties using a simplified controller structure. By endowing the robust-inverse control structure with a bank of dynamic filters, asymptotic LCO regulation is achieved using only pitching and plunging displacement measurements in the feedback loop. The result is an asymptotic SJA-based LCO regulation control design, which does not require velocity measurements, adaptive laws, function approximators, or heavy computations in the feedback loop. A rigorous Lyapunov-based stability analysis is utilized to prove the theoretical result, and numerical simulation results demonstrate that the proposed control law asymptotically regulates LCO in the presence of significant SJA parameter uncertainty.
Chapter 6

A Nonlinear Output Feedback Regulation Method for Limit Cycle Oscillation Suppression using a Sliding Mode Observer

In this chapter, a nonlinear output feedback control method is presented, which achieves asymptotic LCO regulation in an aircraft wing section using SJA. To eliminate the standard requirement that LCO pitching and plunging rates are available for feedback, a finite-time sliding mode observer is utilized to estimate the rates using only measurements of LCO displacements. In Chapter 5, eliminating the velocity measurements from the propose control law was presented by the use of bank of filters, but in this chapter we estimated those velocities using a sliding mode estimator.

A rigorous analysis is used to prove finite-time convergence of the estimation er-
ror, and a Lyapunov-based stability analysis is used to prove asymptotic regulation control of the LCO. Numerical simulation results are also provided which show the performance of the proposed sliding mode observer-based control design in comparison with our recently developed bank of filters-based output feedback LCO control method.

6.1 Dynamic Model

The pitching and plunging dynamics of an airfoil can be expressed as (Elhami & Narab, 2012)

\[ M_s \ddot{p} + C_s \dot{p} + F(p)p + d(t) = \begin{bmatrix} -F_L \\ M \end{bmatrix}, \]  

(6.1)

where \( F_L \in \mathbb{R} \) and \( M \in \mathbb{R} \) denote the control force and the control moment, respectively. In (6.1), \( p(t) \triangleq \begin{bmatrix} h(t) \\ \alpha(t) \end{bmatrix} \in \mathbb{R}^2 \) denotes the state vector, where \( h(t), \alpha(t) \in \mathbb{R} \) are the plunging \([m]\) and pitching \([\text{rad}]\) displacements, respectively; \( M_s \in \mathbb{R}^{2\times2} \) is the inertia matrix; and \( C_s \in \mathbb{R}^{2\times2} \) is damping matrix. Also in (6.1), \( F(p(t)) \in \mathbb{R}^{2\times2} \) denotes a subsequently defined nonlinear stiffness matrix. The term \( d(t) \in \mathbb{R}^2 \) represents a general unknown, norm-bounded, nonvanishing disturbance.

In (6.1), the structural linear mass, \( M_s \), structural linear damping, \( C_s \), and the
6.1. DYNAMIC MODEL

nonlinear stiffness, \( F(p) \), matrices are described as (Elhami & Narab, 2012)

\[
M_s = \begin{bmatrix} m & mx_\alpha b \\ mx_\alpha b & I_\alpha \end{bmatrix}, \quad C_s = \begin{bmatrix} C_h & 0 \\ 0 & C_\alpha \end{bmatrix},
\]

\[
F(p) = \begin{bmatrix} K_h & 0 \\ 0 & K_\alpha(\alpha) \end{bmatrix}
\]

where \( x_\alpha \in \mathbb{R} \) denotes the non-dimensional distance measured from the elastic axis to the center of mass, \( b \in \mathbb{R} \) is the semi-chord of the wing [m], \( m \in \mathbb{R} \) is the mass of the wing section [kg], and \( I_\alpha \in \mathbb{R} \) is the mass moment of inertia of the wing about the elastic axis [kg·m\(^2\)]. The parameter \( C_h \in \mathbb{R} \) denotes the structural damping coefficient in plunge due to viscous damping [kg/s], and \( C_\alpha \in \mathbb{R} \) denotes the structural damping coefficient in pitch due to viscous damping [kg·m\(^2\)/s]. The \( K_h \in \mathbb{R} \) is the structural spring constant in plunge [N/m]; and \( K_\alpha(\alpha(t)) \in \mathbb{R} \) is the nonlinear torsion stiffness coefficient [N·m/rad], which is defined as

\[
K_\alpha = 2.82(1 - 22.1\alpha + 1315.5\alpha^2

- 8580\alpha^3 + 17289.7\alpha^4).
\]

\[
(6.2)
\]

**Remark 10.** The number of significant figures in (6.2) is defined by the accepted model in (S. N. Singh & Brenner, 2003).
6.1. DYNAMIC MODEL

Also in (6.1), the control force $F_L$ and control moment $M$ are defined as

\[
F_L = \rho U^2 s_p b c_{l_\alpha} \left[ \alpha + \frac{\dot{h}}{b} + \left( \frac{1}{2} - a \right) b \frac{\dot{\alpha}}{U} \right] \\
+ \rho U^2 s_p b c_{l_\beta} \beta,
\]

(6.3)

\[
M = \rho U^2 s_p b^2 c_{m_{\alpha}} \left[ \alpha + \frac{\dot{h}}{b} + \left( \frac{1}{2} - a \right) b \frac{\dot{\alpha}}{U} \right] \\
+ \rho U^2 s_p b^2 c_{m_{\beta}} \beta,
\]

(6.4)

where $U \in \mathbb{R}$ denotes forward velocity [m/s], $s_p \in \mathbb{R}$ is the wing span [m], $c_{l_\alpha} \in \mathbb{R}$ is the lift coefficient per angle of attack, $c_{m_{\alpha}} \in \mathbb{R}$ is the moment coefficient per control surface deflection, $c_{l_\beta} \in \mathbb{R}$ is the lift coefficient per control surface deflection, $c_{m_{\beta}} \in \mathbb{R}$ is the moment coefficient per control surface deflection, and $a \in \mathbb{R}$ is the non-dimensional distance from the mid-chord to the elastic axis. In (6.3) and (6.4), the term $\beta (t) \in \mathbb{R}$ denotes the control surface deflection [deg].

After some rearranging of (6.1), the LCO dynamics can be expressed as

\[
M s \ddot{p} = \Psi (t) - d (t) + Bu,
\]

(6.5)

where the unknown, unmeasurable, nonlinear auxiliary signal $\Psi (t) \in \mathbb{R}^2$ is defined as

\[
\Psi (t) \triangleq -C s \dot{p} - F (p) p.
\]

(6.6)

Here $F (p)$ is the nonlinear stiffness. In (6.5), $u (t) \triangleq \left[ u_1 (t) \quad u_2 (t) \right]^T \in \mathbb{R}^2$ denotes the virtual surface deflection angle due to the SJA arrays.

The input control matrix $B \in \mathbb{R}^{2 \times 2}$ can be defined as the following where $B_{1,i}$ and
6.2. SLIDING MODE OBSERVER

\( B_{2,i} \) correspond to the \( i \)th SJA array of the system (Elhami & Narab, 2012)

\[
\begin{align*}
B_{1,i} &= \left( \rho v^2 b_2 c_{m \beta} s_p + \frac{I_{\alpha}}{m x_o b} \rho v^2 b c_{l \beta} s_p \right)_i, \\
B_{2,i} &= \left( -\rho v^2 b c_{l \beta} s_p - \frac{1}{x_a b} \rho v^2 b^2 c_{m \beta} s_p \right)_i.
\end{align*}
\] (6.7)

6.2 Sliding Mode Observer

This section presents a sliding mode observer design to estimate the rates of pitching and plunging caused by the LCO. A reduced-order model for the LCO dynamics of (6.5) can be obtained as

\[
\dot{x} = f(x) - d(t) + \Omega u_d, \quad \text{(6.8)}
\]

\[
y = h(x), \quad \text{(6.9)}
\]

The terms \( f(x) \) and \( h(x) \) can be explicitly defined as

\[
f(x) \triangleq \begin{bmatrix} M_s^{-1} x_2 \\ A(x)x \end{bmatrix},
\]

where \( A(x) = \Psi(t), \ x = [p, M_s \dot{p}]^T \) and \( h(x) \triangleq C x(t) \), with \( C = [I_{2\times2}, 0_{2\times2}] \), i.e., the output measurements in the equation (6.9) are the position measurements, \( h \) and \( \alpha \).

