Optimization of Spacecraft Formations about Lagrange Points for the Next Generation Space Weather Prediction Mission

Roberto Cuéllar Rangel

Follow this and additional works at: https://commons.erau.edu/edt

Part of the Aerospace Engineering Commons, and the Oceanography and Atmospheric Sciences and Meteorology Commons

Scholarly Commons Citation


This Thesis - Open Access is brought to you for free and open access by Scholarly Commons. It has been accepted for inclusion in PhD Dissertations and Master's Theses by an authorized administrator of Scholarly Commons. For more information, please contact commons@erau.edu.
OPTIMIZATION OF SPACECRAFT FORMATIONS ABOUT LAGRANGE POINTS FOR THE NEXT GENERATION SPACE WEATHER PREDICTION MISSION

By

Roberto Cuéllar Rangel

A Thesis Submitted to the Faculty of Embry-Riddle Aeronautical University
In Partial Fulfillment of the Requirements for the Degree of
Master of Science in Aerospace Engineering

July 2020
Embry-Riddle Aeronautical University
Daytona Beach, Florida
OPTIMIZATION OF SPACECRAFT FORMATIONS ABOUT LAGRANGE POINTS
FOR THE NEXT GENERATION SPACE WEATHER PREDICTION MISSION

By

Roberto Cuéllar Rangel

This Thesis was prepared under the direction of the candidate’s Thesis Committee Chair, Dr. Troy Henderson, Department of Aerospace Engineering, and has been approved by the members of the Thesis Committee. It was submitted to the Office of the Senior Vice President for Academic Affairs and Provost, and was accepted in the partial fulfillment of the requirements for the Degree of Master of Science in Aerospace Engineering.

THESIS COMMITTEE

Troy A. Henderson
Chairman, Dr. Troy Henderson

Bogdan Udrea
Member, Dr. Bogdan Udrea

Xuanye Ma
Member, Dr. Xuanye Ma

Katariina Nykyri
Co-Chairman, Dr. Katariina Nykyri

Morad Nazari
Member, Dr. Morad Nazari

Marwan Al-Haik
Graduate Program Coordinator,
Dr. Marwan Al-Haik

Maj Mirmirani
Dean of the College of Engineering,
Dr. Maj Mirmirani

Christopher Grant
Associate Provost of Academic Support,
Dr. Christopher Grant

Date: 09/15/2020

Date: 09/17/2020

Date: 09/17/2020
ACKNOWLEDGMENTS

The completion of this thesis would not have been possible without the support of many important people to me and who I would like to acknowledge. First, I would like to thank my family, especially my parents and my sister, for all the support they have given me, not only since I have been living in the states, but throughout all my life. ¡Gracias hoy y siempre! Next, I would like to thank my advisor, Dr. Troy Henderson, for all the help and support he had provided me, even before he was my advisor and when I only knew him as my professor. I would also like to give my sincerest thanks to my co-advisor, Dr. Katariina Nykyri. Since the first day I met her, she has supported me with everything I have asked. I am grateful that she let me be part of her space physics research group even though I am not a physicist. My most sincere gratitude to Dr. Bogdan Udrea as well. He was my original advisor and the one that brought me into the project. I first started learning about the restricted-three body problem under his guidance. I would also like to thank my other committee members, Dr. Xuanye Ma and Dr. Morad Nazari, for teaching me new things and for all their feedback.

Thanks to Mark Herring, who also contributed to this mission in the astrodynamics field and always supported me. Thanks to Dr. Pedro Llanos, who helped me better understand the design of orbits at libration points. I would also like to thank the ERAU athletic department, especially coach David Paschal, to let me be part of the tennis team, first as a player, and eventually as an assistant coach during my time at Embry-Riddle. I also want to thank all my teammates and friends from Mexico and the US; their friendship means a lot. Thanks to Dean Maj Mimirani and Dean Karen Gaines for giving me the financial support to complete this thesis at NASA JPL, which is where I am currently finishing this work. Finally, I would like to thank Dr. Shyam Bhaskaran, Dr. Reza Karimi, Dr. Jon Sims, and Dr. Roby Wilson from JPL for bringing me in despite all the obstacles due to the world’s current sanitary situation.
ABSTRACT

This thesis’s work serves as proof of concept for the Next Generation Space Weather Prediction Mission, a multi-spacecraft mission at various libration points whose objective is to forecast Space Weather hazards with a 1-2-day warning time. This thesis deals with the design and control of orbits of spacecraft formations at different libration points. The systems studied are Sun-Earth, Sun-Venus, Sun-Mercury, and Sun-Mars. The orbit design and formation keeping control of the spacecraft are solved simultaneously using an optimization software called DIDO. Initial conditions are obtained through two different strategies. The first one, by placing the spacecraft in a tetrahedral formation and using Monte Simulations to find the initial velocities. The second strategy suggests holding velocities fixed while initial locations of the spacecraft are chosen randomly. All results are verified and validated by applying Pontryagin’s principle to the optimal control problem by “hand”, and then comparing the results with the outputs from the optimization software.

