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A thrust equation treats propellers and rotors as aerodynamic cycles and calculates their thrust without resorting to the blade element method

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Introduction

The equation for calculating the lift L by a wing during the predesign stage is (Anderson, 2007)

$$L = \frac{1}{2}\rho v_\infty{}^2\, C_L S_{ref} \tag{1}$$

Currently there is no equivalent "napkin-friendly" equation for calculating of the thrust T of propellers and rotors as one must resort to labor-intensive computational algorithms like the vortex lattice method (VLM), the blade element method (BEM), or specialized CFD packages, tools that are indispensable during detailed design but time consuming during preliminary design. Two of the current legacy thrust equations for a rotor (Burgers, 2012) and a propeller (Burgers, 2016) lack the relevant physical parameters involved in the generation of thrust (i.e., v_∞ and ω) and use reference areas that are dimensionally proper but physically improper as they are not physically capable of exerting work onto the flow field during the generation of thrust, namely, the disk area A of rotors, and D^2 of propellers:

$$T_{rotor} = \rho\, v_{tip}{}^2 C_T\, A \qquad\qquad T_{propeller} = \rho\, C_T\, (n^2\, D^2) D^2 \tag{2}$$

A third legacy equation for calculating the thrust T of a rotor does account for relevant physical parameters (i.e., v_∞ and v_{tip}) and a physically proper reference area, the total blade area S_b (Harris, 2011):

$$T = \left(\frac{1}{4}\, v_\infty{}^2 + \frac{1}{6}\, v_{tip}{}^2\right)\, \rho\, C_L\, S_b \tag{3}$$

This paper shows a slight difference between the above equation and the energy-based thrust equation derived in Section 5. This paper follows the current practice of naming both the forward force of a propeller and the upward vertical aerodynamic force generated by a rotor as thrust T. Transonic blade phenomena and blade kinematics during forward speeds (i.e., cyclic pitch) will not be addressed.

Energy and Work of an Aerodynamic Cycle

Both the propeller and rotor are assumed to operate as aerodynamic cycles with their available kinetic energy and work as input and output respectively, a treatment admittedly borrowed from thermodynamics. These concepts are covered next.

Kinetic Energy: The *dynamic pressure* q_∞, introduced by Prandtl in 1921, is the kinetic energy per unit volume *of a fluid parcel* as it translates at a velocity v_∞ *relative to a stationary lifting surface* (Prandtl, 1921):

$$q_\infty = \frac{1}{2}\rho v_\infty{}^2 \tag{4}$$

This term expression is reinterpreted as the specific kinetic energy $\frac{1}{2}v_\infty{}^2$ *of a lifting surface as it translates relative to a static fluid parcel of density* ρ *placed at infinity*. This reinterpretation is referenced as the *kinematic pressure* Q_∞:

$$Q_\infty = \left(\frac{1}{2}\, v_\infty{}^2\right)\rho = e_{k\,trans}\,\rho \tag{5}$$

When applied to a roto-translating propeller or rotor, the existing specific kinetic energy due to translation $e_{k\,trans}$ (per unit mass of the system) is algebraically added to its specific rotational kinetic energy $e_{k\,rot}$ (also per unit mass of the system):

$$Q_\infty = \sum e_{k\,i}\,\rho = (e_{k\,trans} + e_{k\,rot})\,\rho = \left(\frac{1}{2}\, v_\infty{}^2 + \frac{1}{2}\frac{I}{m}\,\omega^2\right)\rho \tag{6}$$

The kinematic pressure Q_∞ is the algebraic sum of the two sources of kinetic energies available at the roto-translating blades *of propellers, and main and tail rotors*, whereas the density ρ *accounts for the physics of the surrounding flow field*. For a propeller generating static thrust or a hovering rotor, both at $v_\infty = 0$, the kinematic pressure Q_∞ is:

$$Q_\infty = \left(\frac{1}{2}\frac{I}{m}\,\omega^2\right)\rho = e_{k\,rot}\,\rho \tag{7}$$

Work: During the generation of thrust T, the propeller and rotor exert work w onto the flow field by means of their reference area, the *total blade area* S_b. Throughout this paper, the reference area S_{ref} is defined as the summation of all physical aerodynamic surfaces as these (i) contribute to the generation of the aerodynamic force in an additive (wing of tail-configured airplane) or subtractive sense (i.e., its horizontal tail), (ii) while exerting work onto the flow field, and (iii) found (close to) perpendicular to the aerodynamic force, be it lift L, thrust T or drag D. Note that this definition of S_{ref} does not necessarily follow the legacy definition: the reference area S_{ref} of a wing is its planform area S_w; for a tail-configured airplane, its S_{ref} is the sum of the wing *and horizontal tail planform areas*, S_p ($=S_{w+ht}$), and for propellers, main and tail rotors, it is their total blade area S_b. This definition of a reference area S_{ref} does not necessarily negate other possible useful areas used in aerospace (i.e., the exclusive use of the wing area for a tail-configured aircraft or its wetted area), but their use when calculating figures of merit as C_L or C_D will be rendered limited as frequently, these will be able to be used to compare dissimilar systems nor be read on a stand-alone basis (Burgers, 2016).

The reference area S_{ref} is a relevant parameter in the calculation of the specific work w during the generation of thrust T:

$$w = \frac{T}{\rho\, S_b} = \frac{[N]\,[m]}{[\text{kg}]} = \left[\frac{m^2}{s^2}\right] \tag{8}$$

The blade loading T/S_b of a propeller or rotor is an important parameter that can be easily varied during the pre-design stage to fine-tune their operating condition and prevent their stall in much the same way one estimates the wing area of a lifting surface L/S_{ref} using Equation (1).

