Adaptive Commanding of Control Moment Gyroscopes with Backlash

Justin G. Bourke

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ADAPTIVE COMMANDING OF CONTROL MOMENT GYROSCOPES

WITH BACKLASH

A Thesis

Submitted to the Faculty

of

Embry-Riddle Aeronautical University

by

Justin G. Bourke

In Partial Fulfillment of the

Requirements for the Degree

of

Master of Science in Aerospace Engineering

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Daytona Beach, Florida
ADAPTIVE COMMANDING OF CONTROL MOMENT GYROSCOPES
WITH BACKLASH

by

Justin G. Bourke

A Thesis prepared under the direction of the candidate's committee chairman, Dr. Bogdan Udrea, Department of Aerospace Engineering, and has been approved by the members of the thesis committee. It was submitted to the School of Graduate Studies and Research and was accepted in partial fulfillment of the requirements for the degree of Master of Science in Aerospace Engineering.

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**SYMBOLS**

V<sub>in</sub>  Motor Voltage (V)
L  Motor Inductance (H)
R  Motor Resistance (Ohms)
V<sub>emf</sub>  Motor Back EMF (V)
k  Motor Torque Constant
ω  Motor Speed (rad/s)
J  Motor Moment of Inertia (kg·m<sup>2</sup>)
b  Motor Coefficient of Viscous Friction (N·m·rad/s)
T<sub>s</sub>  Shaft torque (N·m)
θ<sub>s</sub>  Shaft twist (rad)
J<sub>g</sub>  Gimbal Moment of Inertia (kg·m<sup>2</sup>)
b<sub>g</sub>  Gimbal Coefficient of Viscous Friction (N·m·rad/s)
ω<sub>g</sub>  Gimbel Speed (rad/s)
G<sub>r</sub>  Gear Ratio 1 : G<sub>r</sub>
k<sub>s</sub>  Shaft stiffness coefficient (N·m/rad)
c<sub>s</sub>  Shaft dampening coefficient (N·m·s/rad)
θ, θ<sub>g</sub>  Rotor and gimbal positions, respectively (rad)
θ<sub>p</sub>  Backlash Angle (rad)
α  Half of the deadband width (rad)
K<sub>p</sub>  Proportional control gain (V/rad/s)
K<sub>i</sub>  Integral control gain (V/rad)
r  Command signal (rad/s)
τ  Time constant (s)
ζ  Damping ratio
ω<sub>n</sub>  Natural Frequency (rad/s)
K<sub>dc</sub>  DC or Low Frequency Gain (rad/s/V)
σ  Adaptive Weighting
<table>
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<tr>
<td>DC</td>
<td>Direct Current</td>
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<td>EMF</td>
<td>Electromotive Force</td>
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<td>CMG</td>
<td>Control Moment Gyroscope</td>
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<td>PI</td>
<td>Proportional Integral Controller</td>
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<td>PID</td>
<td>Proportional Integral Derivative Controller</td>
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<td>MRAC</td>
<td>Model Referencing Adaptive Controller</td>
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<td>LTI</td>
<td>Linear Time Invariant</td>
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ABSTRACT


The existence of backlash in mechanical systems provides significant challenges when attempting to control these systems to a high degree of precision. The imperfect meshing of gear or belt teeth deteriorates the performance of position controllers and tracking of small commands, producing unacceptable steady-state offsets, increased rise and settling times. Agile spacecraft often use control moment gyroscopes (CMGs) equipped with gear trains to efficiently provide torque for the fine attitude adjustments used in docking and precision stabilization maneuvers. A theoretical examination and a practical model is developed to study the effectiveness of both proportional-integral (PI) and model referencing adaptive controllers (MRAC) in overcoming the non-linearity introduced by gear lash. A Lyapunov analysis of the system’s equations of motion provides knowledge of its convergence, the tracking of ideal trajectories, and the rejections of disturbances. The objective is to create an adaptive control law that rejects the non-linearity and maintains acceptable performance with small torque commands. This control law is then validated in Simulink using a discontinuous backlash model.
1. Introduction

The purpose of this thesis is to analyze and demonstrate the superior performance of a Model Reference Adaptive Controller (MRAC) actuating a DC motor with a gearbox. With a geared system, an undesirable phenomenon called backlash arises which disrupts attempts at precise control with traditional control methods. The aim of this demonstration is to test and validate the MRAC’s capability over a PI controller in the presence of this non-linearity. The analysis is performed within context of such a controller used on a Control Moment Gyroscope (CMG) in a spacecraft’s attitude control system.

CMGs have found use primarily in attitude control systems of spacecraft to dampen tumbling and control spacecraft pointing maneuvers. Much like regular gyroscopes, CMGs take advantage of the conservation of angular momentum of a rotating disk, the momentum wheel, that is mounted to a gimbal arm to allow free rotation around a second axis. The difference being that a gyroscope’s gimbal spins as response to acceleration, whereas a CMG’s gimbal is spun to induce an angular acceleration. An array of a minimum of 3 CMGs, but usually 4, are typically arranged in the spacecraft in such a way as to produce torques around all 3 principle axes of a spacecraft. The torque produced by an individual CMG is related to the gimbal spin speed.

\[
T = \vec{\delta} \times \vec{h}_{sc}
\]  

(1.1)
where $\hat{\delta}$ is the angular velocity of the gimbal and $\vec{h}_{sc}$ is the angular momentum of the momentum wheel in the spacecraft’s body frame.

