Non-Linear Bending Analysis of Functionally Graded Beams with Spring Constraints and Thermal Effects

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NON-LINEAR BENDING ANALYSIS OF FUNCTIONALLY GRADED BEAMS
WITH SPRING CONSTRAINTS AND THERMAL EFFECTS

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A Thesis prepared under the direction of the candidate's committee chairman, Dr. Habib Eslami, Department of Aerospace Engineering, and has been approved by the members of the thesis committee. It was submitted to the School of Graduate Studies and Research and was accepted in partial fulfillment of the requirements for the degree of Master of Science in Aerospace Engineering.

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# TABLE OF CONTENTS

ACKNOWLEDGMENTS .................................................................................... iii

TABLE OF CONTENTS ................................................................................. iv

LIST OF TABLES ........................................................................................ vi

LIST OF FIGURES ....................................................................................... ix

ABSTRACT .................................................................................................... xiv

1. INTRODUCTION ...................................................................................... 1

1.1. Functionally Graded Materials .............................................................. 1

1.1.1. Applications of Functionally Graded Materials ............................ 3

1.2. Problem Statement .............................................................................. 4

1.3. Objectives .......................................................................................... 5

2. LITERATURE REVIEW ........................................................................... 7

2.1. Previous Research in ERAU ................................................................. 11

3. MATERIAL MODELING ......................................................................... 14

4. GOVERNING EQUATIONS ..................................................................... 17

4.1. Classical Theory ................................................................................ 17

4.1.1. Assumptions ................................................................................ 17

4.1.2. Strain-Displacement Relations in the Beam ................................. 19

4.1.3. Equations of Equilibrium for the Beam ........................................ 22

4.1.4. Stress-Strain Relations of the Beam ............................................ 28

4.1.5. Force-Displacement Relations in the Beam ................................. 28

4.1.6. Solution to the fourth-order differential equation ....................... 30

4.2. First Order Shear Deformation Theory (FSDT) ................................. 32

4.2.1. Assumptions ................................................................................. 32

4.2.2. Shear and Rotary effects added to the beam ................................. 34

4.2.2. Strain-Displacement Relations in the beam ................................. 37

4.2.3. Stress-Strain Relations of the Beam ............................................ 38

4.2.4. Force-Displacement Relations in the Beam ................................. 39

4.2.5. Solution to the fourth-order differential equation ....................... 41

5. NUMERICAL COMPUTATIONS ............................................................. 43

5.1. Simply-Supported Beam .................................................................... 43

5.1.1. Solution of Classical Theory ......................................................... 43

5.1.2. Newton Raphson Method ............................................................. 45

5.1.3. Solution of First Order Shear Deformation Theory .................... 46

5.2. Clamped-Clamped Beams ................................................................. 48

5.2.1. Solution of Classical Theory ......................................................... 49

5.2.2. Solution of First Order Shear Deformation Theory .................... 51

5.3. Elastic support boundary conditions ................................................. 53

5.3.1. Solution of Classical Theory ......................................................... 57
5.3.2. Solution of First Order Shear Deformation Theory ........................................ 61

6. RESULTS ...................................................................................................................... 65

6.1. Simply Supported Beam (SS beams) ........................................................................ 65
6.1.1. Solutions obtained on applying Classical Theory to SS beams ......................... 66
6.1.2. Solutions obtained on applying FSDT to simply supported beams ...................... 70
6.2. Clamped-Clamped Beams (CC Beams) .................................................................... 74
6.2.1. Solutions obtained on applying Classical Theory to CC beams ......................... 74
6.2.2. Solutions obtained on applying FSDT to CC beams ........................................... 78
6.3. Spring constraint boundary conditions .................................................................... 82
6.3.1. Solution to an FG beam with springs by Classical Theory-SS beam .................. 83
6.3.2. Application of FSDT to FG Beams with Springs-SS beam replicate ..................... 86
6.3.3. Solutions to a FG beam with springs by Classical Theory-CC beam .................. 89
6.3.4. Application of FSDT to FG beams with springs-CC beam replicate ..................... 93
6.4. Validation of Results with ABAQUS ....................................................................... 95
6.4.1. Comparison of the analytical results to ABAQUS results .................................... 99

7. THERMAL EFFECTS .................................................................................................... 101

7.1. Calculation of the critical buckling temperature for Classical Theory ...................... 101
7.2. Non-Linear Bending of C-C beams with thermal load for Classical Theory .......... 108
7.2.1. Results obtained by material modeling using Rule of Mixtures ......................... 110
7.2.2. Comparison of the analytical results to ABAQUS results ................................... 113
7.2.3. Results obtained for different temperature ratios at gradation ‘k’ ...................... 114
7.3. Calculation of critical buckling temperature for FSDT ........................................... 120
7.4. Non-Linear Bending Analysis of CC beams with thermal loads for FSDT .......... 128
7.4.1. Results obtained by material modeling using Rule of Mixtures ......................... 130
7.4.2. Results obtained for different temperature ratios at gradation ‘k’ ...................... 133
7.5. Classical and FSDT results of CC beams under both loads, Mori-Tanaka ............... 139

8. CONCLUSIONS .......................................................................................................... 144

9. FUTURE WORK .......................................................................................................... 147

REFERENCES .................................................................................................................. 149

Appendix A. In-Plane stress resultant N0 vs P (N/mm²) .................................................. 156
LIST OF TABLES

Table 6.1 Geometric properties of the beam................................................................. 65
Table 6.2 Material properties of the beam ................................................................. 65
Table 6.3 Maximum deflection values of simply supported beam for different ‘k’
using rule of mixtures................................................................................................. 67
Table 6.4 Maximum deflection values of simply supported beam for different values
of k using Mori-Tanaka method................................................................................. 69
Table 6.5 comparison of $w_{max}$ results of simply supported beam for different values
of k using rule of mixtures and Mori-Tanaka method under 0.01 N/mm$^2$ .... 69
Table 6.6 Maximum deflection values of simply supported beam for different values
of k using rule of mixtures and applying FSDT .................................................... 71
Table 6.7 Maximum deflection values of SS beam for different values of k using
Mori-Tanaka method and FSDT ............................................................................... 73
Table 6.8 comparison of $w_{max}$ results of SS beam for different values of k using rule
of mixtures and Mori-Tanaka method by applying FSDT under 0.01 N/mm$^2$. .......................................................... 73
Table 6.9 Deflections seen over the length of clamped-clamped beam for different
values of k using rule of mixtures under 0.01 N/mm$^2$ ........................................... 76
Table 6.10 Deflections seen over the length of clamped-clamped beam for different
values of k using Mori-Tanaka method under 0.01 N/mm$^2$. ................................. 77
Table 6.11 comparison of $w_{max}$ results of clamped-clamped beam for different values
of k using rule of mixtures and Mori-Tanaka method under 0.01 N/mm$^2$. .... 78
Table 6.12 Maximum deflection values of clamped-clamped beam for different
values of k using rule of mixtures and applying FSDT ........................................... 80
Table 6.13 Maximum deflection values of clamped-clamped beam for different
values of k using Mori-Tanaka method and applying FSDT ................................. 81
Table 6.14 comparison of $w_{max}$ results of clamped-clamped beam for different values
of k using rule of mixtures and Mori-Tanaka method by applying FSDT under 0.01 N/mm$^2$. .......................................................... 82
Table 6.15 Comparison of results between springs and s-s beam for rule of mixtures .... 83
Table 6.16 Comparison of results between springs and s-s beam for Mori-Tanaka
method ....................................................................................................................... 85
Table 6.17 Comparison of results between springs and SS beam for rule of mixtures
and FSDT theory ....................................................................................................... 86
Table 6.18 Comparison of results between springs and s-s beam for Mori Tanaka and
FSDT theory ............................................................................................................. 88
Table 6.19 Comparison of results between springs and c-c beam for rule of mixtures... 90
Table 6.20 Comparison of results between springs and CC beam for Mori-Tanaka method ................................................................. 91
Table 6.21 Comparison of results between springs and c-c beam for rule of mixtures and FSDT theory ................................................................. 93
Table 6.22 Comparison of results between springs and c-c beam for Mori-Tanaka and FSDT theory ................................................................. 94
Table 6.23 Comparison of analytical and FEM results for simply supported FG beam ................................................................. 99
Table 6.24 Comparison of analytical and FEM results for clamped-clamped FG beam ................................................................. 100
Table 7.1 Maximum deflection values for different values of k in a clamped-clamped beam subjected to thermal and mechanical loads ............................................ 111
Table 7.2 Comparison of Maximum deflection values for different values of k in a clamped-clamped Beam subjected to thermal and mechanical loads. ........ 114
Table 7.3 wmax for different $\Delta T/\Delta T_{cr}$ ratios at each gradation parameter k for rule of mix, classical theory ................................................................. 114
Table 7.4 Maximum deflection values for different values of k in a clamped-clamped beam subjected to thermal and mechanical loads .................................. 132
Table 7.5 wmax for different $\Delta T/\Delta T_{cr}$ ratios at each gradation parameter k for rule of mix, First order shear deformation theory ........................................ 133
Table 7.6 comparison between classical theory results and Mori-Tanaka results .................. 142
Table 9.1 In-plane stress resultant $N_{0}$ for different values of k of a simply-supported beam using rule of mixtures and applying classical theory ...................... 156
Table 9.2 In-plane stress resultant $N_{0}$ for different values of k of a simply-supported beam using Mori-Tanaka method and applying classical theory .............. 158
Table 9.3 In-plane stress resultant $N_{0}$ for different values of k of a simply-supported beam using rule of mixtures and applying first order shear deformation theory ................................................................. 160
Table 9.4 In-plane stress resultant $N_{0}$ for different values of k of a simply-supported beam using Mori-Tanaka method and applying first order shear deformation theory ................................................................. 162
Table 9.5 In-plane stress resultant $N_{0}$ for different values of k of a clamped-clamped beam using rule of mixtures and applying classical theory ...................... 165
Table 9.6 In-plane stress resultant $N_{0}$ for different values of k of a clamped-clamped beam using Mori-Tanaka method and applying classical theory .............. 167
Table 9.7 In-plane stress resultant $N_{0}$ for different values of k of a clamped-clamped beam using rule of mixtures and applying FSDT ........................................ 169
Table 9.8 In-plane stress resultant $N_{0}$ for different values of k of clamped-clamped beam using Mori-tanaka method and applying FSDT .................................. 172
Table 9.9 In-plane stress resultant N0 for different values of k of clamped-clamped beam using rule of mixtures and applying classical theory, for temperature difference 1.2K ................................................................. 174

Table 9.10 In-plane stress resultant N0 for different values of k of a clamped-clamped subjected to temperature difference of 1.2K and mechanical loads using rule of mixtures and applying FSDT theory .................................................. 176

Table 9.11 In-plane stress resultant N0 for different values of k of a clamped-clamped beam subjected to both thermal and mechanical loads, using mori-tanaka and applying classical theory (temperature difference of 1.2K) .............................................................. 178

Table 9.12 In-plane stress resultant N0 for different values of k of a clamped-clamped beam subjected to both thermal and mechanical loads, using mori-tanaka and applying FSDT theory (temperature difference of 1.2K) .. 181
LIST OF FIGURES

Figure 1.1 Variation in microstructure of functionally graded beams (Asiri, 2015) ............ 3
Figure 1.2 Applications of Functionally graded materials (Verma, 2016) ...................... 4
Figure 3.1 Influence of $k$ on the volume fraction of ceramic, along thickness of beam... 14
Figure 4.1 Coordinates and dimensions of the beam (Eslami, 2019) ......................... 17
Figure 4.2 Bernoulli-Euler beam cross section undergoing bending (Eslami, 2018) ....... 18
Figure 4.3 Displacements that a section of plate undergoes (Eslami, 2018) ............... 19
Figure 4.4 Beam element undergoing large deflections (Eslami, 2019) ...................... 21
Figure 4.5 Internal stresses acting on the beam element in x-direction (Eslami, 2019) .. 23
Figure 4.6 Internal stresses acting in z-direction on the beam element (Eslami, 2019) ... 24
Figure 4.7 Beam undergoing shear deformation only (Harrevelt, 2012) ...................... 34
Figure 4.8 Beam undergoing different deformations (Harrevelt, 2012) ...................... 35
Figure 4.9 Beam undergoing total deformation (Harrevelt, 2012) ................................... 36
Figure 4.10 Geometric assumptions in case of First-order shear deformation theory .... 37
Figure 5.1 Simply supported beam (Eslami, Chitikela, & Thivend) ......................... 43
Figure 5.2 Illustration of Newton's method (Liu, 2015) ............................................. 46
Figure 5.3 A schematic diagram of clamped-clamped beams (Eslami, Chitikela, & Thivend) ........................................................................................................ 49
Figure 5.4 spring boundary conditions considered in this thesis .................................. 53
Figure 5.5 Positive directions for deflection and slope (Donaldson, 2008) ................... 54
Figure 5.6 Free body diagram of the left end of the beam (Donaldson, 2008) ............. 55
Figure 5.7 Free body diagram of right end of the beam (Donaldson, 2008) ............... 57
Figure 6.1 In-plane stress resultant for different values of $k$ of a SS beam using rule of mixtures ................................................................. 66
Figure 6.2 Deflections of SS beam for different values of $k$ using rule of mixtures under 0.01 N/mm$^2$ ................................................................. 67
Figure 6.3 In-plane stress resultant for different values of $k$ of a simply supported beam using Mori-Tanaka Method .............................................. 68
Figure 6.4 Deflections over the length of SS beam for different values of $k$ using Mori-Tanaka method under 0.01 N/mm$^2$ ............................................. 68
Figure 6.5 In-plane stress resultant for different values of $k$ of a simply supported beam by applying first order shear deformation theory and using rule of mixtures ............................................................................ 70
Figure 6.6 Deflections of SS beam for different values of $k$ by FSDT and using rule of mixtures method under 0.01 N/mm$^2$ ............................................. 71
Figure 6.7 In-plane stress resultant for different values of k of a simply supported beam applying FSDT and using Mori-Tanaka Method .......................................................... 72
Figure 6.8 Deflections of SS beam for different ‘k’ using FSDT and Mori-Tanaka method under 0.01 N/mm\(^2\) ........................................................................................................ 72
Figure 6.9 In-plane stress resultant for different values of k of a clamped-clamped beam using rule of mixtures ............................................................ 75
Figure 6.10 Deflections seen over the length of clamped-clamped beam for different values of k using rule of mixtures under 0.01 N/mm\(^2\) ...................... 75
Figure 6.11 In-plane stress resultant for different values of k of a clamped-clamped beam using Mori-Tanaka Method to model the beam and by applying classical theory. ............................................................................................................................ 76
Figure 6.12 Deflections seen over the length of clamped-clamped beam for different values of k by applying classical theory and using Mori-Tanaka method under 0.01 N/mm\(^2\). .................................................................................................................. 77
Figure 6.13 In-plane stress resultant for different values of k of a clamped-clamped beam using rule of mixtures and applying FSDT ........................................... 79
Figure 6.14 Deflections seen over the length of clamped-clamped beam for different values of k by applying FSDT and using rule-of mixtures method under 0.01 N/mm\(^2\) .................................................................................................................. 79
Figure 6.15 In-plane stress resultant for different values of k of a clamped-clamped beam applying FSDT and using Mori-Tanaka Method ........................................ 80
Figure 6.16 Deflections seen over the length of clamped-clamped beam for different values of k by applying FSDT theory and using Mori-Tanaka method under 0.01 N/mm\(^2\) .................................................................................................................. 80
Figure 6.17 Maximum deflection values vs gradation parameter k for a FG beam with springs that replicate simply supported FG beam ........................................... 84
Figure 6.18 Maximum deflection values vs gradation parameter k for an FG beam modeled using Mori-Tanaka method and with spring boundary conditions that replicate simply supported beam ......................................................... 86
Figure 6.19 Maximum deflection values vs gradation parameter k for a FG beam with springs and modeled using rule of mixtures and FSDT theory that replicate SS beam .................................................................................................................. 88
Figure 6.20 Maximum deflection values vs gradation parameter k for a FG beam with springs and modeled using Mori-Tanaka and FSDT theory, replicating SS beams .............................................................. 89
Figure 6.21 Maximum deflection values vs gradation parameter k for a FG beam with springs replicating clamped-clamped beams and modeled using rule of mixtures .................................................................................................................... 91
Figure 6.22 Maximum deflection values vs gradation parameter k for an FG beam modeled using Mori-Tanaka method and with spring boundary conditions
replicating CC beams ................................................................. 92

Figure 6.23 Maximum deflection values vs gradation parameter k for a FG beam with springs and modeled using rule of mixtures and FSDT theory replicating CC beams ................................................................. 94

Figure 6.24 Maximum deflection values vs gradation parameter k for a FG beam with springs and modeled using Mori-Tanaka and FSDT theory ............... 95

Figure 6.25 ABAQUS results obtained for a simply supported isotropic ceramic beam (k=0) ........................................................................................................ 96

Figure 6.26 ABAQUS results obtained for a simply supported metal beam (k=99999) . 97

Figure 6.27 ABAQUS results obtained for a simply supported functionally graded beam modeled using rule of mixtures and keeping power index gradation parameter k=1 ........................................................................................................ 97

Figure 6.28 ABAQUS results obtained for a clamped-clamped isotropic ceramic beam (k=0) ........................................................................................................ 98

Figure 6.29 Results obtained for a clamped-clamped isotropic metal beam (k=999999) ........................................................................................................ 98

Figure 6.30 ABAQUS results obtained for a clamped-clamped functionally graded beam modeled using rule of mixtures and keeping power index gradation parameter k=1 ........................................................................................................ 99

Figure 7.1 Variation of $\lambda_{cr}$ for various L/h ratios, for metal ........................................ 107

Figure 7.2 Max. deflection values that occur for a temperature difference of 1.2 K when rule of mixtures and classical theory are applied............................. 110

Figure 7.3 Variation of critical temperature for different ‘k’, rule of mixtures............. 111

Figure 7.4 Variation of in plane loads for different values of transverse loads, rule of mixtures and classical theory ................................................................. 111

Figure 7.5 ABAQUS results for a clamped-clamped ceramic beam subjected to thermal and mechanical loads. Classical theory and rule of mixtures are used ........................................................................................................ 112

Figure 7.6 ABAQUS results for a clamped-clamped metal beam subjected to thermal and mechanical loads. Classical theory and rule of mixtures are used ....... 113

Figure 7.7 ABAQUS results for a clamped-clamped FG beam with k=1, subjected to thermal and mechanical loads. Classical theory and rule of mixtures are used ........................................................................................................ 113

Figure 7.8 Deflection graph for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter k= 0 for rule of mixtures ........................................................................................................ 116

Figure 7.9 Deflection graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter k=0.5 for rule of mixtures ........................................................................................................ 116

Figure 7.10 Deflection graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter k=1 for rule of mixtures ........................................................................................................ 117
Figure 7.11 Deflection graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=10$ for rule of mixtures ........................................... 117

Figure 7.12 Deflection graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=99$ for rule of mixtures ........................................... 118

Figure 7.13 $N_0$ vs pressure $P$ graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=0$ ......................................................... 118

Figure 7.14 $N_0$ vs pressure $P$ graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=0.5$ ......................................................... 119

Figure 7.15 $N_0$ vs pressure $P$ graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=1$ ......................................................... 119

Figure 7.16 $N_0$ vs pressure $P$ graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=10$ ......................................................... 120

Figure 7.17 $N_0$ vs pressure $P$ graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=99$ ......................................................... 120

Figure 7.18 Variation of $\lambda_{cr}$ for various $L/h$ ratios in case of FSDT theory with rule of mixtures, when metal is considered. ........................................... 127

Figure 7.19 Max. deflection values that occur for a temperature difference of 1.2K when rule of mixtures and fsdt theory are applied ......................... 131

Figure 7.20 Variation of critical temperature over different values of $k$, rule of mix .... 131

Figure 7.21 Variation of in plane loads for different values of transverse loads, rule of mix and first order shear deformation theory ......................... 132

Figure 7.22 Deflection graph for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=0$ for rule of mixtures ........................................... 134

Figure 7.23 Deflection graph for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=0.5$ for rule of mixtures ........................................... 135

Figure 7.24 Deflection graph for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=1$ for rule of mixtures ........................................... 135

Figure 7.25 Deflection graph for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=10$ ......................................................... 136

Figure 7.26 Deflection graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=99$ for rule of mixtures ........................................... 136

Figure 7.27 $N_0$ vs $P$ graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=0$ . 137

Figure 7.28 $N_0$ vs $P$ graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=0.5$, for rule of mixtures ........................................... 137

Figure 7.29 $N_0$ vs $P$ graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=1$, for rule of mixtures ........................................... 138

Figure 7.30 $N_0$ vs $P$ graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=10$, for rule of mixtures ........................................... 138
Figure 7.31 N0 vs P graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=99$ 139

Figure 7.32 Variation of lambda critical over different values of k, Mori-Tanaka, for metal ................................................................................................................................. 140

Figure 7.33 Max deflection values that occur for a temperature difference of 1.2K when Mori-Tanaka method and classical theory are applied ......................... 140

Figure 7.34 Variation of lambda critical over different values of k, Mori-Tanaka, for metal ................................................................................................................................. 141

Figure 7.35 Max deflection values that occur for a temperature difference of 1.2K when Mori-Tanaka method and fsdt theory are applied................................. 141

Figure 7.36 Variation of inplane forces vs mechanical load P when temperature difference of 1.2K, classical theory and Mori-Tanaka is applied............... 142

Figure 7.37 Variation of inplane forces vs mechanical load P when temperature difference of 1.2K, FSDT theory and Mori-Tanaka is applied ................. 143
ABSTRACT

Suresh, Swetha MSAE, Embry-Riddle Aeronautical University, May 2019. Non-Linear Bending Analysis of Functionally Graded Beams with Spring Constraints and Thermal Effects

Functionally graded materials, a subcategory of Advanced Composite Materials, is characterized by variation in microstructure and properties across the thickness of the beam. The unique advantage of Functionally Graded Materials (FGM) is the smooth and continuous change in properties of constituent materials from one layer to its adjacent layer in comparison to sharp changes in material properties as seen in composites. This unique attribute of functionally graded materials thereby, reduces the stress concentrations, shear and thermal stresses that occur at the interference of layers. Functionally graded materials can, thus, find applications in areas subjected to high mechanical loads and thermal stresses. The scope of this thesis is twofold: first, to study the nonlinear static analysis of FGM beams subjected to uniformly distributed mechanical transverse pressure load with both conventional and unconventional boundary conditions. The conventional boundary conditions considered here, are simply-supported and clamped-clamped with immovable edges, and unconventional boundary conditions considered are translational and rotational springs. The reason for considering unconventional boundary conditions is that in practice, it might be very difficult to achieve rigidly simply-supported or rigidly clamped boundaries. The effect of first order shear deformation theory is also considered. Second, is to study the nonlinear bending analysis of FGM beams subjected to both thermal loads and uniformly distributed mechanical transverse pressure load, for clamped-clamped beams with immovable edges.
Volume fraction of component materials is varied using power law across the thickness. Material modeling has been done using two different models, namely: rule of mixtures and Mori-Tanaka model. Nonlinear governing equations were obtained using the von Karman geometric nonlinearity and first-order shear deformation theory. Results are obtained for variations with different gradation patterns. A few of the obtained results are compared with Finite Element Results that are obtained using ABAQUS software.
1. INTRODUCTION

1.1. Functionally Graded Materials

Materials, since the evolution of humans have always played a crucial role in bolstering the human needs. For example, raw materials such as wood and stone were used to fabricate weapons and tools that were in turn used to survive predators and acquire food. The necessity to discover or invent new materials, however, increased, the reason being is the conflicting property requirements with the enhancement in knowledge and technology. Man, thus, started to engineer his own materials from the existing raw materials. Materials were initially combined with one another in their molten state that gives properties different from parent materials and the process was termed as conventional alloying (Mahamood, Akinlabi, Shukla, & Pityana, 2012). For example, copper was alloyed with tin and bronze and thus, first alloy was created (Mahamood et al., 2012). This method, however, was proven to be ineffective when large quantities of alloying material was to be dissolved as a result of thermodynamic equilibrium limit (Mahamood et al., 2012).

Materials were then combined in powdered form through a process called Powdered Metallurgy. This method again was proven to be ineffective when intricate shapes and features were to be produced (Mahamood et al., 2012). A new method was therefore, sought for and one that was found to be highly effective is producing composite materials by combining two individual materials in their solid state that offer excellent combination of properties, different from individual parent materials (Mahamood et al., 2012). Over time, huge advancements were observed in composites such that sophisticated composites like fiber reinforced plastics were introduced to the industrial world.
Although composites were found to be successful due to advantageous properties such as lightweight, high strength to weight ratio, low thermal conductivity and easy fabrication (note: different from parent material properties), their biggest drawback of delamination that in turn occurs due to presence of stress concentrations, developed on account of sharp changes in material properties at the interference of layers, paved the way for another breed of composite materials, named, functionally graded materials (Tarlochan, 2013).

Functionally graded materials (a subcategory of Advanced Composite Materials) are defined as the materials, characterized by variation in microstructure across preferred material axis orientation (Mahamood et al., 2012) and thereby, the properties across the thickness of the beam. The sharp interfaces in composites were thus, replaced by gradient interface in FGMs, which in turn leads to a smooth transition from one material to the next (Mahamood et al., 2012). It is interesting to note that many substances in nature could be considered as FGMs. To name a few, human or animal teeth, bones, mollusk shells, and bamboos (Verma, 2016). A unique characteristic of FGM is its ability to retain its parent material characteristics in comparison to composites where completely different properties are possessed. This is mainly because of significant proportions of FGM containing pure form of each element (Verma, 2016). For example, incompatible properties like a metal’s toughness can be mated with refractoriness of ceramic with need for compromise being eliminated (Verma, 2016).

A structural unit of an FGM is termed as an element or as a material ingredient (Verma, 2016). On the basis of material ingredients, FGMs can be composed in a geometrical (granule, rod, fiber, orientation and so on), biological (tissue, cell, and
complex macromolecule), chemical (metal, polymer, ceramic, organic and inorganic) and in physical (ionic state, crystalline state, electronic state and so on) manner (Verma, 2016).

![Diagram](image)

*Figure 1.1* Variation in microstructure of functionally graded beams (Asiri, 2015)

### 1.1.1. Applications of Functionally Graded Materials

Functionally graded materials are applicable to a variety of fields in engineering. For example, they are found to have potential applications in energy conversion systems, machine parts, transport systems, semiconductors, bio systems, cutting tools and so on (Verma, 2016).

Some of the applications in the various fields are as follows:

i. **Engineering:** In manufacturing of shaft, pressure vessels, cutting tools, musical instruments, wind turbine blades, helmets, MRI scanner cryogenic tubes, drilling motor shafts and so on (Verma, 2016).

ii. **Aerospace and Sub-marine Industry:** In heat exchange panels, rocket engine components, solar panels, space shuttles, nose caps, cylindrical pressure hulls and so on (Verma, 2016).

iii. **Biomaterials:** In generation of artificial skin system, implantation and so on (Verma, 2016).

Figure 1.2 Applications of Functionally graded materials (Verma, 2016)

1.2. Problem Statement

The scope of this thesis is twofold: The first part of this thesis deals with the study of nonlinear static analysis of FGM beams subjected to uniformly distributed mechanical transverse pressure load with both conventional and unconventional boundary conditions. The conventional boundary conditions considered here, are simply-supported and clamped-clamped with immovable edges and unconventional boundary conditions are translational and rotational springs supports at the edges. Volume fraction of component materials is varied using power law across the thickness. Material modeling is done using two different models, namely: rule of mixtures and Mori-Tanaka model. Nonlinear governing equations are obtained first, by using Classical theory and then by implementing First Order Shear Deformation theory (FSDT). von Karman geometric
nonlinearity is used in this thesis to account for the large deflection’s theory.

The second part of this thesis deals with the study of the nonlinear bending analysis of FGM beams subjected to both thermal loads and uniformly distributed mechanical transverse pressure loads, for clamped-clamped beams with immovable edges. The critical buckling temperature is first determined to study the pre-thermal buckling behavior of the beam subjected to an external load. In this thesis, the constant temperature rise is considered only.

Different material property models have been utilized in the governing equations of motion in this thesis, namely: rule of mixtures and Mori Tanaka models. The nonlinear governing equations are obtained based on the von Karman geometric nonlinearity. Results are obtained for variations with different gradation patterns. Some of the results are compared with results obtained using Finite Element Software like ABAQUS.

1.3. Objectives

The first objective of this thesis is to employ two different models: rule of mixtures and Mori-Tanaka to obtain the various material constants of the beam. The next objective is to formulate the nonlinear governing equations using Bernoulli-Euler Beam theory as well as First-Order Shear Deformation Theory in conjunction with von Karman Geometric nonlinearity. It is then necessary to find an appropriate numerical scheme and obtain general solutions for conventional as well as unconventional boundary conditions (translational and rotational springs).

