# Abstract

In this research, we discuss some important properties of half line Titchmarsh-Weyl m functions associated to the vector-valued discrete Schrödinger operators induced by the second order difference expression. The Titchmarsh-Weyl m functions provide explicit description of absolutely continuous, singular continuous and pure point spectrum of corresponding Schrödinger operators. The Remling’s theorem utilizes these m functions to describe the absolutely continuous spectrum. We have established that these m functions are matrix-valued Herglotz functions mapping complex upper half plane to Siegel Space, a generalization of complex upper half plane. We then define a metric on the Siegel space as a generalization of the hyperbolic metric on complex upper half plane. Then we establish the distance decreasing property of these m functions with respect to the metric we defined, for vector-valued discrete Schrödinger operators. This property of these m functions is essential to prove the Remling’s theorem, an ongoing research project.

## Introduction

The vector-valued discrete Schrödinger operators associated to the equations

$$y(n + 1) + y(n - 1) + B(n)y(n) = zy(n); \ z \in \mathbb{C}$$

(1)

where \( y(n) \in \mathbb{C}^d \) and the potential \( B(n) \) is a \( d \times d \) matrix is defined by

\[ Iy(n) = y(n + 1) + y(n - 1) + B(n)y(n) \]

If the potential \( B(n) \) is bounded symmetric matrix then \( J \) is a self adjoint operator. Definition. Let \( z \in \mathbb{C}^+ \). The Titchmarsh-Weyl m function is defined as the unique complex matrix \( M(z) \) such that

\[ F(n; z) = U(n; z) + V(n; z)M(z) \]

where \( U(n; z); V(n; z) \) are matrix valued solutions consisting of \( d \) linearly independent solutions with some initial values and the matrix valued solution \( F(n; z) \) is a set of \( d \) linearly independent solutions of that are in \( l^2(N; \mathbb{C}^d) \). If \( F \) is a \( d \times d \) matrix valued solution whose columns are linearly independent solutions of (1.1) that are in \( l^2(N; \mathbb{C}^d) \) then these functions are given by

\[ M(z) = F(1; z)F(0; z)^{-1} \]

Moreover,

\[ M(z) = \left( m_{ij}(z) \right)_{d \times d} \in \mathbb{C}^{d \times d}, \ m_{ij}(z) = \delta_j \ (j - z)^{-1}\delta_i. \]

As a continuation, we extend the theory of Titchmarsh-Weyl m functions from [1] to the one on half lines: \( N = \{ 0, 1, 2, \ldots, \ldots, n \} \) and \( N^+ = \{ n + 1, n + 2, \ldots, \ldots, \ldots \} \)

For \( z \in \mathbb{C}^+ \), the half-line m functions are defined by

\[ M_z(n, z) = \mathcal{F}_z(n + 1, z)F_z(n; z)^{-1} \] where \( F_z(n; z) \) are solutions of (1) such that

\[ F_z(0, z) = 0 \]

Results

**Theorem 1** For \( z \in \mathbb{C}^+ \) \( M_z(n, z) \) are given in terms of resolvent operator as

\[ M_z(n, z) = (\delta_{n+1}, (J_z - zI)^{-1}h_{n+1}) = \left( \delta_{n+1}, (J_z - zI)^{-1}h_{n+1} \right) \]

where \( \Delta_n = [0, 0, \ldots, \ldots, 0, 1, 0 \ldots, 0] \)

These functions \( M_z(n, z) \) are matrix valued fractional transformation. More precisely

\[ M_z(n, z) = T_z(B(n), n)M_z(n - 1) \quad \text{where} \quad T_z(B, n) = \begin{pmatrix} zI - B & \pm I \\ \mp I & zI \end{pmatrix} \]

These transfer matrices \( T_z(B, n) \) satisfy the following symplectic identity:

**Lemma 2.3.** For all \( n \) and \( B \), \( T_z(B, n) \) satisfy

\[ (1) T_z(B, n)^TJT_z(B, n) = J \quad \text{where} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \]

\[ (2) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} T_z(B, n) \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = T_z(B, n) \]

This shows that \( T_z(B, n) \in S_l(2d, \mathbb{C}) \), a group of \( 2d \times 2d \) symplectic matrices. We denote the imaginary part of \( M_z(n, z) \) by

\[ \text{Im} \, M_z(n, z) = \frac{1}{2i} (M - M^*) \]

Let \( S_d \) be the set of all \( d \times d \) symmetric matrices with positive definite imaginary parts. These Weyl m functions \( M_z: \mathbb{C} \rightarrow \mathbb{C}^d \) are matrix valued Herglotz functions. In addition, these are symmetric mixed with positive definite imaginary parts. Therefore \( M_z \in S_d \). We define a metric on \( S_d \) by

\[ d_w(Z_1, Z_2) = \inf_{Z(t)} \int_0^1 \| F_{Z(t)}(Z(t)) \| dt \quad \text{where} \quad F_{Z(t)}(W) = \left\| Y^{-1} W Y^2 \right\| \]

The infimum is taken over all differentiable paths \( Z(t) \) joining \( Z_1 \) and \( Z_2 \).

Let \( P_z(n, z) = T_z(n, z)T_z(n - 1, z) \ldots \ldots \ldots T_z(1, z) \). We have the following theorem.

**Theorem 2.** Let \( z \in \mathbb{C}^+ \), then

\[ d_w(M_-, (n, z), P_z(n, z)M_z(0, z)) \leq \frac{1}{(1 + y^{2n})} d_w(M_-(0, z), M_z(0, z)) \]

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## References


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