To take care the uncertainty in (6.8) in the input-multiplicative matrix \( \Omega \), \( u_d(t) \) can be designed as

\[
u_d(t) = \hat{\Omega}^{-1} \mu(t), \quad \text{(6.10)}
\]
6.2. SLIDING MODE OBSERVER

where \( \hat{\Omega} \in \mathbb{R}^2 \) is a constant feedforward estimate of \( \Omega \). After substituting (6.10) into (6.8), the open loop system can be expressed as

\[
\dot{x} = f(x) - d(t) + \hat{\Omega} u_d.
\]  

(6.11)

where \( \hat{\Omega} \triangleq \hat{\Omega} \hat{\Omega}^{-1} \in \mathbb{R}^{n \times n} \). The uncertain matrix \( \hat{\Omega} \) represents the deviation between the actual SJA parameters \( \theta^* \) and their constant estimates \( \hat{\theta} \), for \( i = 1, ..., m \).

**Property 8.** The uncertain matrix \( \hat{\Omega} \) can be decomposed as

\[
\hat{\Omega} = I_n + \Delta,
\]  

(6.12)

where \( S \in \mathbb{R}^{n \times n} \) is a positive definite symmetric matrix and denotes the identity matrix, and \( \Delta \in \mathbb{R}^{n \times n} \) denotes a constant uncertain “mismatch” matrix.

**Assumption 10.** Approximate model knowledge is available such that the mismatch matrix \( \Delta \) satisfies

\[
\| \Delta \|_{i\infty} < \varepsilon < 1,
\]  

(6.13)

where \( \varepsilon \in \mathbb{R}^+ \) is a known bounding constant, and \( \| \cdot \|_{i\infty} \) denotes the induced infinity norm of a matrix. The inequality (6.13) can be interpreted as the assumption of approximate SJA model knowledge.

Substituting (6.12) into (6.11), the SJA-based LCO model can be expressed as

\[
\dot{x} = f(x) - d(t) + \mu(t) + \Delta\mu(t).
\]  

(6.14)

For the subsequent observer design and analysis, a vector \( H(x, \mu) \in \mathbb{R}^n \) of
output derivatives is defined as (MacKunis et al., 2011; S. V. Drakunov, 1992a; S. V. Drakunov & Reyhanoglu, 2011)

\[
H(x, \mu) \triangleq \begin{bmatrix} h_1(x) & h_2(x, \mu) & \cdots & h_n(x, \mu) \end{bmatrix}^T,
\]

(6.15)

where \( h_1(x) \triangleq h(x) \); and \( h_{i+1} \triangleq L_f h(x), i = 1, \ldots, n-1 \), denotes \( i^{th} \) Lie derivative of the output function \( h(x) \) along the direction of the vector field \( f(x) \) (e.g., \( L_f = \frac{\partial h}{\partial x} f(x) \)). If \( x(t) \) is a solution to the system described in (6.14), then

\[
\frac{d}{dt} h_i(x(t)) = h_{i+1}(x(t)), \quad i = 1, \ldots, n-1.
\]

(6.16)

**Remark 11. (Sufficiently Differentiable Control Input)** Based on the definitions in (6.15) and (6.16) and the subsequent analysis, it must be assumed that the control input function \( \mu(t) \) be sufficiently differentiable. This assumption can be ensured in implementation by approximating the discontinuous signum function using an “equivalent value operator” of a discontinuous function in sliding mode (S. V. Drakunov, 1992a; Sánchez-Torres, Loukianov, Moreno, & Drakunov, 2012).

To design an observer for the actuated system in (6.8), the dynamic model must satisfy the following observability condition and matching condition.

**Condition 1. (Observability)** The system given in (6.14) must satisfy the observability condition

\[
\text{rank}(O(x, \mu)) = n, \quad \forall \ x \in \mathbb{R}^n,
\]

(6.17)

where the observability matrix \( O(x) \triangleq \frac{\partial H(x)}{\partial x} \in \mathbb{R}^{n \times n} \). The observability condition in (6.17) can be ensured by using judicious placements of the sensors.
Using Condition 1, an observer that estimates the full state $x(t)$ of the system in (6.8) using only measurements of $y(t)$ which are pitching and plunging can be designed as (Sánchez-Torres et al., 2012)

$$\dot{\hat{x}} = \mathcal{O}^{-1}(x, \mu) M(\hat{x}, \mu) \{\text{sgn}(V(t) - H(\hat{x}))\}_{eq} + \mu(t)(\hat{x}),$$

(6.18)

where $\{\text{sgn}(\cdot)\}_{eq}$ denotes a smooth, continuous “equivalent value operator” of the discontinuous signum function (S. V. Drakunov, 1992a). In (6.18), $V(t) = [v_1(t), ..., v_n(t)]^T$ is defined via the recursive form

$$v_1(t) = h_1(x),$$

(6.19)

$$v_{i+1}(t) = m_i \{\text{sgn}(v_i(t) - h_i(\hat{x}(t)))\}_{eq} + \frac{\partial h_i(\hat{x})}{\partial x} \mu(t),$$

(6.20)

for $i = 1, ..., n - 1$. The observer design in (6.18) will compensate for the SJA actuator parametric uncertainty through the design of the sliding gain term $M(\hat{x}, \mu) \in \mathbb{R}^n$ described as

$$M(\hat{x}, \mu) = \text{diag}[m_1(\hat{x}, \mu), ..., m_n(\hat{x}, \mu)],$$

(6.21)

where $m_i(\hat{x}, \mu) \in \mathbb{R}$, $i = 1, ..., n$ are user designed robust estimation terms. By the appropriate design of the gain matrix $M(\hat{x}, \mu)$, it will be shown that the sliding mode observer in (6.18) achieves finite time estimation of the velocity states $\dot{h}$ and $\dot{\alpha}$ using only the position measurements of the output signal $y(t)$. The complete proof of convergence of the observer in (6.18) can be found in (Kidambi, MacKunis, Ramos-Pedroza, & Drakunov, 2017) and is omitted here for brevity.
6.3 Control Development

The control objective is to design the control signal \( u_d \) to regulate the state vector \( x(t) \) to a desired reference point \( x_d(t) \), using only the state estimates \( \hat{x}(t) \) as feedback measurements. This is to ensure the velocity variables are well estimated.

6.3.1 Open-Loop Error System

To quantify the control objective, a tracking signal \( e(t) \in \mathbb{R}^2 \) and the auxiliary tracking error \( r(t) \in \mathbb{R}^2 \). They are defined as

\[
e = p - p_d, \quad (6.22)
\]
\[
r = \dot{e} + \alpha_f e, \quad (6.23)
\]

where \( \alpha_f \in \mathbb{R}^{2 \times 2} \) is a positive definite, diagonal control gain matrix; the desired plunging and pitching states \( p_d \triangleq [h_d, \alpha_d]^T = [0, 0]^T \) for the LCO suppression objective. The control objective can be stated as

\[
e(t) \to 0. \quad (6.24)
\]

Note that the auxiliary tracking error signal \( r(t) \) is not directly measurable since it depends on the velocity variables.