![Diagram of a CMG and the torque vector produced](Penn, 2015)

Figure 1.1: Illustration of a CMG and the torque vector produced (Penn, 2015)

The angular momentum of the momentum wheel can be manipulated to produce torques, but it will be assumed to be of constant magnitude as a variable speed CMG is outside the scope of this thesis. Instead, the angular speed of the gimbal will be manipulated. The whole gimbal and wheel assembly can be considered to be an inertial load attached to the shaft of the DC motor.

The system is a standard speed control problem except for the presence of backlash. Backlash, or gear lash, is a phenomenon that occurs in mechanical systems that experience some amount of ”play” between moving parts when in operation. In this case, it is advantageous to place a set of gears between the motor and the gimbal to conserve power, and limit power spikes, on the spacecraft’s electrical system and minimize the size and
weight of the system by allowing a smaller motor to be installed. The disadvantage of this strategy is the introduction of the nonlinear dynamics of the backlash phenomenon.

![Illustration of the backlash effect. (Grendelkhan, 2008)](image)

*Figure 1.2: Illustration of the backlash effect. (Grendelkhan, 2008)*

The research of (Penn, 2015) demonstrates the issues that plague an attitude control system when gear lash exists in the system. Penn simulated the use of 4 CMG array on a frictionless spacecraft. Penn’s simulation and modeling of the backlash phenomenon reveals that backlash caused a cumulative pointing error over the course of multiple maneuvers controlled by a standard PID controller. Thus, the necessity of a control system that accounts for the backlash is demonstrated.

Some methods exist to account for backlash in similar mechanical systems. (Friedland & Davis, 1997) models backlash in a belt driven system powered by a motor on one end and turning an inertial load on the other. The backlash effect was introduced via the toothed belt drive between the motor and the load. Friedland made use of a dead-band model for backlash, which is common in analysis, as well as a continuous approximation of the effect. By building a state estimator to observe the backlash directly and designing a
switching mode control law using linear-quadratic methods, Freidland was able to reduce
the backlash effect to varying levels of success depending on the method of feedback.

(Tao & Kokotovic, 1992) designed a backlash inverse to inform the controller to negate
the backlash effect. The backlash inverse function is placed at the input of the plant to
correct the command signal in a way that filters out the backlash effect.

Both methods, however, required somehow estimating or observing the states of the
backlash or its parameters. This is undesirable if the design of an attitude control system
is to remain simple, and does not make full use of the capabilities of a Model Reference
Adaptive Controller.

Adaptive control is a long studied control scheme that is an alternative to the tradi-
tional PID feedback control used extensively in industry. By contrast, an adaptive con-
trol law is a method of feedback control that uses variable feedback gains that ”adapt”
over time to ensure that tracking performance is maintained, especially if the plant is time
varying. Adaptive control laws are particularly strong in the cases where the plant has an
extreme degree of uncertainty or is subject to persistent disturbances.

A model reference adaptive control law takes a step further by attempting to track an
ideal trajectory for all the plant states. The ideal trajectory is defined by the states of a
reference model. The reference model is usually an ideal, often linear and reduced order,
version of the real plant model that is simulated on-line. The objective the adaptive con-
troller tries to achieve is to make the non-ideal plant behave as closely as possible to the
ideal model, despite the differences between them. As an example, this thesis makes use
of an adaptive control law that uses a 2\textsuperscript{nd} order model without backlash as a reference to
inform the adaptive gain behavior controlling the real plant that has a backlash effect and is a $4^{th}$ order system.
2. Dynamics Modeling

The simplest motor to control is a brushed DC motor, and it is also the most often used in analysis. In practical application, the motor is controlled by varying the voltage applied across the motor’s terminals. The dynamics model of the motor circuit can be obtained using Kirchoff’s voltage law.

\[ -V_{in} + RL + L\dot{I} + V_{emf} = \sum_{k=1}^{n} V_k = 0 \]  
\[ L\dot{I} + RI + V_{emf} = V_{in} \]  

\[ (2.1) \]

The back EMF represents the interplay between the mechanics of the motor and the electronic circuit. Both are represent by 2\textsuperscript{nd} order differential equations. It can be proven that the back EMF current is proportional the rotational velocity of the motor’s shaft.

\[ V_{emf} = k_1 \omega \]  
\[ (2.2) \]
The constant of proportionality is referred to as the motor torque constant. The equation describing the rotor’s spin is found by summation of the torques around the motor’s shaft. The torque applied to the rotor can be shown to be proportional to the current present in the circuit. Using the convention that positive spin is in the counter clockwise direction:

\[ \Sigma T = J\dot{\omega} = -b\omega + k_2I \]  

(2.3)

\[ J\dot{\omega} + b\omega = k_2I \]

This model is accurate when it is assumed that the motor shaft is rigid and the viscous friction is proportional to the spin speed. When deriving the motor torque constants, both constants are found to be numerically equal to each other.

\[ k_1 = k_2 = k \]  

(2.4)

This relationship is true when the constants are expressed in SI units and when we assume that any magnetic field external to the motor is fixed. The whole system of a direct current motor is described by equations 2.1 and 2.3. Fortunately, the model is already linear as a result of what was assumed. Therefore, the system is easily composed into a state-space model of the form:

\[ \dot{x} = A_m\vec{x} + B_m\vec{u} \]  