Some of the other important objectives are, to demonstrate the results for different gradation patterns and compare the proposed model with isotropic cases, to compare a
few of the generated results using FE software like ABAQUS, to obtain critical buckling temperatures for different theories and material modeling methods, to perform non-linear bending analysis, as in part 1 of this thesis to clamped-clamped functionally graded beams with immovable edges subjected to both uniformly distributed mechanical transverse pressure loads and pre-buckling thermal effects, To obtain graphs for different gradation patterns and different ratios of $\frac{\Delta T}{\Delta T_{cr}}$ and to compare a few of the thermal results with FE results.
2. LITERATURE REVIEW

The study of functionally graded materials and the structural analysis of such materials has been and will always be an area of fascination for researchers all over the world. Engineers and researchers are interested in these materials because of their unique advantage of smooth and continuous change in properties of constituent materials from one layer to its adjacent layer in comparison to sharp changes in material properties as seen in composites. It is also highly preferred due to their excellent heat resistance properties in thermal environments.

Plenty of research on the vibrations, buckling, stability, thermal effects and nonlinear bending of functionally graded materials in the form of various structural elements like beams, plates and shells have been conducted over the years. Though there has been ample research done on plates, limited research has been conducted on functionally graded beams. The current literature review mainly deals with the work done on functionally graded beams over the years including this thesis.

The concept of FGMs was first proposed by Japanese material scientists in 1984 (Jha, Kant, & Singh, 2013). In recent years, Bohidar, Sharma and Mishra (2014), in their journal paper provided the readers a detailed insight on the concept, background, processing techniques, and applications of functionally graded materials (Bohidar, Sharma, & Mishra, 2014). Jha, Kant and Singh (2013) did a thorough literature survey on vibration, thermo-elastic, static, and stability analyses of functionally graded plates since 1998 (Jha et al., 2013). Chauhan and Khan (2014) documented the research done on functionally graded beams over the recent years (Chauhan, & Khan, 2014). Zhang et al. (2019) also provided an extensive literature survey on the stability, buckling and free-
vibrational analysis of functionally graded beams (Zhang et al., 2019). These two papers turned out to be a good source of information for the author to conduct further research on functionally graded beams.

Thai and Vo (2012) studied the bending and free vibration of functionally graded beams using higher order shear deformation beam theories. This paper states that the theories developed by the authors account for higher order variation of transverse shear strain and that the shear correction factor can thus be neglected (Thai & Vo, 2012). Sankar (2001) developed Euler-Bernoulli theory for functionally graded beams and has also obtained an elasticity solution (Sankar, 2001). Sankar and Tzeng (2002) have obtained the axial stress distribution by solving the thermoelastic equilibrium equations in closed form. The temperature and thermoelastic constants of the beam are assumed to be varying exponentially through the thickness (Sankar & Tzeng, 2002). She, Yuan and Ren (2017) worked on the nonlinear bending, thermal buckling and post-buckling analysis of functionally graded material tubes with two clamped ends by using a refined beam theory (She, Yuan, & Ren, 2017).

Sun and Chin (1988) solved the cylindrical bending problems of asymmetric composite laminates subjected to large deflections by introducing von-Karman geometric nonlinearity term and reducing the governing equations to linear differential equations with nonlinear boundary conditions, thereby yielding in a simple solution procedure (Sun & Chin, 1988). Chen and Shu (1991) developed large deflection shear deformation theory for unsymmetrical composite laminates and employed the same procedure as Sun and Chin (1988) to solve the problem (Chen & Shu, 1991). Eslami, Park and Gangadharan (2002), then used the same theory and introduced, developed a solution for
unsymmetrical laminates with arbitrary angle of orientation of plies (Eslami, Park, & Gangadharan, 2002).

Li (2008) presented a unified approach to analyze the static and dynamic behaviors of FGM beams with rotary and shear deformation effects. An interesting aspect of this paper is that Bernoulli-Euler and Rayleigh beams are utilized to analytically solve the Timoshenko beams (Li, 2008). Yaghoobi and Torabi (2013) published an interesting paper where the non-linear vibration and post buckling analysis of FGM beams that rest on non-linear elastic foundations and subjected to axial force are studied. Conventional boundary conditions like simply supported and clamped-clamped beams are analyzed. Bernoulli-Euler beam theory with von-Karman geometric non-linearity are used to obtain the governing partial differential equation and this was in turn transformed to an ordinary differential equation by using the Galerkin’s method. The resulting ordinary differential equations are then solved using variational iteration method (Yaghoobi & Torabi, 2013).

Ke, Yang and Kitiporningchai (2010) also studied the nonlinear vibration of functionally graded beams. This paper uses direct numerical integration and Runge-Kutta methods to obtain the non-linear vibrational response of FGM beams with different end supports (Ke, Yang, & Kitiporningchai, 2010). Lin et al. (2018) studied the nonlinear bending deformation of functionally graded beams with variable thickness using meshless Smoothed Hydrodynamic Particle (SPH) method (Lin et al., 2018).

Kumar, Mitra and Roy (2015) worked on the large amplitude free vibration analysis of axially functionally graded (AFG) tapered slender beams with different boundary conditions. In this paper, the static problem was first solved using an iterative numerical scheme followed by solving the dynamic problem which was in turn solved as standard
eigen value problem (Kumar, Mitra, & Roy, 2015). Akbas (2013) worked on the non-linear static analysis of edge-cracked cantilever Timoshenko beams composed of functionally graded materials subjected to non-follower transversal point load at free end of the beam. Large displacements and large rotations were also considered. An interesting part in this paper is the way in which the cracked beam is modeled (Akbas, 2013).

Ebrahimi and Zia (2015) worked on obtaining the nonlinear vibration characteristics of functionally graded clamped-clamped Timoshenko beams made of porous material (Ebrahimi & Zia, 2015). Kien and Gan (2014) studied the large deflections of tapered functionally graded beams subjected to end forces by using finite element method (Kien & Gan, 2014). Wattanasakulpong and Unghakorn (2014) investigated the linear and non-linear vibrational behavior of elastically end restrained beams made of functionally graded materials with porosities. Differential transform method was used to solve for the vibrational responses (Wattanasakulpong & Unghakorn, 2014).

Jiao et al. (2018) studied the linear bending of functionally graded beams. Bernoulli-Euler and First order shear deformation theories were used to obtain the governing equations. Differential quadrature method was then used to obtain the solutions (Jiao et al., 2018). Arefi & Rahimi (2013) worked on non-linear analysis of functionally graded beams with variable thickness and under combined loads. Bernoulli-Euler theory was employed to obtain the governing differential equations and then Adomian's Decomposition Method (ADM) was used to obtain the solution (Arefi & Rahimi, 2013). Fu, Wang and Mao (2012) carried out nonlinear analysis of buckling, free vibration and dynamic stability for piezoelectric functionally graded beams in thermal environment (Fu, Wang, & Mao, 2012).
Tahani & Mirzababaee (2009) used a layer-wise theory to determine displacements and stresses in functionally graded plates undergoing cylindrical bending and subjected to thermomechanical loads (Tahani & Mirzababaee, 2009). Yang & Shen (2003) studied the large deflection and post-buckling responses of functionally graded plates under transverse and in plane loads by using a semi-analytical approach. Solutions were obtained by employing a perturbation technique in conjunction with one-dimensional differential quadrature approximation and Galerkin’s method (Yang & Shen, 2003).

Shen, Lin and Xiang (2017) studied the nonlinear bending and thermal post-buckling behaviors of nanocomposite beams in thermal environments and supported by an elastic foundation (Shen, Lin, & Xiang, 2017). Niknam, Fallah and Aghdam (2014) studied the nonlinear bending analysis of tapered functionally graded beams subjected to both thermal and mechanical loads. Generalized differential quadrature method was used to obtain the solution (Niknam, Fallah, & Aghdam, 2014).

Fallah and Aghdam (2011) obtained simple analytical expressions for large amplitude free vibration and post-buckling analysis of functionally graded beams resting on nonlinear elastic foundations and subjected to axial force. He’s variational method was used to obtain approximate closed form solutions of the nonlinear governing equations (Fallah & Aghdam, 2011). Kiani & Eslami (2010) worked on the post-buckling analysis of beams subjected to three types of thermal loading, namely, uniform temperature rise, linear, and nonlinear temperature distribution through the thickness (Kiani & Eslami, 2010).

2.1. **Previous Research in ERAU**

The cylindrical bending and post buckling analysis of functionally graded beams with
conventional boundary conditions was initially done by Deschilder, Eslami, Thivend and Zhao (Deschilder, Eslami, Thivend, & Zhao) under the guidance of Dr. Habib Eslami. FSDT theory was used to derive the governing equations. The approach followed for the same was that by Sun & Chin (1988) (Sun & Chin, 1988) and Chen and Shu (1991) (Chen & Shu, 1991) as most gradation patterns in FGMs are unsymmetrical, thereby inducing a bending-extension coupling stiffness (Eslami, Chitikela, & Thivend). It was then continued by Eslami, Chitikela and Thivend (Eslami, Chitikela, & Thivend) under the guidance of Dr. Habib Eslami where Chitikela introduced cylindrical bending in Bernoulli-Euler beams and thereby obtained the governing differential equations using classical plate theory. He also worked on validating the results obtained in case of cylindrical bending by using a Finite element software.

The author of this thesis has regenerated Chitikela’s results by rewriting a complete set of new MatLab codes, introduced spring boundary conditions, matched the results obtained using spring boundary conditions with that of conventional boundary conditions and performed non-linear bending analysis of clamped-clamped beams with immovable edges subjected to both temperature and mechanical loads.

The analysis was done using two different material models and both the beam theories, namely classical plate theory and first order shear deformation theory. A few of the results obtained are compared with finite element results in turn obtained using commercially available software like ABAQUS. It can thus be concluded to the best of author’s knowledge that, the work presented in this thesis is unique. The method employed in deriving the governing equations of motions and the solution of transverse deflection is also a fairly simple and accurate one. It was employed by Sun and Chin
(2008) while working with unsymmetrical laminated composites (Sun & Chin, 2008).
3. MATERIAL MODELING

The volume fraction used in this thesis is similar to the one used by Javaheri and Eslami (2002), where \( V \) refers to the volume fraction of the phase material, and the subscripts \( m \) and \( c \) refer to the ceramic and metal phases, respectively, and \( V_c \), is given by the power law, expressed as follows: (Javaheri & Eslami, 2002), (Eslami, Chitikela, & Thivend).

\[
V_c = \left( \frac{z}{h} + \frac{1}{2} \right)^k
\]

(3.1)

And \( k \) is the power index that defines the gradation patterns. Also, the notations \( h \) and \( z \) are:

- \( h \) is the thickness of the beam, and
- \( z \) is the distance from the mid-plane of the beam

The volume fractions of ceramic and metal phases are in turn related by:

\[
V_c + V_m = 1
\]

(3.2)

Therefore,

\[
V_m = 1 - V_c
\]

(3.3)

Figure 3.1 shows the influence of \( k \) on the volume fraction of ceramic, along the thickness of the beam.
thickness of the beam (Eslami, Chitikela, & Thivend). For limit values of \( k \), the beam is made out of isotropic ceramic at \( k=0 \) and isotropic metal at \( k=\infty \). It is to be noted that for computational purposes, \( k=99 \) is used in order to represent \( k=\infty \).

The FGM beam is first modeled using rule of mixtures material schema. The upper bound of rule of mixtures is used, as the direction of loading is similar to the direction of variation of effective material properties (along \( z \)-axis). Accordingly, Young’s modulus is given by,

\[
E(z) = E_c V_c + E_m V_m
\]  \hspace{1cm} (3.4)

Where, \( E_c \) is Young’s modulus of ceramic

\( E_m \) is Young’s modulus of metal

\( V_c \) is volume fraction of ceramic

\( V_m \) is volume fraction of metal

Poisson’s ratio is given by:

\[
\nu(z) = \nu_c V_c + \nu_m V_m
\]  \hspace{1cm} (3.5)

Where, \( \nu_c \) is Poisson’s ratio of ceramic

\( \nu_m \) is Poisson’s ratio of metal

\( V_c \) is volume fraction of ceramic

\( V_m \) is volume fraction of metal

Shear modulus is given by:

\[
G(z) = \frac{V_c}{G_c} + \frac{V_m}{G_m}
\]  \hspace{1cm} (3.6)

Where, \( G_c \) is shear modulus of ceramic,

\( G_m \) is shear modulus of metal,

\( V_c \) is volume fraction of ceramic, and
$V_m$ is volume fraction of metal

Even though these estimates are proven to be accurate in the case of laminated composites, their usage in the case of FGMs is an over simplification of FGMs behavior and particulate composites in general. Thus, in order to take into account the effect of microstructure of the material, estimation schemes such as Mori Tanka or Wakashima Tsukamoto (both are Eshelby’s method) are used (Eslami, Chitikela, & Thivend). Mori Tanaka material schema is defined as follows (Prakash, Singha, & Ganapathi, 2007).

Accordingly, Bulk modulus and Shear modulus is given by,

$$K = K_m + \frac{V_c(K_c-K_m)}{1+(1-V_c)\frac{2(K_c-K_m)}{3K_m+4G_m}}; \quad G = G_m + \frac{V_c(G_c-G_m)}{1+(1-V_c)\frac{2(G_c-G_m)}{G_m+f_1}}$$

(3.7)

Where,

$$f_1 = \frac{G_m(9K_m+8G_m)}{6(K_m+2G_m)}$$

The Young’s modulus and Poisson’s ratio are given by:

$$E(z) = \frac{9KG}{3K+G}; \quad \nu(z) = \frac{3K-2G}{2(3K+G)}$$

(3.8)

Where, $K_m$ is the bulk modulus of metal

$K_c$ is the bulk modulus of ceramic

$G_c$ is the shear modulus of ceramic

$G_m$ is the shear modulus of metal
4. GOVERNING EQUATIONS

4.1. Classical Theory

The governing differential equations in terms of displacements for a uniform cross section of functionally graded beams, undergoing large deflections and subjected to a transverse uniform distributed load is derived here. It is to be noted that the effect of transverse shear will be neglected in the classical theory. The strain that occurs due to large deflections in the beam is taken care of by using the non-linear von-Karman large deflection term in the definition of strains.

The derivation of equations starts with assumptions being stated followed by derivation of strain-displacement equations, the equations of equilibrium for a beam, stress-strain relations (constitutive equations) and thereby the resultant force-strain relations. The force-strain relations are then simultaneously solved for resultant internal force and moment, in terms of beam stiffnesses and displacements and then substituted back in the equations of equilibrium to finally obtain non-linear governing equations in terms of displacements (Eslami, 2018).

![Figure 4.1](image-url) Coordinates and dimensions of the beam (Eslami, 2019)

4.1.1. Assumptions

- The beam is much larger in one of the directions than the other two dimensions and the cross section of the beam is uniform.
- The in-plane displacements $u$ and $v$ are insignificant compared to the transverse displacement $w$. However, $w$ is small compared to the beam thickness $h$.
- In plane strains $\varepsilon_x$, $\varepsilon_z$, and $\gamma_{xz}$ are small compared to unity.
- Plane sections remain plane after deformations and a line originally normal to the plane of cross-section remains perpendicular. (Kirchoff’s Hypothesis)
- Transverse shear strains $\gamma_{xz}$ and $\gamma_{yz}$ are neglected. Tangential displacements or in-plane displacements $u$ and $v$ are linear functions of the $z$ coordinate.
- The transverse normal strain $\varepsilon_z$ is negligible. Therefore, it must be noted that the problem is a plane stress type problem and also that the transverse deflection is a function of position in the $x$ direction only.

$$w = w(x, t)$$

- Shear deformation effect is neglected in classical theory.
- The angle of rotation is assumed to be small. Thus, by applying small angle approximation:

$$\sin \theta \simeq \theta = \frac{\partial w}{\partial x} ; \cos \theta \simeq 1$$

It is also to be noted that the type of beam being dealt with is an Bernoulli- Euler beam as the shear effects have been neglected and as plane sections remain plane after deformation (Eslami, 2018).

\[\text{Figure 4.2 Bernoulli-Euler beam cross section undergoing bending (Eslami, 2018)}\]
4.1.2. Strain-Displacement Relations in the Beam

Strain-Displacement relations in a plate:

The in-plane displacements \( u \) and \( v \) in the \( x \) and \( y \) directions, respectively at any point \( z \) in case of a plate, through its thickness is represented by:

\[
\begin{align*}
   u &= u_0(x, y, t) - z \frac{\partial w}{\partial x} \quad (4.1) \\
   v &= v_0(x, y, t) - z \frac{\partial w}{\partial y} \quad (4.2)
\end{align*}
\]

where \( u_0, v_0 \) and \( w \) are the geometrical midplane displacements in the \( x, y \) and \( z \) directions respectively.

\[
w = w(x, y, t) \quad (4.3)
\]

\textit{Figure 4.3} Displacements that a section of plate undergoes (Eslami, 2018)

The normal and shear strains in terms of displacements \( u \) and \( v \) are therefore,

\[
\varepsilon_x = \frac{(u + \frac{\partial u}{\partial x}dx) - u}{dx} = \frac{\partial u}{\partial x} \quad (4.4)
\]
\[ \varepsilon_y = \left( v + \frac{\partial v}{\partial y} dy \right) - v = \frac{\partial v}{\partial y} \]  
(4.5)

\[ \gamma_{xy} = \left( \frac{u + \partial u}{\partial x} dx \right) - u + \left( v + \frac{\partial v}{\partial y} dy \right) - v = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \]  
(4.6)

It is to be noted that von-Karman geometric non-linearity is yet to be added in the formulations and will be done when the above equations are modified for beams.

Substituting displacements from Equations (4.1, 4.2, and 4.3) into Equations (4.4, 4.5, and 4.6) yield,

\[ \varepsilon_x = \frac{\partial u_0}{\partial x} - 2z \frac{\partial^2 w}{\partial x^2} \]  
(4.7)

\[ \varepsilon_y = \frac{\partial v_0}{\partial y} - 2z \frac{\partial^2 w}{\partial y^2} \]  
(4.8)

\[ \gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} \]  
(4.9)

These equations represent the normal and shear strains in terms of midplane displacements and curvatures. In matrix form,

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} + z
\begin{bmatrix}
K_x \\
K_y \\
K_{xy}
\end{bmatrix}
\]
(4.10)

Thus,

\[
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial u_0}{\partial x} \\
\frac{\partial v_0}{\partial y} \\
\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}
\end{bmatrix} ; 
\begin{bmatrix}
K_x \\
K_y \\
K_{xy}
\end{bmatrix} =
- \begin{bmatrix}
\frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^2 w}{\partial y^2} \\
2 \frac{\partial^2 w}{\partial x \partial y}
\end{bmatrix}
\]
(4.11)

are the midplane strains and curvatures (Eslami, 2018).

Strain-Displacement relations in a beam:

The beam considered here, is long in the \( x \)- direction in comparison to dimensions in the \( y \)- and \( z \)- directions (Gudmundsson, 1995). The mid-plane displacements \( u_0, v_0 \) and the transverse displacement, \( w \), thus, are a function of position \( x \) and time \( t \) only. It is also
to be noted that this thesis deals only with static analysis and hence all displacements are treated only as functions of $x$. A one-dimensional analysis is thus rendered. The above displacements thus simplify to,

$$u(x, z) = u_0(x) - z \frac{\partial w(x)}{\partial x}$$  \hspace{1cm} (4.12) \\
$$v(x) = v_0(x)$$  \hspace{1cm} (4.13)

where $u_0$, $v_0$ and $w$ are the geometrical midplane displacements of the beam in the $x$, $y$, and $z$ directions respectively.

$$w = w(x)$$  \hspace{1cm} (4.14)

We now add von-Karman Large Deflection Geometric Non-Linearity to the above displacements. The derivation of the same is discussed as follows:

![Figure 4.4 Beam element undergoing large deflections (Eslami, 2019)](image)

This figure shows the large deflection that takes place in the beam. The axial strain due to the large deflection $w$ is determined as follows: (Eslami, 2019).

$$ds^2 = dx^2 + \left( \frac{\partial w}{\partial x} \right)^2 dx^2$$  \hspace{1cm} (4.15) \\
$$\frac{ds}{dx} = \sqrt{1 + \left( \frac{\partial w}{\partial x} \right)^2} \approx 1 + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2$$  \hspace{1cm} (4.16)
\[ ds - dx = \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 dx \]  

(4.17)

The displacements at any point along the thickness of the beam thus become,

\[ u = \left( u_0(x) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) - z \frac{\partial w}{\partial x} \]  

(4.18)

\[ w = w(x) \]  

(4.19)

On taking differentials of the same, the strains become:

\[ \varepsilon_x = \frac{\partial \left( u_0(x) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right)}{\partial x} - z \frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_z = 0 \]  

(4.20)

\[ \gamma_{xz} = \frac{\partial \left( u_0(x) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right)}{\partial z} \]  

(4.21)

It is to be noted here that all differentials with respect to y are 0 as all the displacements and curvatures are only functions of x, in beam theory. It is also to be noted that \( \gamma_{xz} \) is 0 in case of Bernoulli-Euler theory. The strains at any point in the beam are thus given by: (Eslami, n.d.-a)(Gudmundsson, 1995).

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_z \\
\gamma_{xz}
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_x^0 \\
0 \\
0
\end{bmatrix} + z
\begin{bmatrix}
K_x \\
0 \\
0
\end{bmatrix}
\]  

(4.22)

Thus,

\[ \{ \varepsilon^0 \} = \begin{bmatrix}
\frac{\partial (u_0(x) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2)}{\partial x} \\
0 \\
0
\end{bmatrix}; \{ \kappa \} = - \begin{bmatrix}
\frac{\partial^2 w}{\partial x^2} \\
0 \\
0
\end{bmatrix} \]  

(4.23)

is the midplane strain in case of Bernoulli-Euler beam theory.

**4.1.3. Equations of Equilibrium for the Beam**

Consider an element of the beam as shown in the following figure:

Part a:

Consider Newton’s second Law:
\[ \Sigma F = ma \] (4.24)

Where, \( \Sigma F \) is the sum of all forces applied on the beam

\( m \) is the mass of the beam

\( a \) is the beam’s acceleration

The acceleration in this beam is zero as the beam is in static equilibrium. The above equation thus, becomes:

\[ \Sigma F = 0 \] (4.25)

\[ \sigma_x + \frac{\partial \sigma_x}{\partial x} \, dx \, dy \, dz - \sigma_x \, dy \, dz + \left( \tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} \, dz \right) \, dx \, dy = 0 \]

After simplifying,

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \] (4.26)

There are no forces acting in y-direction. Equations of equilibrium in the z-direction is obtained by substituting all forces in \( \Sigma F_z = 0 \), from the following elements (b, c, d)
Considering the resultant force acting in the z-direction due to $\sigma_x$ alone, as shown in Fig. b:

$$F_{z}^I = \left( \sigma_x + \frac{\partial \sigma_x}{\partial x} \right) dydz \ast \sin \left( \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx \right) - \sigma_x dydz \ast \sin \frac{\partial w}{\partial x}$$  \quad (4.27)

On taking small angle approximations $\sin \theta \approx \theta$, simplifying and neglecting higher order terms

$$F_{z}^I = \left( \sigma_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial \sigma_x}{\partial x} \frac{\partial w}{\partial x} \right) dx dydz$$  \quad (4.28)

Product rule is given by:

$$\frac{d(uy)}{dx} = u \frac{dy}{dx} + v \frac{du}{dx}.$$  It can now be seen that the right-hand side of the product rule equation is same as the right-hand side of equation (4.28).

Equation (4.28) can thus be simplified to:

$$F_{z}^I = \frac{\partial}{\partial x} \left( \sigma_x \frac{\partial w}{\partial x} \right) dx dydz$$  \quad (4.29)

Considering the resultant force acting in the z-direction due to $\tau_{zx}$ alone, as shown in Fig. c:
\[ F_{z}^{2} = (\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \, dz) \, dxdy \ast \sin \left( \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x \partial z} \, dz \right) - \tau_{zx} \, dxdy \ast \sin \left( \frac{\partial w}{\partial x} \right) \] (4.30)

On simplifying Equation (4.30) in a similar fashion as in the previous equations, we get,

\[ F_{z}^{2} = \frac{\partial}{\partial z} \left( \tau_{zx} \frac{\partial w}{\partial x} \right) dxdydz \] (4.31)

Considering the resultant force acting in the z-direction due to \( \tau_{xz} \) alone, as shown in Fig. d:

\[ F_{z}^{3} = (\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} \, dx) \, dydz - \tau_{xz} \, dydz \] (4.32)

On simplifying Equation (4.32), we get,

\[ F_{z}^{3} = \frac{\partial \tau_{xz}}{\partial x} \, dxdydz \] (4.33)

Considering the resultant force acting in the z-direction due to \( \sigma_{z} \) alone, as shown in Fig. (b):

\[ F_{z}^{4} = (\sigma_{z} + \frac{\partial \sigma_{z}}{\partial z} \, dz) \, dxdy - \sigma_{z} \, dxdy \] (4.34)

Simplifying gives,

\[ F_{z}^{4} = \frac{\partial \sigma_{z}}{\partial z} \, dxdydz \] (4.35)

Adding all components \( F_{z}^{1}, F_{z}^{2}, F_{z}^{3} \) and \( F_{z}^{4} \) of \( F_{z} \) and substituting in \( \Sigma F_{z} = 0 \).

\[ \frac{\partial}{\partial x} \left( \sigma_{x} \frac{\partial w}{\partial x} \right) dxdydz + \frac{\partial}{\partial z} \left( \tau_{zx} \frac{\partial w}{\partial x} \right) dxdydz + \frac{\partial \tau_{xz}}{\partial x} \, dxdydz + \frac{\partial \sigma_{x}}{\partial z} \, dxdydz = 0 \] (4.36)

The above equation finally simplifies to:

\[ \frac{\partial}{\partial x} \left( \sigma_{x} \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial z} \left( \tau_{zx} \frac{\partial w}{\partial x} \right) + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_{x}}{\partial z} = 0 \] (4.37)

The internal in-plane force resultant \( N_{x} \), acting on the geometric middle surface of the beam is obtained by multiplying stress resultant \( \sigma_{x} \) by \( dz \) and integrating over the thickness,
\[ N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x dz \]  
(4.38)

The resultant transverse shear \( Q_x \), acting on the geometric middle surface of the beam is obtained by multiplying stress resultant \( \tau_{zx} \) by \( dz \) and integrating over the thickness,

\[ Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zx} dz \]  
(4.39)

The resultant bending moments \( M_x \), acting on the geometric middle surface of the beam is obtained by multiplying stress resultant \( \sigma_x \) by \( zdz \) and integrating over the thickness,

\[ M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x zdz \]  
(4.40)

Thus, when equation (4.26) is multiplied by \( dz \) on both sides and integrated over the thickness,

\[ \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_x}{\partial x} dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \tau_{zx}}{\partial z} dz = 0 \]  
(4.41)

Interchanging the order of differentiation and integration of first term and using equation (4.38)

\[ \frac{\partial N_x}{\partial x} + (\tau_{zx}) \frac{h}{2} = 0 \]  
(4.42)

Since the transverse shear stress vanishes on the free surface, the above equation reduces to,

\[ \frac{\partial N_x}{\partial x} = 0 \]  
(4.43)

Therefore,

\[ N_x = \text{constant} \]  
(4.44)

When equation (4.26) is multiplied by \( zdz \) on both sides and integrated over the thickness,
\[
\int_{-h/2}^{h/2} \left( \frac{\partial \sigma_x}{\partial x} z \right) dz + \int_{-h/2}^{h/2} \left( \frac{\partial \tau_{xz}}{\partial z} z \right) dz = 0
\]  
(4.45)

Interchanging the order of differentiation and integration for first term.

\[
\frac{\partial}{\partial x} \int_{-h/2}^{h/2} (\sigma_x z) dz + \int_{-h/2}^{h/2} \left( \frac{\partial \tau_{xz}}{\partial z} z \right) dz = 0
\]  
(4.46)

Integrating the second term by parts, by considering \( v = \frac{\partial \tau_{xz}}{\partial z} \) and \( u = z \).

\[
\int_{-h/2}^{h/2} \left( \frac{\partial \tau_{xz}}{\partial z} z \right) dz = \left( z \tau_{xz} \right)_{-h/2}^{h/2} - \int_{-h/2}^{h/2} \frac{dz}{dz} \tau_{xz} dz
\]  
(4.47)

As shear stresses are zero on the free surfaces of the beam the first term in the R.H.S of the equation must be zero.