Aerodynamic Cycle: Throughout this paper, a propeller and rotor are assumed to operate as cycles with their specific kinetic energy e_k as input and specific work w exerted on the flow field as output:

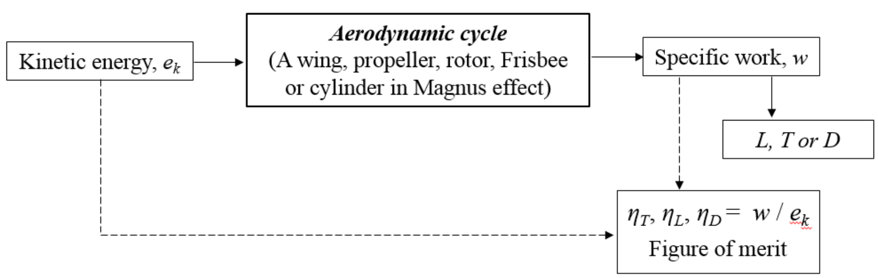

Figure 1. An aerodynamic system as a cycle and the corresponding figures of merit.

The ability of an aerodynamic cycle to generate thrust T is quantified by the ratio of work w, and available kinetic energy e_k, a ratio referred to as the normalized thrust, η_T:

$$\eta_T = \frac{w}{\sum e_{k\,i}} = \frac{\frac{T}{\rho\, S_b}}{\frac{1}{2}v_\infty{}^2 + \frac{1}{2}\frac{I}{m}\omega^2} \tag{9}$$

The normalized thrust η_T (as well as the accompanying normalized lift η_L and drag η_D) can be derived by using the work-energy equation (Burgers & Alexander, 2012) or the Buckingham π theorem. Solving for thrust *T*, the thrust equation of a propeller and a rotor is:

$$T = \left(\frac{1}{2} v_\infty{}^2 + \frac{1}{2}\frac{I}{m}\ \omega^2\right) \rho\ \eta_T\ S_b \tag{10}$$

For the case of $v_\infty = 0$, this equation calculates the static thrust T_{st} of a propeller, or the thrust *T* of a hovering main rotor in or out of ground effect, or the thrust of a static tail rotor. For the case of zero angular velocity, $\omega = 0$, the above equation morphs into the ubiquitous lift equation of a translating lifting surface with its wing planform S_w acting as a reference area, where the normalized lift η_L *equals* the legacy lift coefficient C_L (as both share the same definition of S_{ref}). This equality is not valid for a tail-configured airplane, as its normalized lift η_L uses S_{ref} as the total planform area S_p (= $S_w + S_{ht}$) whereas the lift coefficient C_L uses $S_{ref} = S_w$ (Burgers, 2016).

Energy Sources

Propellers and rotors generate thrust *T* by converting an amount η_T of their available kinetic energy e_k into work *w*. According to this definition, Equation (9), a value of, say, $\eta_T = 0.2$ should be interpreted as converting 20% of the kinetic energy e_k available at the system to work *w*. Obviously, this rationale is inconvenient when the maximum value of the normalized thrust is found to be $\eta_{T\ max} > 1$, which means there is more work *w* exerted on the flow field than kinetic energy e_k available at the system, a physical impossibility. A reason for this may be that the summation of energy available Σe_{ki} in the denominator of Equation (9) is limited for practical purposes to two sources: kinetic energy due to transition $e_{k\ trans}$ ($=\frac{1}{2} v_\infty^2$), and rotation $e_{k\ rot}$ ($=\frac{1}{2} I/m\ \omega^2$), as these sources provide the bulk of the total energy available at the propeller and rotor.

Other kinematic and elastic energy sources relevant to a main rotor could be accounted for: the kinetic energy due to cyclic pitch, blade flapping, blade's lead-lag and the elastic energy contained within the blades due to bending, tension and torsion, these last three sources clearly not of a kinetic nature. By adding these terms, a more accurate thrust equation of a propeller and rotor is arrived at:

$$T = \left(\sum e_{ki}\right) \rho\ \eta_L\ S_b = \left(\frac{1}{2} v_\infty{}^2 + \frac{1}{2}\frac{I}{m}\ \omega^2 + e_{k\ cyc.pitch} + e_{k\ flap} + e_{k\ lead-lag} + e_{k\ elastic}\right) \rho\ \eta_L\ S_b \tag{11}$$

The addition of these supplemental sources contributes a negligible amount of energy. For example, the blade's kinetic energy due to the blade's cyclic pitch $e_{k\ cyc.pitch}$ about its longitudinal axis is analogous to the cyclic pitch of bird and insect flapping wings (i.e., causing the wing to pronate and supinate). The energy due to the flapping wing cyclic pitch has been shown to be negligible (Burgers, 2019).

Accounting for these supplemental energy sources *may* result in a maximum normalized thrust $\eta_{T\ max}$ lower than 1, a statement that deserves more analysis (Moran & Shapiro, 1988):

$$\eta_{T\ max} = \frac{w}{\sum e_k} = \frac{T_{max}}{\left(\frac{1}{2} v_\infty{}^2 + \frac{1}{6} v_{tg}{}^2 + e_{k\ cyc.pitch} + e_{k\ flap} + e_{k\ lead-lag} + e_{k\ elastic}\right)\rho\ S_b} < 1\ ? \tag{12}$$

Making the normalized thrust η_T behave similarly to the thermodynamic efficiency η_{th} is a desirable goal that comes at the cost of complicating the equation, as shown above. As a result, only two energy sources, $e_{k\ trans}$ and $e_{k\ rot}$, are suggested for roto-translating systems, as shown in Equation (9). *The consistent adoption of this two-source energy term for roto-translating systems does not affect the meaningful comparison of the normalized thrust, lift and drag of a wide variety of aerodynamic systems* (e.g., propellers and rotors, flapping wings, cylinders in Magnus effect, Frisbees) as long as this rationale is adopted consistently. Of course, the normalized lift η_L of a system that operates with a single source of energy $e_{k\ trans}$ (i.e., a translating wing) can be compared meaningfully with the normalized thrust η_T of a system that operates with two sources of energy, $e_{k\ trans}$ and $e_{k\ rot}$ (i.e., a roto-translating main rotor of a helicopter).