(2.5)

\[ y = C_m\vec{x} \]
Taking our states to be the angular velocity and current of the motor, $\vec{x} = [\omega \ i]^T$, and $\omega$ as our observed variable, equations 2.1 and 2.3 turn into:

$$
\begin{bmatrix}
\dot{\omega} \\
\dot{i}
\end{bmatrix} =
\begin{bmatrix}
-b/J & k/J \\
-k/L & -R/L
\end{bmatrix}
\begin{bmatrix}
\omega \\
i
\end{bmatrix} +
\begin{bmatrix}
0 \\
1/L
\end{bmatrix} V_{in} \quad (2.6)
$$

**Addition of Flexible Shaft**

To start modeling the motor-gimbal system with backlash, the assumption that the shaft connecting the load (gimbal) to the motor is perfectly rigid is removed. A shaft that is allowed to flex means that the dynamics of the gimbal must be modeled separately from the rotor. Consequently, $\omega \neq G_r \omega_t$. The flexible shaft is modeled as a torsion spring, the resisting torque increases proportionately with the twist angle of the shaft. This creates a new torque on the rotor, $T_s$, that must be modeled as an endogenic disturbance from the perspective of the motor. Equation 2.3 becomes:

$$
J \ddot{\omega} + b \dot{\omega} = k I - T_s \quad (2.7)
$$

The gyroscope can be modeled as a flat disk with radius $r$. By assuming the gimbal itself has negligible moment of inertia, the gyroscope assembly has a moment of inertia, $J_l$, around the \( \hat{\zeta} \) axis in Figure 1.1 that is used for the dynamics modeling.

With the introduction of a gearbox (without backlash) to the flexible shaft as in Figure 2.2, the presence of a gear ratio is also produced. One property of the gearbox is that it acts as a torque multiplier in proportion to the ratio of teeth between the output gear and
the input gear. If the motor is producing torque $T_s$, then the gearbox is outputting torque $G_rT_s$. The gearbox is "seeing" torque $G_rT_s$. The differential equation for the gimbal dynamics can be found the same way as the rotor’s: by summing the torques around the $x$ axis of the gimbal.

$$\sum T = J_l\dot{\omega}_l = -b_l\omega + G_rT_s$$

$$J_l\dot{\omega}_l + b_l\omega = G_rT_s$$  \hspace{1cm} (2.8)

(Nordin et al., 1997) suggests modeling the gearbox with a flexible, inertialess shaft with internal dampening. As the shaft is loaded, it twists and produces a restoring torque, $T_s$, that is proportional to the twist amount, $\theta_s$, much like a torsion spring. The full model includes a term for dampening as well:

$$T_s = k_s\theta_s + c_s\dot{\theta}_s$$  \hspace{1cm} (2.9)

2.1 Addition of Backlash

Adding a gearbox to the model simply increases the total moment of inertia and friction of the system. However, including the assumption that the load is connected to the motor via a geared shaft complicates the matter by not only introducing a gear ratio, but also a phenomenon known as gear lash or backlash. Due to imperfect construction or tooth wear, a gap can exist between corresponding gear teeth, illustrated in Figure 2.2. As a result of this gap, both shafts are permitted to spin independent from each other until the teeth surfaces come in contact again. In particular, backlash occurs when the input shaft is
spinning up from a stop or changing direction. While the gearbox is within the backlash zone, torque is not transmitted.

\[ T_s = k_s (\theta_d - \theta_b) + c_s \left( \dot{\theta}_d - \dot{\theta}_b \right) \]  

(2.10)

\textbf{Figure 2.2}: Analogous illustration of gear teeth meshing and shaft twist. (Nordin et al., 1997)

In a system with no backlash, \( T_s \) is continuous and depends only on the motor and gimbal angles. If we define the total angular displacement between the motor and gimbal as \( \theta_d = \theta - \theta_l \), then \( \theta_s = \theta_d \). \( \theta_s \) is the angular displacement between the motor and the end of the flexible motor shaft, which is the amount of twist of the shaft. However, the displacement caused by backlash does not contribute to the torsion of the shaft, so with backlash \( \theta_s \neq \theta_d \). Instead, the backlash angle \( \theta_b \) is introduced. \( \theta_b \) represents the angular displacement between the end of the shaft and the non-flexible load shaft. Combining all three angles gives an expression that can be used to solve for \( \theta_s \): \( \theta_d = \theta_s + \theta_b \). The expression for \( \theta_s \) is substituted into 2.9 for the new expression of shaft torque.
The severity of the backlash in a motor can be determined by fixing one shaft in place and measuring the full range of angular motion of the other shaft, without twisting of the flexible shaft. This quantity is the deadband width, which relates to the maximum width of the gap between teeth, and is a measure of the amount of play in the system. If the gearbox output is moving freely within the deadband and the shafts are disengaged, it stands to reason that the shafts can only rotate so far before the teeth come into contact and begin transmitting torque again. This observation imposes a maximum value of the backlash angle:

\[ |\theta_b| \leq \alpha = \frac{D_b}{2} \quad \forall t \]  

(2.11)

Once again (Nordin et al., 1997) provides a useful model of the shaft torque that includes backlash ability. Referencing the convention established in Figure 2.2, \( T_s > 0 \) implies contact with the right side of the backlash gap while \( T_s < 0 \) implies contact with the left side of the backlash gap. When the gear teeth are in backlash, the backlash angle is saturated at \( \alpha \), which is reasonably assumed to be a constant and thus \( \dot{\theta}_b = 0 \).

\[
T_s = k_s (\theta_d - \alpha) + c_s \dot{\theta}_d \quad T_s > 0
\]