After premultiplying (6.23) by \( M_s \), taking the time derivative of the result, and utilizing (6.5), the open loop error dynamics are obtained as

\[
M_s \dot{r} = \tilde{N}(t) + N_d(t) + \Omega u_d - Se, \quad (6.25)
\]
where $S \in \mathbb{R}^{n \times n}$ is a subsequently defined uncertain matrix (i.e., see Property 1), and the unknown, unmeasurable auxiliary signals $\tilde{N}(t), N_d(t) \in \mathbb{R}^2$ are defined as

$$
\tilde{N}(t) \triangleq \Psi(t) - M_s \alpha^2_f e + Se \quad (6.26)
$$

$$
N_d(t) \triangleq -d(t) + M_s \alpha_f r. \quad (6.27)
$$

The parameters (6.26) and (6.27) can be upperbounded as

$$
\|\tilde{N}(t)\| \leq \rho(\|z\|) \|z\|, \quad \|N_d(t)\| \leq \zeta_{d1}, \quad \|\dot{N}_d(t)\| \leq \zeta_{d2}, \quad (6.28)
$$

where $\rho(\cdot) \in \mathbb{R}$ is a positive, globally invertible non-decreasing function; $\zeta_{d1}, \zeta_{d2} \in \mathbb{R}^+$ are known bounding constants; and $z(t) \in \mathbb{R}^4$ is defined as

$$
z(t) \triangleq \begin{bmatrix} e^T(t) & r^T(t) \end{bmatrix}^T. \quad (6.29)
$$

A similar matrix decomposition as Property 1 on (6.25) can be used on $\Omega$. The results is given as

$$
M_d \dot{r} = \tilde{N}_1(t) + N_{d1}(t) + \mu + \Delta \mu - e, \quad (6.30)
$$

where $M_d = S^{-1} M_s, \tilde{N}_1 \triangleq S^{-1} \tilde{N}, N_{d1} \triangleq S^{-1} N_d$. Since $S$ is constant and positive definite, $\tilde{N}_1(t)$ and $N_{d1}(t)$ satisfy the inequalities

$$
\|\tilde{N}_1\| \leq \rho_1(\|z\|) \|z\|, \quad \|N_{d1}\| \leq \zeta_{N_{d1}}, \quad \|\dot{N}_{d1}\| \leq \zeta_{\dot{N}_{d1}}, \quad (6.31)
$$

where $\rho_1(\cdot) \in \mathbb{R}$ is a positive, globally invertible non-decreasing function; $\zeta_{N_{d1}}, \zeta_{\dot{N}_{d1}} \in \mathbb{R}$. 

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\( \mathbb{R}^+ \) are known bounding constants. We also have

\[
M_d \dot{r} = \tilde{N}_2(t) + N_{d1}(t) + \mu - e, \quad (6.32)
\]

where

\[
\tilde{N}_2 = \tilde{N}_1 + \Delta \mu. \quad (6.33)
\]

Based on (6.13), (6.31), and (6.33), \( \tilde{N}_2(t) \) satisfies the inequality

\[
\| \tilde{N}_2 \| \leq \rho_2(\| z \|) \| z \|, \quad (6.34)
\]

where \( \rho_2(\cdot) \in \mathbb{R} \) is positive, globally invertible nondecreasing function.

### 6.3.2 Closed-loop Error System

Using the open loop error system, the auxiliary control signal \( \mu(t) \) can be designed as

\[
\mu = -(k_s + 1)\hat{r} - \beta \text{sgn}(\hat{e}), \quad (6.35)
\]

where \( k_s, \beta \in \mathbb{R} \) are positive constant control gains, and where \( \hat{e}, \hat{r} \in \mathbb{R}^n \) are designed as

\[
\hat{e}(t) \triangleq \dot{p}(t) - p_d(t), \quad (6.36)
\]

\[
\hat{r}(t) \triangleq \dot{\hat{e}} + \alpha_f \hat{e}. \quad (6.37)
\]
Using (6.32) and (6.35), the closed loop error system can be obtained as

\[ M_d \dot{r} = \tilde{N}_2 + N_{d1} - e - (ks + 1) \hat{r} - \beta \text{sgn}(\hat{e}). \]  

To facilitate the following stability analysis, the control gain \( \beta_f \) introduced in (6.35) is selected to satisfy the sufficient condition

\[ \beta > \zeta_{N_{d1}} + \frac{1}{\alpha_f} \zeta_{\dot{N}_{d1}}, \]  

where \( \zeta_{N_{d1}} \) and \( \zeta_{\dot{N}_{d1}} \) are introduced in (6.31), and \( \alpha_f \) is introduced in (6.23).

### 6.4 Stability Analysis

**Theorem 6. (Finite-time Estimation)** The observer in (6.18) achieves finite-time estimation of the state \( x(t) \) in the sense that

\[ V(t) \equiv H(\hat{x}) \Rightarrow \dot{V}(t) \equiv x(t), \quad \text{for } t \geq t_0 \geq 0. \]  

Proof of Theorem 1 can be found in (Kidambi et al., 2017).

To facilitate the following stability analysis, let \( D \subset \mathbb{R}^5 \) be a domain containing \( y(t) = 0 \), where \( y(t) \subset \mathbb{R}^5 \) is defined as

\[ y(t) \triangleq \begin{bmatrix} z^T(t) \sqrt{P(t)} \end{bmatrix}^T. \]  

In (6.41), the non-negative auxiliary function \( P(t) \subset \mathbb{R} \) is the generalized solution to
6.4. STABILITY ANALYSIS

the differential equation

\[ \dot{P}(t) = -L(t), \quad (6.42) \]

\[ P(0) = \beta Q |e(0) + \eta(0)| - (e(0) + \eta(0))^T N_{d1}(0), \quad (6.43) \]

where the auxiliary function \( L(t) \in \mathbb{R} \) is defined as

\[ L(t) \triangleq r^T(t)(N_{d1}(t) - \beta \text{sgn}(\dot{e})). \quad (6.44) \]

**Lemma 8.** Provided the sufficient condition in (6.39) is satisfied, the following inequality can be obtained:

\[ \int_0^t L(\tau) \, d\tau \leq \beta Q |e(0)| - (e(0))^T N_{d1}(0). \quad (6.45) \]

Hence, (6.42), (6.43), and (6.45) can be used to conclude that \( P(t) \geq 0 \).

Proof of Lemma 8 can be found in Appendix A.

**Theorem 7.** The robust output feedback control law given in (6.22), and (6.35) ensures asymptotic regulation of LCO in the sense that

\[ \|e(t)\| \to 0 \quad \text{as} \quad t \to \infty \quad (6.46) \]

provided the control gain, \( k_s \), introduced in (6.35) is selected sufficiently large (see the subsequent proof), and \( \beta_f \) is selected to satisfy the sufficient condition in (6.39).