Blade Velocity

The *specific moment of inertia I/m* in equation (10) relates to each blade of radius *R* of a rotor or propeller:

$$\frac{I}{m} = \frac{I_{single\ blade}}{m_{single\ blade}} = \frac{k\ m_{single\ blade}\ R^2}{m_{single\ blade}} = k\ R^2 = \frac{1}{3} R^2 \tag{13}$$

The constant *k* characterizes the distribution of mass along the length of the blade, assumed here to be a rod of constant cross-section as it rotates about its end. The value of *k* is 1/3 (Halliday, 1970) and has no aerodynamic implications. Replacing *I/m* from Equation (13) into equation (10) results in:

$$T = \left(\frac{1}{2} v_\infty{}^2 + \frac{1}{2}\frac{1}{3} R^2\ \omega^2\right)\rho\ \eta_T\ S_b = \frac{1}{2}\ v_b{}^2\ \rho\ \eta_L\ S_b \tag{14}$$

Based on the above equation, the square of a roto-translating blade v_b^2 is defined as:

$$v_b{}^2 = v_\infty{}^2 + \left(\sqrt{\frac{1}{3}}\; R\;\omega\right)^2 = \; v_\infty{}^2 + 0.577^2\; R^2\;\omega^2 = \; v_\infty{}^2 + \frac{1}{3}\; v_{tip}{}^2 \qquad (15)$$

Replacing the product $R\ \omega$ by v_{tip}, the velocity of the blade's tip, we obtain:

$$T = \left(\frac{1}{2}\; v_\infty{}^2 + \frac{1}{6}\; v_{tip}{}^2\right)\; \rho\; \eta_L\; S_b \qquad (16)$$

Based on the above equation, the tangential velocity v_{tg} of the blade is defined as $0.577R\ v_{tip}$, and acting at a radial blade station $r = 0.577R$.

Despite their scalar origin, the sum of the square velocities mimics the sum of perpendicular vectors $\bar{v}_\infty$ and $\bar{v}_{tip}$, which is true for a propeller blade, but not for rotor blades at their azimuth positions $\psi = 90^o$ and $\psi = 270^o$ where these velocity vectors are collinear! This fact notwithstanding, these velocities *quantify the blades' translational and rotational kinetic energies, and are not dependent on the blades' azimuth*.

Comparing the Legacy and the Energy-based Thrust Equations

The definition of the *energy-derived* blade velocity squared v_b^2, Equation (15), is different than the legacy, *vector-based* definition of v_b^2, shown below as a function of the tangential velocity v_{tip}, and the translational velocity v_∞, a function of its azimuth angle ψ (Harris, 2011):

$$v_b{}^2 = \; \left(v_{tip} + \; v_\infty \sin\psi\;\right)^2 \qquad (17)$$

In the above equation, the translation velocity v_∞ *is zero at the azimuth angles* $\psi = 0^o$ and $\psi = 180^o$, and so, its average value over one blade revolution is expected to be somewhat smaller than the energy-based value of v_b^2 in Equation (15).

The legacy thrust equation of a rotor with b blades, a blade airfoil with a lift slope a_∞, a blade chord c, and an angle of attack θ uses expression (17) for v_b^2. The product $b{\cdot}c{\cdot}R$ in the equation below equals the total blade area S_b, and the second product $a_\infty{\cdot}\theta$ is the lift coefficient C_L of the blade (Harris, 2011). The resulting legacy thrust equation, obtained by BEM and same as equation (3), is repeated below:

$$T = \frac{b\;\rho\;a_\infty\;c\;R\;\theta}{6}\left[\frac{3}{2}v_\infty{}^2 + (\omega\;R)^2\right] = \left(\frac{\mathbf{1}}{\mathbf{4}}\; v_\infty{}^2 + \frac{1}{6}\; v_{tip}{}^2\right)\; \rho\; C_L\; S_b \qquad (18)$$

For ease of comparison, the energy-based thrust equation (16) is repeated below:

$$T = \left(\mathbf{\frac{1}{2}}\ v_{\infty}{}^{2} + \frac{1}{6}\ v_{tip}{}^{2}\right) \rho\ \eta_T\ S_b \tag{19}$$

Note the discrepancy **in bold** between the two equations: the presence of the ¼ in Equation (18) compared to the ½ in Equation (19). This difference does not affect the calculation of static thrust by a propeller or thrust by a hovering rotor, as in both cases $v_{\infty} = 0$, and as a result, the thrust equation is:

$$T = \frac{1}{6}\ v_{tip}{}^{2}\ \rho\ \eta_T\ S_b \tag{20}$$

Next, the thrust T of a single roto-translating blade (as $b = 1$, S_{ref} is the area of that single blade) is calculated next using the legacy equation (18) and the energy-based equation (19). The blade has a length R of 9.14 m. (30 ft.), a chord c of 0.3 m. (1 ft.), resulting in a reference area S_b of 2.78 sq. m. (30 sq. ft.), as it rotates at 300 rpm (an angular velocity ω of 31.42 1/s) and a forward speed v_{∞} that varies from hover to a forward speed of, say, 45.72 m/s (150 ft/s) while operating at a constant lift coefficient C_L in Equation (18), and a constant normalized thrust η_T in Equation (19) of 0.7, while operating at sea level ($\rho = 1.225$ kg/m^3 = 0.002378 slugs/ft^3). Results are shown in Figure 2.