$L_{4}$ and $L_{5}$ points of any systems are the easiest to work with due to their inherent stability. In most cases, the use of the $L^1$-norm of control as the cost function yields the lowest station-keeping cost. The culminating remark is that the Next Generation Space Weather Prediction Mission is feasible from an astrodynamics perspective.
TABLE OF CONTENTS

ACKNOWLEDGMENTS ......................................................... iii

ABSTRACT ........................................................................ iv

LIST OF FIGURES .............................................................. vii

LIST OF TABLES ................................................................. xv

NOMENCLATURE ............................................................... xviii

1 Introduction ................................................................. 1
  1.1. Space Weather ......................................................... 2
      1.1.1. Consequences ................................................... 5
      1.1.2. Current Status of Space Weather Prediction ............... 9
      1.1.3. Previous Missions ............................................... 10
  1.2. The Next Generation Space Weather Prediction Mission ....... 12
      1.2.1. Mission Overview .............................................. 13
      1.2.2. Goals and Objectives ......................................... 14
      1.2.3. Banana Region .................................................. 14
  1.3. The Circular Restricted Three Body Problem .................... 16
      1.3.1. Problem History ............................................... 16

2 Background ..................................................................... 18
  2.1. CRTBP ................................................................... 18
  2.2. Scaling Units .......................................................... 24
  2.3. Equilibrium Points ................................................... 25
      2.3.1. Stability of Libration Points ................................. 28
  2.4. Spacecraft Formation Design and Control ...................... 32
      2.4.1. Standard Procedure ........................................... 32
  2.5. Pseudospectral Methods in Optimal Control ................... 35
      2.5.1. Pseudospectral Discretization .............................. 35
  2.6. PS Theory .............................................................. 38
  2.7. Poincaré Plots ......................................................... 41

3 Theory and Procedures .................................................. 48
  3.1. Dynamics and Control of the Spacecraft ....................... 48
  3.2. Cost Function ........................................................ 50
  3.3. Constraints ............................................................ 52
  3.4. Solving the Optimal Control Problem .......................... 53
  3.5. Initial Conditions .................................................... 55
      3.5.1. Monte Carlo ..................................................... 56
  3.6. Delta-V Calculation .................................................. 60
  3.7. Verification and Validation ....................................... 60
      3.7.1. Pontryagin’s Principle ........................................ 60
      3.7.2. V & V on the Proposed Optimal Control Problem ....... 64

4 Simulations and Results .................................................. 72
4.1. Sun-Mercury ........................................................................... 73
   4.1.1. \( L_1 \) ........................................................................... 73
   4.1.2. \( L_3 \) ........................................................................... 77
   4.1.3. \( L_4 \) ........................................................................... 81
   4.1.4. \( L_5 \) ........................................................................... 85
4.2. Sun-Venus ............................................................................... 89
   4.2.1. \( L_1 \) ........................................................................... 89
   4.2.2. \( L_3 \) ........................................................................... 93
   4.2.3. \( L_4 \) ........................................................................... 97
   4.2.4. \( L_5 \) ......................................................................... 101
4.3. Sun-Earth .............................................................................. 105
   4.3.1. \( L_1 \) ........................................................................... 105
   4.3.2. \( L_4 \) ........................................................................... 109
4.4. Sun-Mars ............................................................................... 113
   4.4.1. \( L_1 \) ........................................................................... 113
   4.4.2. \( L_3 \) ........................................................................... 117
   4.4.3. \( L_4 \) ........................................................................... 121
   4.4.4. \( L_5 \) ......................................................................... 125

5 Conclusion ............................................................................. 129

REFERENCES ........................................................................... 131

A APPENDIX - More Results ...................................................... 136
   A.1. Sun-Mercury ................................................................. 136
       A.1.1. \( L_3 \) ................................................................. 136
       A.1.2. \( L_4 \) ................................................................. 139
       A.1.3. \( L_5 \) ................................................................. 142
   A.2. Sun-Venus ................................................................. 145
       A.2.1. \( L_3 \) ................................................................. 145
       A.2.2. \( L_4 \) ................................................................. 148
       A.2.3. \( L_5 \) ................................................................. 151
   A.3. Sun-Mars ................................................................. 154
       A.3.1. \( L_3 \) ................................................................. 154
       A.3.2. \( L_5 \) ................................................................. 157
**LIST OF FIGURES**