**Proof.** Let \( V(y,t) : \mathcal{D} \times [0,\infty) \to \mathbb{R} \) be a radially unbounded, positive definite
function defined as

\[ V = \frac{1}{2} e^T e + \frac{1}{2} r^T M_d r + P, \]  

(6.47)

which satisfies the inequalities

\[ U_1 (y) \leq V (y, t) \leq U_2 (y) \]  

(6.48)

provided the sufficient condition in (6.39) is satisfied. In (6.48), the continuous positive definite functions \( U_1 (y), U_2 (y) \in \mathbb{R} \) are defined as

\[ U_1 (y) \triangleq \| y \|^2, \quad U_2 (y) \triangleq \| y \|^2. \]  

(6.49)

After taking the time derivative of (6.47) and utilizing (6.23), (6.38), (6.42), and (6.44) and completing the squares, \( \dot{V} (t) \) can be upper bounded as

\[ \dot{V} \leq - \left( \lambda_0 - \rho_3^2 (\| z \|) \right) \| z \|^2 \]  

(6.50)

where \( \lambda_0 \triangleq \min \{ \lambda_{\min} (\alpha_f) , 1 \} \), where \( \lambda_{\min} (\cdot) \) denotes the minimum eigenvalue of the argument. Then (6.50) can be used to show that

\[ \dot{V} \leq - U (y), \]  

(6.51)

where \( U (y) = c \| z \|^2 \), for some positive constant \( c \in \mathbb{R} \), provided \( k_s \) is selected as large enough and defined on the domain

\[ \mathcal{D} \triangleq \left\{ y (t) \in \mathbb{R}^5 | \| y \| \leq \rho_3^{-1} \left( 2 \sqrt{k_s \lambda_0} \right) \right\}. \]  

(6.52)
The expressions in (6.48) and (6.51) can be used to prove that $V(y, t) \in L_\infty$ in $\mathcal{D}$; hence $e(t), r(t) \in L_\infty$ in $\mathcal{D}$. Given that $e(t), r(t) \in L_\infty$, a standard linear analysis technique can be used along with (6.22) and (6.23) to show that $\dot{e}(t) \in L_\infty$ in $\mathcal{D}$. Since $e(t), \dot{e}(t) \in L_\infty$ and using (6.22) and (6.23) with the assumption that $x_d(t), \dot{x}_d(t) \in L_\infty$ to prove that $x(t), \dot{x}(t) \in L_\infty$ in $\mathcal{D}$. Given that $e(t), \dot{e}(t) \in L_\infty$, the expression in (6.35) can be used along with (1.1) and (1.2) to prove that the control signals $u_{di}(t), v_i(t), u(t) \in L_\infty$ in $\mathcal{D}$. Since $e(t), r(t) \in L_\infty$, (6.31), (6.32), and (6.34) can be used to prove that $\dot{r}(t) \in L_\infty$ in $\mathcal{D}$. Given that $\dot{e}(t), \dot{r}(t) \in L_\infty$, it follows that $e(t), r(t)$, are uniformly continuous in $\mathcal{D}$; thus, $z(t)$ is uniformly continuous throughout closed-loop controller operation. Hence, the definitions of $U(y)$ and $z(t)$ can be used to prove that $U(y)$ is uniformly continuous in $\mathcal{D}$.

Let $\mathcal{S} \subset \mathcal{D}$ denote a set defined as follows:

$$\mathcal{S} \triangleq \left\{ y(t) \subset \mathcal{D} \mid U(y(t)) \leq \eta_1 \left( \rho^{-1} \left( 2\sqrt{k_s\lambda_0} \right) \right)^2 \right\}. \quad (6.53)$$

Theorem 8.4 of (Khalil, 2002) can now be invoked to state that

$$c \| z(t) \|^2 \to 0 \quad \text{as} \quad t \to \infty \quad \forall \ y(t_0) \in \mathcal{S}. \quad (6.53)$$

Based on the definition of $z(t)$, (6.53) can be used to show that

$$\| e(t) \| \to 0 \quad \text{as} \quad t \to \infty \quad \forall \ y(t_0) \in \mathcal{S}. \quad (6.53)$$

Thus, asymptotic regulation of the pitching and plunging displacements and its velocities can be achieved, provided the initial conditions are within the set $\mathcal{S}$, where
6.5 Simulation Results

A numerical simulation was created to demonstrate the performance of the proposed control law. For a realistic result, the following simulation results are based on detailed dynamic parameters and specifications. The simulation is based on the dynamic model given in (6.1)–(6.4). The dynamic parameters utilized in the simulation are summarized in Table 6.2 and were obtained from (Elhami & Narab, 2012). The actual parameters $\theta_1^*$ and $\theta_2^*$ and their estimates $\hat{\theta}_{1,i}$ and $\hat{\theta}_{2,i}$ are described in Table 6.3 were obtained from (MacKunis et al., 2013). The observer gains are user-defined gains that were selected as shown in Table 6.1.

The following simulation results were obtained using control gains selected as

$$\beta = \text{diag}(0.01, 0.8), \quad k_s = \text{diag}(0.1, 3), \quad \alpha_f = \text{diag}(10, 2).$$

(6.54)

Table 6.1: The gains for the observer used in the system.

<table>
<thead>
<tr>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$m_3$</th>
<th>$m_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

The nonvanishing external disturbance used in the simulation is

$$d = 0.01 \begin{bmatrix} 0.2\sin(0.5t) & 0.1\sin(0.5t) + 0.2\sin(0.5t) \end{bmatrix}^T$$
6.5. SIMULATION RESULTS

displacements in the open-loop scenario.

Figures 6.1 and 6.2 show the performance of the proposed control law to regulate and estimate the pitching and plunging displacements and velocities.

Figure 6.3 shows the virtual surface deflection angle control inputs during closed-loop controller operation of the proposed sliding mode estimation-based method.

Table 6.2: Dynamic parameters and geometric dimensions of the wing section

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\rho)</td>
<td>1.225 kg/m³</td>
</tr>
<tr>
<td>(a)</td>
<td>-0.6</td>
</tr>
<tr>
<td>(c_{m_\alpha})</td>
<td>-0.635</td>
</tr>
<tr>
<td>(m)</td>
<td>12.387 kg</td>
</tr>
<tr>
<td>(b)</td>
<td>0.125 m</td>
</tr>
<tr>
<td>(v)</td>
<td>14 m/s</td>
</tr>
<tr>
<td>(C_\alpha)</td>
<td>0.036 kg·m²/s</td>
</tr>
<tr>
<td>(c_{l_\beta})</td>
<td>3.358</td>
</tr>
<tr>
<td>(s_p)</td>
<td>0.6 m</td>
</tr>
<tr>
<td>(I_\alpha)</td>
<td>0.065 kg·m</td>
</tr>
<tr>
<td>(C_h)</td>
<td>27.43 kg/s</td>
</tr>
<tr>
<td>(c_{l_\alpha})</td>
<td>6.28</td>
</tr>
<tr>
<td>(K_h)</td>
<td>2844.4 N/m</td>
</tr>
<tr>
<td>(c_{m_\beta})</td>
<td>-0.635</td>
</tr>
<tr>
<td>(x_a)</td>
<td>0.2847</td>
</tr>
</tbody>
</table>

Table 6.3: SJA parameters and their estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\theta_{1,i}^*)</td>
<td>32.9 Volt-deg</td>
</tr>
<tr>
<td>(\theta_{1,i})</td>
<td>32.7 Volt-deg</td>
</tr>
<tr>
<td>(\theta_{2,i}^*)</td>
<td>16.5 deg</td>
</tr>
<tr>
<td>(\theta_{2,i})</td>
<td>16.4 deg</td>
</tr>
<tr>
<td>(\theta_{2,i}^*)</td>
<td>29.8 Volt-deg</td>
</tr>
<tr>
<td>(\theta_{2,i})</td>
<td>29.7 Volt-deg</td>
</tr>
<tr>
<td>(\theta_{2,i}^*)</td>
<td>15.9 deg</td>
</tr>
<tr>
<td>(\theta_{2,i})</td>
<td>15.8 deg</td>
</tr>
</tbody>
</table>
6.5. SIMULATION RESULTS

![Graph showing closed-loop response of state estimates and actual LCO displacements.](image1)

Figure 6.1: Closed-loop response of the state estimates (red) and actual LCO displacements (blue) using the sliding mode observer.