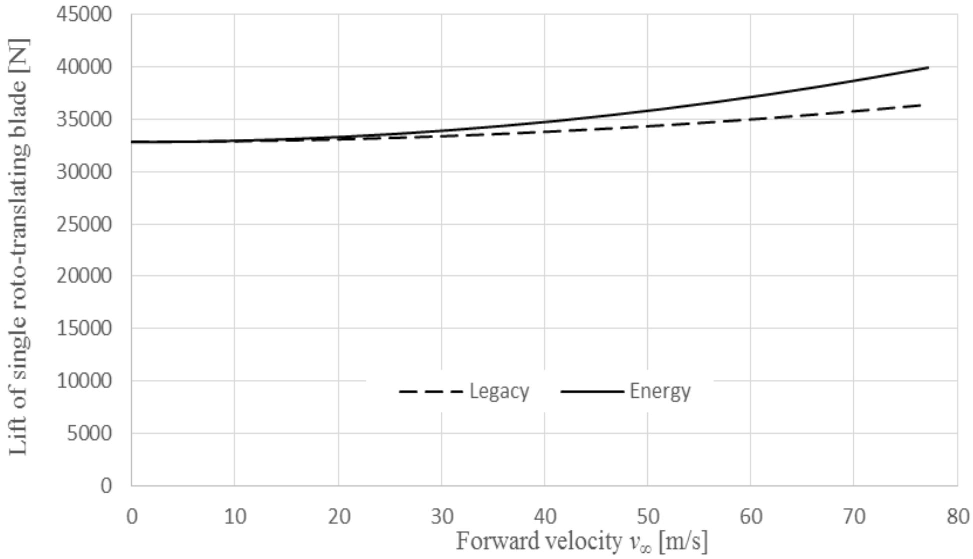

Figure 2. Comparison of legacy and energy-based thrust equation.

During hover, both equations yield the same result. At a forward speed of 30 m/s (100 ft/s), the energy-based equation yields a thrust value that is 2% higher, and at 75 m/s (250 ft/s), 10% higher.

The lift coefficient C_L in Equation (18) does not have a formal physics-based definition and so, using the vector-based definition of the square of the velocity v_b^2, Equation (17), the numerical value of C_L that is not restricted by a physical definition. This same rationale is also applicable when calculating the legacy thrust coefficient C_T of a propeller using the square of its "velocity" n^2D^2, as the resulting value of C_T is not expected to satisfy a physical definition. When using Equation (9), one arrives at a normalized thrust η_T that is defined as the ratio of work and kinetic energy, w/e_k and so, it is important to use the square of the velocity v_b^2 that is indicative of the amount of kinetic energy e_k available at the propeller and rotor. The differences between these two numbers is that the normalized thrust can be used as a figure of merit with a physical meaning that allows for a meaningful comparison between dissimilar systems.

Advance Ratio and a Modifier

The advance ratio J of a propeller or rotor of diameter d, translating at velocity v_∞, and rotating at n rpm is (McCormick, 1979):

$$J = \frac{v_\infty}{n\,d} \tag{21}$$

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Introducing the following geometric and kinematic-based equalities:

$$R^2 = \frac{d^2}{4}\ ; \quad \omega^2 = 4\pi^2 n^2\ ; \quad d^2\, n^2 = \frac{v_\infty{}^2}{J^2} \tag{22}$$

From the above equations, the square of the tangential velocity $R^2\omega^2$ equals:

$$R^2\,\omega^2 = \left(\frac{d^2}{4}\right)(4\,\pi^2\,n^2) = \pi^2\,\frac{v_\infty{}^2}{J^2} \tag{23}$$

Replacing $R^2\omega^2$ (= $v_{tip}{}^2$) by $\pi^2\, v_\infty{}^2/\,J^2$ in equation (15) and moving the term $v_\infty{}^2$ out of the parentheses as a common factor, we define the *blade velocity* v_b as:

$$v_b = \left(v_\infty{}^2 + \frac{1}{3}v_{tip}{}^2\right)^{\frac{1}{2}} = \left(v_\infty{}^2 + \frac{1}{3}R^2\,\omega^2\right)^{\frac{1}{2}} = v_\infty\left[1 + \frac{1}{3}\left(\frac{\pi}{J}\right)^2\right]^{\frac{1}{2}} \tag{24}$$

For lack of a better name, the term in brackets is referred to as the *modifier* as it modifies the blade's translation velocity v_∞ to the roto-translating blade velocity v_b. The thrust of a propeller is written next as a function of the modifier:

$$T = \frac{1}{2}v_\infty{}^2\left[1 + \frac{1}{3}\left(\frac{\pi}{J}\right)^2\right]\rho\,\eta_T\,S_b = \frac{1}{2}v_b{}^2\rho\,\eta_T\,S_b = Q_\infty\,\eta_T\,S_b \tag{25}$$

For a translating wing, $J = \infty$, and making $\pi/J = 0$ and replacing T by L, η_T by η_L and S_b by the wing planform area S_w, the equation morphs to the lift equation of a lifting surface (with $S_{ref} = S_w$) or that of a tail-configured aircraft (with $S_{ref} = S_w + S_{ht}$) (Burgers, 2016).

A more concise equation can be written by borrowing the definition of *equivalent thrust area* f_T (= $\eta_T\, S_b$) from aircraft design (Roskam, 1983):

$$T = Q_\infty\, f_T \tag{26}$$

The drag D of a roto-translating, windmilling propeller ($v_\infty \neq 0, \neq \omega$) is:

$$D = \frac{1}{2}\,v_\infty{}^2\left[1 + \frac{1}{3}\left(\frac{\pi}{J}\right)^2\right]\rho\,\eta_D\,S_b = Q_\infty\,\eta_D\,S_b \tag{27}$$

The drag D of a static wind turbine (assuming wind speed $v_\infty \approx 0$), related to power extraction, is:

$$D = \frac{1}{6}\, v_{tip}^{\;\;2}\, \rho\, \eta_D\, S_b \tag{28}$$

Note that the thrust T and lift L and related normalized thrust η_T and lift η_T are evaluated when work w is exerted on the flow field, whereas the drag D and the normalized drag η_D is evaluated when the flow field exerts work on the system.