(2.12)

\[
T_s = k_s (\theta_d + \alpha) + c_s \dot{\theta}_d \quad T_s < 0
\]
As mentioned previously, $T_s = 0$ and $\theta_b < \alpha$ when inside the deadband. Equation 2.10 becomes an homogeneous differential equation in this case, which can be solved for $\dot{\theta}_b$ and $\theta_b$.

\[
\begin{align*}
0 &= k_s (\theta_d - \theta_b) + c_s \left( \dot{\theta}_d - \dot{\theta}_b \right) \\
\dot{\theta}_b - \dot{\theta}_d &= \frac{k_s}{c_s} (\theta_d - \theta_b) \\
\theta_b - \theta_d(t) &= (\theta_d(t) - \theta_b(t_0)) e^{-\frac{k}{c}(t-t_0)}
\end{align*}
\]

(2.13)

As $\theta_d(t)$ is known (it is calculated and fed back in the simulation), equation 2.13 yields a solution for $\theta_b$. Combining equations 2.13, 2.11, and 2.10, we can construct a switching model to determine $\theta_b$, $\dot{\theta}_b$, and thus $T_s$. (Nordin et al., 1997)

\[
\dot{\theta}_b = \begin{cases} 
\max(0, \dot{\theta}_d + \frac{k_s}{c_s} (\theta_d - \theta_b)) & \theta_b = \alpha \quad T_s \leq 0 \\
\dot{\theta}_d + \frac{k_s}{c_s} (\theta_d - \theta_b) & |\theta_b| < \alpha \quad T_s = 0 \\
\min(0, \dot{\theta}_d + \frac{k_s}{c_s} (\theta_d - \theta_b)) & \theta_b = -\alpha \quad T_s \geq 0
\end{cases}
\]

(2.14)

$\dot{\theta}_b$ is integrated to find $\theta_b$ and both are fed into equation 2.10 to calculate the shaft torque.
3. Controller Design

3.1 Design of the Reference Model Controller

A PI controller is used to establish a performance benchmark for the adaptive controller and an ideal controller for the reference model. The system loop is closed by feeding back the CMG speed to the controller \( y = \omega_l \), resulting in a typical plant-controller architecture illustrated in Figure 3.1, where the feedback control law is of the form:

\[
V_{in} = u = K_p y + K_i \int (r - y) \, dt
\]  

\( (3.1) \)

\[\text{Figure 3.1: Basic system architecture for reference model and bench-marking.}\]

Critical dampening of the ideal system is desired for the fastest rise time without overshoot. Tuning a PI controller for a critically damped response is a well known procedure. The ideal model in equation 2.6 can be approximated by a 1\(^{st}\) order transfer function:

\[
\frac{\Omega_l(s)}{V_{in}} = \frac{K_{dc}}{\tau s + 1}
\]  

\( (3.2) \)
Combined with equation 3.1 in a closed loop provides the ideal, simplified system transfer function:

$$\frac{\Omega_l(s)}{R(s)} = \frac{K_{dc}K_i}{\tau s^2 + (1 + K_{dc}K_p)s + K_{dc}K_i} \quad (3.3)$$

Furthermore, the transfer function for a second order LTI system in terms of the system’s damping ratio and natural frequency is:

$$G(s) = \frac{K_{dc}\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (3.4)$$

Matching and rearranging the coefficients of 3.3 and 3.4 solves for $K_p$ and $K_i$ in terms of the damping coefficient and natural frequency.

$$K_p = \frac{2\tau\zeta\omega_n - 1}{K_{dc}} \quad (3.5)$$

$$K_i = \frac{\tau\omega_n^2}{K_{dc}}$$

The time constant for a DC motor is given as

$$\tau = \frac{J_{eq}R}{k^2} \quad (3.6)$$

For the case of a system with torque multiplication via a gearbox with one set of gears, the total moment of inertia is expressed as:

$$J_{eq} = J + G_r^2J_l \quad (3.7)$$
In addition, the low frequency gain for a DC motor in relation to its parameters is simply the inverse of the motor torque constant.

$$K_{dc} = \frac{1}{k}$$  \hspace{1cm} (3.8)

Finally, the damping ratio for a critically damped system is 1 by definition. The proportional and integral gains of the system that yields a critically damped response can be calculated using the known parameters of the system.

$$K_p = (2\omega_n \tau - 1)k$$  \hspace{1cm} (3.9)

$$K_i = \omega_n^2 \tau k$$

Requiring only that a reasonable value for $\omega_n$, now the desired natural frequency, is selected.

### 3.2 Design of the Adaptive Controller

As the name suggests, a model reference adaptive control law uses an ideal model of the system to adjust the feedback control gains so that the output of the non-ideal plant tracks the output of the ideal plant, despite the differences between the plant and the model. Mathematically expressed, the objective is for $y \rightarrow y_m$ as $t \rightarrow \infty$.\(^1\) A feedback control law is selected to reference the state, output, and control force of the ideal model.

$$V_{in} = u = G_x e + G_x x_m + G_u u_m + G_d \phi_d$$  \hspace{1cm} (3.10)

\(^1\)Note: for clarification, the subscript 'm' refers to the reference model, not the motor.
where

\[
\begin{align*}
\dot{G}_e &= -e^2 \sigma_e \quad \sigma_e > 0 \\
\dot{G}_{x_m} &= -e_{x_m} \sigma_x \quad \sigma_m > 0 \\
\dot{G}_{u_m} &= -e_{u_m} \sigma_u \quad \sigma_u > 0 \\
\dot{G}_d &= -e_{u_d} \sigma_d \quad \sigma_d > 0
\end{align*}
\]

(3.11)

and \(e = y - y_m\). The adaptive gain, \(G_e\) is associated with the stabilization of the system, while the other adaptive gains are implemented for model tracking and persistent disturbance rejection. The controller designed in section 3.1 provides the ideal controller for the adaptive controller to mimic, and the ideal dynamics (without backlash) from chapter 2 are used.