![Graph showing closed-loop response of state estimates and actual LCO rates.](image2)

Figure 6.2: Closed-loop response of the estimates (red) and actual LCO rates (blue) using the sliding mode observer.
6.6 Conclusion

A new result for limit cycle oscillation suppression using synthetic jet actuators and a sliding mode estimation has been presented. The sliding mode observer achieves finite-time estimation for the rates of pitching and plunging. A rigorous Lyapunov-based stability analysis is utilized to prove that the sliding mode observer based control law drives the limit cycle oscillation displacements to zero. Numerical simulation results are provided to demonstrate the performance of the proposed sliding mode observer in comparison to a closed-loop system employing a bank of filters.

Figure 6.3: Virtual surface deflection angle commanded during closed-loop operation using the sliding mode observer.
Chapter 7

Conclusion

In this dissertation an investigation was done using different nonlinear synthetic jet actuator-based control methods to suppress the limit cycle oscillations (i.e $\alpha \to 0$, $h \to 0$) and their rates in a small unmanned aerial vehicle’s wings. The first result was an extension (MacKunis et al., 2013) where an added model of a wind gust was included as well as using the Monte Carlo method to pseudorandomize the uncertain parameters of the SJA dynamics in the system.

The second approach was a method limit cycle oscillations suppression while simultaneously achieving asymptotic small unmanned aerial vehicles trajectory tracking control. The third method addresses the need to eliminate for limit cycle oscillations pitching and plunging rate measurements using a bank of dynamic filters in the feedback control law, as well as including the Monte Carlo method to pseudorandomize the uncertain parameters of the SJA dynamics in the system. The final method is an alternative method to eliminate LCO rate measurements by utilizing a sliding mode observer to estimate the pitching and plunging rates.
Rigorous Lyapunov-based stability analyses were utilized to prove the theoretical results, and numerical simulation results are provided to demonstrate the performance of the different proposed control laws. Future work would be to use and control the flow dynamics around the airplane’s wing while simultaneously suppressing the limit cycle oscillations in the aircraft wings and maintaining aircraft trajectory tracking using synthetic jet actuators. In addition, another future work would be to include a well accepted 3 degrees of freedom limit cycle oscillation dynamic model (i.e, include wing rock to the system) to test the controllers present in this dissertation.
Appendix A

Lemma 7 proof

The proof for this lemma widely use in the dissertation is as shown. This is based on Chapter 5 but it can be similarly used for Chapters 3, 4 and 6.

Lemma 7 provided the sufficient gain condition in (5.41) is satisfied, the following inequality can be obtained:

\[
\int_0^t L(\tau) \, d\tau \leq \beta f Q |e(0) + \eta(0)| - (e(0) + \eta(0))^T N_{d1}(0) \in \mathbb{R}.
\] (A.1)

Hence, (A.1) can be used to conclude that \( P(t) \geq 0 \), where \( P(t) \) is defined in (5.43) and (5.44).

To facilitate the following proof, the expression in (5.45) will be rewritten in a more advantageous form as follows:

\[
L(t) = \sum_{i=1}^{m} \left( r_i(t) \left( N_{d1i}(t) - \sum_{j=i}^{m} T_{ij} \mu_{ij}(t) \right) \right) \in \mathbb{R}
\] (A.2)

where \( m = 2 \) for the LCO regulation objective in the current result. In (A.2), \( r_i(t) \),
\[ N_{d_1i}(t), \mu_{1i}(t) \in \mathbb{R} \text{ for } i = 1, \ldots, m \] denote the \( i \)th elements of the vectors \( r(t), N_{d_1}(t) \), and \( \mu_1(t) \); and \( T_{ij} \in \mathbb{R} \text{ for } i = 1, \ldots, m \text{ and } j = 1, \ldots, m \) denote the \((i, j)\)th elements of the matrix \( T \).

**Proof.** Integrating both sides of (A.2) yields

\[
\int_0^t L(\tau) \, d\tau = \int_0^t \sum_{i=1}^m r_i(\tau) \left( N_{d_1i}(\tau) - \sum_{j=i}^m T_{ij}\mu_{1j}(\tau) \right) \, d\tau \in \mathbb{R}. \tag{A.3}
\]

Based on the expressions in (5.13)–(5.17), the integral in (A.3) can be expressed as

\[
\int_0^t L(\tau) \, d\tau = \int_0^t \sum_{i=1}^m (\dot{e}_i(\tau) + \dot{\eta}_i(\tau)) \left( N_{d_1i}(\tau) - \sum_{j=i}^m T_{ij}\mu_{1j}(\tau) \right) \, d\tau \\
+ \int_0^t \sum_{i=1}^m (\phi_i e_i(\tau) + \phi_i \eta_i(\tau)) \left( N_{d_1i}(\tau) - \sum_{j=i}^m T_{ij}\mu_{1j}(\tau) \right) \, d\tau \tag{A.4}
\]

where \( \phi_i \in \mathbb{R} \) denotes the \( i \)th diagonal element of the control gain matrix \( \phi \). By defining an auxiliary variable \( \nu(t) \in \mathbb{R}^m \) as

\[
\nu(t) \triangleq e(t) + \eta(t). \tag{A.5}
\]

The expression in (A.4) can be rewritten as

\[
\int_0^t L(\tau) \, d\tau = \int_0^t \sum_{i=1}^m \frac{\partial \nu_i(\tau)}{\partial \tau} N_{d_1i}(\tau) d\tau \\
- \int_0^t \sum_{i=1}^m \frac{\partial \nu_i(\tau)}{\partial \tau} \sum_{j=i}^m T_{ij}\mu_{1j}(\tau) d\tau \\
+ \int_0^t \sum_{i=1}^m \phi_i \nu_i(\tau) \left( N_{d_1i}(\tau) - \sum_{j=i}^m T_{ij}\mu_{1j}(\tau) \right) d\tau. \tag{A.6}
\]
By evaluating the first integral in (A.6) using integration by parts, equation (A.6) can be expressed as

\[
\int_0^t L(\tau) \, d\tau = \sum_{i=1}^m \nu_i(t) N_{d1i}(t) - \sum_{i=1}^m \nu_i(0) N_{d1i}(0) \\
- \int_0^t \sum_{i=1}^m \nu_i(\tau) \frac{\partial N_{d1i}(\tau)}{\partial \tau} \, d\tau - \int_0^t \sum_{i=1}^m \frac{\partial \nu_i(\tau)}{\partial \tau} \sum_{j=i}^m T_{ij} \mu_{1j}(\tau) \, d\tau \\
+ \int_0^t \sum_{j=1}^m \phi_i \nu_i(\tau) \left( N_{d1i}(\tau) - \sum_{j=i}^m T_{ij} \mu_{1j}(\tau) \right) \, d\tau. \tag{A.7}
\]

After substituting the definition of the auxiliary control term \( \mu_1(t) \) given in (5.34) and rearranging, (A.7) can be expressed as

\[
\int_0^t L(\tau) \, d\tau = \sum_{i=1}^m \nu_i(t) N_{d1i}(t) - \sum_{i=1}^m \nu_i(0) N_{d1i}(0) \\
- \int_0^t \sum_{i=1}^m \frac{\partial \nu_i(\tau)}{\partial \tau} \sum_{j=i}^m T_{ij} \beta_f \nu_j(\tau) \, d\tau \\
+ \int_0^t \sum_{i=1}^m \phi_i \nu_i(\tau) \left( N_{d1i}(\tau) - \sum_{j=i}^m T_{ij} \beta_f \nu_j(\tau) \right) \, d\tau. \tag{A.8}
\]