The modifier also corrects the Reynolds number Re_∞ of a propeller blade translating at a speed v_∞ to its proper Reynolds number Re_b by accounting for its actual blade speed v_b:

$$Re_b = \frac{\rho\, v_b\, c}{\mu} = \frac{\rho\, v_\infty\, c}{\mu}\left[1 + \frac{1}{3}\left(\frac{\pi}{J}\right)^2\right]^{\frac{1}{2}} = Re_\infty\left[1 + \frac{1}{3}\left(\frac{\pi}{J}\right)^2\right]^{\frac{1}{2}} \tag{29}$$

The length of the chord c is measured at a blade station $r = 0.577 \cdot R$.

The following table reviews how translation-related parameters are modified to account for their rotation.

Table 1
Translation-related parameters modified to roto-translating parameters

Parameter	***multiplied by***	***converts to***	***Equation (#)***	***Observations***
q_∞	$1 + 1/3\ (\pi/J)^2$	Q_∞	(25)	Dynamic pressure to kinetic pressure
v_∞	$[1 + 1/3\ (\pi/J)^2]^{1/2}$	v_b	(24)	Translation blade velocity to roto-trans. blade velocity
ek_{trans}	$1 + 1/3\ (\pi/J)^2$	$ek_{trans} + ek_{rot}$	—	KE due to translation to KE due to roto-translation
Re_∞	$[1 + 1/3\ (\pi/J)^2]^{1/2}$	Re_b	(29)	Re of translating blade to Re roto-transl. blade
$f_{T\,prop}$	$1 + 1/3\ (\pi/J)^2$	$f_{D\,af}$	(38)	Equivalent thrust area to equivalent drag area

The relationship between the equivalent thrust area $f_{T\,prop}$ of a propeller and the equivalent drag area $f_{D\,af}$ of an aircraft's airframe is shown in the above table.

Numerical Characterization of a Propeller

A series of small propellers where tested in the UIUC wind tunnel (Uhlig & Selig, 2008). One particular propeller test showed a clear blade stall while operating at the highest possible blade angle pitch setting of 39°, measured at 75% radius: a two-bladed *Ramoser varioPROP* 9.9D of diameter d of 25.14 cm (9.9").

This propeller has the following attributes: a blade length of 125.57 mm (0.412 ft), a specific moment of inertia, *I/m,* of 5258.31 mm^2 (0.0566 ft^2), calculated using Equation (13). The area of a single blade S_{1b}, is 2461.93 mm^2 (0.0265 ft^2) is obtained from the blade's chord evolution along the blade, r/R in (Uhlig & Selig, 2008). The total blade area of the propeller, S_b, of 4942.44 mm^2 (0.0532 ft^2), is used as S_{ref}.

Side and planform views

Figure 3. Side and plan views of the Ramoser varioPROP 9.9D.

During this particular test, the propeller operated at ≈ 4000 rpm and a blade pitch angle of 39° as the incoming translation velocity in the wind tunnel, v_∞, is varied from 0 m/s to 24.61 m/s (80.77 ft/s). At an advance ratio J (= v_∞/nd) of zero, the propeller has zero forward speed, $v_\infty = 0$, and the corresponding available specific kinetic energies are $e_{k\ trans} = 0\ m^2/s^2$, and $e_{k\ rot} = 462.27\ m^2/s^2$ (4975.93 ft^2/s^2) while generating a static thrust T of 2.685 N (0.6037 lb.). At an advance ratio J of 1.46, its maximum translational velocity v_∞ is 24.61 m/s (80.77 ft/s), and the available specific kinetic energies are $e_{k\ trans} = 303\ m^2/s^2$ (3262 ft^2/s^2), and $e_{k\ rot}$ of 467.11 m^2/s^2 (5028 (ft^2/s^2) as the propeller generates a thrust T of -0.3 N (-0.068 lb.).

A useful parameter from wing design is the blade loading T/S_b. At an advance ratio J of 0.4, the blades operate at a normalized thrust η_T is 0.95 and the thrust T is 2.79 N (0.28 kg/0.628 lb.). The resulting blade loading T/S_b is 56.6 kg/m^2 (11.8 lb/sq. ft.), comparable to the wing loading of a sailplane.

The following figure shows the normalized thrust η_T and the legacy thrust coefficient C_T plotted against the advance ratio:

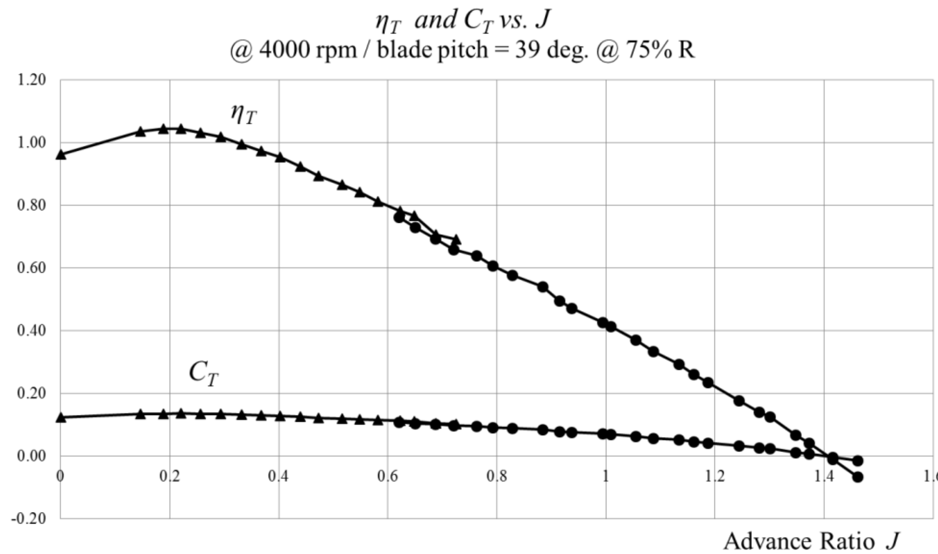

Figure 4. Normalized thrust and the legacy thrust coefficient against J for a 39° blade pitch.