Therefore,

\[
\int_{-h/2}^{h/2} \left( \frac{\partial \tau_{xz}}{\partial z} z \right) dz = - \int_{-h/2}^{h/2} \frac{dz}{dz} \tau_{xz} dz
\]  
(4.48)

Equation (4.48) thus becomes,

\[
\frac{\partial M}{\partial x} - Q_x = 0
\]  
(4.49)

Consider equation (4.37) and expanding the same using product rule,

\[
\frac{\partial}{\partial x} \left( \sigma_x \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial z} \left( \tau_{xz} \frac{\partial w}{\partial x} \right) + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_x}{\partial z} = 0
\]  
(4.50)

\[
\frac{\partial \sigma_x}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x \partial x} \sigma_x + \frac{\partial \tau_{xz}}{\partial z} \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial z \partial x} \tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_x}{\partial z} = 0
\]  
(4.51)

Multiplying by \( dz \) and integrating over thickness,

\[
\left( \frac{\partial}{\partial x} \int_{-h/2}^{h/2} \sigma_x dz \right) \frac{\partial w}{\partial x} + \left( \int_{-h/2}^{h/2} \sigma_x dz \right) \frac{\partial^2 w}{\partial x \partial x} + \frac{\partial}{\partial x} \left( \int_{-h/2}^{h/2} \tau_{xz} dz \right) + \frac{\partial}{\partial x} \left( \int h \int_{-h/2}^{h/2} \tau_{xz} dz \right) + \int_{-h/2}^{h/2} \frac{\partial \sigma_x}{\partial x} dz = 0
\]  
(4.52)

Substituting \( N_x, M_x, Q_x \) considering the fact that shear strains are zero at free surfaces of the beam and that \( P(x) = \sigma_z \frac{h}{2} \), the above equation becomes,
\[
\frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} + N_x \frac{\partial^2 w}{\partial x^2} + 0 + 0 + \frac{\partial Q_{xz}}{\partial x} + P(x) = 0
\] (4.53)

We thus have,

\[
\frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} + N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial Q_{xz}}{\partial x} + P(x) = 0
\] (4.54)

From (4.43), \(\frac{\partial N_x}{\partial x} = 0\). The above equation thus becomes,

\[
N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial Q_{xz}}{\partial x} + P(x) = 0
\] (4.55)

The equations of equilibrium are thus: (Eslami, n.d.-c)

\[
\frac{\partial N_x}{\partial x} = 0
\]

\[
\frac{\partial M}{\partial x} - Q_x = 0
\]

\[
N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial Q_{xz}}{\partial x} + P(x) = 0
\]

### 4.1.4. Stress-Strain Relations of the Beam

According to Hooke’s Law,

\[
\sigma = E\varepsilon
\] (4.56)

Where, \(\sigma\) is the longitudinal stress

\(E\) is the young’s modulus

\(\varepsilon\) is the longitudinal strain

In case of Functionally Graded Beams,

\[
\sigma_x = E(z)\varepsilon_x
\]

\[
\sigma_x = E(z) \left[ \frac{\partial \left( u_0(x) + \frac{1}{2} \frac{\partial w}{\partial x} \right)}{\partial x} - z z \frac{\partial^2 w}{\partial x^2} \right]
\] (4.57)

### 4.1.5. Force-Displacement Relations in the Beam

Substituting equation (4.57) in \(N_x\) and \(M_x\),
\[ N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \left[ \frac{\partial}{\partial x} \left( u_0(x) + \frac{1}{2} \frac{\partial w}{\partial x} \right) \right] dx - z \frac{\partial^2 w}{\partial x^2} \right] \right] dz \] (4.58)

\[ M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \left[ \frac{\partial}{\partial x} \left( u_0(x) + \frac{1}{2} \frac{\partial w}{\partial x} \right) \right] dz \] (4.59)

In matrix form,

\[ \begin{pmatrix} M_x \\ N_x \end{pmatrix} = \begin{pmatrix} E_1 & E_2 \\ E_0 & E_1 \end{pmatrix} \begin{pmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\ -\frac{\partial^2 w}{\partial x^2} \end{pmatrix} \] (4.60)

Where,

\[ E_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \] (4.61)

\[ E_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} (E(z)z) \] (4.62)

\[ E_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} (E(z)z^2) \] (4.63)

\( N_x \) is replaced with \( N_0 \) (as \( N_0 \) is a constant value and independent of \( x \)),

Let \( \alpha = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \); \( \kappa_x = -\frac{\partial^2 w}{\partial x^2} \) (4.64)

Thus,

\[ M_x = E_1 \alpha - E_2 \gamma \] (4.65)

\[ N_0 = E_0 \alpha - E_1 \gamma \] (4.66)

Equation (4.65) and (4.66) are simultaneously solved for \( M_x \) and is as follows,

\[ M_x = \begin{pmatrix} E_0 E_2 - E_1^2 \end{pmatrix} \frac{\partial^2 w}{\partial x^2} + \frac{N_0 E_1}{E_0} \] (4.67)

We know that,

\[ \frac{\partial M}{\partial x} = Q_x \] (4.68)

On differentiating equation (4.64)
\[ Q_x = \frac{\partial M}{\partial x} = - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial^3 w}{\partial x^3} \] (4.65)

Differentiating (4.65) and substituting in \( N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial Q_x}{\partial x} + P(x) = 0 \)

\[ N_0 \frac{\partial^2 w}{\partial x^2} - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial^4 w}{\partial x^4} + P(x) = 0 \] (4.66)

Rearranging the above equation,

\[ \frac{\partial^4 w}{\partial x^4} - \frac{N_0}{\beta} \frac{\partial^2 w}{\partial x^2} - \frac{P(x)}{\beta} = 0 \] (4.67)

If \( \beta = \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \),

\[ \frac{\partial^4 w}{\partial x^4} - \frac{N_0}{\beta} \frac{\partial^2 w}{\partial x^2} = \frac{P(x)}{\beta} \] (4.68)

Therefore, (Eslami, Chitikela, & Thivend, n.d.)

\[ \frac{\partial^4 w}{\partial x^4} - \zeta^2 \frac{\partial^2 w}{\partial x^2} = \varphi \] (4.69)

### 4.1.6. Solution to the fourth-order differential equation

(a) Homogenous Solution

Let us initially assume a solution for transverse deflection

\[ w(x) = ce^{mx} \] (4.70)

On differentiating the above solution four times and substituting the same in (4.71), the characteristic equation is thus obtained to be,

\[ m^4 - \zeta^2 m^2 = 0 \] (4.71)

The roots are thus, \( m = 0, 0, \zeta \) and \(-\zeta\).

The homogenous solution thus becomes,
\[ w(x) = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{\xi x} + c_4 e^{-\xi x} \]  
(4.72)

We know that \( e^x = \cosh x + \sinh x \). Substituting the same in (4.74)

\[ w(x) = c_1 + c_2 x + c_3 (\cosh \xi x + \sinh \xi x) + c_4 (\cosh \xi x - \sinh \xi x) \]  
(4.73)

On rearranging the above equation,

\[ w(x) = c_1 + c_2 x + (c_3 + c_4) \cosh \xi x + (c_3 - c_4) \sinh \xi x \]  
(4.74)

As the sum of two constants is again a constant, setting,

\[ C_3 = (c_3 + c_4) \]
\[ C_4 = (c_3 - c_4) \]

Also setting,

\[ C_1 = c_1 \quad ; \quad C_2 = c_2 \]

Equation (4.74) thus becomes,

\[ w(x) = C_1 + C_2 x + C_3 \cosh \xi x + C_4 \sinh \xi x \]  
(4.75)

Equation (4.75) is the homogenous solution of the fourth-order differential equation.

(b) Particular solution

A particular solution is assumed of the form

\[ w_p(x) = b x^2 \]  
(4.76)

Where, \( b \) is a random unknown constant. \( x^2 \) is multiplied to \( b \) in the particular solution to account for \( x \) in the complementary solution.

On substituting equation (4.76) in equation (4.69)

\[ 0 - 2\xi^2 b = \varphi \]  
(4.77)

On rearranging equation (4.77), \( b \) is obtained to be

\[ b = -\frac{\varphi}{2\xi^2} \]  
(4.78)

Substituting \( b \) in equation (4.76),
\[ w_p(x) = -\frac{n}{2\xi^2}x^2 \]  

Transverse Deflection is thus,

\[ w(x) = C_1 + C_2x + C_3 \cosh(\xi x) + C_4 \sinh(\xi x) - \frac{\phi}{2\xi^2}x^2 \]  

Where \( C_1, C_2, C_3 \) and \( C_4 \) depend not only on the boundary conditions, but on \( N_0 \) as well.

Now,

\[ N_0 = E_0 \left[ \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] - E_1 \frac{\partial^2 w}{\partial x^2} \]  

Multiply by \( dx \) and integrating over the length, (Eslami, Chitikela, & Thivend)

\[ N_0 dx = E_0 \left[ \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] dx - E_1 \frac{\partial^2 w}{\partial x^2} dx \]

\[ \int_{-a}^{a} N_0 dx = E_0 \int_{-a}^{a} \frac{\partial u_0}{\partial x} dx + \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - E_1 \int_{-a}^{a} \frac{\partial^2 w}{\partial x^2} dx \]  

4.2. **First Order Shear Deformation Theory (FSDT)**

Thus far, shear deformation terms was neglected in case of Bernoulli-Euler. It can thus be concluded that Bernoulli-Euler theory works well for thin beams but fails for thick beams as shear deformation effect must not be ignored in thick beams. Timoshenko was the first to come up with the idea of including these two terms for thick beam (Eslami, 2019). The governing differential equations in terms of displacements for a uniform cross section of functionally graded beams, undergoing large deflections and subjected to a transverse uniform distributed load is derived including the transverse shear effect.

The strain that occurs due to large deflections in the beam is taken care by carrying the non-linear von-Karman large deflection term in the definition of strains. The derivation of equations starts with assumptions being stated followed by derivation of strain-displacement equations, the equations of equilibrium for a beam, stress-strain relations (constitutive equations) and thereby the resultant force-strain relations. The derivation of
equations of equilibrium is neglected in this section as it remains the same as obtained in the previous section. The force-strain relations are then simultaneously solved for resultant internal force and moment, in terms of beam stiffnesses and displacements and then substituted back in the equations of equilibrium to finally obtain non-linear governing equations in terms of displacements (Deschilder, Eslami, Thivend, & Zhao).

4.2.1. Assumptions

- The beam is much larger in one of the directions than the other two dimensions and the cross section of the beam is uniform.
- The in-plane displacements $u$ and $v$ are insignificant compared to the transverse displacement $w$. However, $w$ is small compared to the beam thickness $h$.
- In plane strains $\varepsilon_x, \varepsilon_z$ and $\gamma_{xz}$ are small compared to unity.
- Transverse shear strain $\gamma_{yz}$ is neglected.
- Tangential displacements or in-plane displacements $u$ and $v$ are linear functions of the $z$ coordinate.
- The transverse normal strain $\varepsilon_z$ is negligible. Therefore, it must be noted that the problem is a plane stress type problem and also that the transverse deflection is a function of position in the $x$ direction only.

$$w = w(x,t)$$

- Shear deformation effect is taken into account in first order shear deformation theory.
- Plane sections remain plane after deformations but a line originally normal to the plane of cross-section does not remain perpendicular.
- The angle of rotation is assumed to be small. Thus, by applying small angle
approximation: (Eslami, 2018)

\[
\sin \theta \approx \theta = \frac{\partial w}{\partial x} \quad ; \quad \cos \theta \approx 1
\]

4.2.2. Shear and Rotary effects added to the beam

Timoshenko Beam theory as mentioned before includes the effect of both shear deformation and rotary inertia. Rotary inertia, however, is not of importance in case of static analysis of beams. A beam undergoing only shear deformation is first considered (Eslami, 2019).

![Figure 4.7 Beam undergoing shear deformation only (Harrevelt, 2012)](image)

A vertical cross section PQ, before deformation remains vertical (P’Q’) after deformation but moves by a distance \( w \) in the \( z \) direction. The components of displacement of a point in the beam are thereby given by:

\[
u = v = 0 \text{ and } w = w(x,t)
\] (4.83)

Components of strain \( \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = \varepsilon_{yz} = 0 \) are thus all zero. However,

\[
\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 + \frac{\partial w}{\partial x}
\] (4.84)

Shear strain \( \gamma_{zx} \) is found to be the same as the rotation \( \theta = \frac{\partial w}{\partial x} \), experienced by any fiber located parallel to the centerline of the beam. (Harrevelt, 2012)
The components of stress $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = 0$ are all found to be zero.

Shear stress $\tau_{xz}$ is thus given by

$$\tau_{xz} = \tau_{zx} = G \frac{\partial w}{\partial x} \quad (4.85)$$

Equation (4.85) states that $\tau_{xz}$ is uniform at every point in the cross section of the beam.

As this is not true in reality, Timoshenko used a constant $k_s$, termed as the shear correction factor. $\tau_{zx}$ thus becomes, (Harrevelt, 2012)

$$\tau_{zx} = k_s G \frac{\partial w}{\partial x} \quad (4.86)$$

---

**Figure 4.8** Beam undergoing different deformations (Harrevelt, 2012)

The beam that undergoes both rotary and shear deformations is as below:
The total transverse displacement of the centerline of the beam is thus given to be:

$$w = w_s + w_b \quad (4.87)$$

Where, $w_s$ is the shear displacement of the centerline

$w_b$ is the bending displacement of the center line

The total slope of the deflected centerline of the beam is thus approximated to be (Harrevelt, 2012).

$$\frac{\partial w}{\partial x} = \frac{\partial w_s}{\partial x} + \frac{\partial w_b}{\partial x} \quad (4.88)$$

Therefore, the slope due to bending from this equation can be expressed as:

$$\frac{\partial w_b}{\partial x} = \frac{\partial w}{\partial x} - \frac{\partial w_s}{\partial x} \quad (4.89)$$

On assuming $\frac{\partial w_b}{\partial x} = \phi_x$; $\frac{\partial w_s}{\partial x} = \theta$

Thus,

$$\phi_x = \frac{\partial w}{\partial x} - \theta \quad (4.90)$$

$\theta$ is in turn equal to shear deformation or shear angle $\gamma_{zx}$ as shown in equation (4.86).

$$\phi_x = \frac{\partial w}{\partial x} - \gamma_{zx} \quad (4.91)$$

It is also to be noted that an element of fiber located at a distance $z$ undergoes axial displacement only due to rotation of cross section as shear deformation does not cause any axial displacements (Harrevelt, 2012).
4.2.2. Strain-Displacement Relations in the beam

Figure 4.10 Geometric assumptions in case of First-order shear deformation theory

The displacements at any point along the thickness of the beam after considering von-Karman geometric nonlinearity and setting all the differentials with respect to y equal to 0 as shown in classical theory (equation (4.18)), thus become,

\[ u = \left[ u_0(x) + \frac{1}{2} (\frac{\partial w}{\partial x})^2 \right] dx - z \phi_x \]  

(4.92)

\[ w = w_0(x) \]  

(4.93)

taking differentiating u with respect to x, the strains become:

\[ \varepsilon_x = \frac{\partial (u_0(x) + \frac{1}{2} (\frac{\partial w}{\partial x})^2)}{\partial x} - z \frac{\partial \phi_x}{\partial x} \]  

(4.94)

\[ \varepsilon_x = \frac{\partial w}{\partial x} = 0 \]  

(4.95)

It is to be noted here that \( \gamma_{xz} \neq 0 \) as shear deformation effect is considered in Timoshenko Beam Theory (Deschilder, Eslami, Thivend, & Zhao).

\[ \gamma_{xz} = \frac{\partial w}{\partial x} - \phi_x \]  

(4.96)

Where, \( \frac{\partial w}{\partial x} \) is the total deformation

\( \phi_x \) is the rotary deformation
The strains at any point in the beam are thus given by:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_z \\
\gamma_{xz}
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} +
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_z^0 \\
\gamma_{xz}^0
\end{bmatrix}
\]

(4.97)

Thus,

\[
\{e^0\} = \begin{bmatrix}
\frac{\partial(u_0(x)+\frac{1}{2}(\frac{\partial w}{\partial x})^2}{\partial x} \\
0 \\
\frac{\partial w}{\partial x} - \phi_x
\end{bmatrix}; \{\kappa\} = -\begin{bmatrix}
\frac{\partial \phi_x}{\partial x} \\
0 \\
0
\end{bmatrix}
\]

(4.98)

(Eslami, 2018) (Gudmundsson, 1995) (Eslami, 2019)

The equations of equilibrium have been previously derived in classical theory and are obtained to be:

\[
\frac{\partial N_x}{\partial x} = 0
\]

\[
\frac{\partial M}{\partial x} - Q_x = 0
\]

\[
N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial q_{xz}}{\partial x} + P(x) = 0
\]

4.2.3. Stress-Strain Relations of the Beam

According to Hooke’s Law,

\[
\sigma = E\varepsilon
\]

(4.99)

Where, \(\sigma\) is the longitudinal stress.

\(E\) is the young’s modulus

\(\varepsilon\) is the longitudinal strain

In case of Functionally Graded Beams,

\[
\sigma_x = E(z) * \varepsilon_x
\]
\[ \sigma_x = E(z) \left[ \frac{\partial \left( u_0(x) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right)}{\partial x} - z \frac{\partial \phi_x}{\partial x} \right] \]  

(4.100)

### 4.2.4. Force-Displacement Relations in the Beam

Substituting equation (4.100) in \( N_x \) and \( M_x \),

\[ N_x = \int_{-h/2}^{h/2} E(z) \left[ \frac{\partial \left( u_0(x) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right)}{\partial x} - z \frac{\partial \phi_x}{\partial x} \right] dz \]  

(4.101)

\[ M_x = \int_{-h/2}^{h/2} \left( E(z) \left[ \frac{\partial \left( u_0(x) + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right)}{\partial x} - z \frac{\partial \phi_x}{\partial x} \right] - z \right) z dz \]  

(4.102)

In matrix form,

\[
\begin{pmatrix}
M_x \\
N_x
\end{pmatrix} = 
\begin{pmatrix}
E_1 & E_2 \\
E_0 & E_1
\end{pmatrix} 
\begin{pmatrix}
\frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
-\frac{\partial \phi_x}{\partial x}
\end{pmatrix}
\]  

(4.103)

Where,

\[ E_0 = \int_{-h/2}^{h/2} E(z) dz \]

\[ E_1 = \int_{-h/2}^{h/2} (E(z) \cdot z) dz \]

\[ E_2 = \int_{-h/2}^{h/2} (E(z) \cdot z^2) dz \] are the beam stiffnesses.

\( N_x \) is replaced with \( N_0 \) (as \( N_0 \) is a constant value and independent of \( x \)),

Let \( \alpha = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \)

; \( \gamma = \frac{\partial \phi_x}{\partial x} \)

Thus,

\[ M_x = E_1 \alpha - E_2 \gamma \]  

(4.105)

\[ N_0 = E_0 \alpha - E_1 \gamma \]  

(4.106)

Equation (4.105) and (4.106) are simultaneously solved for \( M_x \) and is as follows,

\[ M_x = \left[ - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial \phi_x}{\partial x} + N_0 \frac{E_1}{E_0} \right] \]  

(4.107)

We know that shear force is given by,
\[ Q_{xz} = Q_x = \int_{\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \, dz \quad (4.108) \]

From Hooke’s law and by taking into account the shear correction factor \( k_s \), we have,

\[ Q_{xz} = Q_x = k_s \int_{\frac{h}{2}}^{\frac{h}{2}} G(z) \, dz \, \gamma_{xz} \quad (4.109) \]

Where, \( G_0 = \int_{\frac{h}{2}}^{\frac{h}{2}} G(z) \, dz \) is the shear modulus through the thickness.

\[ \gamma_{xz} = \frac{\partial w}{\partial x} - \phi_x \] is the shear strain

Differentiating equation (4.109) with respect to \( x \) and substituting the same in

\[ N_x \frac{\partial^2 w}{\partial x} + \frac{\partial Q_{xz}}{\partial x} + P(x) = 0 \] to obtain \( \frac{\partial \phi_x}{\partial x} \)

\[ \frac{\partial \phi_x}{\partial x} = \left( 1 + \frac{N_0}{k_s G_0} \right) \frac{\partial^2 w}{\partial x^2} + \frac{p}{k_s G_0} \quad (4.110) \]

Differentiating Equation (4.110) again with respect to \( x \),

\[ \frac{\partial^2 \phi_x}{\partial x^2} = \left( 1 + \frac{N_0}{k_s G_0} \right) \frac{\partial^3 w}{\partial x^3} \quad (4.111) \]

On differentiating equation (4.107) with respect to \( x \), we get

\[ \frac{\partial M_x}{\partial x} = \left[ - \frac{E_0 E_2 - E_1^2}{E_0} \right] \frac{\partial^2 \phi_x}{\partial x^2} \quad (4.112) \]

We know that \( Q_x = \frac{\partial M_x}{\partial x} \) and \( Q_x = k_s G_0 \gamma_{xz} \). We thus have

\[ k_s G_0 \frac{\partial w}{\partial x} - k_s G_0 \phi_x = \left( - \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial^2 \phi_x}{\partial x^2} \quad (4.113) \]

On rearranging equation (4.111) to obtain \( \phi_x \), we get,

\[ \phi_x = \left[ \frac{1}{k_s G_0} \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \right] \frac{\partial^2 \phi_x}{\partial x^2} + \frac{\partial w}{\partial x} \quad (4.114) \]

On substituting equation (4.111) in (4.114),

\[ \phi_x = \left[ \frac{1}{k_s G_0} \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) \frac{\partial^3 w}{\partial x^3} \right] + \frac{\partial w}{\partial x} \quad (4.115) \]

Differentiating equation (4.115) with respect to \( x \),
\[
\frac{\partial \phi}{\partial x} = \left[ \frac{1}{k_s G_0} \left( \frac{E_0 E_2 - E_1^2}{E_2} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) \right] \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial x^2} \tag{4.116}
\]

On equating (4.110) and (4.116) and simplifying we get,

\[
\frac{\partial^4 w}{\partial x^4} - \frac{N_0}{\left( \frac{E_0 E_2 - E_1^2}{E_2} \right) \left( 1 + \frac{N_0}{k_s G_0} \right)} \frac{\partial^2 w}{\partial x^2} - \frac{P(x)}{\left( \frac{E_0 E_2 - E_1^2}{E_2} \right) \left( 1 + \frac{N_0}{k_s G_0} \right)} = 0 \tag{4.117}
\]

Let \( \alpha^2 = \frac{N_0}{\left( \frac{E_0 E_2 - E_1^2}{E_2} \right) \left( 1 + \frac{N_0}{k_s G_0} \right)} \); \( \eta = \frac{P(x)}{\left( \frac{E_0 E_2 - E_1^2}{E_2} \right) \left( 1 + \frac{N_0}{k_s G_0} \right)} \)

Equation (4.117) thus becomes, (Deschilder, Eslami, Thivend, & Zhao, n.d.)

\[
\frac{\partial^4 w}{\partial x^4} - \alpha^2 \frac{\partial^2 w}{\partial x^2} = \eta \tag{4.118}
\]

### 4.2.5. Solution to the fourth-order differential equation

(a) Homogenous Solution:

Let us initially assume a solution for transverse deflection

\[ w(x) = ce^{mx} \tag{4.119} \]

On differentiating the above solution four times and substituting the above in (4.118) and the characteristic equation is thus obtained to be,

\[ m^4 - \alpha^2 m^2 = 0 \tag{4.120} \]

The roots are thus, \( m = 0, 0, \alpha \text{ and } -\alpha \).

The homogenous solution thus becomes,

\[ w(x) = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{\alpha x} + c_4 e^{-\alpha x} \tag{4.121} \]

We know that \( e^x = \cosh x + \sinh x \). Substituting the same in (4.121)

\[ w(x) = c_1 + c_2 x + c_3 (\cosh \alpha x + \sinh \alpha x) + c_4 (\cosh \alpha x - \sinh \alpha x) \tag{4.122} \]

On rearranging the above equation,

\[ w(x) = c_1 + c_2 x + (c_3 + c_4) \cosh \alpha x + (c_3 - c_4) \sinh \alpha x \tag{4.123} \]

\[ C_3 := (c_3 + c_4) \]

\[ C_4 := (c_3 - c_4) \]
Also setting,

\[ C_1 = c_1 \]
\[ C_2 = c_2 \]

Equation (4.123) thus becomes,

\[ w(x) = C_1 + C_2 x + C_3 \cosh(\alpha x) + C_4 \sinh(\alpha x) \] (4.124)

Equation (4.124) is the homogenous solution of the fourth-order differential equation.

(b) Particular solution:

A particular solution is assumed of the form

\[ w_p(x) = b x^2 \] (4.125)

Where, \( b \) is a random unknown constant. \( x^2 \) is multiplied to \( b \) in the particular solution to account for \( x \) in the complementary solution.

On substituting equation (4.125) in equation (4.118)

\[ 0 - 2\alpha^2 b_1 = \eta \] (4.126)

On rearranging equation (4.126), \( b \) is obtained to be

\[ b = -\frac{\eta}{2\alpha^2} \] (4.127)

Substituting \( b \) in equation (4.125),

\[ w_p(x) = -\frac{\eta}{2\alpha^2} x^2 \] (4.128)

Transverse Deflection is thus,

\[ w(x) = C_1 + C_2 x + C_3 \cosh(\alpha x) + C_4 \sinh(\alpha x) - \frac{\eta}{2\alpha^2} x^2 \] (4.129)

Where \( C_1, C_2, C_3 \) and \( C_4 \) depend not only on the boundary conditions, but on \( N_0 \) as well.

It is to be noted at this point that the method used to obtain the governing equations is independent of the way the Young’s modulus is estimated (Deschilder, Eslami, Thivend, \& Zhao).
5. NUMERICAL COMPUTATIONS

5.1. Simply-Supported Beam

Consider a simply-supported beam with immovable edges as shown below:

![Simply supported beam](image)

**Figure 5.1 Simply supported beam (Eslami, Chitikela, & Thivend)**

Boundary Conditions of a simply-supported beam are as follows:

\[ w(-a) = w(a) = 0 \]  \hspace{1cm} (5.1)

\[ M(-a) = M(a) = 0 \]  \hspace{1cm} (5.2)

\[ u(-a) = u(a) = 0 \]  \hspace{1cm} (5.3)

5.1.1. Solution of Classical Theory

On substituting boundary condition (5.1) in equation (4.80)

At \( x = -a \); \( w(-a)=0 \)

\[ 0 = C_1 - C_2 a + C_3 \cosh(\zeta a) - C_4 \sinh(\zeta a) - \frac{\varphi}{2\zeta^2} a^2 \]  \hspace{1cm} (5.4)

At \( x = a \); \( w(a)=0 \)

\[ 0 = C_1 + C_2 a + C_3 \cosh(\zeta a) + C_4 \sinh(\zeta a) - \frac{\varphi}{2\zeta^2} a^2 \]  \hspace{1cm} (5.5)

We now differentiate equation (4.80) twice with respect to \( x \),

\[ \frac{\partial^2 w}{\partial x^2} = \zeta^2 C_3 \cosh(\zeta x) + \zeta^2 C_4 \sinh(\zeta x) - \frac{\varphi}{\zeta^2} \]  \hspace{1cm} (5.6)

Substituting equation (5.6) in equation (4.64)
\[ M_x = \left(- \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \right) \left( \zeta^2 C_3 \cosh(\zeta x) + \zeta^2 C_4 \sinh(\zeta x) - \frac{\varphi}{\zeta^2} \right) + N_0 \frac{E_1}{E_0} \] (5.7)

On substituting boundary condition (5.2) in equation (5.7)

At \( x=-a \); \( M(-a)=0 \)

\[ 0 = -\left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \zeta^2 C_3 \cosh(\zeta a) + \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \zeta^2 C_4 \sinh(\zeta x) + \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \frac{\varphi}{\zeta^2} + N_0 \frac{E_1}{E_0} \] (5.8)

At \( x=a \); \( M(a)=0 \)

\[ 0 = -\left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \zeta^2 C_3 \cosh(\zeta a) - \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \zeta^2 C_4 \sinh(\zeta x) + \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \frac{\varphi}{\zeta^2} + N_0 \frac{E_1}{E_0} \] (5.9)

On solving equations (5.8) and (5.9) simultaneously,

\[ C_3 = \frac{P}{N_0 \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \zeta^2 \cosh(\zeta a) + N_0 \frac{E_1}{E_0}} \] (5.10)

\[ C_3 \text{ can thus be obtained to be:} \]

\[ C_3 = \frac{\frac{P}{N_0} + \frac{N_0}{E_0} \frac{E_1}{E_0}}{\zeta^2 \cosh(\zeta a)} \] (5.11)

Substituting equation (5.11) in equation (5.9) gives,

\[ C_4 = 0 \] (5.12)

Substituting equation (5.12) and solving equations (5.4) and (5.5) simultaneously,

\[ 0 = 2C_1 + 2C_3 \cosh(\zeta a) \zeta^2 - \frac{\varphi}{\zeta^2} a^2 \] (5.13)

We thus obtain \( C_1 \) to be,

\[ C_1 = \frac{\varphi}{2\zeta^2} a^2 - C_3 \cosh(\zeta a) \] (5.14)

On substituting \( C_1, C_3 \) and \( C_4 \) in equation (5.5), we get \( C_2 \)

\[ C_2 = 0 \] (5.15)

On substituting all of the above constants back in equation (4.80), we get

\[ w(x) = C_1 + C_3 \cosh(\zeta x) - \frac{\varphi}{2\zeta^2} x^2 \] (5.16)
Differentiating equation (5.16) with respect to \( x \),

\[
\frac{\partial w}{\partial x} = C_3 \zeta \sinh(\zeta x) - \frac{\varphi}{\zeta^2} x \quad (5.17)
\]

Differentiating equation (5.17) again with respect to \( x \),

\[
\frac{\partial^2 w}{\partial x^2} = C_3 \zeta^2 \cosh(\zeta x) - \frac{\varphi}{\zeta^2} \quad (5.18)
\]

Using equation (5.3) in equation (4.82)

\[
\int_{-a}^{a} N_0 dx = E_0 \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - E_1 \int_{-a}^{a} \frac{\partial^2 w}{\partial x^2} dx \quad (5.19)
\]

Substituting equations (5.17) and (5.18) in equation (5.19)

\[
N_0 = \frac{E_0}{4a} \int_{-a}^{a} \left( C_3 \zeta \sinh(\zeta x) - \frac{\varphi}{\zeta^2} x \right)^2 dx - \frac{E_1}{2a} \int_{-a}^{a} \left( C_3 \zeta^2 \cosh(\zeta x) - \frac{\varphi}{\zeta^2} \right) dx \quad (5.20)
\]

It is to be noted here that \( \zeta \) depends on \( N_0 \) and \( \varphi \) depends on \( P \left( \frac{N}{mm^2} \right) \). It is thus obvious that obtaining a closed form expression of \( N_0 \) in terms of \( P \) can be extremely difficult. \( N_0 \) is thus evaluated for discrete values of \( P \) using an iterative process. The iterative process used in this thesis is Newton Raphson Method (Eslami, Chitikela, & Thivend).