![Figure 3.2: Adaptive controller integrated with the plant and reference model.](image)
The non-ideal dynamics differ from the ideal dynamics in only that the backlash model is included. The adaptive weights are experimentally adjusted to determine the weighting profile that yields the best performance.
4. Closed Loop Stability

The aim of this thesis is to demonstrate the closed loop stability of a DC motor can be achieved by an adaptive controller, even in the presence of backlash. The stability of the system will be shown by demonstrating that the model of the system satisfies the Kalman-Yacubovic conditions. Observing that the backlash model can be split into separate terms containing the system and backlash states will allow the model to be rearranged into an LTI state-space model.

\[
T_s = T_f + T_b = k_s \theta_d + c_s \dot{\theta}_d - k_b \theta_b - c_b \theta_b
\]  

(4.1)

with \( \theta_d \) defined as \( \theta_d = \theta - \theta_l \). This allows the addition of half the torque equation into the A Matrix. Now, \( \vec{x} = [\omega \ I \ \omega_l \ \theta_d]^T \). After making this change, an LTI system can be constructed.

\[
\begin{bmatrix}
\dot{\omega} \\
\dot{i} \\
\dot{\omega}_l \\
\dot{\theta}_d
\end{bmatrix} =
\begin{bmatrix}
\frac{-b-c_s}{J} & k & \frac{c_s}{J} & \frac{-k_s}{J} \\
\frac{-k}{L} & \frac{-R}{L} & 0 & 0 \\
\frac{c_s}{J_l} & 0 & \frac{-b_l-c_s}{J_l} & \frac{k_l}{J_l} \\
1 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
\omega \\
I \\
\omega_l \\
\theta_d
\end{bmatrix} +
\begin{bmatrix}
0 \\
1/L \\
0 \\
0
\end{bmatrix} \frac{1}{J_l}
\begin{bmatrix}
V_{in} \\
0 \\
0 \\
0
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} T_b
\]  

(4.2)

\[
\dot{\vec{x}} = A \vec{x} + B u + \Gamma u_D
\]

\[
y = C \vec{x}
\]
where two cases will be studied: a case where the gimbal speed will be fed back with $C = [0 \ 0 \ 1 \ 0]$, and a case where the motor current will be fed back $C = [0 \ 1 \ 0 \ 0]$.

In this way, the torque contributed by the backlash variables, $\theta_b$ and $\dot{\theta}_b$, in 2.10 is represented as a disturbance torque. As shown in 2.13, solutions for $\theta_b$ approach $\theta_d$ asymptotically with the form $c_1 e^{-\lambda t}$, showing that the disturbance from backlash is bounded for $\forall t$ while the gear shafts are not in contact provided $\theta_d$ is bounded. The backlash disturbance is constant outside when the gear shafts are in contact with $\varphi_d = 1$ ($\theta_b = \pm \alpha$ and $\dot{\theta}_b = 0$).

Ultimately, the objective is for the plant output to track the model output, so we define an error signal.

$$y - y_m = e_y \rightarrow 0 \Rightarrow t \rightarrow \infty \quad (4.3)$$

Furthermore, we define a system that yields ideal trajectories

$$\dot{x}_* = A \dot{x}_* + Bu_* + \Gamma u_D$$

$$y_* = C \dot{x}_* = y_m$$

where solutions take the form,

$$\ddot{x}_* = S^{11}_* \ddot{x}_* + S^{12}_* u_* + S^{13}_* u_d$$

$$u_* = S^{21}_* \ddot{x}_* + S^{22}_* u_* + S^{23}_* u_d$$
and make use of the reference model from before.

\[
\begin{align*}
\dot{x}_m &= A_m x_m + B_m u_m \\
y_m &= C_m x_m = \omega_m
\end{align*}
\]  

(4.6)

Of course, the reference model is designed to be stable and critically damped with input \( u_m \) being bounded. Regardless of whether the gimbal speed or motor current is fed back to the adaptive controller, the reference model gimbal speed will be fed back to its own controller for speed control. Naturally, we would like our system to behave ideally, so the state tracking error is defined.

\[
e_* = x - \bar{x}
\]

(4.7)

Furthermore, we define \( \Delta u = u - u_* \) and assemble the error system between the plant and ideal trajectories.

\[
\dot{e}_* = A e_* + B \Delta u
\]

(4.8)

Now, it is necessary to define a hypothetical fixed gain controller for the plant

\[
u = u_* + G_e^* e_y
\]

(4.9)

which is incorporated into the ideal error system.

\[
\dot{e}_* = (A + B G_e^* C) e_* = A_c e_*
\]

(4.10)
If this system is stabilizable with fixed gain $G_e^*$, then 4.10 will produce asymptotic stability, $e_y \to 0$. This fact will be useful later. Next we define:

$$
\Delta G = \begin{bmatrix}
\Delta G_e &= G_e - G_e^* \\
\Delta G_m &= G_m - S_{21}^* \\
\Delta G_u &= G_u - S_{22}^* \\
\Delta G_d &= G_d - S_{23}^*L
\end{bmatrix}
$$