The maximum normalized thrust $\eta_{T\ max}$ and the thrust coefficient $C_{T\ max}$ is the operating point at which the blades generate their maximum thrust, T_{max}, of 2.95 N (0.663 lb.) at an advance ratio, J, of 0.188, and a Reynolds number of ≈ 37,000. Beyond this operating condition, the blades stall. The maximum normalized thrust, $\eta_{T\ max}$ is 1.044 (equal to its C_{Lmax}) is calculated by solving for $\eta_{T\ max}$ in Equation (25):

$$\eta_{T\ max} = \frac{T_{max}}{\frac{1}{2}\rho\, v_b{}^2\, S_b} = \frac{T_{max}}{\frac{1}{2}\rho\, v_\infty{}^2 \left[1+\frac{1}{3}\left(\frac{\pi}{J}\right)^2\right] S_b} = \frac{w_{max}}{e_{k\ avail}} = 1.044 \qquad (30)$$

Whereas C_T lacks a physical interpretation, the $\eta_{T\ max}$ is interpreted as the ratio of the maximum specific work w exerted by the propeller onto the flow field per kinetic energy available at the propeller prior to stall.

Note that this maximum value exceeds 1, and as discussed, comes close to conforming to the expected behavior from the efficiency η_{th} in thermodynamics. As mentioned, the inclusion of supplemental sources of kinetic and elastic sources of energy may correct this, at the expense of impracticality.

The following table compares the normalized thrust, η_T, the normalized torque, η_Q, the normalized power, η_P, and the propeller efficiency η_{prop} with their analogous legacy equations:

Table 2

Comparative table of legacy coefficients and normalized numbers for the thrust T, torque Q, power P and efficiency η_{prop} of a propeller

Parameter	Legacy Coefficient	Normalized Number	Observations
Thrust	$C_T = \frac{T}{\rho\, n^2\, D^4}$	$\eta_T = \frac{T}{\frac{1}{2}\,\rho\, {v_b}^2\, S_b}$	$C_T \neq \eta_T$
Torque	$C_Q = \frac{Q}{\rho\, n^2\, D^5}$	$\eta_Q = \frac{Q}{\frac{1}{2}\,\rho\, {v_b}^2\, S_b\, D}$	$C_Q \neq \eta_Q$
Power	$C_P = \frac{P}{\rho\, n^3\, D^5}$	$\eta_P = \frac{P}{\frac{1}{2}\,\rho\, {v_b}^2\, S_b\, n\, D}$	$C_P \neq \eta_P$
Efficiency	$\eta_{prop} = \frac{T\, v_\infty}{P} = \frac{C_T}{C_P}\, J = \frac{\eta_T}{\eta_P}\, J$		$\frac{C_T}{C_P} = \frac{\eta_T}{\eta_P}$

The last row in this table shows that the propeller efficiency η_{prop} can be calculated by using either the ratio of legacy thrust and power coefficients, or the ratio of energy-based normalized thrust and power, as:

$$\eta_{prop} = \frac{C_T}{C_P}\, J = \frac{\eta_T}{\eta_P}\, J \tag{31}$$

The next figure plots the normalized power η_P and the propeller efficiency η_{prop}, against the advance ratio *J*:

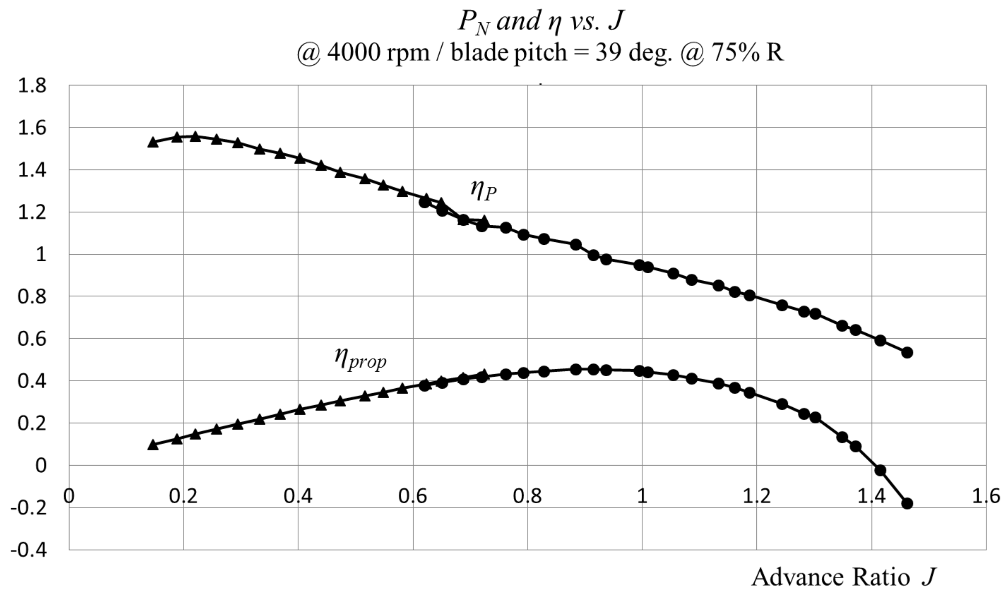

Figure 5. Normalized power, and propeller efficiency η_{prop} plotted against J for a 39° blade pitch.