By using the fact that

\[
\sum_{j=i}^m T_{ij} \beta_f \nu_j(\tau) = \beta_f \left( \text{sgn} (\phi_j \nu_j(\tau)) + \sum_{j=i+1}^m \bar{T}_{ij} \text{sgn} (\nu_j(\tau)) \right), \tag{A.9}
\]
then the bounding inequalities in (5.28) can be used to express (A.8) as

\[
\int_{0}^{t} L(\tau) \, d\tau = \sum_{i=1}^{m} \nu_{i}(t) N_{d1i}(t) - \sum_{i=1}^{m} \nu_{i}(0) N_{d1i}(0)
- \int_{0}^{t} \sum_{i=1}^{m} \frac{\partial \nu_{i}(\tau)}{\partial \tau} \beta_{f} \delta \text{sgn}(\phi_{i}\nu_{i}(\tau)) \, d\tau
+ \int_{0}^{t} \sum_{i=1}^{m} \phi_{i}\nu_{i}(\tau) \left( N_{d1i}(\tau) - \frac{1}{\phi} \frac{\partial N_{d1i}(\tau)}{\partial \tau} \right)
- \delta \beta_{f} \text{sgn}(\phi_{i}\nu_{i}(\tau)) \right) \, d\tau \tag{A.10}
\]

where \( \delta \in (\varepsilon, Q) \) is a positive constant parameter. By using the property

\[
\int_{0}^{t} \frac{\partial \nu(\tau)}{\partial \tau} \text{sgn}(\phi\nu(\tau)) \, d\tau = |v(t)| - |v(0)|, \tag{A.11}
\]

then the expression in (A.10) can be rewritten as

\[
\int_{0}^{t} L(\tau) \, d\tau = - \sum_{i=1}^{m} \nu_{i}(0) N_{d1i}(0) + \sum_{i=1}^{m} \beta_{f} \delta |v_{i}(0)|
+ \sum_{i=1}^{m} \nu_{i}(t) N_{d1i}(t) - \sum_{i=1}^{m} \beta_{f} \delta |v_{i}(t)|
+ \int_{0}^{t} \sum_{i=1}^{m} \phi_{i}\nu_{i}(\tau) \left( N_{d1i}(\tau) - \frac{1}{\phi} \frac{\partial N_{d1i}(\tau)}{\partial \tau} \right)
- \delta \beta_{f} \text{sgn}(\phi_{i}\nu_{i}(\tau)) \right) \, d\tau \tag{A.12}
\]
The expression in (A.12) can be upper bounded as

\[
\int_0^t L(\tau) d\tau \leq -\nu^T(0)N_{d1}(0) + \beta_f Q|\nu(0)|
\]

\[
+ \sum_{i=1}^m (\zeta_{N_{d1}} - \varepsilon \beta_f) |\nu_i(t)|
\]

\[
+ \int_0^t \sum_{i=1}^m \phi_i |\nu_i(\tau)| \left( \zeta_{N_{d1}} - \frac{1}{\phi} \zeta_{N_{d1}} - \varepsilon \beta_f \right) d\tau. \quad (A.13)
\]

Thus, it is clear from (A.13) that if \( \beta_f \) satisfies the sufficient condition in (5.41), then

\[
\int_0^t L(\tau) d\tau \leq \beta_f Q |e(0) + \eta(0)| - (e(0) + \eta(0))^T N_{d1}(0). \quad (A.14)
\]

Hence, \( P(t) \geq 0 \) from (5.43), (5.44), and (A.14).

\[\square\]
Appendix B

Tracking and LCO Code and Simulink

The following $m-$file and Simulink models were used for the work in Chapter 4.

Main_LCO_tracking.m

```matlab
1 clc
2 clear
3
4 % Initial conditions
5 x_0 = [1 2 3 0 0 0 10 10 -10 3 1 -5]';
6 % x_0 = zeros(12,1);
7 tau_0 = [0.5 0.5 0.5]';
8 % tau_0 = zeros(3,1);
9
10 % Mode parameters
11 zeta = 0.02*eye(3);
12 omega = [45 0 0; 0 50 0; 0 0 55];
13
14
```
% A matrix
A = zeros(12);
A(1,4) = 1;
A(2,5) = 1;
A(3,6) = 1;
A(4,5) = -12.92;
A(4,6) = 1.10;
A(5,4) = -9.96;
A(6,5) = -0.10;
A(6,6) = -0.97;
A(7,10) = 1;
A(8,11) = 1;
A(9,12) = 1;
A(10,7) = -omega(1,1)^2;
A(10,10) = -2*omega(1,1)*zeta(1,1);
A(11,8) = -omega(2,2)^2;
A(11,11) = -2*omega(2,2)*zeta(2,2);
A(12,9) = -omega(3,3)^2;
A(12,12) = -2*omega(3,3)*zeta(3,3);

% B matrix
B = zeros(12,3);
B(4,2) = 1.50;
B(4,3) = -0.02;
B(5,1) = -0.98;
B(6,2) = -0.09;
B(6,3) = 0.17;

% F vector
F = zeros(9,1);
F(4) = -37.35;
F(5) = -3.13;
F(6) = 17.03;

% N Vector
```matlab
N = 5*[1 1 1]';
sin_tovector = [1 1 1]';

% Thetas
theta_hat1 = [32.0; 29.0; 20.7];
theta_hat2 = [13.7; 13.5; 12.0];
theta_star1 = [32.9; 29.8; 26.7];
theta_star2 = [14.7; 13.8; 12.8];

% Gains
alpha1 = 7;
alpha2 = 0.09;
beta = 0.5;
k = 100.550;

% Time
dt = 0.001;
Tfinal = 300;

%% Run the sim
sim('LCO_tracking_sim')

%% Plots
set(0,'defaultAxesFontSize', 22)
figure(1)
set(gca,'fontsize',20)
hold on
subplot(2,3,1)
plot(graphs.time,graphs.signals.values(:,1))
ylabel('$$\phi(t)$$','interpreter','latex')
subplot(2,3,2)
plot(graphs.time,graphs.signals.values(:,2))
ylabel('$$\theta(t)$$','interpreter','latex')
subplot(2,3,3)
plot(graphs.time,graphs.signals.values(:,3))
```
ylabel('$$\psi(t)$$','interpreter','latex')
hold off

subplot(2,3,4)
plot(graphs.time,graphs.signals.values(:,4))
ylabel('p(t)')
xlabel('Time (s)')

subplot(2,3,5)
plot(graphs.time,graphs.signals.values(:,5))
ylabel('q(t)')
xlabel('Time (s)')

subplot(2,3,6)
plot(graphs.time,graphs.signals.values(:,6))
ylabel('r(t)')
xlabel('Time (s)')
hold off

figure(2)
set(gca,'fontsize',20)
hold on

subplot(2,3,1)
plot(graphs.time,graphs.signals.values(:,7))
ylabel('$$\eta_1(t)$$','interpreter','latex')

subplot(2,3,2)
plot(graphs.time,graphs.signals.values(:,8))
ylabel('$$\eta_2(t)$$','interpreter','latex')

subplot(2,3,3)
plot(graphs.time,graphs.signals.values(:,9))
ylabel('$$\eta_3(t)$$','interpreter','latex')

subplot(2,3,4)
plot(graphs.time,graphs.signals.values(:,10))
ylabel('$$\dot{\eta}_1(t)$$','interpreter','latex')
xlabel('Time (s)')

subplot(2,3,5)
plot(graphs.time,graphs.signals.values(:,11))
ylabel('$$\dot{\eta}_2(t)$$','interpreter','latex')
xlabel('Time (s)')

subplot(2,3,6)
plot(graphs.time,graphs.signals.values(:,12))
ylabel('$$\dot{\eta}_3(t)$$','interpreter','latex')
xlabel('Time (s)')

hold off

figure(3)
set(gca,'fontsize',20)
hold on
subplot(1,3,1)
plot(graphs2.signals.values(:,1))
ylabel('$$u_1(t)$$','interpreter','latex')
xlabel('Time (s)')

subplot(1,3,2)
plot(graphs2.signals.values(:,2))
ylabel('$$u_2(t)$$','interpreter','latex')
xlabel('Time (s)')

subplot(1,3,3)
plot(graphs2.signals.values(:,3))
ylabel('$$u_3(t)$$','interpreter','latex')
xlabel('Time (s)')

hold off
Figure B.1: Overall system.
Appendix C

Bank of Filters Code and Simulink

The following m-file and Simulink models were used for the work in Chapter 5.