(4.11)

where $L$ is part of the dynamic system relating $u_d$ to its basis, $\phi_d$,

$$
u_d = \Theta z_d$$

(4.12)

$$z_d = L\phi_d$$

and is generally unknown. Using the adaptive gains from 3.10. This leads to the tracking error system:

$$\hat{\epsilon}_e = A_c\epsilon_e + B\Delta G\eta$$

(4.13)

$$e_y = C\epsilon_e$$
where \( \eta = [e_y \ x_m^T \ u_m \ \phi_d]^T \) is a vector containing the inputs to the adaptive controller. As it follows:

\[
\dot{G} = -e_y \eta^T \sigma = \Delta \dot{G} \tag{4.14}
\]

\[
\sigma = \begin{bmatrix}
\sigma_e & 0 & 0 & 0 \\
0 & \sigma_x & 0 & 0 \\
0 & 0 & \sigma_u & 0 \\
0 & 0 & 0 & \sigma_d
\end{bmatrix} > 0 \tag{4.15}
\]

from 3.11. Finally, the stability of the system can be analyzed using Lyapunov’s methods.

**Lyapunov’s Method**

Lyapunov’s method for stability can be used to show that a dynamic system represented as \( \dot{x} = f(x(t), u(t), u_d(t), t) \) is globally asymptotically stable, provided the following are true:

1. There exists a continuous function, \( V(x) \), with the properties:

   (a) \( V(x) = 0 \) when \( x = 0 \) and

   (b) \( V(x) > 0 \ \forall x \)

   (c) \( V(x) \) is radially unbounded

2. \( \dot{V}(x) < 0 \ \forall x \)

   However, the 2\(^{nd}\) property is often difficult to prove with a natural system. Instead, the least restrictive case where \( \dot{V}(x) \leq 0 \ \forall x \) occurs more frequently. In these cases, the only
element that is proven is that the system’s trajectories are bounded. Furthermore, it is not
strictly necessary that the storage function, \( V(\vec{x}) \), be radially unbounded. In this case, the
system would be locally stable.

Suppose we suggest two Lyapunov function candidates:

\[
V_1(e_*) = \frac{1}{2} e_*^T P e_*
\]

\[
V_2(\Delta G) = \frac{1}{2} tr(\Delta G G^{-1} \Delta G^T)
\]

(4.16)

where \( P \geq 0 \) and is the solution to:

\[
-Q = A_C^T P + P A_C
\]

(4.17)

\[
P B = C^T
\]

where \( Q \geq 0 \) as well. These equations, and the satisfaction thereof, are referred to as the
Kalman-Yacubovic conditions. The existence of a symmetric positive definite solution to
these equations is known to be equivalent to the following condition:

\[
T_c = C(sI - A_C)^{-1} B
\]

(4.18)
where the transfer function $T_C$ is strictly positive definite real (Balas & Frost, 2011). That is, the real part of the of $T_C$ is $\geq 0$. Taking the derivatives of 4.16 we get:

$$\dot{V}_1 = -\frac{1}{2}e^*_TQe* + e^*_T(PB)\Delta G\eta$$

$$= -\frac{1}{2}e^*_TQe* + e^*_TC^T\Delta G\eta$$

$$= -\frac{1}{2}e^*_TQe* + e^*_y\nu$$

$$\nu \equiv \Delta G\eta$$

and

$$\dot{V}_2 = tr(\Delta G\sigma^{-1}\Delta G^T)$$

$$= tr(-e_y\eta^T\Delta G^T)$$

$$= tr(-e_y\nu^T) = -e^*_y\nu$$

Thus, $\dot{V}_1 + \dot{V}_2 = -\frac{1}{2}e^*_TQe* = \dot{V}$. Provided $P$ and $Q$ satisfy 4.17, then $\dot{V} \leq 0$. Lyapunov stability methods guarantee that both $e*$ and $\Delta G$ are bounded when $\dot{V} \leq 0$. For the case where the gimbal speed is fed back to the adaptive controller and $C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$, the transfer function $T_c$ is not strictly positive real. The transfer function takes the form

$$\frac{c_yks + kk_s}{C_4s^4 + C_3s^3 + C_2s^2 + C_1s + C_0}$$ (4.21)
where the coefficients are equations containing the system parameters. Since the relative
degree of the transfer function is 3, not 0 or 1, \( T_e \) cannot be strictly positive real and solu-
tions to 4.17 are not guaranteed to exist.

When the motor current is fed back through the adaptive controller, the case where \( C =
\begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix} \), it is rudimentary to demonstrate the satisfaction of the Kalman-Yacubovic
conditions by use of the physical energy storage of the system.

\[
P = \begin{bmatrix}
J & 0 & 0 & 0 \\
0 & L & 0 & 0 \\
0 & 0 & J_l & 0 \\
0 & 0 & 0 & k_s
\end{bmatrix}
\] (4.22)

being the simplest solution to the condition \( PB = C^T \). The corresponding \( Q \) matrix con-
tains the dissipative terms in the system.

\[
Q = \begin{bmatrix}
2(b + c_s) & 0 & 2c_s & 0 \\
0 & 2R & 0 & 0 \\
2c_s & 0 & 2(b_k + c_s) & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] (4.23)

Since \( P \) is diagonal, its eigenvalues are easily shown to be positive. However, \( Q \) is not
diagonal and has a redundant dimension due to the shaft energy dissipation being related
to the other states, $\omega$ and $\omega_l$. Instead, $\dot{V}(\bar{x}) \leq 0$ is easier to show when it is expanded into its scalar form.