**5.1.2. Newton Raphson Method**

Newton Raphson method is a well-known and powerful method. The iterative process starts with assuming an initial estimate \( x_i \) that is quite close to the actual root \( A \) new estimate \( x_{i+1} \) of the actual root is then found using (Liu, 2015)

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\]

This process continues until

\[
|x_{i+1} - x_i| \leq \varepsilon_2 \text{ and/or } |f(x_{i+1})| \leq \varepsilon_2
\]

\( \varepsilon_2 \) is the acceptable error.
Graphical illustration:

Newton’s method can also be obtained from the Taylor series

\[ f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \cdots \]

\[ \text{Figure 5.2 Illustration of Newton's method (Liu, 2015)} \]

Truncating after the first derivative, setting \( f(x_{i+1}) = 0 \) and solving for \( x_{i+1} \) yields

(Liu, 2015):

\[ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \]

5.1.3. Solution of First Order Shear Deformation Theory

Consider equation (4.107)

\[ M_x = \left( -\left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial \phi_x}{\partial x} + N_0 \frac{E_1}{E_0} \right) \]

On substituting equation (4.110) in equation (4.107),

\[ M_x = \left( -\left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) \frac{\partial^2 w}{\partial x^2} + \frac{P}{k_s G_0} \right) + N_0 \frac{E_1}{E_0} \]

Differentiating equation (4.129) twice with respect to \( x \),

\[ \frac{\partial^2 w}{\partial x^2} = C_3 \alpha^2 \cosh(\alpha x) + C_4 \alpha^2 \sinh(\alpha x) - \frac{P}{N_0} \]

On substituting (5.23) in (5.22), we get

\[ M_x = -\left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) \left( C_3 \alpha^2 \cosh(\alpha x) + C_4 \alpha^2 \sinh(\alpha x) - \frac{P}{N_0} \right) - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{P}{k_s G_0} + N_0 \frac{E_1}{E_0} \]
On substituting the boundary condition (5.2) in equation (5.24)

At \(x=-a\) \(M_x=0\)

\[
M_x = -\left(\frac{\delta R - E_0}{E_0}\right) \left(1 + \frac{N_0}{k_0} \frac{P}{N_0}\right) \left(1 + \frac{N_0}{k_0} \frac{P}{N_0}\right) + \frac{p}{k_0} N_0^2 (5.25)
\]

At \(x=a\) \(M_x=0\)

\[
M_x = -\left(\frac{\delta R - E_0}{E_0}\right) \left(1 + \frac{N_0}{k_0} \frac{P}{N_0}\right) \left(1 + \frac{N_0}{k_0} \frac{P}{N_0}\right) + \frac{p}{k_0} N_0^2 (5.26)
\]

Solving equations (5.25) and (5.26) simultaneously and obtaining \(C_3\)

\[
C_3 = \frac{\beta P + E_1 N_0}{N_0 \cosh(\alpha a)} (5.27)
\]

Substituting \(C_3\) in equation (5.26), \(C_4\) is obtained to be

\[
C_4 = 0 (5.28)
\]

We now apply boundary conditions (5.1) in equation (4.129)

At \(x=-a\) \(w=0\)

\[
0 = C_1 - C_2 a + C_3 \cosh(\alpha a) - C_4 \sinh(\alpha a) - \frac{pa^2}{2N_0} (5.29)
\]

At \(x=a\) \(w=0\)

\[
0 = C_1 + C_2 a + C_3 \cosh(\alpha a) + C_4 \sinh(\alpha a) - \frac{pa^2}{2N_0} (5.30)
\]

On solving equations (5.29) and (5.30) simultaneously,

\[
C_1 = \frac{pa^2}{2N_0} - C_3 \cosh(\alpha a) (5.31)
\]

Substituting \(C_1, C_3\) and \(C_4\) in equation (5.29)

\[
C_2 = 0 (5.32)
\]

Equation (4.129) thereby becomes,
\[ w(x) = C_1 + C_3 \cosh(\alpha x) - \frac{\eta}{2\alpha^2} x^2 \] (5.33)

Now,

\[ N_0 = E_0 \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) - E_1 \frac{\partial \phi_x}{\partial x} \] (5.34)

Multiply by \( dx \) and integrating over the length,

\[ N_0 dx = E_0 \int_{-a}^{a} \frac{\partial u_0}{\partial x} dx + \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - E_1 \int_{-a}^{a} \frac{\partial \phi_x}{\partial x} dx \] (5.35)

Noting \( u(-a)=u(a)=0 \) and that \( N_0 \) is a constant (as \( N_0 \) is independent of \( x \)).

\[ N_0(2a) = E_0(u(a) - u(-a)) + \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - E_1 \left( 1 + \frac{N_0}{k_sG_0} \right) \int_{-a}^{a} \frac{\partial^2 w}{\partial x^2} dx - E_1 \frac{p}{k_sG_0} \] (5.36)

Differentiating equation (5.33) with respect to \( x \),

\[ \frac{\partial w}{\partial x} = C_3 \alpha \sinh(\alpha x) - \frac{\eta}{\alpha^2} x \] (5.38)

Differentiating equation (5.38) again with respect to \( x \),

\[ \frac{\partial^2 w}{\partial x^2} = C_3 \alpha^2 \cosh(\alpha x) - \frac{\eta}{\alpha^2} \] (5.39)

Substituting equation (5.38) and (5.39) in equation (5.37), (Deschilder, Eslami, Thivend, & Zhao).

\[ N_0 = \frac{E_0}{4a} \int_{-a}^{a} \left( C_3 \alpha \sinh(\alpha x) - \frac{\eta}{\alpha^2} x \right)^2 dx - \frac{E_1}{2a} \left( 1 + \frac{N_0}{k_sG_0} \right) \int_{-a}^{a} \frac{\partial^2 w}{\partial x^2} dx - E_1 \frac{p}{k_sG_0} \] (5.40)

5.2. Clamped-Clamped Beams

Consider a clamped-clamped beam with immovable edges as shown below:
Boundary Conditions of a clamped-clamped beam are as follows:

\[ w(-a) = w(a) = 0 \]  \hspace{1cm} (5.41)

\[ w_x(-a) = w_x(a) = 0 \]  \hspace{1cm} (5.42)

\[ u(-a) = u(a) = 0 \]  \hspace{1cm} (5.43)

### 5.2.1. Solution of Classical Theory

On substituting boundary condition (5.41) in equation (4.80)

At \( x=-a; w(-a)=0 \)

\[ 0 = C_1 - C_2 a + C_3 \cosh(\zeta a) - C_4 \sinh(\zeta a) - \frac{\varphi}{2\xi^2} a^2 \]  \hspace{1cm} (5.44)

At \( x=a; w(a)=0 \)

\[ 0 = C_1 + C_2 a + C_3 \cosh(\zeta a) + C_4 \sinh(\zeta a) - \frac{\varphi}{2\xi^2} a^2 \]  \hspace{1cm} (5.45)

We now differentiate equation (4.80) with respect to \( x \),

\[ \frac{\partial w}{\partial x} = C_2 + \zeta C_3 \sinh(\zeta x) + \zeta C_4 \cosh(\zeta x) - \frac{\varphi}{\xi^2} x \]  \hspace{1cm} (5.46)

On substituting boundary condition (5.42) in equation (5.46)

At \( x=-a; w_x(-a)=0 \)

\[ 0 = C_2 - \zeta C_3 \sinh(\zeta a) + \zeta C_4 \cosh(\zeta a) + \frac{\varphi}{\xi^2} a \]  \hspace{1cm} (5.47)

At \( x=a; w_x(a)=0 \)
\[ 0 = C_2 + \zeta C_3 \sinh(\zeta a) + \zeta C_4 \cosh(\zeta a) - \frac{\varphi}{\zeta^2} a \]  

(5.48)

Solving equations (5.47) and (5.48) simultaneously,

\[ C_2 = -\zeta C_4 \cosh(\zeta a) \]  

(5.49)

Substituting equation (5.49) in equation (5.47) gives \( C_3 \)

\[ C_3 = \frac{\varphi}{\zeta^2} \frac{a}{\sinh(\zeta a)} = \frac{p}{\zeta N_0 a} \]  

(5.50)

Solving equations (5.44) and (5.45) simultaneously to obtain \( C_1 \)

\[ C_1 = \frac{p a^2}{2 N_0} - C_3 \cosh(\zeta a) \]  

(5.51)

Substituting equation (5.50) in (5.51)

\[ C_1 = \frac{p a^2}{2 N_0} - \frac{p}{\zeta N_0} a \coth(\zeta a) \]  

(5.52)

Substituting \( C_1, C_3 \) and \( C_2 \) in equation (5.44) gives \( C_4 \),

\[ C_4 = 0 \]  

(5.53)

Therefore, from equation (5.49)

\[ C_2 = 0 \]  

(5.54)

Equation (4.80) thus becomes,

\[ w(x) = C_1 + C_3 \cosh(\zeta x) - \frac{p}{2 N_0} x^2 \]  

(5.55)

Differentiating equation (5.55) with respect to \( x \),

\[ \frac{\partial w}{\partial x} = C_3 \zeta \sinh(\zeta x) - \frac{\varphi}{\zeta^2} x \]  

(5.56)

Differentiating equation (5.56) again with respect to \( x \),

\[ \frac{\partial^2 w}{\partial x^2} = C_3 \zeta^2 \cosh(\zeta x) - \frac{\varphi}{\zeta^2} \]  

(5.57)

Using equation (5.3) in equation (4.82)
\[ \int_{-a}^{a} N_0 \, dx = \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 \, dx - E_1 \int_{-a}^{a} \frac{\partial^2 w}{\partial x^2} \, dx \]  
(5.58)

Substituting equations (5.56) and (5.57) in equation (5.58)

\[ N_0 = \frac{E_0}{4a} \int_{-a}^{a} \left( C_3 \zeta \sinh(\zeta x) - \frac{\varphi}{\xi^2} x \right)^2 \, dx - \frac{E_1}{2a} \int_{-a}^{a} \left( C_3 \zeta \sinh(\zeta x) - \frac{\varphi}{\xi^2} x \right) \, dx \]  
(5.59)

\( N_0 \) is evaluated for discrete values of \( P \) using Newton Raphson Method (Eslami, Chitikela, & Thivend).

### 5.2.2. Solution of First Order Shear Deformation Theory

On substituting boundary condition (5.41) in equation (4.129)

At \( x = -a \); \( w(-a) = 0 \)

\[ 0 = C_1 - C_2 a + C_3 \cosh(aa) - C_4 \sinh(aa) - \frac{\eta}{2a^2} a^2 \]  
(5.60)

At \( x = a \); \( w(a) = 0 \)

\[ 0 = C_1 + C_2 a + C_3 \cosh(aa) + C_4 \sinh(aa) - \frac{\eta}{2a^2} a^2 \]  
(5.61)

We now differentiate equation (4.129) with respect to \( x \),

\[ \frac{\partial w}{\partial x} = C_2 + \alpha C_3 \sinh(ax) + \alpha C_4 \cosh(ax) - \frac{\eta}{a^2} x \]  
(5.62)

On substituting boundary condition (5.42) in equation (5.62)

At \( x = -a \); \( w_x(-a) = 0 \)

\[ 0 = C_2 - \alpha C_3 \sinh(aa) + \alpha C_4 \cosh(aa) + \frac{\eta}{a^2} a \]  
(5.63)

At \( x = a \); \( w_x(a) = 0 \)

\[ 0 = C_2 + \alpha C_3 \sinh(aa) + \alpha C_4 \cosh(aa) - \frac{\eta}{a^2} a \]  
(5.64)

On solving equations (5.63) and (5.64) simultaneously,

\[ C_2 = -\alpha C_4 \cosh(aa) \]  
(5.65)

Substituting equation (5.65) in equation (5.63) gives \( C_3 \)
\[ C_3 = \frac{\eta \alpha^2}{\alpha \sinh(\alpha a)} = \frac{p}{N_0 \alpha} \]  

(5.66)

Solving equations (5.63) and (5.64) simultaneously to obtain \( C_1 \)

\[ C_1 = \frac{pa^2}{2N_0} - C_3 \cosh(\alpha a) \]  

(5.67)

Substituting equation (5.66) in (5.67)

\[ C_1 = \frac{pa^2}{2N_0} - \frac{p}{\alpha N_0} a \coth(\alpha a) \]  

(5.68)

Substituting \( C_1, C_3 \) and \( C_2 \) in equation (5.63) gives \( C_4 \),

\[ C_4 = 0 \]  

(5.69)

Therefore, from equation (5.65)

\[ C_2 = 0 \]  

(5.70)

Equation (4.129) thus becomes,

\[ w(x) = C_1 + C_3 \cosh(\alpha x) - \frac{p}{2N_0} x^2 \]  

(5.71)

Differentiating equation (5.71) with respect to \( x \),

\[ \frac{\partial w}{\partial x} = C_3 \alpha \sinh(\alpha x) - \frac{\eta}{\alpha^2} x \]  

(5.72)

Differentiating equation (5.72) again with respect to \( x \),

\[ \frac{\partial^2 w}{\partial x^2} = C_3 \alpha^2 \cosh(\alpha x) - \frac{\eta}{\alpha^2} \]  

(5.73)

Now,

\[ N_0 = \left[ E_0 \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) - E_1 \frac{\partial \phi_x}{\partial x} \right] \]  

(5.74)

Multiply by \( dx \) and integrating over the length,

\[ N_0 dx = \left[ E_0 \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) dx - E_1 \frac{\partial \phi_x}{\partial x} dx \right] \]

\[ \int_{-a}^{a} N_0 dx = E_0 \int_{-a}^{a} \frac{\partial u_0}{\partial x} dx + \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - E_1 \int_{-a}^{a} \frac{\partial \phi_x}{\partial x} dx \]  

(5.75)
Noting \( u(-a) = u(a) = 0 \) and that \( N_0 \) is a constant (as \( N_0 \) is independent of \( x \)).

\[
N_0(2a) = E_0(u(a) - u(-a)) + \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \, dx - E_1 \left( 1 + \frac{N_0}{k_c g_0} \right) \int_{-a}^{a} \frac{\partial^2 w}{\partial x} \, dx - E_1 \left( 1 + \frac{N_0}{k_c g_0} \right)
\]

(5.76)

Substituting equation (5.72) and (5.73) in equation (5.77), (Deschilder, Eslami, Thivend, 
& Zhao, n.d.).

\[
N_0 = \frac{E_0}{4a} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 \, dx - \frac{E_1}{2a} \left( 1 + \frac{N_0}{k_c g_0} \right) \int_{-a}^{a} \frac{\partial^2 w}{\partial x^2} \, dx - E_1 \left( 1 + \frac{N_0}{k_c g_0} \right)
\]

(5.77)

5.3. Elastic support boundary conditions

\[
N_0 = \frac{E_0}{4a} \int_{-a}^{a} \left( C_3 \alpha \sinh(\alpha x) - \frac{n}{a^2} x \right)^2 \, dx - \frac{E_1}{2a} \left( 1 + \frac{N_0}{k_c g_0} \right) \int_{-a}^{a} \left( C_3 \alpha^2 \cosh(\alpha x) - \frac{n}{a} x \right) \, dx - E_1 \left( 1 + \frac{N_0}{k_c g_0} \right)
\]

(5.78)

Figure 5.4 spring boundary conditions considered in this thesis

The first step here, is to draw the free body diagram (FBD) at the boundaries of the beam end. Figure (5.4) shows the position of the left end of the beam before loading and its assumed position after loading. When springs are part of the boundary conditions, it is always essential to assume positive deflection and slopes irrespective of what the actual deflection and slopes are expected, in response to the applied loadings (Donaldson, 2008).

Thus, based on the coordinate system considered here, the left end of the beam is assumed to move downward and thus, to have a positive lateral deflection. The direction of the positive bending slope is as shown in Figure (5.5).
Bending Slope Sign Convention:

The sign conventions associated with the derivatives of a deflection function in turn depend on the sign conventions of both the deflection function and the positive direction of spatial coordinate used to form these derivatives. Hence, the sign convention of $w'(x)$ depends on both the sign convention of $w(x)$ and the selected positive direction of $x$. In order to determine the positive direction of $w'(x)$ in our case, let the coordinate $x$ be positive to the right and the lateral deflection $w$ be positive downward.

The positive values of $dw$ and $dx$ are thus also positive downwards and to the right, respectively. The positive bending slope is then obtained by putting these differential quantities together in vector form according to their positive directions as shown in the below figure. (Donaldson, 2008)

![Figure 5.5 Positive directions for deflection and slope](Donaldson, 2008)

It can thus, be determined from this figure that the slope $\frac{dw}{dx}$ is positive in the clockwise direction. The left end of the beam is thus, assumed to rotate clockwise and move downwards. The Free Body Diagram of the left end of the beam for a differential element $dx$ is drawn as follows:
Figure 5.6 Free body diagram of the left end of the beam (Donaldson, 2008)

In this figure the sign conventions for the force resultants $V_z$ or $Q_x$ and moment resultant $M_x$ are depicted. Internal force resultant $V_z$, according to these conventions, is assumed to move the right hand face downward with respect to the left hand face for it to be positive and the internal moment $M_x$ is assumed to compress the upper part of the beam end and elongate the lower part for it to be positive (Gere, 2003). It is to be noted, here, that it is always necessary to use positive directions for internal force $V_z$ and moment $M_x$ stress resultants.

In case of the springs, the downward movement of the beam causes the translational spring with spring constant $k_t$ to resist the movement by generating a spring force that acts upwards. The clockwise rotation of the beam causes the rotational spring with spring constant $K_r$ to resist the rotation by developing a spring force in the counter-clockwise direction. (Donaldson, 2008)

Static sign conventions are used when writing equations of equilibrium in this section and these conventions in turn depend on the direction of the coordinate axes (Gere, 2003). It is also to be noted that, since the distributed force per unit length acts on only a
differential length at this beam, the forces and moments created by this load is negligible and hence neglected while writing the differential equations of equilibrium (Donaldson, 2008).

Accordingly, setting \( \sum F_z = 0 \) with \( \downarrow = +ve \) and \( \uparrow = +ve \)

\[-k_t w(-a) + V_z(-a) = 0 \quad (5.79)\]

Therefore,

\[V_z(-a) = k_t w(-a) \quad (5.80)\]

On setting \( \sum M_x = 0 \)

\[M_x(-a) + K_r w'(-a) = 0 \quad (5.81)\]

\[M_x(-a) = -K_r w'(-a) \quad (5.82)\]

The free body diagram of the right end of the beam is considered next, and equilibrium equations are obtained by following the discussions made while working with the left end of the beam. It is also to be noted that the internal shear force \( V_z \) will now be acting on the left face of the differential element taken at the right end and in the upwards direction and the internal moment \( M_x \) will also be acting on the left face of the differential element and in the clockwise direction. These conventions are in line of keeping these internal stress resultants positive.
Therefore,
\[ V_z(a) = -k_t w(a) \]  \hspace{1cm} (5.84)

On setting \( \Sigma M_x = 0 \)
\[ -M_x(a) + K_r w'(a) = 0 \]  \hspace{1cm} (5.85)
\[ M_x(a) = K_r w'(a) \]  \hspace{1cm} (5.86)

**5.3.1. Solution of Classical Theory**

We know from equation (4.64)
\[ M_x = -\left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial^2 w}{\partial x^2} + N_0 \frac{E_1}{E_0} \]  \hspace{1cm} (5.87)

We know that,
\[ \frac{\partial M}{\partial x} = Q_x \]

On differentiating equation (5.87)
\[ Q_x = \frac{\partial M}{\partial x} = -\left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial^3 w}{\partial x^3} \]  \hspace{1cm} (5.88)

At \( x=-a \), equating equation (5.88) to boundary condition in (5.80)
\[ k_t w(-a) = -\left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \frac{\partial^3 w}{\partial x^3} \]  
(5.89)

On differentiating transverse deflection in equation (4.80) thrice with respect to \( x \) and substituting both (4.80) and \( \frac{\partial^3 w}{\partial x^3} \), we get,

\[ k_t (C_1 - C_2a + C_3 \cosh(\zeta a) - C_4 \sinh(\zeta a) - \frac{\varphi}{2\zeta^2} a^2) = -\left(\frac{E_0 E_2 - E_1^2}{E_0}\right) (-C_3 \zeta^3 \sinh(\zeta a) + C_4 \zeta^3 \cosh(\zeta a)) \]  
(5.90)

On expanding equation (5.90), we have,

\( k_t (C_1 - C_2a + C_3 \cosh(\zeta a) - C_4 \sinh(\zeta a) - \frac{\varphi}{2\zeta^2} k_t a^2) = \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) C_3 \zeta^3 \sinh(\zeta a) - \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) C_4 \zeta^3 \cosh(\zeta a) \)  
(5.91)

At \( x=a \), equating equation (5.88) to boundary condition in (5.84)

\[ -k_t w(a) = -\left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \frac{\partial^3 w}{\partial x^3} \]  
(5.92)

On differentiating transverse deflection in equation (4.80) thrice with respect to \( x \) and substituting both (4.80) and \( \frac{\partial^3 w}{\partial x^3} \), we get,

\[ -k_t (C_1 + C_2a + C_3 \cosh(\zeta a) + C_4 \sinh(\zeta a) - \frac{\varphi}{2\zeta^2} a^2) = \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) (C_3 \zeta^3 \sinh(\zeta a) + C_4 \zeta^3 \cosh(\zeta a)) \]  
(5.93)

\[ k_t C_1 + k_t C_2a + k_t C_3 \cosh(\zeta a) + k_t C_4 \sinh(\zeta a) - \frac{\varphi}{2\zeta^2} k_t a^2 = \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) (C_3 \zeta^3 \sinh(\zeta a) + \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) C_4 \zeta^3 \cosh(\zeta a) \)  
(5.94)

On solving equations (5.91) and (5.94) simultaneously we get,

\[ 2k_t C_1 + 2k_t C_3 \cosh(\zeta a) - \frac{\varphi}{\zeta^2} k_t a^2 = 2 \left(\frac{E_0 E_2 - E_1^2}{E_0}\right) C_3 \zeta^3 \sinh(\zeta a) \]  
(5.95)

On simplifying equation (5.95) further, we obtain \( C_1 \)

\[ C_1 = \frac{C_3}{k_t} \left(\left(\frac{E_0 E_2 - E_1^2}{E_0}\right) \zeta^3 \sinh(\zeta a) - k_t \cosh(\zeta a) \right) + \frac{p a^2}{2N_0} \]  
(5.96)

We now substitute \( x=-a \) and equate equations (5.82) and (5.87)
\[-K_r w'(-a) = - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial^2 w}{\partial x^2} + N_0 \frac{E_1}{E_0} \quad (5.97)\]

On differentiating transverse deflection in equation (4.80) and substituting in equation (5.97), we get
\[-K_r \left( C_2 - \zeta C_3 \sinh(\zeta a) + \zeta C_4 \cosh(\zeta a) + \frac{\varphi}{\zeta} a \right) = - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( \zeta^2 C_3 \cosh(\zeta a) - \zeta^2 C_4 \sinh(\zeta a) - \frac{\varphi}{\zeta} \right) + N_0 \frac{E_1}{E_0} \quad (5.98)\]

On simplifying equation (5.98) by expanding, we obtain,
\[
\left( K_r C_2 - K_r \zeta C_3 \sinh(\zeta a) + K_r \zeta C_4 \cosh(\zeta a) + \frac{\varphi}{\zeta} K_r a \right) = \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( \zeta^2 C_3 \cosh(\zeta a) - \frac{\varphi \zeta}{E_0} \right) \theta C_4 \sinh(\zeta a) - \frac{\varphi}{\zeta} + N_0 \frac{E_1}{E_0} \quad (5.99)\]

Rearranging equation (5.99)
\[K_r C_2 = C_3 \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \zeta^2 \cosh(\zeta a) + K_r \zeta \sinh(\zeta a) \right) - C_4 \left( K_r \zeta \cosh(\zeta a) + \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \zeta^2 \sinh(\zeta a) - \frac{\varphi}{\zeta} \right) + \theta C_4 \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \zeta^2 \cosh(\zeta a) + N_0 \frac{E_1}{E_0} \quad (5.100)\]

We now substitute \( x = a \) and equate equations (5.86) and (5.87)
\[K_r w'(a) = - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial^2 w}{\partial x^2} + N_0 \frac{E_1}{E_0} \quad (5.101)\]

On differentiating transverse deflection in equation (4.80) and substituting in equation (5.100), we get
\[K_r \left( C_2 + \zeta C_3 \sinh(\zeta a) + \zeta C_4 \cosh(\zeta a) - \frac{\varphi}{\zeta} a \right) = - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( \zeta^2 C_3 \cosh(\zeta a) + \zeta^2 C_4 \sinh(\zeta a) - \frac{\varphi}{\zeta} \right) + N_0 \frac{E_1}{E_0} \quad (5.102)\]

On simplifying equation (5.101) we get,
\[K_r C_2 + K_r \zeta C_3 \sinh(\zeta a) + K_r \zeta C_4 \cosh(\zeta a) - K_r \frac{\varphi}{\zeta} a = - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \zeta^2 C_3 \cosh(\zeta a) - \frac{\varphi \zeta}{E_0} \right) \theta C_4 \sinh(\zeta a) - \frac{\varphi}{\zeta} + N_0 \frac{E_1}{E_0} \quad (5.102)\]

Rearranging the same gives,
\[K_r C_2 = C_3 \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \zeta^2 \cosh(\zeta a) - K_r \zeta \sinh(\zeta a) \right) + C_4 \left( K_r \zeta \cosh(\zeta a) + \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \zeta^2 \sinh(\zeta a) - \frac{\varphi}{\zeta} \right) + \theta C_4 \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \zeta^2 \cosh(\zeta a) + K_r a + N_0 \frac{E_1}{E_0} \quad (5.103)\]

Equating equations (5.99) and (5.103), \( C_3 \) is obtained.
\[ C_3 = \frac{P}{N_0} \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) + K_r a + N_0 E_0 ( \frac{E_0 E_2 - E_1^2}{E_0} ) \zeta^2 \cosh(\zeta a) + K_r \zeta \sinh(\zeta a) \]  

(5.104)

On substituting equations (5.104) and (5.96) in equation (5.93), \( C_2 \) is obtained

\[ C_2 = \frac{C_4 \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \zeta^3 \cosh(\zeta a) - K_r \sinh(\zeta a) }{k_4 a} \]  

(5.105)

On substituting \( C_2 \) and \( C_3 \) in equation (5.103), \( C_4 \) is obtained to be

\[ C_4 = 0 \]  

(5.106)

Therefore,

\[ C_2 = 0 \]

The transverse deflection, after applying the constants is obtained from equation (4.80) as

\[ w(x) = C_1 + C_3 \cosh(\zeta x) - \frac{P}{2N_0} x^2 \]  

(5.107)

Differentiating equation (5.107) with respect to \( x \),

\[ \frac{\partial w}{\partial x} = C_3 \zeta \sinh(\zeta x) - \frac{\varphi}{\zeta^2} x \]  

(5.108)

Differentiating equation (5.107) again with respect to \( x \),

\[ \frac{\partial^2 w}{\partial x^2} = C_3 \zeta^2 \cosh(\zeta x) - \frac{\varphi}{\zeta^2} \]  

(5.109)

Using equation (5.3) in equation (4.82)

\[ \int_{-a}^{a} N_0 dx = \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - E_1 \int_{-a}^{a} \frac{\partial^2 w}{\partial x^2} dx \]  

(5.110)

Substituting equations (5.108) and (5.109) in equation (5.110)

\[ N_0 = \frac{E_0}{4a} \int_{-a}^{a} \left( C_3 \zeta \sinh(\zeta x) - \frac{\varphi}{\zeta^2} x \right)^2 dx - \frac{E_1}{2a} \int_{-a}^{a} \left( C_3 \zeta^2 \cosh(\zeta x) - \frac{\varphi}{\zeta^2} \right) dx \]  

(5.111)

\( N_0 \) is evaluated for discrete values of \( P \) using Newton Raphson Method (Eslami, Chitikela, & Thivend). MatLab is used for this numerical computation.
5.3.2. Solution of First Order Shear Deformation Theory

Consider equation (4.107)

\[ M_x = \left( -\left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial \phi_x}{\partial x} + N_0 \frac{E_1}{E_0} \right) \]  \hspace{1cm} (5.112)

On substituting equation (4.110) in equation (4.107),

\[ M_x = \left( -\left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_1 G_0} \right) \frac{\partial^2 w}{\partial x^2} + \frac{P}{k_2 g_0} \right) + N_0 \frac{E_1}{E_0} \]  \hspace{1cm} (5.113)

