In our research, we discuss some important properties of half line Titchmarsh-Weyl m functions associated to the vector-valued discrete Schrödinger operators induced by the second order difference expression. The Titchmarsh-Weyl m functions provide explicit description of absolutely continuous, singular continuous and pure point spectrum of corresponding Schrödinger operators. The Remling’s theorem utilizes these m functions to describe the absolutely continuous spectrum. We have established that these m functions are matrix-valued Herglotz functions mapping complex upper half plane to Siegel Space, a generalization of complex upper half plane. We then define a metric on the Siegel space as a generalization of the hyperbolic metric on complex upper half plane. Then we establish the distance decreasing property of these m functions with respect to the metric we defined, for vector-valued discrete Schrödinger operators. This property of these m functions is essential to prove the Remling’s theorem, an ongoing research project.

The vector-valued discrete Schrödinger operators associated to the equations

\[ y(n + 1) + y(n - 1) + B(n)y(n) = zy(n); \quad z \in \mathbb{C} \]  

where \( y(n) \in \mathbb{C}^d \) and the potential \( B(n) \) is a \( d \times d \) matrix defined by

\[ Iy(n) = y(n + 1) + y(n - 1) + B(n)y(n) \]

If the potential \( B(n) \) is bounded symmetric matrix then \( J \) is a self adjoint operator. Let \( z \in \mathbb{C}^+ \). The Titchmarsh-Weyl m function is defined as the unique complex matrix \( M(z) \) such that

\[ F(n; z) = U(n; z) + V(n; z)M(z) \]

where \( U(n; z); V(n; z) \) are matrix valued solutions consisting of \( d \) linearly independent solutions with some initial values and the matrix valued solution \( F(n; z) \) is a set of \( d \) linearly independent solutions of that are in \( L^2(N; \mathbb{C}^d) \). If \( F \) is a \( d \times d \) matrix valued solution whose \( d \) columns are linearly independent solutions of (1.1) that are in \( L^2(N; \mathbb{C}^d) \) then these functions are given by

\[ M(z) = -F(1; z)\overline{F}(0; z)^{-1} \]

Moreover,

\[ M(z) = \left[ m_{ij}(z) \right]_{d \times d} \in \mathbb{C}^{d\times d}, \quad m_{ij}(z) = \{\delta_{ij}, (z^{-1})^{-\delta_{ij}}\} \]

As a continuation, we extend the theory of Titchmarsh-Weyl m functions from [1] to the one on half lines: \( N = \{0, 1, 2, \ldots, n\} \) and \( N_+ = \{n + 1, n + 2, \ldots, \ldots \} \).

For \( z \in \mathbb{C}^+ \), the half-line m functions are defined by

\[ M_\pm(n; z) = -F_\pm(n + 1, z)F_\pm(n; z)^{-1} \quad \text{where} \quad F_\pm(n; z) \text{ are solutions of (1) such that} \]

\[ F_+(0, z) = 0 \quad \text{and} \quad F_+(n, z) \in L^2(N; \mathbb{C}^d) \].

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### Abstract

In this research, we discuss some important properties of half line Titchmarsh-Weyl m functions associated to the vector-valued discrete Schrödinger operators induced by the second order difference expression. The Titchmarsh-Weyl m functions provide explicit description of absolutely continuous, singular continuous and pure point spectrum of corresponding Schrödinger operators. The Remling’s theorem utilizes these m functions to describe the absolutely continuous spectrum. We have established that these m functions are matrix-valued Herglotz functions mapping complex upper half plane to Siegel Space, a generalization of complex upper half plane. We then define a metric on the Siegel space as a generalization of the hyperbolic metric on complex upper half plane. Then we establish the distance decreasing property of these m functions with respect to the metric we defined, for vector-valued discrete Schrödinger operators. This property of these m functions is essential to prove the Remling’s theorem, an ongoing research project.

### Results

**Theorem 1** For \( z \in \mathbb{C}^+ \) \( M_\pm(n, z) \) are given in terms of resolvent operator as

\[ M_\pm(n, z) = (\delta_{n+1}, (J_n - izI)^{-1}\delta_{n+1}) = (\overline{R}^{n+1}, (J_n - izI)^{-1}\overline{R}^{n+1}) \]

where \( \Delta_n = \{0, 0, \ldots, 0, i, 0, \ldots, 0\} \).

These functions \( M_\pm(n, z) \) are matrix valued fractional transformation. More precisely, \( M_\pm(n, z) = T_\pm(B, n)M_\pm(n - 1) \)

\[ T_\pm(B, n) = \begin{pmatrix} iI - B \pm i \end{pmatrix} \]

These transfer matrices \( T_\pm(B, n) \) satisfy the following symplectic identity:

**Lemma 2.3.** For all \( n \) and \( B \), \( T_\pm(B, n) \) satisfy

\[ T_\pm(B, n)^\top J T_\pm(B, n) = J \quad \text{where} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \]

\[ T_\pm(B, n)^\top \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} T_\pm(B, n) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = T_\mp(B, n) \]

This shows that \( T_\pm(B, n) \in Sl(2d, \mathbb{C}) \), a group of \( 2d \times 2d \) symplectic matrices. We denote the imaginary part of \( M_\pm(n, z) \) by

\[ \text{Im} M_\pm(n, z) = \frac{1}{2i} (M - M^\ast) \]

Let \( S_d \) is the set of all \( d \times d \) symmetric matrices with positive definite imaginary parts. These Weyl-m functions \( M_\pm : \mathbb{C} \rightarrow \mathbb{C}^d \) are matrix valued Herglotz functions. In addition, thesea are symmetric matrices with positive definite imaginary parts. Therefore \( M_\pm \in S_d \). We define a metric on \( S_d \) by

\[ d_\infty(Z_1, Z_2) = \inf_{t \in (1)} \int_0^1 F_{Z_1(t)}(Z(t)) \, dt \quad \text{where} \quad F_{Z_1}(W) = \| W^{-\frac{1}{2}} Y W^{\frac{1}{2}} \|. \]

The infimum is taken over all differentiable paths \( Z(t) \) joining \( Z_1 \) and \( Z_2 \).

Let \( P_\pm(n, z) = T_\pm(n - 1, z)T(n - 1, z) \ldots \ldots T(1, z) \). The we have the following theorem.

**Theorem 2.** Let \( z \in \mathbb{C}^+ \), then

\[ d_\infty(M_\pm(n, z), P_\pm(n, z)M_\pm(0, z)) \leq \frac{1}{(1 + y)^2} \quad d_\infty(M_\pm(0, z), M_\pm(0, z)) \]

**References:**