$$
\dot{V}(\bar{x}) = -\frac{1}{2} e^T \dot{Q} e^* = -be_\omega^2 - Re_\omega^2 - b_g e_{\omega_l}^2 - c_s (e_\omega^2 - 2e_\omega e_{\omega_l} + e_{\omega_l}^2)
$$

$$
= -be_\omega^2 - Re_\omega^2 - b_g e_{\omega_l}^2 - c_s (e_\omega - e_{\omega_l})^2
$$

$$
= -be_\omega^2 - Re_\omega^2 - b_g e_{\omega_l}^2 - c_s e_\theta^2 \leq 0 \quad \forall e^*
$$

(4.24)

Since $P, Q \geq 0$ (with the one eigenvalue of $Q = 0$) and satisfy 4.17, $V(\bar{x}) \geq 0$ and $\dot{V}(\bar{x}) \leq 0$, the system satisfies the Kalman-Yakubovic lemma and is globally bounded.

However, asymptotic stability of the error system, $e^*$, is desired for $e^* \rightarrow 0$. The property that $e^*$ is bounded is not a sufficient enough condition to provide this. Further analysis is required with the use of Barbalat’s lemma. The Lemma states:

$$
\lim_{t \rightarrow \infty} \frac{df}{dt} = 0
$$

for the function $f(t)$ if $(t)$ has a finite limit and a uniformly continuous derivative. Applying the lemma to $V(t)$ shows that $\dot{V} \rightarrow 0$ as $t \rightarrow \infty$. From Lyapunov’s methods, we already know that $V(t)$ has a finite limit as $t \rightarrow \infty$, so it must be demonstrated that $\dot{V}(t)$ uniformly continuous. It follows from the definition uniform continuity that a function, $f(t)$ is uniformly continuous if its derivative is bounded. For this case,

$$
\dot{V}(t) = e^T \dot{Q} e^* = e^T QA^e_\epsilon e^* + e^T QB^\Delta G\eta
$$

(4.25)
Lyapunov’s methods and the satisfaction of the Kalman-Yacubovic conditions have proven the $\mathbf{e}_\star$ is bounded and $\eta$, being the vector of inputs into the system, is bounded by construction. Recall that the reference model was constructed to be stable, so $u_m$ and $x_m$ are bounded. Furthermore, it was mentioned earlier that the basis of the backlash disturbance takes the form of $e^{-t}$ within the backlash and a constant when not, so it too is bounded. Ergo, $\hat{V}(t)$ is bounded and $\dot{V}(t)$ is uniformly continuous. Barbalat’s Lemma states that, under these conditions, $\dot{V}(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore,

$$
\lim_{t \rightarrow \infty} \dot{V}(t) = 0 = - \lim_{t \rightarrow \infty} \frac{1}{2} e_\star^T \mathbf{Q} e_\star \Rightarrow e_\star \rightarrow 0 \text{ as } t \rightarrow \infty \quad (4.26)
$$

for $\mathbf{Q} > 0$. This result demonstrates the presence of the desired asymptotic stability to 0 of the error system, $\dot{e}_\star = f(e_\star, \eta)$. Being that the error system was defined as the difference between the trajectory of the real plant and the ideal trajectory, this result proves that the model reference adaptive controller proposed in 3.2 rejects the disturbance caused by the backlash property and forces the system to track the ideal trajectories.
5. Simulated Results

The model described in equation 4.2 was constructed in Simulink using the shaft model described by (Nordin et al., 1997) to connect the motor and the gimbal. The properties of Anaheim Automation’s BLWR112S-36V-10000 motor were substituted in for the motor’s circuit and rotor dynamics. The shaft and gimbal were modeled with the parameters in Table 5.1.

Table 5.1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_l$</td>
<td>0.0125</td>
<td>kg·m²</td>
</tr>
<tr>
<td>$b_l$</td>
<td>0.001</td>
<td>N·m·rad/s</td>
</tr>
<tr>
<td>$k_s$</td>
<td>$7 \times 10^4$</td>
<td>N·m·rad⁻¹</td>
</tr>
<tr>
<td>$c_s$</td>
<td>0.1</td>
<td>N·m·s·rad⁻¹</td>
</tr>
<tr>
<td>$D_b$</td>
<td>5</td>
<td>deg</td>
</tr>
</tbody>
</table>

The model was connected as shown in Figure 5.1 to form the non-linear plant with backlash.

The non-linear plant was copied so that they can be separately controlled in parallel by the critically damped PI controller described in section 3.1 and the adaptive controller from section 3.2 in parallel. The adaptive weights and controller gains are listed in Table 5.2.

The controllers were tasked to track a square wave with an amplitude of $0.1 \frac{\text{rad}}{s}$ and a period of 20 seconds for 500 seconds. The square wave was chosen over the traditional step command to encourage the motor to change direction for a full analysis of the back-
Figure 5.1: Diagram of the nonlinear plant.
lash effect. When the motor current was fed into the adaptive controller, the adaptive controller performed nearly as well as the PI when a disturbance was not being applied to the gimbal.

Table 5.2

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_p$</td>
<td>17.24</td>
</tr>
<tr>
<td>$K_I$</td>
<td>108.34</td>
</tr>
<tr>
<td>$\sigma_e$</td>
<td>1</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td>$\begin{bmatrix} 50 &amp; 0 \ 0 &amp; 50 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\sigma_u$</td>
<td>500</td>
</tr>
<tr>
<td>$\sigma_d$</td>
<td>$\begin{bmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 50 &amp; 0 \ 0 &amp; 0 &amp; 10 \end{bmatrix}$</td>
</tr>
</tbody>
</table>

Figure 5.2: Gimbal speed error signals between each controller and reference model gimbal speed.
Note, the large error spikes occur when the command and reference model change direction. The adaptive controller was also observed to consume slightly less energy than the PI controller.