Differentiating equation (4.129) twice with respect to x,

\[ \frac{\partial^2 w}{\partial x^2} = C_3 \alpha^2 \cosh(\alpha x) + C_4 \alpha^2 \sinh(\alpha x) - \frac{P}{N_0} \]  \hspace{1cm} (5.114)

On substituting (5.114) in (5.113), we get

\[ M_x = -\left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_1 G_0} \right) (C_3 \alpha^2 \cosh(\alpha x) + C_4 \alpha^2 \sinh(\alpha x) - \frac{P}{N_0}) - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{P}{k_2 g_0} + N_0 \frac{E_1}{E_0} \]  \hspace{1cm} (5.115)

Equating (5.115) and boundary condition (5.82) and substituting \( x = -a \)

\[-K_1 w'(-a) = -\left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_1 G_0} \right) (C_3 \alpha^2 \cosh(\alpha x) + C_4 \alpha^2 \sinh(\alpha x) - \frac{P}{N_0}) - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{P}{k_2 g_0} + N_0 \frac{E_1}{E_0} \]  \hspace{1cm} (5.116)

Simplifying equation (5.116), we have

\[ K_1 w'(-a) = \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_1 G_0} \right) C_3 \alpha^2 \cosh(\alpha x) + \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_1 G_0} \right) C_4 \alpha^2 \sinh(\alpha x) - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{P}{k_2 g_0} + N_0 \frac{E_1}{E_0} \]  \hspace{1cm} (5.117)

\[-K_1 \left( C_2 + \alpha C_3 \sinh(\alpha x) + \alpha C_4 \cosh(\alpha x) + \frac{P}{N_0} \right) = \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_1 G_0} \right) C_3 \alpha^2 \cosh(\alpha x) - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{P}{k_2 g_0} + N_0 \frac{E_1}{E_0} \]  \hspace{1cm} (5.118)

Equating (5.115) and boundary condition (5.86) and substituting \( x = a \)

\[ K_1 \left( C_2 + \alpha C_3 \sinh(\alpha x) + \alpha C_4 \cosh(\alpha x) - \frac{P}{N_0} \right) = -\left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_1 G_0} \right) C_3 \alpha^2 \cosh(\alpha x) - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{P}{k_2 g_0} + N_0 \frac{E_1}{E_0} \]  \hspace{1cm} (5.119)
On equating equations (5.118) and (5.119), $C_3$ is obtained to be,

$$
C_3 = \frac{\rho}{N_0} \left( K_r a + \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \right) + N_0 E_1
$$

(5.120)

We know that,

$$
\frac{\partial M}{\partial x} = Q_x
$$

On differentiating equation (5.112) with respect to $x$

$$
Q_x = \frac{\partial M}{\partial x} = - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial^2 \phi}{\partial x^2}
$$

(5.121)

Differentiating equation (4.110) with respect to $x$ and substituting the same in equation (5.121)

$$
Q_x = - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) \frac{\partial^3 \phi}{\partial x^3}
$$

(5.122)

Differentiating equation (5.114) again with respect to $x$ and substituting the same in equation (5.122).

$$
Q_x = - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) \left( C_3 a^3 \sinh(\alpha x) + C_4 a^3 \cosh(\alpha x) \right)
$$

(5.123)

Equating equation (5.123) to boundary condition (5.80) and substituting $x = -a$

$$
k_t w(-a) = - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) \left( -C_3 a^3 \sinh(\alpha a) + C_4 a^3 \cosh(\alpha a) \right)
$$

(5.124)

$$
k_t \left( C_1 - C_2 a + C_3 \cosh(\alpha a) - C_4 \sinh(\alpha a) - \frac{n}{2a^2} \right) = \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) \left( C_3 a^3 \sinh(\alpha a) - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) C_4 a^3 \cosh(\alpha a) \right)
$$

(5.125)

Equating equation (5.123) to boundary condition (5.84) and substituting $x = a$

$$
-k_t \left( C_1 + C_2 a + C_3 \cosh(\alpha a) + C_4 \sinh(\alpha a) - \frac{n}{2a^2} \right) = - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) \left( C_3 a^3 \sinh(\alpha a) - \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) C_4 a^3 \cosh(\alpha a) \right)
$$

(5.126)
Simplifying equation (5.126), we get,

\[ k_t C_1 + k_t C_2 a + k_t C_3 \cosh(\alpha a) + k_t C_4 \sinh(\alpha a) - \frac{k_t}{a^2} a^2 = \left( \frac{E_0 E_2 - E_1}{E_0} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) C_3 a^3 \sinh(\alpha a) + \left( \frac{E_0 E_2 - E_1}{E_0} \right) C_4 a^3 \cosh(\alpha a) \]

(5.127)

On solving equations (5.127) and (5.125) simultaneously, we get,

\[ 2k_t C_1 + 2k_t C_3 \cosh(\alpha a) - \frac{k_t}{a^2} a^2 = 2 \left( \frac{E_0 E_2 - E_1}{E_0} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) C_3 a^3 \sinh(\alpha a) \]

(5.128)

\[ C_1 = \frac{\eta}{2a^2} a^2 + \frac{C_3}{k_t} \left( \frac{E_0 E_2 - E_1}{E_0} \right) \left( 1 + \frac{N_0}{k_s G_0} \right) a^3 \sinh(\alpha a) - k_t \cosh(\alpha a) \]

(5.129)

Substitute \( C_1, C_3 \) in equation (5.127), we get

\[ C_2 = \frac{C_4 \left( \frac{E_0 E_2 - E_1}{E_0} \right) a^3 \cosh(\alpha a) - k_t \sinh(\alpha a) \}}{k_t a} \]

(5.130)

On substituting \( C_2 \) and \( C_3 \) in equation (5.119), \( C_4 \) is obtained to be

\[ C_4 = 0 \]

(5.131)

Therefore,

\[ C_2 = 0 \]

The transverse deflection, after applying the constants is obtained from equation (4.82) as

\[ w(x) = C_1 + C_3 \cosh(\alpha x) - \frac{p}{2N_0} x^2 \]

(5.132)

Now,

\[ N_0 = E_0 \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) - E_1 \frac{\partial \phi_x}{\partial x} \]

(5.133)

Multiply by \( dx \) and integrating over the length,

\[ N_0 dx = E_0 \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) dx - E_1 \frac{\partial \phi_x}{\partial x} dx \]

\[ \int_{-a}^{a} N_0 dx = E_0 \int_{-a}^{a} \frac{\partial u_0}{\partial x} dx + \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - E_1 \int_{-a}^{a} \frac{\partial \phi_x}{\partial x} dx \]

(5.134)

Noting \( u(-a) = u(a) = 0 \) and that \( N_0 \) is a constant (as \( N_0 \) is independent of \( x \)),

\[ N_0(2a) = E_0 (u(a) - u(-a)) + \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - E_1 * \left( 1 + \frac{N_0}{k_s G_0} \right) * \int_{-a}^{a} \frac{\partial \phi_x}{\partial x} dx - E_1 * \frac{p}{k_s G_0} * 2a \]

(5.135)
\[ N_0 = \frac{E_0}{4a} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - \frac{E_1}{2a} \left( 1 + \frac{N_0}{k_c e_0} \right) \int_{-a}^{a} \frac{\partial^2 w}{\partial x^2} dx - E_1 \frac{P}{k_c e_0} \]  \hspace{1cm} (5.136)

Differentiating equation (5.132) with respect to \( x \),

\[ \frac{\partial w}{\partial x} = C_3 \alpha \sinh(\alpha x) - \frac{\eta}{\alpha^2} x \]  \hspace{1cm} (5.137)

Differentiating equation (5.132) again with respect to \( x \),

\[ \frac{\partial^2 w}{\partial x^2} = C_3 \alpha^2 \cosh(\alpha x) - \frac{\eta}{\alpha^2} \]  \hspace{1cm} (5.138)

Substituting equation (5.137) and (5.138) in equation (5.136)

\[ N_0 = \frac{E_0}{4a} \int_{-a}^{a} \left( C_3 \alpha \sinh(\alpha x) - \frac{\eta}{\alpha^2} x \right)^2 dx - \frac{E_1}{2a} \left( 1 + \frac{N_0}{k_c e_0} \right) \int_{-a}^{a} C_3 \alpha^2 \cosh(\alpha x) - \frac{\eta}{\alpha^2} dx - E_1 \frac{P}{k_c e_0} \]  \hspace{1cm} (5.139)

\( N_0 \) is evaluated for discrete values of \( P \) using Newton Raphson Method (Deschilder, Eslami, Thivend, & Zhao). MatLab is used for this numerical computation.
6. RESULTS

The geometric and material properties used in this thesis to validate the analysis are given as: (Eslami, Chitikela, & Thivend)

Table 6.1
Geometric properties of the beam

<table>
<thead>
<tr>
<th>Properties</th>
<th>Metal</th>
<th>Ceramic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (L)</td>
<td>500 mm</td>
<td></td>
</tr>
<tr>
<td>Thickness (h)</td>
<td>1.5 mm</td>
<td></td>
</tr>
<tr>
<td>Width (w)</td>
<td>25.4 mm</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.2
Material properties of the beam

<table>
<thead>
<tr>
<th>Properties</th>
<th>Metal</th>
<th>Ceramic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young’s modulus (E)</td>
<td>70 GPa</td>
<td>380 GPa</td>
</tr>
<tr>
<td>Shear modulus (G)</td>
<td>26.7 GPa</td>
<td>146 GPa</td>
</tr>
<tr>
<td>Poisson’s Ratio ((\nu))</td>
<td>0.31</td>
<td>0.30</td>
</tr>
<tr>
<td>Thermal Expansion Coefficient ((\alpha))</td>
<td>23*10^{-6} K^{-1}</td>
<td>7.4*10^{-6} K^{-1} (M. Darvizeh, A. Darvizeh, Ansari, &amp; Alijani, 2015)</td>
</tr>
</tbody>
</table>

6.1. Simply Supported Beam (SS beams)

This section deals with discussing the results obtained on applying Classical Theory and First Order Shear Deformation Theory to a functionally graded simply supported beam undergoing large deflections when subjected to a uniformly distributed mechanical transverse pressure load. Rule of Mixtures and Mori Tanaka Material Scheme, as
mentioned in Chapter 3 are both used to model the beam for each of the above mentioned theories. MatLab software is used to perform all numerical computations and obtain the results.

6.1.1. Solutions obtained on applying Classical Theory to SS beams

The results obtained by applying classical theory and using both material models are discussed below. Graphs obtained by modeling using rule of mixtures is first discussed. The below graph displays variation of in-plane tensile force resultant \( N_0 \left( \frac{N}{\text{mm}} \right) \) for discrete values of uniformly distributed mechanical transverse pressure load \( P \left( \frac{N}{\text{mm}^2} \right) \) when rule of mixtures method is used to model the functionally graded beam.

![Graph of in-plane stress resultant for different values of k of a SS beam using rule of mixtures](image)

*Figure 6.1* In-plane stress resultant for different values of k of a SS beam using rule of mixtures

The next graph displays the deflections of a simply supported beam for different values of gradation power index \( k \) over the length of the beam under 0.01 \( \frac{N}{\text{mm}^2} \), when rule of mixtures is used to model the beam.
The maximum deflection values for different values of $k$, when rule of mixtures is used are as mentioned below.

**Table 6.3**

*Maximum deflection values of simply supported beam for different ‘k’ using rule of mixtures*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=0$</td>
<td>3.69300150248</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>4.00614452354</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.25425448229</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.65822457864</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.40007743245</td>
</tr>
</tbody>
</table>

Graphs obtained by modeling using Mori-Tanaka method is discussed next. Figure 6.3
displays variation of in-plane tensile force resultant $N_0 \left( \frac{N}{mm} \right)$ for discrete values of pressure $P \left( \frac{N}{mm^2} \right)$ when Mori-Tanaka method is used to model the functionally graded beam.

![Figure 6.3 In-plane stress resultant for different values of k of a simply supported beam using Mori-Tanaka Method](image)

The next graph displays the deflections of a simply supported beam for different values of gradation power index $k$ over the length of the beam under $0.01 \frac{N}{mm^2}$.

![Figure 6.4 Deflections over the length of SS beam for different values of k using Mori-Tanaka method under 0.01 $\frac{N}{mm^2}$](image)
The maximum deflection values for different values of $k$, when Mori-Tanaka method is used are as mentioned below

Table 6.4

*Maximum deflection values of simply supported beam for different values of $k$ using Mori-Tanaka method*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm)</th>
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<tbody>
<tr>
<td>$k=0$</td>
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</tr>
<tr>
<td>$k=0.5$</td>
<td>4.34184079557</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.68755297539</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.99321846335</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.45947222424</td>
</tr>
</tbody>
</table>

The following table portrays a comparison between the maximum deflection results obtained using the two different material models for different values of $k$ under $0.01 \frac{N}{mm^2}$.

Table 6.5

*Comparison of $w_{\text{max}}$ results of simply supported beam for different values of $k$ using rule of mixtures and Mori-Tanaka method under $0.01 \frac{N}{mm^2}$.*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm)</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>RULE OF MIXTURES</td>
<td>MORI-TANAKA METHOD</td>
</tr>
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<td>3.69432592814</td>
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</tr>
<tr>
<td>$k=1$</td>
<td>4.25425448229</td>
<td>4.68755297539</td>
</tr>
<tr>
<td>Gradation Power Index $k$</td>
<td>Maximum Deflection $w_{max}$ (mm)</td>
<td>Maximum Deflection $w_{max}$ (mm)</td>
</tr>
<tr>
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<td>----------------------------------</td>
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<tr>
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<td>RULE OF MIXTURES</td>
<td>MORI-TANAKA METHOD</td>
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<tr>
<td>$k=99$</td>
<td>6.40007743245</td>
<td>6.45947222424</td>
</tr>
</tbody>
</table>

6.1.2. Solutions obtained on applying FSDT to simply supported beams

The results obtained by applying first order shear deformation theory and using both material models are discussed below. Graphs obtained by modeling using rule of mixtures is first discussed. The following graph displays variation of in-plane tensile force resultant $N_0 \left( \frac{N}{mm} \right)$ for discrete values of uniformly distributed pressure load $P \left( \frac{N}{mm^2} \right)$ when first order shear deformation theory is applied, and rule of mixtures method is used to model the functionally graded beam.

![Graph showing in-plane stress resultant for different values of k of a simply supported beam](image)

*Figure 6.5 In-plane stress resultant for different values of k of a simply supported beam by applying first order shear deformation theory and using rule of mixtures*
Figure 6.6 displays the deflections of a simply supported beam for different values of gradation power index $k$ over the length of the beam under $0.01 \frac{N}{mm^2}$.

The maximum deflection values for different values of $k$, when rule of mixtures is used are as mentioned below.

Table 6.6

Maximum deflection values of simply supported beam for different values of $k$ using rule of mixtures and applying FSDT

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=0$</td>
<td>3.69299907676</td>
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<tr>
<td>$k=0.5$</td>
<td>4.00615225677</td>
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<tr>
<td>$k=1$</td>
<td>4.25428244004</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.65824916978</td>
</tr>
</tbody>
</table>
### Gradation Power Index $k$ vs. Maximum Deflection $w_{\text{max}}$ (mm)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$w_{\text{max}}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>99</td>
<td>6.4007349696</td>
</tr>
</tbody>
</table>

Graphs obtained by modeling using Mori-Tanaka method is discussed next. The below graph displays variation of in-plane tensile force resultant $N_0 \left( \frac{N}{\text{mm}} \right)$ for discrete values of uniformly distributed pressure load $P \left( \frac{N}{\text{mm}^2} \right)$ when Mori-Tanaka method is used to model the functionally graded beam.

![Graph of In-plane stress resultant for different values of k](image1)

**Figure 6.7** In-plane stress resultant for different values of $k$ of a simply supported beam applying FSDT and using Mori-Tanaka Method

The following graph displays the deflections of a simply supported beam for different values of gradation power index $k$ over the length of the beam under $0.01 \frac{N}{\text{mm}^2}$.

![Graph of Deflections of SS beam for different ‘k’](image2)

**Figure 6.8** Deflections of SS beam for different ‘$k$’ using FSDT and Mori-Tanaka method under $0.01 \frac{N}{\text{mm}^2}$
The maximum deflection values for different values of $k$, using Mori-Tanaka method are as mentioned below.

Table 6.7

Maximum deflection values of SS beam for different values of $k$ using Mori-Tanaka method and FSDT

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
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</thead>
<tbody>
<tr>
<td>$k=0$</td>
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<tr>
<td>$k=0.5$</td>
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<tr>
<td>$k=1$</td>
<td>4.68774019825</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.99322574633</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.45946764880</td>
</tr>
</tbody>
</table>

The following table portrays a comparison between the maximum deflection results obtained using the two different material models for different values of $k$ under $0.01 \frac{N}{mm^2}$.

Table 6.8

Comparison of $w_{max}$ results of SS beam for different values of $k$ using rule of mixtures and Mori-Tanaka method by applying FSDT under $0.01 \frac{N}{mm^2}$.

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>RULE OF MIXTURES</td>
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<td>4.34185727231</td>
</tr>
<tr>
<td>Gradation Power Index $k$</td>
<td>Maximum Deflection $w_{\text{max}}$ (mm)</td>
<td>Maximum Deflection $w_{\text{max}}$ (mm)</td>
</tr>
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<td>------------------------------------------</td>
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<tr>
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<td>4.68774019825</td>
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<td>$k=99$</td>
<td>6.40007349696</td>
<td>6.45946764880</td>
</tr>
</tbody>
</table>

6.2. Clamped-Clamped Beams (CC Beams)

This section deals with discussing the results obtained on applying Classical Theory and First Order Shear Deformation Theory to a functionally graded clamped-clamped beam undergoing large deflections when subjected to a uniformly distributed mechanical transverse pressure load.

Rule of Mixtures and Mori Tanaka Material Scheme, as mentioned in Chapter 3 are both used to analyze the beam for each of the above mentioned theories.

6.2.1. Solutions obtained on applying Classical Theory to CC beams

The results obtained by applying classical theory and using both material models are discussed below. Graphs obtained by modeling using rule of mixtures is first discussed.

The below graph displays variation of in-plane tensile force resultant $N_0 \left( \frac{N}{mm} \right)$ for discrete values of pressure $P \left( \frac{N}{mm^2} \right)$ when the rule of mixtures method is used to study the functionally graded beam.
The next graph displays the deflections of a clamped-clamped beam for different values of gradation power index $k$ over the length of the beam under $0.01 \dfrac{N}{mm^2}$.

**Figure 6.9** In-plane stress resultant for different values of $k$ of a clamped-clamped beam using rule of mixtures

**Figure 6.10** Deflections seen over the length of clamped-clamped beam for different values of $k$ using rule of mixtures under $0.01 \dfrac{N}{mm^2}$. 
The maximum deflection values for different values of $k$, when rule of mixtures is used are as mentioned below.

Table 6.9

*Deflections seen over the length of clamped-clamped beam for different values of $k$ using rule of mixtures under $0.01 \frac{N}{mm^2}$.*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=0$</td>
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<td>$k=0.5$</td>
<td>3.79251641244</td>
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<tr>
<td>$k=1$</td>
<td>4.10785942048</td>
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<tr>
<td>$k=10$</td>
<td>5.47425530160</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.11158542334</td>
</tr>
</tbody>
</table>

Graphs obtained by modeling using Mori-Tanaka method is discussed next. The below graph displays variation of in-plane tensile force resultant $N_0 \left( \frac{N}{mm} \right)$ for discrete values of pressure $P \left( \frac{N}{mm^2} \right)$ when Mori-Tanaka method is used to model the functionally graded beam.

*Figure 6.11* In-plane stress resultant for different values of $k$ of a clamped-clamped beam using Mori-Tanaka Method to model the beam and by applying classical theory.
The next graph displays the deflections of a clamped-clamped beam for different values of gradation power index \( k \) over the length of the beam under 0.01 \( \frac{N}{mm^2} \).

**Figure 6.12** Deflections seen over the length of clamped-clamped beam for different values of \( k \) by applying classical theory and using Mori-Tanaka method under 0.01 \( \frac{N}{mm^2} \).

The maximum deflection values for different values of \( k \), when Mori-Tanaka method is used are as mentioned below.

Table 6.10

*Deflections seen over the length of clamped-clamped beam for different values of \( k \) using Mori-Tanaka method under 0.01 \( \frac{N}{mm^2} \)*

<table>
<thead>
<tr>
<th>Gradation Power Index ( k )</th>
<th>Maximum Deflection ( w_{max} ) (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k=0 )</td>
<td>3.33063109886</td>
</tr>
<tr>
<td>( k=0.5 )</td>
<td>4.18112781026</td>
</tr>
<tr>
<td>( k=1 )</td>
<td>4.55944328420</td>
</tr>
<tr>
<td>( k=10 )</td>
<td>5.75923112801</td>
</tr>
<tr>
<td>( k=99 )</td>
<td>6.16319623133</td>
</tr>
</tbody>
</table>

The following table portrays a comparison between the maximum deflection results...
obtained using the two different material models for different values of $k$ under $0.01 \frac{N}{mm^2}$.

Table 6.11

*Comparison of $w_{\text{max}}$ results of clamped-clamped beam for different values of $k$ using rule of mixtures and Mori-Tanaka method under $0.01 \frac{N}{mm^2}$*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm)</th>
<th>Maximum Deflection $w_{\text{max}}$(mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RULE OF MIXTURES</td>
<td>MORI-TANAKA METHOD</td>
</tr>
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<td>3.79251641244</td>
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<tr>
<td>$k=1$</td>
<td>4.10785942048</td>
<td>4.55944328420</td>
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<td>5.75923112801</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.11158542334</td>
<td>6.16319623133</td>
</tr>
</tbody>
</table>

6.2.2. Solutions obtained on applying FSDT to CC beams

The results obtained by applying first order shear deformation theory and using both material models are discussed below. Graphs obtained by modeling using rule of mixtures is first discussed. The following graph displays variation of in-plane tensile force resultant $N_0 \left( \frac{N}{mm} \right)$ for discrete values of pressure $P \left( \frac{N}{mm^2} \right)$ when first order shear deformation theory is applied and rule of mixtures method is used to model the functionally graded beam.
Figure 6.13 In-plane stress resultant for different values of k of a clamped-clamped beam using rule of mixtures and applying FSDT

The next graph displays the deflections of a clamped-clamped beam for different values of gradation power index k over the length of the beam under 0.01 $\frac{N}{mm^2}$.

Figure 6.14 Deflections seen over the length of clamped-clamped beam for different values of k by applying FSDT and using rule-of-mixtures method under 0.01 $\frac{N}{mm^2}$.
The maximum deflection values for different values of $k$, when rule of mixtures is used are as mentioned below.

Table 6.12

*Maximum deflection values of clamped-clamped beam for different values of $k$ using rule of mixtures and applying FSDT.*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm)</th>
</tr>
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<tbody>
<tr>
<td>$k=0$</td>
<td>3.32915424058</td>
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</table>

Graphs obtained by modeling using Mori-Tanaka method is discussed next. The below graph displays variation of in-plane tensile force resultant $N_0 \left( \frac{N}{\text{mm}} \right)$ for discrete values of uniformly distributed pressure load $P \left( \frac{N}{\text{mm}^2} \right)$ when Mori Tanaka method is used to model the functionally graded beam.

*Figure 6.15*  In-plane stress resultant for different values of $k$ of a clamped-clamped beam applying FSDT and using Mori-Tanaka Method.
The next graph displays the deflections of a clamped-clamped beam for different values of gradation power index $k$ over the length of the beam under $0.01 \frac{N}{mm^2}$.

![Graph showing deflections of a clamped-clamped beam for different values of $k$](image)

*Figure 6.16* Deflections seen over the length of clamped-clamped beam for different values of $k$ by applying FSDT theory and using Mori-Tanaka method under $0.01 \frac{N}{mm^2}$.

The maximum deflection values for different values of $k$, when Mori-Tanaka method is used are as mentioned below

Table 6.13

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=0$</td>
<td>3.33053180441</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>4.18113015362</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.55948225966</td>
</tr>
</tbody>
</table>
Gradation Power Index $k$ & Maximum Deflection $w_{max}$ (mm) \\
\hline
$k=10$ & 5.75948123258 \\
$k=99$ & 6.16298447163 \\
\hline

The following table portrays a comparison between the maximum deflection of the beam obtained using the two different material models for different values of $k$ subjected to $0.01 \frac{N}{mm^2}$.

**Table 6.14**

*Comparison of $w_{max}$ results of clamped-clamped beam for different values of $k$ using rule of mixtures and Mori-Tanaka method by applying FSDT under 0.01 $\frac{N}{mm^2}$.*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
<th>Maximum Deflection $w_{max}$(mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td><strong>RULE OF MIXTURES</strong></td>
<td><strong>MORI-TANAKA METHOD</strong></td>
</tr>
<tr>
<td>$k=0$</td>
<td>3.32915424058</td>
<td>3.33053180441</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>3.79246158320</td>
<td>4.18113015362</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.10787629235</td>
<td>4.55948225966</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.47428530992</td>
<td>5.75948123258</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.11138796965</td>
<td>6.16298447163</td>
</tr>
</tbody>
</table>

**6.3. Spring constraint boundary conditions**

This section deals with discussing the results obtained by applying Classical Theory and First Order Shear Deformation Theory to a functionally graded beam with spring boundary conditions, undergoing large deflections when subjected to a uniformly
distributed mechanical transverse pressure load. Rule of Mixtures and Mori Tanaka Material Scheme, as mentioned in Chapter 3 are both used for the analysis of the beam for each of the above mentioned theories.

6.3.1. Solution to an FG beam with springs by Classical Theory-SS beam

The results obtained by applying classical theory and using both material models are discussed below. As mentioned before, since conventional boundary conditions are often difficult to replicate, unconventional boundaries like springs are also utilized here, to be able to change the fixity of the edges and maybe replace the conventional boundaries.

A MatLab code was thus built in this regard. The translational spring constant $k_t$ was set to a very high value to replicate infinity and rotational spring constant $K_r$ was set to a very small value of 0.0000001. The boundaries are now thus similar to simply supported beams. Graphs obtained by modeling using rule of mixtures is first discussed. The maximum deflections values obtained are exactly same as the simply supported beams. It can thus be concluded that it is possible to replicate the simply supported beam using springs.

Table 6.15

*Comparison of results between springs and s-s beam for rule of mixtures*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>RULE OF MIXTURES</strong></td>
<td></td>
</tr>
<tr>
<td>Conventional Boundary-Simply Supported beam</td>
<td>Maximum Deflection $w_{max}$ (mm)</td>
</tr>
<tr>
<td>$k=0$</td>
<td>3.69300150248</td>
</tr>
</tbody>
</table>

For springs:

- $k_t = 10^25 \, \frac{N}{mm^2}$
- $K_r = 0.0001 \, \frac{N}{mm \, rad}$

<table>
<thead>
<tr>
<th>Maximum Deflection $w_{max}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Springs</td>
</tr>
<tr>
<td>Gradation Power Index ( k )</td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>( k=0.5 )</td>
</tr>
<tr>
<td>( k=1 )</td>
</tr>
<tr>
<td>( k=10 )</td>
</tr>
<tr>
<td>( k=99 )</td>
</tr>
</tbody>
</table>

A graph of the variation of maximum deflection values for different values of gradation parameter \( k \) of a functionally graded beam attached to spring boundary conditions is as shown below.

*Figure 6.17* Maximum deflection values vs gradation parameter \( k \) for a FG beam with springs that replicate simply supported FG beam.

The beam is next modeled using Mori-Tanaka material scheme and governing
equations are obtained using classical theory. The maximum deflection values obtained using simply-supported boundary conditions are exactly the same as using springs with $k_t \to \infty$ and $K_r \to 0$. It can thus, be concluded that it is possible to replicate the simply supported beam using springs.

Table 6.16

*Comparison of results between springs and s-s beam for Mori-Tanaka method*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm) MORI-TANAKA Conventional Boundary-Simply Supported beam</th>
<th>Maximum Deflection $w_{max}$ (mm) MORI-TANAKA-Springs $k_t = 10^25 \frac{N}{mm^2}$; $K_r = 0.00001 \frac{Nmm}{rad}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=0$</td>
<td>3.69432592814</td>
<td>3.69432592814</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>4.34184075957</td>
<td>4.34184075957</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.6875527539</td>
<td>4.6875527539</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.9932184635</td>
<td>5.9932184635</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.45947222424</td>
<td>6.45947222424</td>
</tr>
</tbody>
</table>

A graph of the variation of max. deflection values for different values of gradation parameter $k$ of a functionally graded beam attached to spring boundary conditions is as shown in the following graph.
Figure 6.18 Maximum deflection values vs gradation parameter k for an FG beam modeled using Mori-Tanaka method and with spring boundary conditions that replicate simply supported beam

6.3.2. Application of FSDT to FG Beams with Springs-SS beam replicate

The beam is modeled using both material models and first order shear deformation theory is applied. The analysis here is exactly the same as above except that the beam here is analyzed using first, the rule of mixtures and then using Mori-Tanaka method along with the first order shear deformation theory is considered to derive the governing equations. As seen before, the maximum deflection values match the values obtained using simply supported beams.

The following table shows a comparison of maximum deflection values between a simply supported FG beam and FG beam with springs, modeled with rule of mixtures and using FSDT theory.

