Half Line Titchmarsh-Weyl m functions of vector-valued discrete Schrodinger operators

Keshav Raj Acharya

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Abstract

In this research, we discuss some important properties of half line Titcchmarsh-Weyl m functions associated to the vector-valued discrete Schrodinger operators induced by the second order difference expression. The Titchmarsh-Weyl m functions provide explicit description of absolutely continuous, singular continuous and pure point spectrum of corresponding Schrodinger operators. The Remling's theorem utilizes these m functions to describe the absolutely continuous spectrum. We have established that these m functions are matrix-valued Herglotz functions maping complex upper half plane to Siegel Space, a generalization of complex upper half plane. We then define a metric on the Siegel space as a generalization of the hyperbolic metric on complex upper half plane. Then we establish the distance decreasing property of these m functions with respect to the metric we defined, for vector-valued discrete Schrodinger operators. This property of these m functions is isssential to prove the Remling's theorem, an ongoing research project.

Introduction

The vector-valued discrete Schrödinger operators associated to the equations

$$y(n+1)+y(n-1)+B(n)y(n) = zy(n);\ z \in \mathbb{C} \qquad (1)$$

where $y(n) \in \mathbb{C}^d$ and the potential B(n) is a $d \times d$ matrix is defined by

$$Jy(n) = y(n+1)+y(n-1)+B(n)y(n)$$

If the potential B(n) is bounded symmetric matrix then J is a self adjoint operator.
Definition. Let $z \in C^+$. The Titchmarsh-Weyl m function is defined as the unique complex matrix $M(z)$ such that

$$F(n;\ z) = U(n;\ z) + V(n;\ z)M(z)$$

where $U(n;\ z)$; $V(n;\ z)$ are matrix valued solutions consisting of d linearly independent solutions with some initial values and the matrix valued solution $F(n;\ z)$ is a set of d linearly independent solutions of that are in $l^2(N; C^d)$. If F is a $d \times d$ matrix valued solution whose d columns are linearly independent solutions of (1.1) that are in $l^2(N, C^d)$ then these functions are given by

$$M(z) = -F(1;\ z)F(0;\ z)^{-1}$$

Moreover,

$$M(z) = \left(m_{ij}(z)\right)_{d\times d} \in \mathbb{C}^{d\times d},\quad m_{ij}(z) = \langle \delta_j, (J-z)^{-1}\delta_i \rangle.$$

As a continuation, we extend the theory of Titchmarsh-Weyl m functions from [1] to the one on helf lines: $N_- = \{0, 1, 2, \dots\dots, n\}$ and $N_+ = \{n+1, n+2, \dots\dots\dots..\}$
For $z \in C^+$, the half-line m functions are defined by
$M_\pm\ (n,z) = -F_\pm(n+1,z)F_\pm(n;\ z)^{-1}$ where $F_\pm(n,z)$ are solutions of (1) such that

$F_-(0,z) = 0$ and $F_+(n,z) \in l^2(N;\ C^{d\times d})$.

Results

Theorem 1 For $z \in \mathbb{C}^+$ $M_\pm\ (n,z)$ are given in terms of resolvent operator as

$$M_+(n,z) = \langle \Delta_{n+1}, (J_+ - zI)^{-1}\Delta_{n+1}\rangle = \left(\langle \delta_i^{n+1}, (J_+ - zI)^{-1}\delta_j^{n+1}\rangle\right)$$

where $\Delta_n = \{0, 0, \cdots\cdots\cdots 0, I, 0 \cdots, 0\}$

These functions $M_\pm\ (n,z)$ are matrix valued fractional transformation. More precisely

$$M_\pm(n,z) = T_\pm(B,n)M_\pm(n-1) \qquad \text{where} \qquad T_\pm(B,n) = \begin{pmatrix} zI - B & \pm I \\ \mp I & 0 \end{pmatrix}$$

These transfer matrices $T_\pm(B,n)$ satisfy the following symplectic identity:

Lemma 2.3. *For all n and B, $T_\pm(B,n)$ satisfy*

(1) $T_\pm(B,n)^T J T_\pm(B,n) = J$ *where* $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

(2) $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} T_+(B,n) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = T_-(B,n)$

This shows that $T_\pm(B,n) \in Sl(2d,\ C)$, a group of of $2d \times 2d$ symplectic matrices. We denote the imaginary part of $M_\pm(n,z)$ by

$$Im\ M_\pm(n,z) = \frac{1}{2i}(M - M^*)$$

Let $\mathcal{S}_d$ is the set of all $d \times d$ symmetric matrices with positive definite imaginary parts.
These Weyl-m functions $M_\pm: \mathbb{C} \to \mathbb{C}^d$ are matrix valued Hergtolz functions. In addition, thesea are symmetric matricws with positive definite imaginary parts. Therefore $M_\pm \in \mathcal{S}_d$. We define a metric on $\mathcal{S}_d$ by

$$d_\infty(Z_1, Z_2) = \inf_{Z(t)} \int_0^1 F_{Z(t)}(\dot{Z}(t))\, dt \quad \text{where } F_{Z(t)}(W) = \left\| Y^{-\frac{1}{2}}\, W\, Y^{\frac{1}{2}} \right\|.$$

The infimum is taken over all differentiable paths $Z(t)$ joining Z_1 and Z_2.

Let $P_-(n,z) = T_-(n,z).T(n-1,z) \dots\dots\dots\dots T(1,z)$. The we have the following theorem.

Theorem 2. Let $z \in \mathbb{C}^+$, then

$$d_\infty(M_-(n,z), P_-(n,z)M_+(0,z)) \le \frac{1}{(1+y^2)^n}\ d_\infty(M_-(0,z), M_+(0,z))$$

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Contact: Embry-Riddle Aeronautical University Department of Mathematics
E-mail: acharyak@erau.edu
Ph: 3862266298

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