The translation velocity v_∞, the tangential velocity v_{tip}, and the resultant blade velocity v_b for a rotating speed of 4,000 rpm are plotted next against advance ratio, J:

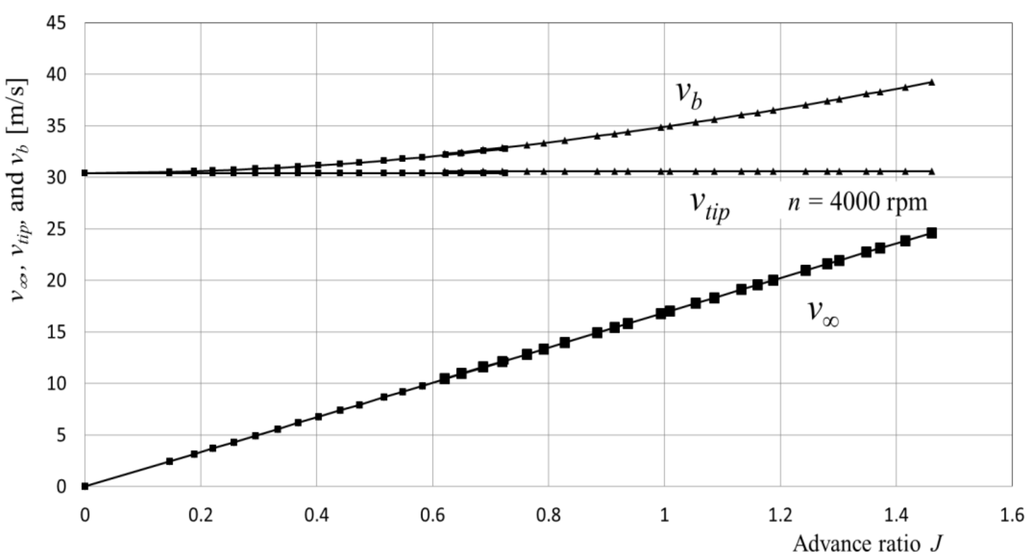

Figure 6. Tip, translation, and blade velocities plotted against the advance ratio, J.

The blade's Reynolds number is calculated using Equation (29) and plotted below against the advance ratio *J*:

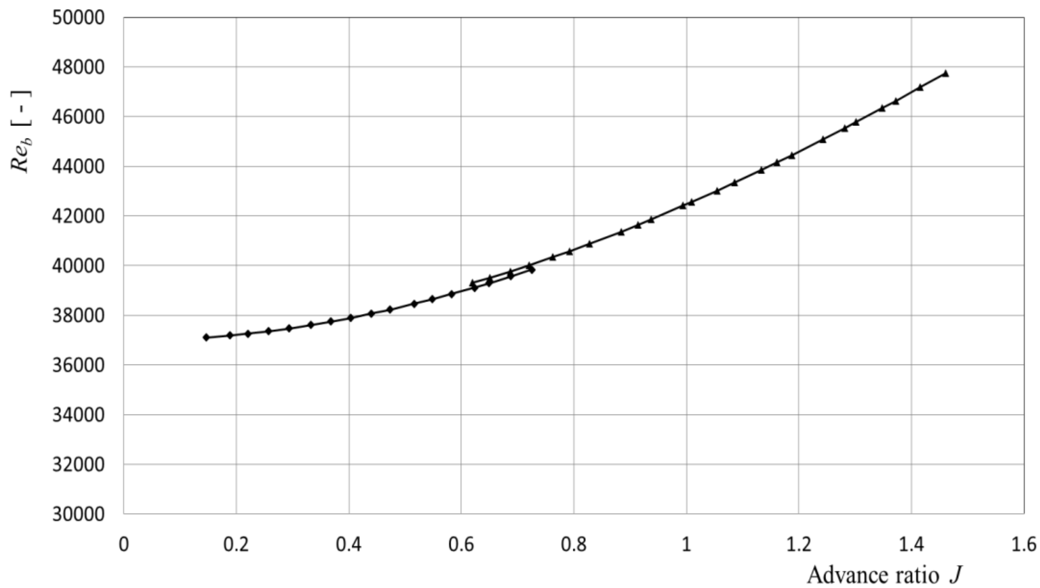

Figure 7. Graph of blade Reynolds number Re_b at $0.577R$ against advance ratio *J*.

It is apparent that a propeller can be designed during its predesign stage in a similar way one would design a wing.

Equilibrium Equations for Propellers and Rotor-driven Vehicles

Various conceptual and numerical applications are presented next.

The Equilibrium Equation for the Rotor Torque of a Helicopter

In order to counter the torque imposed by the main rotor of a helicopter, the following equation can be used to predesign its tail rotor:

$$\begin{aligned}\text{Helicopter torque} &= \text{Sideforce by tail rotor} \times \text{Moment arm} \\ &= K \times \left(\frac{1}{6}\, {v_{tip}}^2\, \rho\, \eta_T\, S_b\right) \times \text{Moment arm} \qquad (32)\end{aligned}$$

The factor *K* multiplying the parentheses containing equation (20) accounts for the effects due to the aerodynamic distortion at the face of the tail rotor due to the main rotor's downwash.

Calculation of the Blade Area of a Propeller

In a case of "point design," the total blade area of the propeller powering a Piper Cherokee PA-28-180 (McCormick, 1979) is calculated next. The aircraft flies at a constant true airspeed of 60.25 m/s (197.7 fps), a constant altitude of 914.4 m (3,000 ft), a density ρ of 1.1209 kg/m^3 (0.002175 slugs/cu.ft.) and has an aerodynamic drag D equal to its thrust T of 1423.4 N (145.1 kg / 320 lb.). Its fixed pitch, single (N = 1) two-bladed propeller, has a diameter of 1.88 m (6.17 ft.), operates at 2,400 rpm (n = 40 1/s), and at an advance ratio J (= v_∞/nD) of 0.8. The normalized thrust of the blade, η_T, is estimated to be 0.7. The equation for the blade area S_b is solved from Equation (41):

$$S_b = \frac{T}{\frac{1}{2}\rho\ v_\infty{}^2\left[1+\frac{1}{3}\left(\frac{\pi}{J}\right)^2\right]\eta_T} = \frac{T}{\frac{1}{2}\rho\, v_\text{b}{}^2\, \eta_T} = \frac{T}{Q_\infty\, \eta_T} \quad (33)$$

The blade area S_b of the propeller is 0.162 m^2 (1.75 sq. ft.).