Table 6.17

Comparison of results between springs and SS beam for rule of mixtures and FSDT theory
A graph of the variation of maximum deflection values for different values of gradation parameter $k$ of a functionally graded beam attached to spring boundary conditions is as shown in the following graph.

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional Boundary-Simply Supported beam</td>
<td>RULE OF MIXTURES</td>
</tr>
<tr>
<td>$k=0$</td>
<td>3.69299907676</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>4.00615225677</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.25428244004</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.658249169786</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.40007349696</td>
</tr>
</tbody>
</table>

Springs

\[
\begin{align*}
k_t &= 10^{25} \frac{N}{mm^2} \\
K_r &= 0.00001 \frac{N mm}{rad}
\end{align*}
\]
The beam is next modeled using Mori-Tanaka method and first order shear deformation theory is considered to derive the governing equations. As seen before, the maximum deflection values match the values obtained using simply supported beams.

Table 6.18
*Comparison of results between springs and s-s beam for Mori Tanaka and FSDT theory*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
<th>$k_t = 10^{25} \frac{N}{mm^2}$</th>
<th>$k_r = 0.00001 \frac{Nm}{rad}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mori-Tanaka</td>
<td>Simply Supported beam</td>
<td></td>
</tr>
<tr>
<td>$k=0$</td>
<td>3.69432350575</td>
<td>3.69432350575</td>
<td></td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>4.34185727231</td>
<td>4.34185727231</td>
<td></td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.68774019825</td>
<td>4.68774019825</td>
<td></td>
</tr>
</tbody>
</table>
Gradation Power Index $k$

<table>
<thead>
<tr>
<th>Maximum Deflection $w_{max}$ (mm)</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mori-Tanaka Simply Supported beam</td>
<td>Springs</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.99322574633</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.45946764880</td>
</tr>
</tbody>
</table>

A graph of the variation of maximum deflection values for different values of gradation parameter $k$ of a functionally graded beam attached to spring boundary conditions and modeled using Mori-Tanaka is as shown in the following figure.

![Graph of the variation of maximum deflection values](image)

*Figure 6.20* Maximum deflection values vs gradation parameter $k$ for a FG beam with springs and modeled using Mori-Tanaka and FSDT theory, replicating SS beams.

### 6.3.3. Solutions to a FG beam with springs by Classical Theory-CC beam

The results obtained by applying classical theory and using both material models are discussed below. As mentioned before, conventional boundary conditions may be
difficult to replicate, unconventional boundaries like springs can be used to be at the liberty of changing the fixities at the boundaries.

A MatLab code was thus built in this regard. The translational spring constant $k_t$ and rotational spring constant $K_r$ are both set to a very high value ($k_t \to \infty$ and $K_r \to \infty$). The boundaries are now similar to clamped-clamped beams. Rule of mixtures is first used to model the FG beam. The maximum deflections values obtained are exactly same as the clamped-clamped beams. It can thus, be concluded that it is possible to replicate the clamped-clamped beam using springs.

Table 6.19

*Comparison of results between springs and c-c beam for rule of mixtures*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RULE OF MIXTURES</td>
<td>Springs</td>
</tr>
<tr>
<td></td>
<td>Conventional Boundary-Clamped Clamped beam</td>
<td>$k_t = 10^{25} \frac{N}{mm^2}$</td>
</tr>
<tr>
<td>$k=0$</td>
<td>3.32925358896</td>
<td>3.32925358896</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>3.79251641244</td>
<td>3.79251641244</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.10785942048</td>
<td>4.10785942048</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.47425530160</td>
<td>5.47425530160</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.11158542334</td>
<td>6.11158542334</td>
</tr>
</tbody>
</table>

A graph of the variation of maximum deflection values for different values of gradation parameter $k$ of a functionally graded beam attached to spring boundary conditions is as shown below.
Figure 6.21 Maximum deflection values vs gradation parameter $k$ for a FG beam with springs replicating clamped-clamped beams and modeled using rule of mixtures.

The beam is next modeled using Mori-Tanaka material scheme and governing equations are obtained using classical theory. The maximum deflections values obtained are exactly same as the springs. It can therefore, be concluded that it is possible to replicate the clamped-clamped beam using springs.

Table 6.20

Comparison of results between springs and CC beam for Mori-Tanaka method

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm) MORI-TANAKA METHOD Conventional Boundary-Clamped-Clamped beam</th>
<th>Maximum Deflection $w_{max}$ (mm) MORI-TANAKA METHOD Springs $k_t = 10^25 \frac{N}{mm^2}$ $k_r = 10^25 \frac{Nmm}{rad}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=0$</td>
<td>3.33063109886</td>
<td>3.33063109886</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>4.18112781026</td>
<td>4.18112781026</td>
</tr>
<tr>
<td>Gradation Power Index $k$</td>
<td>Maximum Deflection $w_{max}$ (mm) MORI-TANAKA METHOD</td>
<td>Maximum Deflection $w_{max}$ (mm) MORI-TANAKA METHOD</td>
</tr>
<tr>
<td>---------------------------</td>
<td>------------------------------------------------------</td>
<td>------------------------------------------------------</td>
</tr>
<tr>
<td></td>
<td>Conventional Boundary-Clamped-Clamped beam</td>
<td>Springs</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$k_t = 10^{25} \frac{N}{mm^2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$K_r = 10^{25} \frac{Nmm}{rad}$</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.55944328420</td>
<td>4.55944328420</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.75923112801</td>
<td>5.75923112801</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.16319623133</td>
<td>6.16319623133</td>
</tr>
</tbody>
</table>

A graph of the variation of maximum deflection values for different values of gradation parameter $k$ of a functionally graded beam attached to spring boundary conditions is as shown as follows:

*Figure 6.22 Maximum deflection values vs gradation parameter $k$ for an FG beam modeled using Mori-Tanaka method and with spring boundary conditions replicating CC beams*
6.3.4. Application of FSDT to FG beams with springs-CC beam replicate

The results obtained by applying first order shear deformation theory and using both material models are discussed below. The analysis here, is exactly the same as above except that the beam is modeled using the rule of mixtures and the first order shear deformation theory is considered to obtain the governing equations. Rule of mixtures is first used to model the FG beam. As seen before, the maximum deflection values match the values obtained in conventional clamped-clamped beams.

Table 6.21

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
<th>Rule of Mixtures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Conventional Boundary-Clamped Clamped Beams</td>
</tr>
<tr>
<td>$k=0$</td>
<td>3.32915424058</td>
<td>3.32915424058</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>3.79246158320</td>
<td>3.79246158320</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.10787629235</td>
<td>4.10787629235</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.658249169786</td>
<td>5.658249169786</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.11138796965</td>
<td>6.11138796965</td>
</tr>
</tbody>
</table>

A graph of the variation of maximum deflection values for different values of gradation parameter $k$ of a functionally graded beam attached to spring boundary conditions is as shown in the following graph.
Figure 6.23 Maximum deflection values vs gradation parameter $k$ for a FG beam with springs and modeled using rule of mixtures and FSDT theory replicating CC beams

The beam is next modeled using Mori-Tanaka method and first order shear deformation theory is considered to derive the governing equations. As seen before, the maximum deflection values match the values obtained using clamped-clamped beams.

Table 6.22

Comparison of results between springs and c-c beam for Mori-Tanaka and FSDT theory

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm)</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mori-Tanaka</td>
<td>Springs</td>
</tr>
<tr>
<td></td>
<td>Conventional Boundary-clamped</td>
<td>$k_t = 10^25 \frac{N}{\text{mm}^2}$</td>
</tr>
<tr>
<td></td>
<td>clamped beam</td>
<td>$k_r = 10^25 \frac{N\text{mm}}{\text{rad}}$</td>
</tr>
<tr>
<td>$k=0$</td>
<td>3.33053180441</td>
<td>3.33053180441</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>4.18113015362</td>
<td>4.18113015362</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.55948225966</td>
<td>4.55948225966</td>
</tr>
</tbody>
</table>
A graph of the variation of maximum deflection values for different values of gradation parameter $k$ of a functionally graded beam attached to spring boundary conditions is as shown in the following figure.

![Graph showing variation of maximum deflection values](image)

Figure 6.24 Maximum deflection values vs gradation parameter $k$ for a FG beam with springs and modeled using Mori-Tanaka and FSDT theory

6.4. **Validation of Results with ABAQUS**

Results obtained using rule of mixtures are compared with the FE solutions, in turn obtained using a commercially available FE software named ABAQUS. The functionally
graded beam is modeled using an 8-node doubly curved shell element (S8R). The material properties gradation of the rectangular FGM is modeled by applying a stepwise approximation. The thickness is divided into 160 layers and the stiffness matrix for each specific layer is calculated and assigned at its centroid.

Material modeling is done using Rule of mixtures approach and ABAQUS results are obtained for three cases: k=0, k=1 and k=9999999. The ABAQUS results obtained in case of a simply supported ceramic (isotropic, k=0) beam is as shown in the following graph. The material properties are assigned per Table 6.2

Material properties of the beam.

![Graph showing ABAQUS results](image)

*Figure 6.25* ABAQUS results obtained for a simply supported isotropic ceramic beam (k=0)

The ABAQUS results obtained in case of a simply supported metal (isotropic, k=99999) beam is as shown below. Material properties are given in Table 6.2.
The ABAQUS results obtained in case of a simply supported functionally graded beam using rule of mixtures theory and keeping power index gradation parameter $k=1$ is as shown below.

The ABAQUS results obtained in case of a clamped-clamped ceramic (isotropic, $k=0$) beam is as shown below. The material properties are assigned as per Table 6.2

Material properties of the beam
The ABAQUS results obtained in case of a clamped-clamped metal (isotropic, \( k=999999 \)) beam is as shown here. The material properties are given in Table 6.2 as well.

The ABAQUS results obtained in case of a clamped-clamped functionally graded beam using rule of mixtures theory and keeping power index gradation parameter \( k=1 \) is as shown in the following
6.4.1. Comparison of the analytical results to ABAQUS results

In this section, results obtained analytically in a simply supported beam which is analyzed using rule of mixtures and classical theory, for a few gradation patterns is compared to the FE results. A table incorporating the results and percentage difference is shown here:

Table 6.23
Comparison of analytical and FEM results for simply supported FG beam

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm) RULE OF MIXTURES Simply-Supported Beam</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm) FE Results Simply-Supported Beam</th>
<th>Percentage Difference $%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=0$</td>
<td>3.693</td>
<td>3.694</td>
<td>0.027</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.254</td>
<td>4.255</td>
<td>0.001</td>
</tr>
</tbody>
</table>
### Gradation Power

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
<th>Percentage Difference %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RULE OF MIXTURES Simply-Supported Beam</td>
<td>FE Results Simply-Supported Beam</td>
<td></td>
</tr>
<tr>
<td>$k=999999$</td>
<td>6.527</td>
<td>6.527</td>
<td>0</td>
</tr>
</tbody>
</table>

The same table set up for a clamped-clamped again using rule of mixtures and classical theory, for a few gradation patterns is compared to the FE results.

Table 6.24

**Comparison of analytical and FEM results for clamped-clamped FG beam**

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
<th>Percentage Difference %</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RULE OF MIXTURES Clamped-clamped Beam</td>
<td>FE Results Clamped-clamped beam</td>
<td></td>
</tr>
<tr>
<td>$k=0$</td>
<td>3.329</td>
<td>3.325</td>
<td>0.12</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.108</td>
<td>4.103</td>
<td>0.12</td>
</tr>
<tr>
<td>$k=999999$</td>
<td>6.221</td>
<td>6.208</td>
<td>0.209</td>
</tr>
</tbody>
</table>

As can be seen in this table, there is a good agreement between the analytical results and the Finite Element results.
7. THERMAL EFFECTS

In this section, a constant temperature is added along with the uniformly distributed mechanical transverse pressure loads, to the functionally graded beam. The boundary conditions taken are clamped-clamped with immovable edges. Analysis of the functionally graded beam is done using both rule of mixtures and Mori – Tanaka models. Classical theory and First order shear deformation theory are both used to derive the governing equations.

7.1. Calculation of the critical buckling temperature for Classical Theory

The purpose of this thesis is to study the effects of temperature on the bending of FG beams before reaching the critical buckling temperature. Therefore, critical buckling is calculated first to avoid setting a temperature beyond the critical buckling temperature. If by any chance the temperature is set above the critical buckling temperature, the beam deflection falls into post-buckling region which can be studied in the future. The analytical model used for non-linear bending analysis is extended in this section to introduce an axial thermal load only (Thivend, Eslami, & Deschilder).

The total strain acting on a beam when subjected to both mechanical and thermal loads is given by

\[ \varepsilon_x^T = \varepsilon_x^M + \varepsilon_x^{TH} \]  

(7.1)

Where,

- \( \varepsilon_x^T \) is the total strain acting on the midplane
- \( \varepsilon_x^M \) is the mechanical strain
- \( \varepsilon_x^{TH} \) is the thermal strain
Total strain is known to be,

$$\varepsilon_x^T = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2}$$

(7.2)

Mechanical strain is thus found to be,

$$\varepsilon_x^T - \varepsilon_x^{TH} = \varepsilon_x^M$$

(7.3)

In which, $\varepsilon_x^{TH}$ is further given to be,

$$\varepsilon_x^{TH} = \alpha(z)[T - T_0]$$

(7.4)

The mechanical strain thus, will be,

$$\varepsilon_x^M = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} - \alpha(z)[T - T_0]$$

(7.5)

From Hooke’s Law,

$$\sigma_x = E(z) \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} - \alpha(z)[T - T_0] \right)$$

(7.6)

On multiplying by $dz$ and integrating over the thickness we have,

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, dz = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \right) - \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \frac{\partial^2 w}{\partial x^2} z - \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \alpha(z) \Delta T(z)$$

(7.7)

From equation (4.38) and considering,

$$E_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \, dz$$

(7.8)

$$E_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} (E(z)z) \, dz$$

(7.9)

$$N^T = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \alpha(z) \Delta T(z)$$

(7.10)

Equation (7.7) becomes,
\[ N_x = E_0 \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) - E_1 \frac{\partial^2 w}{\partial x^2} - N_T \]  

(7.11)

On multiplying equation \( \sigma x = E(z) \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) - z \frac{\partial^2 w}{\partial x^2} - \alpha(z) [T - T_0] \) (7.6) by \( z dz \) and integrating over the thickness we have

\[ M_x = E_1 \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) - E_2 \frac{\partial^2 w}{\partial x^2} - M^T \]  

(7.12)

Where,

\[ E_2 = \int_{-h}^{h} (E(z) \cdot z^2) dz \]  

(7.13)

On putting equations (7.11) and (7.12) in matrix form,

\[ \begin{pmatrix} M_x \\ N_x \end{pmatrix} = \begin{bmatrix} E_1 & E_2 \\ E_0 & E_1 \end{bmatrix} \begin{pmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\ -\frac{\partial^2 w}{\partial x^2} \end{pmatrix} - \begin{pmatrix} M^T \\ N^T \end{pmatrix} \]  

(7.14)

\( N_x \) is replaced with \( N_0 \) (as \( N_0 \) is a constant value and independent of \( x \)),

Let \( \alpha = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \); \( \kappa_x = -\frac{\partial^2 w}{\partial x^2} \)  

(7.15)

Thus,

\[ M_x = E_1 \alpha - E_2 \gamma - M^T \]  

(7.16)

\[ N_0 = E_0 \alpha - E_1 \gamma - N^T \]  

(7.17)

Equation (7.16) and (7.17) are simultaneously solved for \( M_x \) and is as follows,

\[ M_x = \left[ \left( \frac{E_0 E_2 + E_1^2}{E_0} \right) \frac{\partial^2 w}{\partial x^2} + (N_0 + N^T) \frac{E_1}{E_0} - M^T \right] \]  

(7.18)

We know that,

\[ \frac{\partial M}{\partial x} = Q_x \]  

(7.19)

On differentiating equation (4.65) Pg. 29
\[ Q_x = \frac{\partial M}{\partial x} = \left( \frac{-E_0 E_2 + E_1^2}{E_0} \right) \frac{\partial^3 w}{\partial x^3} \]  

(7.20)

Differentiating (7.20) and substituting in \( N_x \frac{\partial^2 w}{\partial x} + \frac{\partial Q_{xx}}{\partial x} + 0 = 0 \). Remember \( P(x) \) is zero initially while calculating buckling temperature.

\[ \frac{\partial^4 w}{\partial x^4} + \frac{N_0}{\left( \frac{-E_0 E_2 + E_1^2}{E_0} \right)} \frac{\partial^2 w}{\partial x^2} = 0 \]  

(7.21)

If \( \beta = \frac{-E_0 E_2 + E_1^2}{E_0} \) and if \( \zeta^2 = \frac{N_0}{\beta} \) equation (7.21) becomes,

\[ \frac{\partial^4 w}{\partial x^4} + \zeta^2 \frac{\partial^2 w}{\partial x^2} = 0 \]  

(7.22)

The above differential equation is solved by assuming a solution for \( w(x) = ce^{mx} \).

Substituting the same in equation (7.22), we get

\[ e^{mx}(m^4 + \zeta^2 m^2) = 0 \]  

(7.23)

As \( e^{mx} \) cannot be zero, setting \( (m^4 + \zeta^2 m^2) = 0 \)

The roots are thus obtained to be,

\[ m = 0, 0, -i\zeta, +i\zeta \]

On substituting the same back in the assumed solution, we have,

\[ w(x) = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{i\zeta x} + c_4 e^{-i\zeta x} \]  

(7.24)

On expanding \( e^{i\zeta x} \) and \( e^{-i\zeta x} \) and rearranging we have the solution to be,

\[ w(x) = C_1 + C_2 x + C_3 \sin(\zeta x) + C_4 \cos(\zeta x) \]  

(7.25)

Where,

\[ C_3 = (c_3 + ic_4) \]

\[ C_4 = (c_3 - ic_4) \]

Also setting,

\[ C_1 = c_1 \quad ; \quad C_2 = c_2 \]

In case of a clamped-clamped beam with immovable edges, boundary conditions are
found to be
\[ w(-a) = w(a) = 0 \] (7.26)
\[ w_x(-a) = w_x(a) = 0 \] (7.27)

@ \( x=0, w(0) = 0 \). Substituting in equation (7.25)
\[ C_1 + C_4 = 0 \] (7.28)

@ \( x=0, w'(0) = 0 \)
\[ C_2 + \zeta C_3 = 0 \] (7.29)

@ \( x=L, w(L) = 0 \)
\[ 0 = C_1 + C_2 L + C_3 \sin(\zeta L) + C_4 \cos(\zeta L) \] (7.30)

@ \( x=L, w'(L) = 0 \)
\[ 0 = C_2 + C_3 \zeta \cos(\zeta L) - C_4 \zeta \sin(\zeta L) \] (7.31)

In matrix form,
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & \zeta & 1 \\
1 & L & \sin(\zeta L) & \cos(\zeta L) \\
0 & 1 & \zeta \cos(\zeta L) & -\zeta \sin(\zeta L)
\end{bmatrix} \begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix} = \{0\} \] (7.32)

On taking the determinant and solving for the roots of this determinant,
\[
\begin{vmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & \zeta & 1 \\
1 & L & \sin(\zeta L) & \cos(\zeta L) \\
0 & 1 & \zeta \cos(\zeta L) & -\zeta \sin(\zeta L)
\end{vmatrix} = 0 \] (7.33)

Using MatLab, the characteristic equation of this determinant is obtained to be,
\[ 2\zeta \cos(\zeta L) - \zeta - \zeta \left( \cos^2(\zeta L) + \sin^2(\zeta L) \right) + \zeta^2 L \sin(\zeta L) = 0 \] (7.34)
On using trigonometric quantities and simplifying we have,

\[
(1 - \cos(\zeta L)) - \frac{\xi_L}{2} \sin(\zeta L) = 0 \tag{7.35}
\]

\[
2\sin^2\left(\frac{\xi_L}{2}\right) - \frac{\xi_L}{2} 2\sin\left(\frac{\xi_L}{2}\right) \cos\left(\frac{\xi_L}{2}\right) = 0 \tag{7.36}
\]

Two roots are obtained from equation (7.36),

\[
sin\left(\frac{\xi_L}{2}\right) = 0 ; \quad tan\left(\frac{\xi_L}{2}\right) = \frac{\xi_L}{2} \tag{7.37}
\]

Smallest non-zero value of \( \frac{\xi_L}{2} \) for which \( sin\left(\frac{\xi_L}{2}\right) = 0 \) is \( \pi \), we thus have

\[
\zeta = \frac{2\pi}{L} \text{ or } \zeta^2 = \frac{4\pi^2}{L^2} \tag{7.38}
\]

\[
\frac{N_0}{-\frac{E_2E_0}{E_0} + \frac{E_1^2}{E_0}} = \frac{4\pi^2}{L^2} \tag{7.39}
\]

But setting \( N_0 = -P \) (axial load in pre-buckling). Equation (7.39) thus becomes

\[
\frac{P}{\frac{E_2E_0}{E_0} - \frac{E_1^2}{E_0}} = \frac{4\pi^2}{L^2} \tag{7.40}
\]

The critical buckling temperature \( \Delta T_{cr} \) is reached when \( N_T = P_{cr} \)

\[
\frac{P_{cr}}{\frac{E_2E_0}{E_0} - \frac{E_1^2}{E_0}} = \frac{4\pi^2}{L^2} \tag{7.41}
\]

\[
P_{cr} = \frac{4\pi^2}{L^2} \left[ \frac{\frac{E_2E_0}{E_0} - \frac{E_1^2}{E_0}}{E_0} \right] \tag{7.42}
\]

We thus have from equation (7.10)

\[
\frac{4\pi^2}{L^2} \left[ \frac{\frac{E_2E_0}{E_0} - \frac{E_1^2}{E_0}}{E_0} \right] = \Delta T_{cr} \int_{\frac{h}{2}}^{\frac{h}{2}} E(z) \alpha(z) dz \tag{7.43}
\]

\( \Delta T_{cr} \) is thus obtained to be (Thivend, Eslami, & Deschilder),

\[
\Delta T_{cr} = \frac{\frac{4\pi^2}{L^2} \left[ \frac{\frac{E_2E_0}{E_0} - \frac{E_1^2}{E_0}}{E_0} \right]}{\int_{\frac{h}{2}}^{\frac{h}{2}} E(z) \alpha(z) dz} \tag{7.44}
\]

On taking thermal expansion coefficients from (Eslami, Chitikela, & Thivend)
Table 6.1

Geometric properties of the beam and using thermal expansion coefficient formula based on rule of mixtures which is as below,

\[ \alpha(z) = \alpha_c V_c + \alpha_m V_m \]  \hspace{1cm} (7.45)

Where,

- \( \alpha_c \) is the thermal expansion coefficient of ceramic
- \( \alpha_m \) is the thermal expansion coefficient of metal

(Tung & Duc, 2014)

and by creating a Matlab code, we obtain \( \lambda_{cr} = \left( \frac{L}{h} \right)^2 \alpha_m \Delta T_{cr} \). This is found to 3.2899 for an Euler Bernoulli Beam and continues to remain constant for various \( \left( \frac{L}{h} \right) \) ratios.

Variation of \( \lambda_{cr} \) for various \( \left( \frac{L}{h} \right) \) ratios is as plotted below.

![Graph showing variation of \( \lambda_{cr} \) for various \( \left( \frac{L}{h} \right) \) ratios](image)

*Figure 7.1* Variation of \( \lambda_{cr} \) for various \( \left( \frac{L}{h} \right) \) ratios, for metal

The critical buckling temperature for aluminum of gradation power index \( k=99999 \)
In case of the beam under consideration with an \( \frac{L}{h} \) ratio of 333.33 is found to be,

\[
\Delta T_{cr} = 1.278 \, K
\]  

(7.46)

I thus pick a value for \( \Delta T \) to be \( 1.2 \, K \). This value is way less than the critical buckling temperatures attained at different gradation power indexes of \( k \).

7.2. Non-Linear Bending of C-C beams with thermal load for Classical Theory

We now perform the non-linear bending analysis of clamped-clamped beams undergoing both thermal and uniformly distributed transverse pressure loading. Rule of mixtures is used to obtain material properties of the beam, and classical theory is used to obtain governing differential equations. Referring to equation (7.18) and differentiating the same. It is to be noted that differentiation of \( N^T \) with respect to \( x \) is zero as \( N^T \) is only a function of \( z \) in this problem. Differentiation of \( N_x = N_0 \) with respect to \( x \) will also be zero.

It is also to be noted that thermal stresses induce zero resultant forces and moments and hence \( \int_{\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, dz = N_0 \) (in-plane stress resultant) is purely due to mechanical stresses.

It can thus, be concluded that thermal loads can induce only strains and not any stresses. (Kaw, 1997) Differentiation of \( M_T \) with respect to \( x \) is also zero.

We thus have, (Eslami, Chitikela, & Thivend )

\[
Q_x = \frac{\partial M}{\partial x} = \left( -\frac{E_0 E_2 + E_1^2}{E_0} \right) \frac{\partial^3 w}{\partial x^3} + 0 + 0
\]  

(7.47)

Differentiating (7.47) and substituting it into \( N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial Q_{xx}}{\partial x} + P(x) = 0 \) (the same as equation (4.55)), yields,
$N_0 \frac{\partial^2 w}{\partial x^2} - \left( \frac{E_0E_2-E_1^2}{E_0} \right) \frac{\partial^4 w}{\partial x^4} + P(x) = 0 \quad (7.48)$

Rearranging this equation,

$$\frac{\partial^4 w}{\partial x^4} - \frac{N_0}{\beta} \frac{\partial^2 w}{\partial x^2} - \frac{P(x)}{\beta} = 0$$ \quad (7.49)

Assuming $\beta = \left( \frac{E_0E_2-E_1^2}{E_0} \right)$, we will obtain

$$\frac{\partial^4 w}{\partial x^4} - \frac{N_0}{\beta} \frac{\partial^2 w}{\partial x^2} = \frac{P(x)}{\beta} \quad (7.50)$$

Knowing that $\frac{N_0}{\beta} = \zeta^2$, gives,

$$\frac{\partial^4 w}{\partial x^4} - \zeta^2 \frac{\partial^2 w}{\partial x^2} = \varphi \quad (7.51)$$

The solution for this equation is found by following the same steps in section (4.1.6),

$$w(x) = C_1 + C_2x + C_3 \cosh(\zeta x) + C_4 \sinh(\zeta x) - \frac{\varphi}{2\zeta^2} x^2$$

Where $C_1, C_2, C_3$ and $C_4$ depend not only on the boundary conditions, but on $N_0$ as well.

Now, however, here is

$$N_0 = E_0 \left[ \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] - E_1 \frac{\partial^2 w}{\partial x^2} \quad (7.52)$$

Multiply by $dx$ and integrating over the length,

$$N_0 dx = E_0 \left[ \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] dx - E_1 \frac{\partial^2 w}{\partial x^2} dx - N_T$$

$$\int_{-a}^{a} N_0 dx = E_0 \int_{-a}^{a} \frac{\partial u_0}{\partial x} dx + \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - E_1 \int_{-a}^{a} \frac{\partial^2 w}{\partial x^2} dx - N_T (2a) \quad (7.53)$$

On using the same steps from equation (5.44) to (5.57) Pg.52 in section 5.2.1 and substituting the same in the above equation, we have,
\[ N_0 = \frac{E_a}{4a} \int_{-a}^{a} \left( C_3 \zeta \sinh(\zeta x) - \frac{\varphi}{\zeta^2} x \right)^2 dx - \frac{E_1}{2a} \int_{-a}^{a} \left( C_3 \zeta^2 \cosh(\zeta x) - \frac{\varphi}{\zeta^2} \right) dx \cdot N_T \] (7.54)

\[ N_0 \] is evaluated for discrete values of \( P \) using Newton Raphson Method. (Eslami, Chitikela, & Thivend). \( C_3 \) and \( C_1 \) are found to be,

\[ C_3 = \frac{\frac{\varphi}{\zeta^2} a}{\zeta \sinh(\zeta a)} = \frac{P}{N_0} a \frac{1}{\zeta \sinh(\zeta a)} \]

\[ C_1 = \frac{Pa^2}{2N_0} - C_3 \cosh(\zeta a) \]

### 7.2.1. Results obtained by material modeling using Rule of Mixtures

The graph displays the deflections of a clamped-clamped beam for different values of gradation power index \( k \) over the length of the beam under 0.01 \( \frac{N}{mm^2} \).

![Graph of deflections](image)

*Figure 7.2* Max. deflection values that occur for a temperature difference of 1.2 K when rule of mixtures and classical theory are applied
Figure 7.3 Variation of critical temperature for different ‘k’, rule of mixtures.

Figure 7.4 Variation of in plane loads for different values of transverse loads, rule of mixtures and classical theory.

Table 7.1

Maximum deflection values for different values of k in a clamped-clamped beam subjected to thermal and mechanical loads.
<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{max}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clamped-Clamped beam</td>
<td>RULE OF MIXTURES</td>
</tr>
<tr>
<td>$k=0$</td>
<td>3.41181223889</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>3.90660271481</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.23007500880</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.60998832964</td>
</tr>
<tr>
<td>$k=999999$</td>
<td>6.36068964207</td>
</tr>
</tbody>
</table>

These results are in turn compared with results obtained using FE software ABAQUS. A temperature of 1.2K is introduced to the beam. The results are as follows:

In case of a pure ceramic case, $k=0$, we have

*Figure 7.5 ABAQUS results for a clamped-clamped ceramic beam subjected to thermal and mechanical pressure loads. Classical theory and rule of mixtures are used.*
In case of a pure metal case, \( k = 99999 \), we have

**Figure 7.6** ABAQUS results for a clamped-clamped metal beam subjected to thermal and mechanical pressure loads. Classical theory and rule of mixtures are used.