Main_file_OFB2.m

```matlab
1 clear
2 clc
3
4 %% Initial Conditions
5 T = 50;
6 dt = 0.001;
7 t = 0:dt:T;
8 N = length(t);
9
10 %% Wing section model parameters
11 b = 0.125; % Semi chord [m]
12 Ca = 0.036; % Structural damping coefficient in pitch [kg\cdot m^2/s]
13 c_{lb} = 3.358; % Lift coefficients per control surface deflection
14 m = 12.387; % Mass [kg]
15 s_p = 0.6; % Wing span [m]
16 K_h = 2844.4; % Structural spring constant in plunge [N/m]
17 rho = 1.225; % Density of air [kg/m^3]
```
\[ I_a = 0.065; \text{ % Mass moment of inertia of the wing about the elastic axis \left[ \text{kg} \cdot \text{m}^2 \right] } \]
\[ C_h = 27.43; \text{ % Structural damping coefficient in plunge \left[ \text{kg/s} \right] } \]
\[ c_{la} = 6.28; \text{ % Lift coefficient per angle of attack } \]
\[ a = -0.6; \text{ % Non-dimensional distance from the mid chord to the elastic axis } \]
\[ c_{mb} = -0.635; \text{ % Moment coefficient per control surface deflection } \]
\[ x_a = 0.2847; \text{ % Non-dimensional distance measured from the elastic axis to center of mass } \]
\[ c_{ma} = -0.635; \text{ % Moment coefficient per angle of attack } \]
\[ v = 12; \text{ % Velocity \left[ \text{m/s} \right] } \]

%% D coefficients
\[ d_1 = I_a/(m \cdot x_a \cdot b); \]
\[ d_2 = 1/(x_a \cdot b); \]
\[ d = m \cdot (x_a \cdot b - d_1); \]

%% B coefficients
\[ b_1 = (\rho \cdot v^2 \cdot b^2 \cdot c_{mb} \cdot s_p + d_1 \cdot \rho \cdot v^2 \cdot b \cdot c_{lb} \cdot s_p)/d; \]
\[ b_2 = (-\rho \cdot v^2 \cdot b \cdot c_{lb} \cdot s_p - d_2 \cdot \rho \cdot v^2 \cdot b^2 \cdot s_p \cdot c_{mb})/d; \]

%% B matrix
\[ B(1,1) = b_1; \]
\[ B(2,2) = b_2; \]
\[ B = \text{diag([0,0])}; \]

%% A matrix coefficients without nonlinear terms
\[ a_1 = d_1 \cdot K_h / d; \]
\[ a_2 = (\rho \cdot v^2 \cdot b^2 \cdot c_{ma} \cdot s_p + d_1 \cdot \rho \cdot v^2 \cdot b \cdot c_{la} \cdot s_p)/d; \]
\[ a_3 = (\rho \cdot v \cdot b^2 \cdot c_{ma} \cdot s_p + d_1 \cdot (C_h + \rho \cdot v \cdot b \cdot s_p \cdot c_{la}))/d; \]
\[ a_4 = (-C_h \cdot a + \rho \cdot v \cdot b \cdot 3 \cdot c_{ma} \cdot (1/2 - a) \cdot s_p + d_1 \cdot \rho \cdot v \cdot b^2 \cdot c_{la} \cdot s_p \cdot (1/2 - a))/d; \]
\[ c_1 = -K_h / d; \]
\[ c_2 = (-\rho \cdot v^2 \cdot b \cdot c_{la} \cdot s_p + d_2 \cdot (\rho \cdot v^2 \cdot b^2 \cdot s_p \cdot c_{ma})/d; \]
\[ c_3 = (-C_h - \rho \cdot v \cdot b \cdot c_{la} \cdot s_p + d_2 \cdot \rho \cdot v \cdot b^2 \cdot s_p \cdot c_{ma})/2; \]
\[ c_4 = (-\rho \cdot v \cdot b^2 \cdot c_{la} \cdot s_p \cdot (1/2 - a) + d_2 \cdot (C_h - \rho \cdot v \cdot b \cdot 3 \cdot c_{ma} \cdot s_p \cdot (1/2 - a))/d; \]
%% A Matrix population
A(1,3) = 1;
A(2,4) = 1;
A(3,1) = a1;
A(3,2) = a2;
A(3,3) = a3;
A(3,4) = a4;
A(4,1) = c1;
A(4,2) = c2;
A(4,3) = c3;
A(4,4) = c4;

%% Synthetic jet voltage approximation
theta_star1 = [32.9; 29.8];
theta_hat1 = [32.7; 29.7];
theta_star2 = [16.5; 15.9];
theta_hat2 = [16.4; 15.8];

%% Gains
k_f = diag([0.05, 0.015]);
beta_f = diag([0.068, 0.0178]);
a_f = diag([0.05, 0.015]);

%% Initial x
x_initial = [0.02; 0.02; 0; 0];
e_initial = [0; 0];
p_initial = [0; 0];
ef_initial = [0; 0];
nu_initial = [0; 0];

%% Run simulink
sim('OFB_sim3')
t = tout;

set(0,'DefaultAxesFontSize',20)
figure(1)
subplot(2,1,1)
grid on
plot(t, u_actual3(:,3))
xlabel('Time (s)','FontSize',12)
ylabel('Control Force [N]','FontSize',12)
xlim([0 T])
set(gca,'fontsize',15)

subplot(2,1,2)
grid on
plot(t, u_actual3(:,4))
xlabel('Time (s)','FontSize',12)
ylabel('Control Moment [Nm]','FontSize',12)
xlim([0 T])
set(gca,'fontsize',15)

figure(2)
subplot(2,1,1)
grid on
plot(t, x(:,1))
xlabel('Time (s)')
ylabel('h [m]')
xlim([0 T])
ylim([-0.02 0.02])
set(gca,'fontsize',15)

subplot(2,1,2)
grid on
plot(t, x(:,2))
xlabel('Time (s)')
ylabel('$\alpha$ [rad]')
Figure C.1: Overall system.

Figure C.2: The plant of the system.
Figure C.3: The controller of the system with the bank of filters shown.

Figure C.4: The disturbance added to the system.
Appendix D

Estimation Code and Simulink

The following $m-$file and Simulink models were used for the work in Chapter 6.

