



### Abstract

In this research, we discuss some important properties of half line Titchmarsh-Weyl  $m$  functions associated to the vector-valued discrete Schrodinger operators induced by the second order difference expression. The Titchmarsh-Weyl  $m$  functions provide explicit description of absolutely continuous, singular continuous and pure point spectrum of corresponding Schrodinger operators. The Remling's theorem utilizes these  $m$  functions to describe the absolutely continuous spectrum. We have established that these  $m$  functions are matrix-valued Herglotz functions mapping complex upper half plane to Siegel Space, a generalization of complex upper half plane. We then define a metric on the Siegel space as a generalization of the hyperbolic metric on complex upper half plane. Then we establish the distance decreasing property of these  $m$  functions with respect to the metric we defined, for vector-valued discrete Schrodinger operators. This property of these  $m$  functions is issential to prove the Remling's theorem, an ongoing research project.

### Introduction

The vector-valued discrete Schrödinger operators associated to the equations

$$y(n + 1) + y(n - 1) + B(n)y(n) = zy(n); z \in \mathbb{C} \quad (1)$$

where  $y(n) \in \mathbb{C}^d$  and the potential  $B(n)$  is a  $d \times d$  matrix is defined by

$$Jy(n) = y(n + 1) + y(n - 1) + B(n)y(n)$$

If the potential  $B(n)$  is bounded symmetric matrix then  $J$  is a self adjoint operator.

Definition. Let  $z \in \mathbb{C}^+$ . The Titchmarsh-Weyl  $m$  function is defined as the unique complex matrix  $M(z)$  such that

$$F(n; z) = U(n; z) + V(n; z)M(z)$$

where  $U(n; z); V(n; z)$  are matrix valued solutions consisting of  $d$  linearly independent solutions with some initial values and the matrix valued solution  $F(n; z)$  is a set of  $d$  linearly independent solutions of that are in  $l^2(N; \mathbb{C}^d)$ . If  $F$  is a  $d \times d$  matrix valued solution whose  $d$  columns are linearly independent solutions of (1.1) that are in  $l^2(N, \mathbb{C}^d)$  then these functions are given by

$$M(z) = -F(1; z)F(0; z)^{-1}$$

Moreover,

$$M(z) = \left( m_{ij}(z) \right)_{d \times d} \in \mathbb{C}^{d \times d}, \quad m_{ij}(z) = \langle \delta_j, (J - z)^{-1} \delta_i \rangle.$$

As a continuation, we extend the theory of Titchmarsh-Weyl  $m$  functions from [1] to the one on half lines:  $N_- = \{0, 1, 2, \dots, n\}$  and  $N_+ = \{n + 1, n + 2, \dots, \dots\}$

For  $z \in \mathbb{C}^+$ , the half-line  $m$  functions are defined by

$$M_{\pm}(n, z) = -F_{\pm}(n + 1, z)F_{\pm}(n; z)^{-1} \text{ where } F_{\pm}(n, z) \text{ are solutions of (1) such that}$$

$$F_-(0, z) = 0 \text{ and } F_+(n, z) \in l^2(N; \mathbb{C}^{d \times d}).$$

### Results

**Theorem 1** For  $z \in \mathbb{C}^+$   $M_{\pm}(n, z)$  are given in terms of resolvent operator as

$$M_+(n, z) = \langle \Delta_{n+1}, (J_+ - zI)^{-1} \Delta_{n+1} \rangle = \left( \langle \delta_i^{n+1}, (J_+ - zI)^{-1} \delta_j^{n+1} \rangle \right)$$

where  $\Delta_n = \{0, 0, \dots, 0, I, 0, \dots, 0\}$

These functions  $M_{\pm}(n, z)$  are matrix valued fractional transformation. More precisely

$$M_{\pm}(n, z) = T_{\pm}(B, n)M_{\pm}(n - 1) \quad \text{where} \quad T_{\pm}(B, n) = \begin{pmatrix} zI - B & \pm I \\ \mp I & 0 \end{pmatrix}$$

These transfer matrices  $T_{\pm}(B, n)$  satisfy the following symplectic identity:

**Lemma 2.3.** For all  $n$  and  $B$ ,  $T_{\pm}(B, n)$  satisfy

$$(1) T_{\pm}(B, n)^T J T_{\pm}(B, n) = J \text{ where } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$(2) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} T_+(B, n) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = T_-(B, n)$$

This shows that  $T_{\pm}(B, n) \in Sl(2d, \mathbb{C})$ , a group of  $2d \times 2d$  symplectic matrices. We denote the imaginary part of  $M_{\pm}(n, z)$  by

$$Im M_{\pm}(n, z) = \frac{1}{2i}(M - M^*)$$

Let  $\mathcal{S}_d$  is the set of all  $d \times d$  symmetric matrices with positive definite imaginary parts.

These Weyl- $m$  functions  $M_{\pm}: \mathbb{C} \rightarrow \mathbb{C}^d$  are matrix valued Herglotz functions. In addition, these are symmetric matrices with positive definite imaginary parts. Therefore  $M_{\pm} \in \mathcal{S}_d$ . We define a metric on  $\mathcal{S}_d$  by

$$d_{\infty}(Z_1, Z_2) = \inf_{Z(t)} \int_0^1 F_{Z(t)}(Z(t)) dt \text{ where } F_{Z(t)}(W) = \|Y^{-\frac{1}{2}} W Y^{\frac{1}{2}}\|.$$

The infimum is taken over all differentiable paths  $Z(t)$  joining  $Z_1$  and  $Z_2$ .

Let  $P_-(n, z) = T_-(n, z).T(n - 1, z) \dots \dots \dots T(1, z)$ . The we have the following theorem.

**Theorem 2.** Let  $z \in \mathbb{C}^+$ , then

$$d_{\infty}(M_-(n, z), P_-(n, z)M_+(0, z)) \leq \frac{1}{(1+y^2)^n} d_{\infty}(M_-(0, z), M_+(0, z))$$

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