In case of a FGM case, \( k = 1 \), we have

**Figure 7.7** ABAQUS results for a clamped-clamped FG beam with \( k = 1 \), subjected to thermal and mechanical pressure loads. Classical theory and rule of mixtures are used.

### 7.2.2. Comparison of the analytical results to ABAQUS results

In this section, the results obtained analytically in a clamped-clamped beam which is in turn modeled using rule of mixtures and classical theory, for a few gradation patterns is compared to the FE results. Mechanical uniformly distributed pressure load and
temperature of $1.2K$ is introduced. A table incorporating the results and percentage
difference is as shown below:

Table 7.2

*Comparison of Maximum deflection values for different values of $k$ in a clamped-clamped Beam subjected to thermal and mechanical pressure loads.*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm)</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm)</th>
<th>Percentage Difference $%$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RULE OF MIXTURES</td>
<td>FE Results Clamped-clamped</td>
<td></td>
</tr>
<tr>
<td>$k=0$</td>
<td>3.4118</td>
<td>3.407</td>
<td>0.141</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.2301</td>
<td>4.228</td>
<td>0.05</td>
</tr>
<tr>
<td>$k=999999$</td>
<td>6.360</td>
<td>6.352</td>
<td>0.126</td>
</tr>
</tbody>
</table>

7.2.3. Results obtained for different temperature ratios at gradation ‘$k$’

Table 7.3

$w_{\text{max}}$ for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at each gradation parameter $k$ for rule of mix, classical theory

$w_{\text{max}}$ for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at each gradation parameter $k$.

For $k=0$ with $\Delta T_{cr} = 4.0012$, $w_{\text{max}}$ are given below

<table>
<thead>
<tr>
<th>$\frac{\Delta T}{\Delta T_{cr}} = 0.1$</th>
<th>$\frac{\Delta T}{\Delta T_{cr}} = 0.5$</th>
<th>$\frac{\Delta T}{\Delta T_{cr}} = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.35769</td>
<td>3.466097</td>
<td>3.574843</td>
</tr>
</tbody>
</table>

For $k=0.5$ with $\Delta T_{cr} = 2.2670$, $w_{\text{max}}$ are given below
$w_{max}$ for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at each gradation parameter $k$.

<table>
<thead>
<tr>
<th>$\frac{\Delta T}{\Delta T_{cr}}$</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.1</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.5</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.814033</td>
<td>3.900268</td>
<td>3.9866979</td>
<td></td>
</tr>
</tbody>
</table>

For $k=1$ with $\Delta T_{cr} = 1.8588$, $w_{max}$ are given below

<table>
<thead>
<tr>
<th>$\frac{\Delta T}{\Delta T_{cr}}$</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.1</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.5</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1267617</td>
<td>4.2024825</td>
<td>4.278341</td>
<td></td>
</tr>
</tbody>
</table>

For $k=10$ with $\Delta T_{cr} = 1.7478$, $w_{max}$ are given below

<table>
<thead>
<tr>
<th>$\frac{\Delta T}{\Delta T_{cr}}$</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.1</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.5</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.493982459</td>
<td>5.57303745</td>
<td>5.65231502</td>
<td></td>
</tr>
</tbody>
</table>

For $k=99$ with $\Delta T_{cr} = 1.2875$, $w_{max}$ are given below

<table>
<thead>
<tr>
<th>$\frac{\Delta T}{\Delta T_{cr}}$</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.1</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.5</th>
<th>$\frac{\Delta T}{\Delta T_{cr}}$ = 0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.2364673</td>
<td>6.29739128</td>
<td>6.3584936</td>
<td></td>
</tr>
</tbody>
</table>

The below graphs display the variation of transverse deflection $w$ for various $\frac{\Delta T}{\Delta T_{cr}}$ ratios at each gradation parameter $k$ for clamped-clamped beams modeled using rule of mixtures and classical theory.
Figure 7.8 Deflection graph for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at gradation parameter $k=0$ for rule of mixtures

Figure 7.9 Deflection graphs for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at gradation parameter $k=0.5$ for rule of mixtures
**Figure 7.10** Deflection graphs for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at gradation parameter $k=1$ for rule of mixtures

**Figure 7.11** Deflection graphs for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at gradation parameter $k=10$ for rule of mixtures
The next set of graphs display the inplane force $N_0$ vs uniformly distributed mechanical transverse pressure load $P$ for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at each gradation parameter $k$. It can be seen from the above graphs that deflections increase with increase in the $\frac{\Delta T}{\Delta T_{cr}}$ ratio.
Figure 7.14 $N_0$ vs pressure $P$ graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=0.5$

Figure 7.15 $N_0$ vs pressure $P$ graphs for different $\Delta T/\Delta T_{cr}$ ratios at gradation parameter $k=1$
It can be seen from the above graphs that in plane forces $N_0$ reduces with increase in the $\frac{\Delta T}{\Delta T_{cr}}$ ratio.

### 7.3. Calculation of critical buckling temperature for FSDT

The analytical model used for non-linear bending analysis is extended in this section.
to introduce an axial thermal load. The total strain acting on a beam when subjected to both mechanical and thermal loads is given by

\[ \varepsilon_x^T = \varepsilon_x^M + \varepsilon_x^{TH} \]  

(7.55)

Where,

- \( \varepsilon_x^T \) is the total strain acting on the midplane
- \( \varepsilon_x^M \) is the mechanical strain
- \( \varepsilon_x^{TH} \) is the thermal strain

Total strain, when considering shear deformation, is in turn found from equation (4.97) to be

\[ \varepsilon_x^T = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 - z \frac{\partial \phi_x}{\partial x} \]  

(7.56)

Mechanical strain is thus found to be,

\[ \varepsilon_x^T - \varepsilon_x^{TH} = \varepsilon_x^M \]  

(7.57)

\( \varepsilon_x^{TH} \) is further given to be,

\[ \varepsilon_x^{TH} = \alpha(z)[T - T_0] \]  

(7.58)

The mechanical strain is thus given to be,

\[ \varepsilon_x^M = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 - z \frac{\partial \phi_x}{\partial x} - \alpha(z)[T - T_0] \]  

(7.59)

From Hooke’s Law,

\[ \sigma_x = E(z) \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 - z \frac{\partial \phi_x}{\partial x} - \alpha(z)[T - T_0] \right) \]  

(7.60)

On multiplying by \( dz \) and integrating over the thickness we have,

\[ \int_{-h/2}^{h/2} \sigma_x dz = \int_{-h/2}^{h/2} E(z) \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \right) dz - \int_{-h/2}^{h/2} E(z) \frac{\partial \phi_x}{\partial x} dz - \int_{-h/2}^{h/2} E(z) \alpha(z) \Delta T(z) dz \]  

(7.61)
From equation (4.38) and considering,

\[
E_0 = \int_{-h/2}^{h/2} E(z) \, dz
\]  

(7.62)

\[
E_1 = \int_{-h/2}^{h/2} (E(z) \ast z) \, dz
\]  

(7.63)

\[
N^T = \int_{-h/2}^{h/2} E(z) \alpha(z) \Delta T(z)
\]  

(7.64)

Equation (7.61) becomes,

\[
N_x = E_0 \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) - E_1 \frac{\partial \phi_x}{\partial x} - N^T
\]  

(7.65)

It is also to be noted here that \( \gamma_{xz} \neq 0 \) as shear deformation effect is considered in Timoshenko Beam Theory.

\[\gamma_{xz} = \frac{\partial w}{\partial x} - \phi_x \]  

(7.66)

Where, \( \frac{\partial w}{\partial x} \) is the total deformation

\( \phi_x \) is the rotary deformation

Multiplying equation (7.61) by \( zdz \) and integrating over the thickness we have

\[
M_x = E_1 \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) - E_2 \frac{\partial \phi_x}{\partial x} - M^T
\]  

(7.67)

Where,

\[
E_2 = \int_{-h/2}^{h/2} (E(z) z^2) \, dz
\]  

(7.68)

Putting (7.65) and (7.67) in matrix form,

\[
\begin{pmatrix}
M_x \\
n_x
\end{pmatrix} =
\begin{bmatrix}
E_1 & E_2 \\
E_0 & E_1
\end{bmatrix}
\begin{pmatrix}
\frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
- \frac{\partial \phi_x}{\partial x}
\end{pmatrix} -
\begin{pmatrix}
M^T \\
n^T
\end{pmatrix}
\]  

(7.69)

Replacing \( N_x \) with \( N_0 \) (as \( N_0 \) is a constant value and independent of \( x \)),
Let \[ \alpha = \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial \omega}{\partial x} \right)^2 \] ; \[ \kappa_x = -\frac{\partial \phi_x}{\partial x} \] (7.70)

Thus,
\[ M_x = E_1 \alpha - E_2 \gamma - M^T \] (7.71)
\[ N_0 = E_0 \alpha - E_1 \gamma - N^T \] (7.72)

Equation (7.71) and (7.72) are simultaneously solved for \( M_x \) and is as follows,
\[ M_x = \left[ \left( \frac{-E_0 E_2 + E^2}{E_0} \right) \frac{\partial \phi_x}{\partial x} + (N_0 + N_T) \frac{E_1}{E_0} - M^T \right] \] (7.73)

We know that,
\[ \frac{\partial M}{\partial x} = Q_x \] (7.74)

Differentiating equation (7.73)
\[ Q_x = \frac{\partial M}{\partial x} = \left( \frac{-E_0 E_2 + E^2}{E_0} \right) \frac{\partial^2 \phi_x}{\partial x^2} \] (7.75)

We know that shear force is also given by,
\[ Q_{xz} = Q_x = \int_{-h/2}^{h/2} \tau_{xz} \text{d}z \] (7.76)

From Hooke’s law and by taking into account the shear correction factor \( k_s \), we have,
\[ Q_{xz} = Q_x = k_s \int_{-h/2}^{h/2} G(z) \text{d}z \gamma_{xz} \] (7.77)

Where, \( G_0 = \int_{-h/2}^{h/2} G(z) \text{d}z \) is the shear modulus through the thickness.
\[ \gamma_{xz} = \frac{\partial w}{\partial x} - \phi_x \] is the shear strain

Differentiating equation (7.77) with respect to \( x \) and substituting the same in
\[ N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial Q_{xz}}{\partial x} + P(x) = 0 \] to obtain \( \frac{\partial \phi_x}{\partial x} \)
\[ \frac{\partial \phi_x}{\partial x} = \left( 1 + \frac{N_0}{k_s G_0} \right) \frac{\partial^2 w}{\partial x^2} + \frac{P}{k_s G_0} \] (7.78)

Differentiating Equation (7.78) again with respect to \( x \).
\[
\frac{\partial^2 \phi_x}{\partial x^2} = \left(1 + \frac{N_0}{k_s G_0}\right) \frac{\partial^3 w}{\partial x^3}
\] (7.79)

Substituting equation (7.79) and \( Q_x = k_s G_0 \gamma_{xz} \) in equation (7.75).

\[
k_s G_0 \frac{\partial w}{\partial x} - k_s G_0 \phi_x = \left(- \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \frac{\partial^3 \phi_x}{\partial x^3} \right)
\] (7.80)

Rearranging equation (7.80) and substituting equation (7.79) in the same,

\[
\phi_x = \left[ \frac{1}{k_s G_0} \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left(1 + \frac{N_0}{k_s G_0} \right) \frac{\partial^3 w}{\partial x^3} \right] + \frac{\partial w}{\partial x}
\] (7.81)

Differentiating equation (7.81) again with respect to x,

\[
\frac{\partial \phi_x}{\partial x} = \left[ \frac{1}{k_s G_0} \left( \frac{E_0 E_2 - E_1^2}{E_0} \right) \left(1 + \frac{N_0}{k_s G_0} \right) \frac{\partial^4 w}{\partial x^4} \right] + \frac{\partial^2 w}{\partial x^2}
\] (7.82)

Equating equations (7.78) and (7.82) and assuming \( P(x) = 0 \).

\[
\frac{\partial^4 w}{\partial x^4} + \frac{N_0}{\left( -\frac{E_0 E_2 + E_1^2}{E_0} \right) \left(1 + \frac{N_0}{k_s G_0} \right)} \frac{\partial^2 w}{\partial x^2} = 0
\] (7.83)

If \( \beta = \frac{-E_2 E_0 + E_1^2}{E_0} \left(1 + \frac{N_0}{k_s G_0} \right) \) and if \( \zeta^2 = \frac{N_0}{\beta} \) equation (7.83) becomes,

\[
\frac{\partial^4 w}{\partial x^4} + \zeta^2 \frac{\partial^2 w}{\partial x^2} = 0
\] (7.84)

The above differential equation is solved by assuming a solution for \( w(x) = ce^{mx} \).

Substituting the same in equation (7.84), we get

\[
e^{mx}(m^4 + \zeta^2 m^2) = 0
\] (7.85)

As \( e^{mx} \) cannot be zero, setting \( m^4 + \zeta^2 m^2 = 0 \)

The roots are thus obtained to be,

\[
m = 0, 0, -i\zeta, +i\zeta
\]

Substituting the same back in the assumed solution, we have,

\[
w(x) = c_1 e^{0x} + c_2 x e^{0x} + c_3 e^{i\zeta x} + c_4 e^{-i\zeta x}
\] (7.86)

Expanding \( e^{i\zeta x} \) and \( e^{-i\zeta x} \) and rearranging we have the solution to be,
\[ w(x) = C_1 + C_2x + C_3 \sin(\zeta x) + C_4 \cos(\zeta x) \quad (7.87) \]

Where,

\[ C_3 = (c_3 + ic_4) \]
\[ C_4 = (c_3 - ic_4) \]

Also setting,

\[ C_i = c_1 \quad ; \quad C_2 = c_2 \]

In case of a clamped-clamped beam with immovable edges, boundary conditions are found to be

\[ w(-a) = w(a) = 0 \quad (7.88) \]
\[ \frac{dw}{dx}(-a) = \frac{dw}{dx}(a) = 0 \quad (7.89) \]

@ \( x=0 \), \( w(0) = 0 \). Substituting in equation (7.88 and 7.89)

\[ C_1 + C_4 = 0 \quad (7.90) \]

@ \( x=0 \), \( w'(0) = 0 \)

\[ C_2 + \zeta C_3 = 0 \quad (7.91) \]

@ \( x=L \), \( w(L) = 0 \)

\[ 0 = C_1 + C_2L + C_3 \sin(\zeta L) + C_4 \cos(\zeta L) \quad (7.92) \]

@ \( x=L \), \( w'(L) = 0 \)

\[ 0 = C_2 + C_3 \zeta \cos(\zeta L) - C_4 \zeta \sin(\zeta L) \quad (7.93) \]

In matrix form,
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & \zeta & 1 \\
1 & L & \sin(\zeta L) & \cos(\zeta L) \\
0 & 1 & \zeta \cos(\zeta L) & -\zeta \sin(\zeta L)
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{bmatrix}
= \{0\}
\] (7.94)

Using MatLab, the characteristic equation of this determinant is obtained to be,
\[
2\zeta \cos(\zeta L) - \zeta - \zeta (\cos^2(\zeta L) + \sin^2(\zeta L)) + \zeta^2 L \sin(\zeta L) = 0
\] (7.95)

Using trigonometric quantities and simplifying we have,
\[
\left(1 - \cos(\zeta L)\right) - \frac{\zeta L}{2} \sin(\zeta L) = 0
\] (7.96)

\[
2 \sin^2\left(\frac{\zeta L}{2}\right) - \frac{\zeta L}{2} 2 \sin\left(\frac{\zeta L}{2}\right) \cos\left(\frac{\zeta L}{2}\right) = 0
\] (7.97)

Two roots are obtained from equation (7.97),
\[
\sin\left(\frac{\zeta L}{2}\right) = 0 \quad ; \quad \tan\left(\frac{\zeta L}{2}\right) = \frac{\zeta L}{2}
\] (7.98)

Smallest non-zero value of \(\frac{\zeta L}{2}\) for which \(\sin\left(\frac{\zeta L}{2}\right) = 0\) is \(\pi\), We thus have
\[
\zeta = \frac{2\pi}{L} \text{ or } \zeta^2 = \frac{4\pi^2}{L^2}
\] (7.99)

\[
\frac{N_0}{-E_2 E_0 + E_1^2} \left(1 + \frac{N_0}{k_s G_0}\right) = \frac{4\pi^2}{L^2}
\] (7.100)

\[
\frac{N_0}{-E_2 E_0 + E_1^2} \frac{1}{N_0 \left(1 + \frac{1}{k_s G_0}\right)} = \frac{4\pi^2}{L^2}
\]

But setting \(N_0 = -P\) (axial load in pre-buckling) and solving the above equation for \(P\), we have
\[
\frac{k_s G_0}{1 + \frac{L^2}{4\pi^2 E_2 E_0 - E_1^2}} = P
\] (7.101)

The critical buckling temperature \(\Delta T_{cr}\) is reached when \(N_T = P_{cr}\) (Thivend, Eslami, & Deschilder).
Using thermal expansion coefficient formula based on rule of mixtures,

\[ \alpha(z) = \alpha_c V_c + \alpha_m V_m \]  \hspace{1cm} (7.103)

Where,

\[ \alpha_c \] is the thermal expansion coefficient of ceramic

\[ \alpha_m \] is the thermal expansion coefficient of metal \((\text{Tung & Duc, 2014})\)

In which \(\alpha_c\) and \(\alpha_m\) can be found from Table 6.1.

and by creating a Matlab code, we obtain \(\lambda_{cr} = \left(\frac{L}{h}\right)^2 \alpha_m \Delta T_{cr}\). This is found to be slowing increasing for a Timoshenko beam and finally saturating at 3.2899 at high ratios of \(\left(\frac{L}{h}\right)\).

Variation of \(\lambda_{cr}\) for various \(\left(\frac{L}{h}\right)\) ratios is as plotted below.

\[ \text{Figure 7.18 Variation of } \lambda_{cr} \text{ for various } \left(\frac{L}{h}\right) \text{ ratios in case of FSDT theory with rule of mixtures, when metal is considered.} \]
7.4. Non-Linear Bending Analysis of CC beams with thermal loads for FSDT.

We now perform non-linear bending analysis of clamped-clamped beam undergoing both thermal and uniformly distributed transverse pressure loading. Rule of mixtures is used to obtain material properties of the beam and first order shear deformation theory is used to obtain governing differential equations.

Referring back to equation (7.73) and differentiating the same. It is to be noted here that differentiation of $N^T$ with respect to $x$ is zero as $N^T$ is only a function of $z$ in this problem. Differentiation of $N_x = N_0$ with respect to $x$ will also be zero considering the fact that, $N_x$ is a constant from $\frac{\partial N_x}{\partial x} = 0$. It is to be noted here that thermal stresses induce zero resultant forces and moments and hence $\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x \, dz = N_0$ (in-plane stress resultant) is purely due to mechanical stresses only. It can be thus be concluded that thermal loads can induce only free expansion strains and not any stresses(Kaw, 1997). Differentiation of $M_T$ with respect to $x$ is also zero.

We thus have,

$$Q_x = \frac{\partial M}{\partial x} = \left(-\frac{E_0 E_2 + E_1^2}{E_0}\right) \frac{\partial^2 \phi_x}{\partial x^2}$$

(7.104)

We know that shear force is also given by,

$$Q_{xz} = Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \, dz$$

(7.105)

From Hooke’s law and by taking into account the shear correction factor $k_s$, we have,

$$Q_{xz} = Q_x = k_s \int_{-\frac{h}{2}}^{\frac{h}{2}} G(z) \, dz$$

(7.106)

Where, $G_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} G(z) \, dz$ is the shear modulus through the thickness.
\[
\gamma_{xz} = \frac{\partial w}{\partial x} - \phi_x \text{ is the shear strain} \quad (7.107)
\]

Differentiating equation (7.104) with respect to x and substituting the same in

\[
N_x \frac{\partial^2 w}{\partial x^2} + \frac{\partial Q_{xz}}{\partial x} + P(x) = 0 \quad \text{to obtain} \quad \frac{\partial \phi_x}{\partial x}
\]

\[
\frac{\partial \phi_x}{\partial x} = \left(1 + \frac{N_0}{k_sG_0}\right) \frac{\partial^2 w}{\partial x^2} + \frac{P}{k_sG_0} \quad (7.108)
\]

Differentiating Equation (7.108) again with respect to x,

\[
\frac{\partial^2 \phi_x}{\partial x^2} = \left(1 + \frac{N_0}{k_sG_0}\right) \frac{\partial^3 w}{\partial x^3} \quad (7.109)
\]

Substituting equation (7.79) and \(Q_x = k_sG_0\gamma_{xz}\) in equation (8.75).

\[
k_sG_0 \frac{\partial w}{\partial x} - k_sG_0 \phi_x = \left(- \left(\frac{E_0E_2-E_1^2}{E_0}\right) \frac{\partial^2 \phi_x}{\partial x^2}\right) \quad (7.110)
\]

Rearranging equation (7.80) and substituting equation (7.79) in the same,

\[
\phi_x = \left(\frac{1}{k_sG_0} \left(\frac{E_0E_2-E_1^2}{E_0}\right) \left(1 + \frac{N_0}{k_sG_0}\right) \frac{\partial^3 w}{\partial x^3} + \frac{\partial w}{\partial x}\right) \quad (7.111)
\]

Differentiating equation (7.111) again with respect to x,

\[
\frac{\partial \phi_x}{\partial x} = \left(\frac{1}{k_sG_0} \left(\frac{E_0E_2-E_1^2}{E_0}\right) \left(1 + \frac{N_0}{k_sG_0}\right) \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial x^2}\right) \quad (7.112)
\]

On equating (7.108) and (7.112) and simplifying we get,

\[
\frac{\partial^4 w}{\partial x^4} - \frac{N_0}{\left(\frac{E_0E_2-E_1^2}{E_0}\right) \left(1 + \frac{N_0}{k_sG_0}\right)} \frac{\partial^2 w}{\partial x^2} - \frac{P(x)}{\left(\frac{E_0E_2-E_1^2}{E_0}\right) \left(1 + \frac{N_0}{k_sG_0}\right)} = 0
\]

Let \(\alpha^2 = \frac{N_0}{\left(\frac{E_0E_2-E_1^2}{E_0}\right) \left(1 + \frac{N_0}{k_sG_0}\right)}\); \(\eta = \frac{P(x)}{\left(\frac{E_0E_2-E_1^2}{E_0}\right) \left(1 + \frac{N_0}{k_sG_0}\right)}\)

Equation (7.112) thus becomes,

\[
\frac{\partial^4 w}{\partial x^4} - \alpha^2 \frac{\partial^2 w}{\partial x^2} = \eta
\]

Transverse Deflection is thus,

\[
w(x) = C_1 + C_2x + C_3 \cosh(\alpha x) + C_4 \sinh(\alpha x) - \frac{\eta}{\alpha^2} x^2
\]

Where \(C_1, C_2, C_3\) and \(C_4\) depend not only on the boundary conditions, but on \(N_0\) as well.

It is obtained by following the steps in section 5.3.2.
Now

\[ N_0 = E_0 \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) - E_1 \frac{\partial \phi}{\partial x} \]  

(7.113)

Multiply by \( dx \) and integrating over the length,

\[ N_0 dx = \left[ E_0 \left( \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) dx - E_1 \frac{\partial \phi}{\partial x} dx \right] \]

\[ \int_{-a}^{a} N_0 dx = E_0 \int_{-a}^{a} \frac{\partial u_0}{\partial x} dx + \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - E_1 \int_{-a}^{a} \frac{\partial \phi}{\partial x} dx \]

Noting \( u(-a)=u(a)=0 \) and that \( N_0 \) is a constant (as \( N_0 \) is independent of \( x \)).

\[ N_0(2a) = E_0 \left( u(a) - u(-a) \right) + \frac{E_0}{2} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - E_1 (1 + \frac{N_0}{k_0 \epsilon_0}) \int_{-a}^{a} \frac{\partial^2 w}{\partial x^2} dx - E_1 \frac{p}{k_0 \epsilon_0} 2a \]

\[ N_0 = \frac{E_0}{4a} \int_{-a}^{a} \left( \frac{\partial w}{\partial x} \right)^2 dx - \frac{E_1}{2a} \left( 1 + \frac{N_0}{k_0 \epsilon_0} \right) \int_{-a}^{a} \frac{\partial^2 w}{\partial x^2} dx - E_1 \frac{p}{k_0 \epsilon_0} \]  

(7.114)

\( N_0 \) is evaluated for discrete values of \( P \) using Newton Raphson Method. \( C_3 \) and \( C_1 \) are found to be (Eslami, Chitikela, & Thivend).

\[ C_3 = \frac{\varphi a}{\zeta \sinh(\zeta a)} = \frac{P \epsilon}{N_0 \epsilon} a = \frac{P a^2}{2N_0} - C_3 \cosh(\zeta a) \]

\[ C_1 = \frac{P a^2}{2N_0} - C_3 \cosh(\zeta a) \]

7.4.1. Results obtained by material modeling using Rule of Mixtures

The graph displays the deflections of a clamped-clamped beam for different values of gradation power index \( k \) over the length of the beam under 0.01 \( \frac{N}{mm^2} \).
Figure 7.19 Max. deflection values that occur for a temperature difference of 1.2K when rule of mixtures and fsdt theory are applied.

Figure 7.20 Variation of critical temperature over different values of k, rule of mix.
Figure 7.21 Variation of in-plane loads for different values of transverse loads, rule of mix and first order shear deformation theory

Table 7.4

Maximum deflection values for different values of $k$ in a clamped-clamped beam subjected to thermal and mechanical loads.

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RULE OF MIXTURES</td>
</tr>
<tr>
<td></td>
<td>Clamped-Clamped beam (FSDT theory)</td>
</tr>
<tr>
<td>$k=0$</td>
<td>3.41171601472</td>
</tr>
<tr>
<td>$k=0.5$</td>
<td>3.90655681563</td>
</tr>
<tr>
<td>$k=1$</td>
<td>4.23010252852</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.61003743559</td>
</tr>
</tbody>
</table>
### 7.4.2. Results obtained for different temperature ratios at gradation ‘k’

Table 7.5

$w_{\text{max}}$ for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at each gradation parameter $k$ for rule of mix, First order shear deformation theory

<table>
<thead>
<tr>
<th>$\Delta T$ for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at each gradation parameter $k$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>For $k=0$ with $\Delta T_{cr} = 4.00081$, $w_{\text{max}}$ are given below</td>
</tr>
<tr>
<td>$\frac{\Delta T}{\Delta T_{cr}} = 0.1$</td>
</tr>
<tr>
<td>3.3575898</td>
</tr>
<tr>
<td>For $k=0.5$ with $\Delta T_{cr} = 2.266683$, $w_{\text{max}}$ are given below</td>
</tr>
<tr>
<td>$\frac{\Delta T}{\Delta T_{cr}} = 0.1$</td>
</tr>
<tr>
<td>3.813977841</td>
</tr>
<tr>
<td>For $k=1$ with $\Delta T_{cr} = 1.8586144$, $w_{\text{max}}$ are given below</td>
</tr>
<tr>
<td>$\frac{\Delta T}{\Delta T_{cr}} = 0.1$</td>
</tr>
<tr>
<td>4.12677801</td>
</tr>
</tbody>
</table>
\( w_{max} \) for different \( \frac{\Delta T}{\Delta T_{cr}} \) ratios at each gradation parameter \( k \).

For \( k=10 \) with \( \Delta T_{cr} = 1.747578 \), \( w_{max} \) are given below

<table>
<thead>
<tr>
<th>( \frac{\Delta T}{\Delta T_{cr}} )</th>
<th>( \frac{\Delta T}{\Delta T_{cr}} )</th>
<th>( \frac{\Delta T}{\Delta T_{cr}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.5</td>
<td>0.9</td>
</tr>
<tr>
<td>5.4940125905</td>
<td>5.57306819419</td>
<td>5.6523466108</td>
</tr>
</tbody>
</table>

For \( k=99 \) with \( \Delta T_{cr} = 1.2872339 \), \( w_{max} \) are given below

<table>
<thead>
<tr>
<th>( \frac{\Delta T}{\Delta T_{cr}} )</th>
<th>( \frac{\Delta T}{\Delta T_{cr}} )</th>
<th>( \frac{\Delta T}{\Delta T_{cr}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.5</td>
<td>0.9</td>
</tr>
<tr>
<td>6.2363343168</td>
<td>6.2972532908</td>
<td>6.3583510888</td>
</tr>
</tbody>
</table>

The below graphs display the variation of transverse deflection \( w \) for various \( \frac{\Delta T}{\Delta T_{cr}} \) ratios at each gradation parameter \( k \) for clamped-clamped beams modeled using rule of mixtures and fsdt theory.