The Equilibrium Equation of a Propeller-Driven Aircraft

The drag equation of an aircraft's airframe is a function of the normalized drag η_D (or its drag coefficient C_D if frontal area S_f is used as its reference area):

$$D = \frac{1}{2}\,\rho\, v_\infty{}^2\, \eta_D\, S_f \quad (34)$$

The above drag equation D of an airplane's airframe translating at a speed v_∞ is equalized to the thrust equation T, Equation (25), generated by a number N of propellers. Simplifying terms (½ρ $v_\infty{}^2$) results in the following equilibrium equation:

$$\left[1+\frac{1}{3}\left(\frac{\pi}{J}\right)^2\right]\eta_T\, S_b\ N = \eta_D\, S_f \quad (35)$$

This equation can be rewritten next by using the *equivalent thrust area* $f_{T\ prop}$ of the propellers (= $\eta_T\, S_b$) and the *equivalent drag area* $f_{D\ af}$ of the aircraft's airframe (= $\eta_D\, S_f$):

$$\left[1+\frac{1}{3}\left(\frac{\pi}{J}\right)^2\right] f_{T\,prop}\, N = f_{D\,af} \quad (36)$$

The modifier is seen modifying the equivalent thrust area of the propeller to the equivalent drag area of the airframe, or it can be seen acting as a figure of merit:

$$\left[1 + \frac{1}{3}\left(\frac{\pi}{J}\right)^2\right] N = \frac{f_{Daf}}{f_{T\,prop}} \tag{37}$$

This ratio can be fixed by defining the advance ratio J and N. Its use is illustrated in the next section.

Next, the normalized drag η_D (or drag coefficient $C_{D\bullet}$, using Hoerner's nomenclature (Hoerner, 1965) of the same aircraft, a Piper Cherokee (McCormick, 1979), is calculated, with a wing area, S_p, of 14.86 m^2 (160 sq. ft.), a total frontal area, S_f, of 3.56 m^2 (36.38 sq. ft.) and same propeller characteristics detailed in the prior subsection. Solving for the normalized drag η_D in Equation (35) for $N = 1$, we obtain:

$$\eta_D = C_{D\bullet} = \left[1 + \frac{1}{3}\left(\frac{\pi}{J}\right)^2\right] \eta_T \left(\frac{S_b}{S_f}\right) = 0.196 \tag{38}$$

Two design parameters can be estimated simultaneously from the above equation: (i) the equivalent drag area, $f_{D\,ac}$, (= $\eta_D S_f = C_{D\bullet} S_f$), which equals 0.69 m^2 (7.52 sq.ft.), and (ii) the equivalent thrust area f_T (= $\eta_T \cdot S_b$) which equals 0.11 m^2 (1.23 sq. ft.).

Conclusion

The application of the concepts of energy, work and cycle in the field of aerodynamics results in a thrust equation suited to the predesign of propellers and rotors, and can be extended to the calculation of lift, thrust and drag of, say, flapping wings and rotating cylinders in Magnus effect. These equations, which share a common lineage with the ubiquitous lift equation of a translating lifting surface, do not require prior knowledge of the surrounding flow field (as when applying the BEM), and the respective normalized force η_F can be read on a stand-alone basis (estimating it as a high or low value, relative to a maximum value close to 1, depending on its Reynolds number), and can be used as a figure of merit to compare the ability to generate aerodynamic forces of a diverse range of systems, regardless of their kinematics.

The empirical nature of the thrust equation acts as a convenient nexus between experiment (i.e., by obtaining, say, the normalized thrust η_T of coaxial rotors in ground effect or contra-rotating propellers by means of flight testing or wind tunnel testing and matching these values with theoretical tools (i.e., BEM, CFD).

This energy-based perspective may help reduce the existing compartmentalization within aerodynamics (e.g., between rotors and propellers as discussed earlier), facilitate cross-pollination between aerospace, biomechanics, and marine propulsion, and contribute to the mathematical groundwork of nascent sciences (e.g., paleoaerodynamics) (Burgers, 2019).

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Nomenclature

b = number of blades of propeller or rotor

C_D = parasite drag coefficient

$C_{D\bullet}$ = parasite drag coefficient, referenced to frontal area, S_f

C_L = lift coefficient

C_P = power coefficient, $P / \rho_\infty n^3 D^5$

C_Q = torque coefficient, $Q / \rho_\infty n^2 D^5$

C_T = thrust coefficient, $T / \rho_\infty n^2 D^4$

c = chord of blade measured at 0.577 R

D = parasite drag

d = propeller and rotor diameter = $2R$

e_k = specific kinetic energy per unit mass available at the system

F = aerodynamic force, lift, thrust or drag

$f_{D\ af}$ = equivalent drag area of the aircraft's airframe

$f_{T\ prop}$ = equivalent propulsive area

I = moment of inertia of blade

J = advance ratio of propeller = v_∞ / nd

KE = kinetic energy

L = lift

m = mass

n = revolutions per second

N = number of propellers

P = power

η_P = normalized power

Q = propeller torque

η_Q = normalized torque

q_∞ = dynamic pressure, $½ \cdot \rho_\infty \cdot v_\infty^2$

Q_∞ = kinetic pressure, $½ \cdot \rho_\infty \cdot v_b^2$

Re_∞ = Reynolds number of translating (non-rotating) propeller blade

Re_b = Reynolds number of roto-translating propeller blade

r = radius at given blade station, $0 < r < R$

R = propeller or rotor radius

S_b = reference blade area of propeller or rotor

S_{ht} = area of horizontal tail of tail-configured aircraft

S_p = total wing area, including wing, horizontal tail and canard, if applicable

S_{ref} = reference area

S_w = wing area

S_{wet} = wetted area

T = thrust

v_b = blade velocity

v_{tip} = blade tip tangential velocity

v_∞ = forward (translation) velocity

η_{prop} = propeller efficiency

η_D = normalized drag

η_L = normalized lift

η_T = normalized thrust

η_{th} = thermodynamic efficiency

ρ = density of fluid

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ω = angular velocity
θ = angle of attack of blade