![Energy consumption graph](image)

*Figure 5.3: Total energy consumed by the motor over time.*

However, the performance of the adaptive controller degrades when a $1 \text{ rad/sec}$ sinusoidal disturbance torque is applied to the gimbal. The adaptive disturbance gain is observed to be ineffective at filtering away the offending frequency. The PI controller was not tuned or filtered to reject this disturbance, and yet the effect of the disturbance is amplified by the adaptive controller by comparison.
Figure 5.4: Gimbal speed error signals between each controller and reference model gimbal speed with disturbance.

Figure 5.5: Dynamic response of the adaptive gains. The initial conditions, $G_0$, were selected experimentally to produce the best performance.

In Chapter 4, it was shown that the system with gimbal speed feedback was not strictly positive real due to the relative order of the transfer function, $T_c$. However, the scenario
was simulated as it was observed that the adaptive controller performed just as well as when it was given the motor current, but was also able to reject the 1 \( \frac{\text{rad}}{s} \) disturbance.

![Figure 5.6: Gimbal speed feedback with disturbance rejection](image)

Naturally, a PI controller is ill-equipped to filter sinusoidal disturbances, especially of such low frequency. The adaptive gains, however, were able to react to the disturbance. The adaptive gain associated with the sine wave basis was singled out in Figure 5.7, as it had the most impact.
Figure 5.7: Adaptive gains rejecting the periodic disturbance

Figure 5.8: Energy consumed by the adaptive controller mitigating the sinusoidal disturbance

The consumed energy in Figure 5.8 is significantly higher as both controllers are constantly trying to stabilize the system. The adaptive controller, however, is able to mitigate
the disturbance while the PI controller fails to do so. However, due to the non-passive nature of this configuration, higher frequency disturbances cause the output and adaptive gains to grow unbounded.

*Figure 5.9:* A $100 \frac{\text{rad}}{s}$ frequency is not filtered out and causes unbounded growth.

*Figure 5.10:* Adaptive gains reacting to $100 \frac{\text{rad}}{s}$ frequency.
To establish a metric for controller performance, accumulated output error was calculated as:

\[
e_a = \int_{t_0}^{t_f} \sqrt{(r(t) - y(t))^2} \, dt
\]  

(5.1)

for both the adaptive and PI controllers. The results of this analysis are summarized in Table 5.3.

Table 5.3

*Accumulated command signal error in radians*

<table>
<thead>
<tr>
<th>Controller</th>
<th>Without Disturbance</th>
<th>With Disturbance</th>
</tr>
</thead>
<tbody>
<tr>
<td>PI</td>
<td>0.24</td>
<td>3.4</td>
</tr>
<tr>
<td>Adaptive current feedback</td>
<td>0.33</td>
<td>8.1</td>
</tr>
<tr>
<td>Adaptive speed feedback</td>
<td>0.25</td>
<td>0.62</td>
</tr>
</tbody>
</table>
6. Conclusion

Control moment gyroscopes remain the gold standard for fine spacecraft attitude control. With the cost of more weight and complexity than a reaction wheel control system comes superior accuracy in pointing maneuvers using less power. With the inclusion of a gearbox on the gimbal actuator of the CMG, the power efficiency improves while producing the same magnitude of torque. For the foreseeable future, a gear driven system will always challenge the ability of control systems to accurately track pointing and spin rate commands in a micro-gravity environment, necessitating the design of control systems robust to the effects of backlash.

A system was designed and modeled using Simulink in order to test a novel controller for fine attitude control. Implementing a model of a DC motor, flexible shaft, backlash, and moment gyroscope provided a test bed that served to compare the performance of a standard controller as well as an adaptive controller in feedback. The adaptive controller was proven to be stable in feedback with the system with backlash using Lyapunov methods combined with the application of Barbalat’s lemma when the adaptive controller was placed in a motor current feedback loop. The controller was implemented in a Simulink model to demonstrate the performance in tracking a square wave input and rejecting a sinusoidal disturbance. Despite challenges presented in the simulation, notably the increased accumulated output error, it provided an opportunity for alternative analysis and demonstrated that the MRAC was capable of rejecting persistent disturbances.
In conclusion, the theoretical and simulated results signifies the possibility of the use of an adaptive controller on a system with mechanical play present in its mechanics. However, this analysis demonstrates the need for further research into this application of the MRAC before implementing in practice.
7. Recommendations

There remains much research to be done in the area of adaptive control methods. As previously mentioned, in (Penn, 2015) a study was performed to measure the amount of “drift” in pointing accuracy experienced by a simulated spacecraft when actuated by CMGs. The drift was revealed to be the consequence of gear lash in the planetary gear assemblies that torqued the CMGs. It is hypothesized that the drift in pointing accuracy can be reduced or eliminated if the spacecraft’s attitude control system were guided by a MRAC. To test this theory, a 1 dimensional spacecraft simulation could be constructed using a single equation of rotational motion, \( I\dot{\omega} = T \). The objective would be to position control this system through a series of pointing maneuvers actuated by a single CMG, which can be controlled by an MRAC. Alternatively or additionally, a MRAC can be used to calculate the spacecraft torque, \( T \), to command to the CMG. In either case, the MRAC would be tested against a traditional PID controller to see which accumulates more pointing error over time.
REFERENCES