**Figure 7.22** Deflection graph for different \( \frac{\Delta T}{\Delta T_{cr}} \) ratios at gradation parameter \( k=0 \) for rule of mixtures
Figure 7.23 Deflection graph for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at gradation parameter $k=0.5$ for rule of mixtures

Figure 7.24 Deflection graph for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at gradation parameter $k=1$ for rule of mixtures
Figure 7.25 Deflection graph for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at gradation parameter $k=10$

Figure 7.26 Deflection graphs for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at gradation parameter $k=99$ for rule of mixtures

It can be seen from the above graphs that deflections increase with increase in the $\frac{\Delta T}{\Delta T_{cr}}$ ratio. The next set of graphs display the inplane force $N_0$ vs pressure load $P$ for different
\[ \frac{\Delta T}{\Delta T_{cr}} \] ratios at each gradation parameter \( k \).

**Figure 7.27** \( N_0 \) vs pressure \( P \) graphs for different \( \frac{\Delta T}{\Delta T_{cr}} \) ratios at gradation parameter \( k=0 \)

**Figure 7.28** \( N_0 \) vs pressure \( P \) graphs for different \( \frac{\Delta T}{\Delta T_{cr}} \) ratios at gradation parameter \( k=0.5 \), for rule of mixtures.
Figure 7.29 $N_0$ vs pressure $P$ graphs for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at gradation parameter $k=1$, for rule of mixtures

Figure 7.30 $N_0$ vs pressure $P$ graphs for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at gradation parameter $k=10$, for rule of mixtures
Figure 7.31 $N_0$ vs pressure $P$ graphs for different $\frac{\Delta T}{\Delta T_{cr}}$ ratios at gradation parameter $k=99$

It can be seen from the above graphs that inplane forces $N_0$ reduces with increase in the $\frac{\Delta T}{\Delta T_{cr}}$ ratio.

### 7.5. Classical and FSDT results of CC beams under both loads, Mori-Tanaka.

The formula used for thermal expansion coefficient is given by:

$$\alpha = \frac{(\alpha_c - \alpha_m) \left( \frac{1}{K_c} - \frac{1}{K_m} \right)}{\frac{1}{K_c} - \frac{1}{K_m}}$$

(7.114)

Where,

- $K$ is the bulk modulus of the entire material
- $K_c$ is the bulk modulus of ceramic
- $K_m$ is the bulk modulus of the metal
- $\alpha_c$ is the thermal expansion coefficient of ceramic
- $\alpha_m$ is the thermal expansion coefficient of metal  
  (Sevostianov, 2012)

The Mori-Tanaka results for classical theory are first shown.
Figure 7.32 Variation of lambda critical over different values of k, Mori-Tanaka, for metal

The maximum deflection results in case of applying Mori-Tanaka method and classical theory are:

Figure 7.33 Max deflection values that occur for a temperature difference of 1.2K when Mori-Tanaka method and classical theory are applied
The Mori-Tanaka results for first order shear deformation theory are next shown

![Graph](image)

**Figure 7.34** Variation of lambda critical over different values of k, Mori-Tanaka, for metal.

The maximum deflection results in case of applying Mori-Tanaka method and first order shear deformation theory are:

![Graph](image)

**Figure 7.35** Max deflection values that occur for a temperature difference of 1.2K when Mori-Tanaka method and fsdt theory are applied.
The comparison between classical theory results and Mori-Tanaka results are given below:

Table 7.6

*Comparison between classical theory results and Mori-Tanaka results*

<table>
<thead>
<tr>
<th>Gradation Power Index $k$</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm) Classical Theory</th>
<th>Maximum Deflection $w_{\text{max}}$ (mm) FSDT Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k=0$</td>
<td>3.41316027415</td>
<td>3.413064099</td>
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<tr>
<td>$k=0.5$</td>
<td>4.27519337154</td>
<td>4.27520121052</td>
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<td>$k=1$</td>
<td>4.66053463119</td>
<td>4.6605802229</td>
</tr>
<tr>
<td>$k=10$</td>
<td>5.89851608052</td>
<td>5.8986605394</td>
</tr>
<tr>
<td>$k=99$</td>
<td>6.30390933457</td>
<td>6.3037132622</td>
</tr>
</tbody>
</table>

The inplane forces in case of classical theory is as shown below:

*Figure 7.36* Variation of inplane forces vs pressure $P$ when temperature difference of 1.2K, classical theory and Mori-Tanaka is applied.
The inplane forces in case of FSDT theory is as shown below:

*Figure 7.37 Variation of inplane forces vs pressure P when temperature difference of 1.2K, FSDT theory and Mori-Tanaka is applied*
8. CONCLUSIONS

A brief summary of the conclusions is discussed as follows. Two different methods have been employed namely, rule of mixtures and Mori-Tanaka model. There exists coupling between the bending moment and in-plane tensile stress (Eslami, Chitikela, & Thivend). Due to their asymmetric nature of the FGMs, the non-linear effects cannot be neglected when studying the FGM beams. The von-Karman Geometric non-linearity is thus, taken into consideration (Eslami, Chitikela, & Thivend).

Results and Graphs for in-plane stress versus applied load and deflection versus length of the beam have been obtained. Results of the Bernoulli-Euler beam theory and First Order Shear Deformation (FSDT) beam theory have been obtained and compared. Solutions were obtained for both conventional and unconventional boundary conditions.

When k tends to either 0 or infinity the beam behavior replicates to that of an isotropic beam (Thivend, Eslami, & Deschilder). The maximum deflection obtained between classical theory and first order shear deformation theory are found to be extremely close, the reason why is that the \( \frac{L}{h} \) ratio for the beam under consideration is very high (333.33 in our case). The effect of shear deformation is generally prominent when the \( \frac{L}{h} \) is close to 20 or less.

Maximum deflection values obtained using unconventional boundary conditions matched exactly the same as the one obtained using conventional boundary conditions. It can thus, be concluded that it is possible to replicate conventional boundary conditions in the industry using spring constraints. A few of maximum deflection values obtained using rule of mixtures and classical theory were compared with Finite element results. The percentage difference was minimal and this proves that the results obtained
analytically match with those of finite element method.

Nonlinear bending analysis of functionally graded beams subjected to both, transverse load and constant thermal load for clamped-clamped immovable edges was analyzed next. Critical buckling temperatures were determined, before setting the temperature of the beam in order to avoid the beam from falling into the post-buckling range. Rule of mixtures and Mori-Tanaka were used to model the beam. Governing differential equations were obtained using both, classical theory and first order shear deformation theory.

Since we are considering constant temperature, only additional thermal in-plane load is therefore added, no transverse thermal load, and in-plane force contributes to the maximum deflection. Thus, maximum deflection will increase. It is also to be noted that when applied mechanical transverse load is set close to zero, residual stresses still remain in the beam and this phenomenon is visible in the $N_0$ vs $P$ graphs, which is presented in the thermal effects chapter.

When gradation parameter $k$ is set constant for different ratios of $\frac{\Delta T}{\Delta T_{cr}}$, it can be seen that $N_0$ decreases with an increase in $\frac{\Delta T}{\Delta T_{cr}}$ ratio. The maximum deflection on the other hand increases with an increase in $\frac{\Delta T}{\Delta T_{cr}}$ ratio. As seen before, the maximum deflection results obtained for classical theory and first order shear deformation theory are very close as our beam is thin. Maximum deflection results obtained for the clamped-clamped beam subjected to both temperature and transverse loads are found to match the results obtained using Finite element.
To validate our results, ABAQUS was used for which the functionally graded beam was modeled by considering 160 layers whose properties are considered isotropic. The stiffness matrix for each specific layer was calculated and assigned at its centroid. For this purpose, 8-node doubly curved shell element (S8R) was used to model the beam. It can thus, be concluded that the objectives of this thesis were achieved with logically sound observations.
9. FUTURE WORK

The following are some of the recommendations for future work. The method used to obtain the governing equations and solve the problem can further be extended to other complicated structural elements like functionally graded plates, shells, stiffened plates and so on.

The effect of temperature on the bending of the beam in this thesis was studied for clamped-clamped beams only. However, we can extend this method to include beams of any type of boundary conditions. Thermal post buckling analysis of functionally graded beams can also be studied for conventional boundary conditions as well as for spring constraints.

Linear and nonlinear vibrational analysis of functionally graded beams, plates and shells can be considered. Functionally graded beams can be fabricated using 3D-printing technology and be experimentally tested. If successful, the same concept can be extended to plates and shells.

In this analysis, we only considered variation of material properties in the direction of thickness (in the z direction). As future work, the material properties can be varied along the axial direction (along x-axis) as well. Furthermore, in this thesis, we only considered $\Delta T$ to be constant. As future work the temperature gradation can be considered to vary as a function of x and z or even three dimensionally (as a function of x, y, and z). Also, $\Delta T$
could vary as function of time. Free and forced vibrations of functionally graded beams can also be considered as future work.
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Appendix A. In-Plane stress resultant $N_0$ vs $P$

Table 9.1

In-plane stress resultant $N_0$ for different values of $k$ of a simply-supported beam using rule of mixtures and applying classical theory

<table>
<thead>
<tr>
<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.01</td>
<td>81.0565</td>
<td>0.01</td>
<td>81.0565</td>
</tr>
<tr>
<td>-0.008</td>
<td>69.4320</td>
<td>0.008</td>
<td>69.4320</td>
</tr>
<tr>
<td>-0.006</td>
<td>56.8229</td>
<td>0.006</td>
<td>56.8229</td>
</tr>
<tr>
<td>-0.004</td>
<td>42.6999</td>
<td>0.004</td>
<td>42.6999</td>
</tr>
<tr>
<td>-0.002</td>
<td>25.8788</td>
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<td>25.8788</td>
</tr>
<tr>
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<td></td>
</tr>
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</table>

For power index $k=0$ and $k=0.5$

<table>
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<th>$N_0 \left( \frac{N}{mm} \right)$</th>
<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>73.5995</td>
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<td>62.8980</td>
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<td>63.1615</td>
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<td>$N_{mm}$</td>
<td>$P\left(\frac{N}{mm^2}\right)$</td>
<td>$N_0\left(\frac{N}{mm}\right)$</td>
<td>$P\left(\frac{N}{mm^2}\right)$</td>
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<tr>
<td>--------</td>
<td>-------------------------------</td>
<td>-------------------------------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>0.00000001</td>
<td>0.0183</td>
<td>0.0237</td>
<td>0.0183</td>
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</table>

For power index $k=1$

<table>
<thead>
<tr>
<th>$N_{mm}$</th>
<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
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</thead>
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</tr>
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<td>59.1117</td>
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<td>0.0237</td>
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For power index $k=10$

<table>
<thead>
<tr>
<th>$N_{mm}$</th>
<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
</thead>
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<td>28.0898</td>
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<td>0.0237</td>
</tr>
</tbody>
</table>

For power index $k=99$

| $N_{mm}$ | $P\left(\frac{N}{mm^2}\right)$ | $N_0\left(\frac{N}{mm}\right)$ | $P\left(\frac{N}{mm^2}\right)$ | $N_0\left(\frac{N}{mm}\right)$ |
Table 9.2

In-plane stress resultant \( N_0 \) for different values of \( k \) of a simply-supported beam using Mori-Tanaka method and applying classical theory

<table>
<thead>
<tr>
<th>( k )</th>
<th>( N_0 (\frac{N}{mm}) )</th>
<th>( P (\frac{N}{mm^2}) )</th>
<th>( N_0 (\frac{N}{mm}) )</th>
<th>( P (\frac{N}{mm^2}) )</th>
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<td>0.008</td>
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<td>0.0000001</td>
<td>0.0064</td>
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For power index \( k = 0 \)

For power index \( k = 0.5 \)

<table>
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<th>( P (\frac{N}{mm^2}) )</th>
<th>( N_0 (\frac{N}{mm}) )</th>
<th>( P (\frac{N}{mm^2}) )</th>
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For power index $k=1$

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For power index $k=10$

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</table>

For power index \( k=99 \)

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<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
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<td>47.5701</td>
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<td>25.5598</td>
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Table 9.3

*In-plane stress resultant \( N_0 \) for different values of \( k \) of a simply-supported beam using rule of mixtures and applying first order shear deformation theory*

<p>| | | | | |</p>
<table>
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For power index $k=0.5$

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<th>$N_0\left(\frac{N}{mm}\right)$</th>
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<tbody>
<tr>
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<td>-0.008</td>
<td>62.8948</td>
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<td>63.1608</td>
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<td>51.5544</td>
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<td>38.8488</td>
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<td>23.7198</td>
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For power index $k=1$

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<td>58.7983</td>
<td>0.008</td>
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<td>48.2238</td>
<td>0.006</td>
<td>48.5547</td>
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<tr>
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<td>36.3849</td>
<td>0.004</td>
<td>36.7204</td>
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</table>
For power index \( k = 10 \)

<table>
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<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
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<td>52.3240</td>
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<td>-0.008</td>
<td>44.9430</td>
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<td>-0.006</td>
<td>36.9140</td>
<td>0.006</td>
<td>37.0927</td>
</tr>
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<td>-0.004</td>
<td>27.9234</td>
<td>0.004</td>
<td>28.0921</td>
</tr>
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<td>-0.002</td>
<td>17.2136</td>
<td>0.002</td>
<td>17.3884</td>
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<td>0.0030</td>
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For power index \( k = 99 \)

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<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
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</thead>
<tbody>
<tr>
<td>-0.01</td>
<td>47.8558</td>
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<td>47.8783</td>
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<td>41.1518</td>
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<td>41.1723</td>
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<td>33.8593</td>
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<td>33.8791</td>
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<td>25.6921</td>
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<td>15.9771</td>
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Table 9.4

In-plane stress resultant \( N_0 \) for different values of \( k \) of a simply-supported beam using Mori-Tanaka method and applying first order shear deformation theory
For power index $k=0$

<table>
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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
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<tr>
<td>-0.01</td>
<td>81.0301</td>
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<td>81.0301</td>
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<td>-0.008</td>
<td>69.4097</td>
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<td>69.4097</td>
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<td>-0.006</td>
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<td>56.8051</td>
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<td>42.6870</td>
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<td></td>
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For power index $k=0.5$

<table>
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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
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<tr>
<td>-0.01</td>
<td>67.2908</td>
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<td>-0.008</td>
<td>57.7441</td>
<td>0.008</td>
<td>58.0332</td>
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<td>47.3612</td>
<td>0.006</td>
<td>47.6505</td>
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<td>-0.004</td>
<td>35.7342</td>
<td>0.004</td>
<td>36.0387</td>
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<td>21.8843</td>
<td>0.002</td>
<td>22.2084</td>
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<tr>
<td>0.00000001</td>
<td>0.0040</td>
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<td></td>
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</tbody>
</table>

For power index $k=1$

<table>
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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
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<td>$P\left(\frac{N}{mm^2}\right)$</td>
<td>$N_0\left(\frac{N}{mm}\right)$</td>
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<td>----------------</td>
<td>------------------------</td>
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<td>----------------</td>
</tr>
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<td>53.4011</td>
<td>0.008</td>
<td>53.6732</td>
</tr>
<tr>
<td>-0.006</td>
<td>43.8191</td>
<td>0.006</td>
<td>44.0943</td>
</tr>
<tr>
<td>-0.004</td>
<td>33.0905</td>
<td>0.004</td>
<td>33.3827</td>
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<tr>
<td>-0.002</td>
<td>20.3146</td>
<td>0.002</td>
<td>20.6192</td>
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<tr>
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<td>0.0040</td>
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</tr>
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For power index $k=10$

<table>
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<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
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<tbody>
<tr>
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<td>0.01</td>
<td>50.2252</td>
</tr>
<tr>
<td>-0.008</td>
<td>43.0838</td>
<td>0.008</td>
<td>43.1776</td>
</tr>
<tr>
<td>-0.006</td>
<td>35.4138</td>
<td>0.006</td>
<td>35.5098</td>
</tr>
<tr>
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<td>26.8246</td>
<td>0.004</td>
<td>26.9211</td>
</tr>
<tr>
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<td>16.5897</td>
<td>0.002</td>
<td>16.6906</td>
</tr>
<tr>
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<td>0.0023</td>
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<td></td>
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For power index $k=99$

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<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
</thead>
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<td>-0.01</td>
<td>47.5601</td>
<td>0.01</td>
<td>47.5701</td>
</tr>
<tr>
<td>-0.008</td>
<td>40.9035</td>
<td>0.008</td>
<td>40.9140</td>
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<tr>
<td>-0.006</td>
<td>33.6605</td>
<td>0.006</td>
<td>33.6715</td>
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<tr>
<td>-0.004</td>
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<td>0.004</td>
<td>25.5598</td>
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Table 9.5

In-plane stress resultant $N_0$ for different values of $k$ of a clamped-clamped beam using rule of mixtures and applying classical theory

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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.002</td>
<td>15.8795</td>
<td>0.002</td>
<td>15.8922</td>
</tr>
<tr>
<td>0.00000001</td>
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<td></td>
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For power index $k=0$

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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
</tr>
</thead>
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<tr>
<td>-0.008</td>
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<td>52.7009</td>
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<td>41.5011</td>
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<tr>
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<td>29.1649</td>
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For power index $k=0.5$

<table>
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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
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<td>50.5478</td>
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<td>40.2762</td>
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<td>28.9444</td>
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</table>

For power index \( k=1 \)

<table>
<thead>
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<th>( N_0 \left( \frac{N}{mm} \right) )</th>
<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
</tr>
</thead>
<tbody>
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<td>57.4416</td>
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<tr>
<td>-0.008</td>
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<td>48.6223</td>
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<tr>
<td>-0.002</td>
<td>15.6985</td>
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<td>15.7518</td>
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</table>

For power index \( k=10 \)

<table>
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<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>44.8539</td>
<td>0.01</td>
<td>44.8701</td>
</tr>
<tr>
<td>-0.008</td>
<td>38.0139</td>
<td>0.008</td>
<td>38.0354</td>
</tr>
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<td>-0.006</td>
<td>30.5630</td>
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<td>22.3604</td>
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<td>22.4004</td>
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<td>-0.002</td>
<td>12.7522</td>
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<td>0.0032</td>
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</table>

For power index \( k=99 \)
Table 9.6

*In-plane stress resultant* $N_0$ for different values of $k$ of a clamped-clamped beam using Mori-Tanaka method and applying classical theory

<table>
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<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>21.1451</td>
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For power index $k=0$

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<th>$P \left( \frac{N}{\text{mm}^2} \right)$</th>
<th>$N_0 \left( \frac{N}{\text{mm}} \right)$</th>
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<td>38.1451</td>
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For power index $k=1$

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<th>$P \left( \frac{N}{\text{mm}^2} \right)$</th>
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<td>36.0035</td>
<td>0.006</td>
<td>36.0466</td>
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<td>-0.004</td>
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<td>0.004</td>
<td>26.2671</td>
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For power index $k=10$

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<th>$P \left( \frac{N}{\text{mm}^2} \right)$</th>
<th>$N_0 \left( \frac{N}{\text{mm}} \right)$</th>
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<td>36.6371</td>
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For power index $k=99$

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<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
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Table 9.7

In-plane stress resultant $N_0$ for different values of $k$ of a clamped-clamped beam using rule of mixtures and applying FSDT.

For power index $k=0$

<table>
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<th>$N_0\left(\frac{N}{mm}\right)$</th>
<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
</thead>
<tbody>
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<td>0.01</td>
<td>62.9572</td>
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<tr>
<td>( P \left( \frac{N}{mm^2} \right) )</td>
<td>( N_0 \left( \frac{N}{mm} \right) )</td>
<td>( P \left( \frac{N}{mm^2} \right) )</td>
<td>( N_0 \left( \frac{N}{mm} \right) )</td>
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<tr>
<td>-----</td>
<td>-----</td>
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<td>-----</td>
</tr>
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<td>52.6975</td>
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<td>41.4985</td>
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<td>29.1633</td>
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For power index \( k=0.5 \)

<table>
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<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
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<tr>
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<td>40.2562</td>
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<td>40.2712</td>
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<td>28.8596</td>
<td>0.004</td>
<td>28.9411</td>
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</table>

For power index \( k=1 \)

<table>
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<th>( N_0 \left( \frac{N}{mm} \right) )</th>
<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
</tr>
</thead>
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<tr>
<td>-0.01</td>
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<td>57.4326</td>
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<td>( P \left( \frac{N}{mm^2} \right) )</td>
<td>( N_0 \left( \frac{N}{mm} \right) )</td>
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For power index \( k=10 \)

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<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
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</thead>
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<td>38.0287</td>
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<td>30.6135</td>
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<td>0.004</td>
<td>22.3970</td>
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<td>0.002</td>
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For power index \( k=99 \)

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<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
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Table 9.8

*In-plane stress resultant* $N_0$ *for different values of* $k$ *of clamped-clamped beam using Mori-tanaka method and applying FSDT*

<table>
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<th>$P \left( \frac{N}{mm^2} \right)$</th>
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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
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<td>$P\left(\frac{N}{mm^2}\right)$</td>
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</tr>
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For power index $k=10$

<table>
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<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>43.1860</td>
<td>0.01</td>
<td>43.2066</td>
</tr>
<tr>
<td>-0.008</td>
<td>36.6235</td>
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<td>36.6321</td>
</tr>
<tr>
<td>-0.006</td>
<td>29.5094</td>
<td>0.006</td>
<td>29.5526</td>
</tr>
<tr>
<td>-0.004</td>
<td>21.6485</td>
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<td>21.6900</td>
</tr>
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<td>-0.002</td>
<td>12.4383</td>
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<td>12.4294</td>
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For power index $k=99$

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<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
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Table 9.9

**In-plane stress resultant** $N_0$ for **different values of k of clamped-clamped beam using rule of mixtures and applying classical theory, for temperature difference 1.2K**

<table>
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<tr>
<th>$k$</th>
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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
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</thead>
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</tr>
<tr>
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</tr>
<tr>
<td></td>
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<tr>
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</table>

For power index $k=0$

For power index $k=0.5$

<table>
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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>57.7643</td>
<td>0.01</td>
<td>57.7602</td>
</tr>
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<td></td>
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<tr>
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<td>0.008</td>
<td>48.2236</td>
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<tr>
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<td>0.006</td>
<td>38.0609</td>
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<td>26.6436</td>
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</tr>
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<td></td>
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For power index $k=1$

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<th>$N_0\left(\frac{N}{mm}\right)$</th>
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<td>0.01</td>
<td>55.4500</td>
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<tr>
<td>-0.008</td>
<td>46.5421</td>
<td>0.008</td>
<td>46.5146</td>
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<td>36.9257</td>
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<td>0.004</td>
<td>26.1685</td>
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For power index $k=10$

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<th>$N_0\left(\frac{N}{mm}\right)$</th>
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<tr>
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<td>43.5573</td>
<td>0.01</td>
<td>43.5355</td>
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<td>36.7416</td>
<td>0.008</td>
<td>36.7619</td>
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<tr>
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<td>29.3218</td>
<td>0.006</td>
<td>29.3455</td>
</tr>
<tr>
<td>-0.004</td>
<td>21.0640</td>
<td>0.004</td>
<td>21.1110</td>
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For power index $k=99$

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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
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</thead>
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<td>-0.01</td>
<td>40.6537</td>
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<td>40.6669</td>
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<tr>
<td>-0.008</td>
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<td>0.008</td>
<td>34.4444</td>
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<td>0.006</td>
<td>27.6496</td>
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<td>11.1925</td>
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Table 9.10

In-plane stress resultant $N_0$ for different values of $k$ of a clamped-clamped subjected to temperature difference of 1.2K and mechanical loads using rule of mixtures and applying FSDT theory

For power index $k=0$

<table>
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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.01</td>
<td>60.9499</td>
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<td>60.9499</td>
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<td>0.008</td>
<td>50.5439</td>
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<td>39.4219</td>
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For power index $k=0.5$

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<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
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<tbody>
<tr>
<td>-0.01</td>
<td>57.7603</td>
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<tr>
<td>-0.008</td>
<td>48.2485</td>
<td>0.008</td>
<td>48.2172</td>
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<tr>
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<td>38.0277</td>
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<td>38.0562</td>
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<td>26.6147</td>
<td>0.004</td>
<td>26.6404</td>
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<td>13.1550</td>
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<td>13.2369</td>
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<tr>
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For power index $k=1$

<table>
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<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
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<tbody>
<tr>
<td>-0.01</td>
<td>55.4943</td>
<td>0.01</td>
<td>55.4410</td>
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<td>-0.008</td>
<td>46.5396</td>
<td>0.008</td>
<td>46.5074</td>
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<tr>
<td>-0.006</td>
<td>36.9030</td>
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<td>36.9202</td>
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<tr>
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<td>26.1473</td>
<td>0.004</td>
<td>26.1648</td>
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For power index \( k=10 \)

<table>
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<tr>
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<td>29.3198</td>
<td>0.006</td>
<td>29.3406</td>
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<tr>
<td>-0.004</td>
<td>21.0627</td>
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<td>21.1078</td>
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<td>0.002</td>
<td>11.3755</td>
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For power index \( k=99 \)

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<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
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<tr>
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<tr>
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<td>34.4304</td>
<td>0.008</td>
<td>34.4404</td>
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<tr>
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<td>27.6500</td>
<td>0.006</td>
<td>27.6466</td>
</tr>
<tr>
<td>-0.004</td>
<td>20.0642</td>
<td>0.004</td>
<td>20.1039</td>
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<tr>
<td>-0.002</td>
<td>11.1997</td>
<td>0.002</td>
<td>11.1915</td>
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Table 9.11

In-plane stress resultant \( N_0 \) for different values of \( k \) of a clamped-clamped beam subjected to both thermal and mechanical loads, using mori-tanaka and applying classical theory (temperature difference of 1.2K)
For power index $k=0$

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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
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<tr>
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<td>0.01</td>
<td>60.9433</td>
</tr>
<tr>
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<td>50.5397</td>
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<td>39.4202</td>
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<td>26.9838</td>
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<td>0.002</td>
<td>12.4803</td>
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<tr>
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<td></td>
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</table>

For power index $k=0.5$

<table>
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<th>$N_0 \left( \frac{N}{mm} \right)$</th>
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<tbody>
<tr>
<td>-0.01</td>
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<td>45.9966</td>
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<tr>
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<tr>
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<td>26.0629</td>
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<tr>
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<td>13.6993</td>
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<tr>
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<td>-4.0263</td>
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For power index $k=1$

<table>
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<th>$P \left( \frac{N}{mm^2} \right)$</th>
<th>$N_0 \left( \frac{N}{mm} \right)$</th>
</tr>
</thead>
<tbody>
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<td>$P\left(\frac{N}{mm^2}\right)$</td>
<td>$N_0\left(\frac{N}{mm}\right)$</td>
</tr>
<tr>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
</tr>
<tr>
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<td>0.008</td>
<td>43.3865</td>
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<tr>
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<td>34.6507</td>
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<td>13.3469</td>
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<tr>
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For power index $k=10$

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<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
</thead>
<tbody>
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<td>42.0471</td>
</tr>
<tr>
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<td>35.5425</td>
<td>0.008</td>
<td>35.5450</td>
</tr>
<tr>
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<td>28.4520</td>
</tr>
<tr>
<td>-0.004</td>
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<tr>
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For power index $k=99$

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<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
</thead>
<tbody>
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<td>40.5497</td>
</tr>
<tr>
<td>-0.008</td>
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<td>34.3696</td>
</tr>
<tr>
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<td>27.6054</td>
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<tr>
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<td>0.004</td>
<td>20.0723</td>
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</table>
Table 9.12

In-plane stress resultant $N_0$ for different values of $k$ of a clamped-clamped beam subjected to both thermal and mechanical loads, using mori-tanaka and applying FSDT theory (temperature difference of 1.2K)

<table>
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<th>$N_0\left(\frac{N}{mm}\right)$</th>
<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
</thead>
<tbody>
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For power index $k=0$

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<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
</thead>
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<td>60.9433</td>
</tr>
<tr>
<td>-0.008</td>
<td>50.5363</td>
<td>0.008</td>
<td>50.5363</td>
</tr>
<tr>
<td>-0.006</td>
<td>39.4179</td>
<td>0.006</td>
<td>39.4179</td>
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<tr>
<td>-0.004</td>
<td>26.9822</td>
<td>0.004</td>
<td>26.9822</td>
</tr>
<tr>
<td>-0.002</td>
<td>12.4796</td>
<td>0.002</td>
<td>12.4796</td>
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<tr>
<td>0.0000001</td>
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<td></td>
</tr>
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</table>

For power index $k=0.5$

<table>
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<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
</tr>
</thead>
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<td>54.8211</td>
<td>0.01</td>
<td>54.7810</td>
</tr>
<tr>
<td>-0.008</td>
<td>46.0401</td>
<td>0.008</td>
<td>45.9920</td>
</tr>
<tr>
<td>-0.006</td>
<td>36.5946</td>
<td>0.006</td>
<td>36.6097</td>
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<td>( P \left( \frac{N}{mm^2} \right) )</td>
<td>( N_0 \left( \frac{N}{mm} \right) )</td>
</tr>
<tr>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
<td>-----------------</td>
</tr>
<tr>
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<td>26.0605</td>
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<tr>
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For power index \( k=1 \)

<table>
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<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
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For power index \( k=10 \)

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<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
<th>( P \left( \frac{N}{mm^2} \right) )</th>
<th>( N_0 \left( \frac{N}{mm} \right) )</th>
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For power index $k=99$

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<th>$P\left(\frac{N}{mm^2}\right)$</th>
<th>$N_0\left(\frac{N}{mm}\right)$</th>
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