Main_file_Estimation_Control_SJAUncertainty.m

```matlab
1 clear all
2 close all
3 clc
4
5 % Initial Conditions
6 T = 10;
7
8 dt = 0.001;
9
10 N = 1;
11
12 x1 = zeros(2,floor(T/dt));
13 x2 = zeros(2,floor(T/dt));
14 x1_hat = zeros(2,floor(T/dt));
15 x2_hat = zeros(2,floor(T/dt));
16
17 h1 = zeros(2,floor(T/dt));
```
h2 = zeros(2,floor(T/dt));
h1_xh = zeros(2,floor(T/dt));
h2_xh = zeros(2,floor(T/dt));
y = zeros(4,floor(T/dt));
X = zeros(4,floor(T/dt));
X_hat = zeros(4,floor(T/dt));
H = zeros(4,floor(T/dt));
A = zeros(4,4);
v1 = zeros(2,floor(T/dt));
v2 = zeros(2,floor(T/dt));
x1_dot = zeros(2,floor(T/dt));
x2_dot = zeros(2,floor(T/dt));
x1_hatdot = zeros(2,floor(T/dt));
x2_hatdot = zeros(2,floor(T/dt));
mu = zeros(4,floor(T/dt));
mu_dot = zeros(4,floor(T/dt));
e = zeros(4,floor(T/dt));
e_dot = zeros(4,floor(T/dt));

%% Initialziation
x1(:,1) = [0.05; 0];
x2(:,1) = [0.02; 0.2];
x_initial = [0.02; 0.2; 0; 0];
x1_hat(:,1) = [0; 0];
x2_hat(:,1) = [0; 0];
mu(:,1) = [0; 0; 0.02; 0.2];

%% Synthetic jet voltage approximation
theta_star1 = [32.9; 29.8];
theta_star2 = [16.5; 15.9];
## Synthetic jet voltage parameters

theta\_hat1 = [32.7; 29.7];

theta\_hat2 = [16.4; 15.8];

tanhgain = 10;

for i = 1:N

%% Wing section model parameters

b = 0.125; % Semi chord [m]

C\_a = 0.036; % Structural damping coefficient in pitch [kg·m\(^2\)/s]

c\_lb = 3.358; % Lift coefficients per control surface deflection

m = 12.387; % Mass [kg]

s\_p = 0.6; % Wing span [m]

K\_h = 2844.4; % Structural spring constant in plunge [N/m]

rho = 1.225; % Density of air [kg/m\(^3\)]

I\_a = 0.065; % Mass moment of inertia of the wing about the elastic axis [kg·m\(^2\)]

C\_h = 27.43; % Structural damping coefficient in plunge [kg/s]

c\_la = 6.28; % Lift coefficient per angle of attack

a = -0.6; % Non-dimensional distance from the mid chord to the elastic axis

c\_mb = -0.635; % Moment coefficient per control surface deflection

x\_a = 0.2847; % Non-dimensional distance measured from the elastic axis to center ...

of mass

c\_ma = -0.635; % Moment coefficient per angle of attack

v = 14; % Velocity [m/s]

%% D coefficients

d1 = I\_a/(m*x\_a*b);

d2 = 1/(x\_a*b);

d = m*(x\_a*b-d1);

%% B coefficients

b1 = (rho*v\(^2\)+b\(^2\)+c\_mb*s\_p + d1*rho*v\(^2\)+b*c\_lb+s\_p)/d;

b2 = (-rho*v\(^2\)+b*c\_lb+s\_p - d2*rho*v\(^2\)+b\(^2\)+s\_p+c\_mb)/d;

%% B matrix
B(1,1) = b1;
B(2,2) = b2;

%B = [0.9 0.1; -0.1 1.1];

%%Estimate Initial Conditions
xhat_initial = [0.01; 0.1];
x2hat_initial = [0.0; 0];

%% Synthetic jet voltage approximation
theta_hat1 = [32.7; 29.7];
theta_hat2 = [16.4; 15.8];

theta_star1(:,:,i+1) = theta_star1(:,:,1) + 0.03 - 0.06*rand;
theta_star2(:,:,i+1) = theta_star2(:,:,1) + 0.03 - 0.06*rand;

%% Observer Gains
beta11 = 5;
beta12 = 3;
beta21 = 3;
beta22 = 1;

%% Gains
k_f = diag([0.1, 3]);
beta_f = diag([0.01, 0.8]);
a_f = diag([10, 2]);

%% Observer Stuff
Q_inv = inv(eye(2));

%% Desired States
x_d1 = zeros(2,1);
x_d2 = zeros(2,1);
x_d = [x_d1;x_d2];

%% Initial x
x_initial = [0.02;0.2;0;0];
e_initial = [0; 0];
p_initial = [0; 0];
ef_initial = [0; 0];
u_initial = [0; 0];

%% Run simulink
sim('Estimation_OFB_code')
% sim('Estimation_Control_OFB_code_KKB_NRP')
sim('OFB_Estimation_Control_SJAUncertainty')

e_out(:,:,i) = x;
e_out1(:,:,i) = xhat;
u_out(:,:,i) = u_actual3;
u_surf(:,:,i) = u;
% v_surf1(:,:,i) = vol1;
% v_surf2(:,:,i) = vol2;

end

t = tout;

figure(1)
```matlab
subplot(2,1,1)
plot(tout,squeeze(e_out(:,1,:)))
hold on;plot(tout,squeeze(e_out1(:,1,:)),'r')
xlabel('Time [s]','FontSize',12)
ylabel('Plunging [m]','FontSize',12)
grid on
subplot(2,1,2)
plot(tout,squeeze(e_out(:,2,:)))
hold on;plot(tout,squeeze(e_out1(:,2,:)),'r')
xlabel('Time [s]','FontSize',12)
ylabel('Pitching [deg]','FontSize',12)
grid on

figure(2)
subplot(2,1,1)
plot(tout,squeeze(e_out(:,3,:)))
hold on;plot(tout,squeeze(e_out1(:,3,:)),'r')
xlabel('Time [s]','FontSize',12)
ylabel('Velocity [m/s]','FontSize',12)
grid on
subplot(2,1,2)
plot(tout,squeeze(e_out(:,4,:)))
hold on;plot(tout,squeeze(e_out1(:,4,:)),'r')
xlabel('Time [s]','FontSize',12)
ylabel('Pitch rate [deg]','FontSize',12)
grid on

figure(2)
subplot(2,1,1)
plot(tout,squeeze(u_surf(:,1,:)))
xlabel('Time [s]','FontSize',12)
ylabel('Virtual Deflection Angle [deg]','FontSize',12)
grid on
subplot(2,1,2)
```
% plot(tout,squeeze(u.surf(:,2,:)))
% xlabel('Time [s]','FontSize',12)
% ylabel('Virtual Deflection Angle [deg]','FontSize',12)
% grid on

% figure(3)
% subplot(2,1,1)
% plot(tout,squeeze(v.surf1(:,1,:)))
% xlabel('Time [s]','FontSize',12)
% ylabel('Voltage 1 [Volt]','FontSize',12)
% grid on
% subplot(2,1,2)
% plot(tout,squeeze(v.surf2(:,1,:)))
% xlabel('Time [s]','FontSize',12)
% ylabel('Voltage 2 [Volt-deg]','FontSize',12)
% grid on

figure(4)
subplot(2,1,1)
plot(tout,squeeze(u.out(:,3,:)))
xlabel('Time [s]','FontSize',12)
ylabel('Control Force [N]','FontSize',12)
grid on
subplot(2,1,2)
plot(tout,squeeze(u.out(:,4,:)))
xlabel('Time [s]','FontSize',12)
ylabel('Control Moment [Nm]','FontSize',12)
grid on

figure(3)
subplot(221)
plot(tout,e.out(:,1,:)-e.out1(:,1,:));grid on
Figure D.1: Overall system.
Figure D.2: The plant of the system.

Figure D.3: The controller of the system.

Figure D.4: The estimation algorithm with the sliding mode estimator.
Figure D.5: The disturbance added to the system.


1441–1453.


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