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Monthly average of sunspots count from solar cycle 12 to 24, spanning the years 1880 to 2020 (Hathaway, 2015)</td>
<td>4</td>
</tr>
<tr>
<td>1.2</td>
<td>2D structure of the Parker spiral</td>
<td>5</td>
</tr>
<tr>
<td>1.3</td>
<td>Heliospheric current sheet (NASA, 2013b)</td>
<td>5</td>
</tr>
<tr>
<td>1.4</td>
<td>When the solar magnetic field lines (yellow) are anti-parallel to Earth’s magnetic field (blue), the fields can reconnect, and Earth’s magnetic shield (magnetopause) gets penetrated by the SW plasma (JAXA, 2016)</td>
<td>6</td>
</tr>
<tr>
<td>1.5</td>
<td>Consequences produced by space weather (ESA, 2018)</td>
<td>9</td>
</tr>
<tr>
<td>1.6</td>
<td>Heliophysics System Observatory (NASA, 2020a)</td>
<td>13</td>
</tr>
<tr>
<td>1.7</td>
<td>Banana Region: Traces of 9 years of the SW magnetic field, which eventually reaches the Sun-Earth $L_1$ point. The color bar represents the logarithmic number of the counts (X. Ma, personal communication, July 2, 2020)</td>
<td>15</td>
</tr>
<tr>
<td>2.1</td>
<td>Geometry of the CRTBP</td>
<td>19</td>
</tr>
<tr>
<td>2.2</td>
<td>Relationship between mass and distance</td>
<td>21</td>
</tr>
<tr>
<td>2.3</td>
<td>Different Hill’s regions configurations for various values of the Jacobi constant $C$ of a system with a mass ratio of 0.1. The white domains correspond to the Hill’s regions, the gray shaded domains indicate the energetically forbidden regions, and the thick black lines depict the Zero Velocity Curves. The yellow and green points represent the primary bodies, while the crosses represent the locations of the five libration points</td>
<td>23</td>
</tr>
<tr>
<td>2.4</td>
<td>Location of equilibrium points in the CRTBP</td>
<td>27</td>
</tr>
<tr>
<td>2.5</td>
<td>Recreation of Richardson’s ISEE-3 halo reference orbit as described in <em>Halo Orbit Formulation for the ISEE-3 Mission</em> (Richardson, 1980)</td>
<td>33</td>
</tr>
<tr>
<td>2.6</td>
<td>Schematic of dualization, discretization and the Covector Mapping Principle (Ross &amp; Fahroo, 2002)</td>
<td>39</td>
</tr>
<tr>
<td>2.7</td>
<td>Complete Covector Mapping Principle (Ross, 2015)</td>
<td>40</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>2.8</td>
<td>Paths for solving Lambert’s problem. For the long way, the transfer angle is</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td>greater than 180°, and vice versa (Vallado, 2007).</td>
<td></td>
</tr>
<tr>
<td>2.9</td>
<td>Geometry of Lambert’s problem (Schaub &amp; Junkins, 2009)</td>
<td>43</td>
</tr>
<tr>
<td>2.10</td>
<td>Multiple Earth-Mars porkchop plots from 2030-2040 showing change in</td>
<td>44</td>
</tr>
<tr>
<td></td>
<td>shape and structure</td>
<td></td>
</tr>
<tr>
<td>2.11</td>
<td>Multiple Earth-Venus porkchop plots from 2030-2040 showing change in</td>
<td>45</td>
</tr>
<tr>
<td></td>
<td>shape and structure</td>
<td></td>
</tr>
<tr>
<td>3.1</td>
<td>Controls of the spacecraft</td>
<td>48</td>
</tr>
<tr>
<td>3.2</td>
<td>Representation of an unknotted optimal control problem with 40 nodes</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td>(Gong &amp; Ross, 2008)</td>
<td></td>
</tr>
<tr>
<td>3.3</td>
<td>Representation of a knotted optimal control problem with 40 nodes (Gong</td>
<td>55</td>
</tr>
<tr>
<td></td>
<td>&amp; Ross, 2008)</td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td>DIDO takes Problem $B$ as an input and outputs a candidate solution to</td>
<td>56</td>
</tr>
<tr>
<td></td>
<td>Problem $B^\lambda$. The candidate solution can then be tested for</td>
<td></td>
</tr>
<tr>
<td></td>
<td>optimality through Pontryagin’s principle (Ross, 2015)</td>
<td></td>
</tr>
<tr>
<td>3.5</td>
<td>Initial locations for a spacecraft formation at Sun-Mars $L_4$</td>
<td>58</td>
</tr>
<tr>
<td>3.6</td>
<td>Example of how the Hamiltonian evolution should look for the optimal</td>
<td>67</td>
</tr>
<tr>
<td></td>
<td>control problem of one spacecraft</td>
<td></td>
</tr>
<tr>
<td>4.1</td>
<td>Sun-Mercury $L_4$: 3D solution</td>
<td>73</td>
</tr>
<tr>
<td>4.2</td>
<td>Sun-Mercury $L_4$: Separations between spacecraft throughout time</td>
<td>74</td>
</tr>
<tr>
<td>4.3</td>
<td>Sun-Mercury $L_4$: Velocity profiles in $x, y, z$</td>
<td>74</td>
</tr>
<tr>
<td>4.4</td>
<td>Sun-Mercury $L_4$: Control profiles in $x, y, z$</td>
<td>75</td>
</tr>
<tr>
<td>4.5</td>
<td>Sun-Mercury $L_4$: Thrust directions. The arrows represent the direction</td>
<td>75</td>
</tr>
<tr>
<td></td>
<td>where the spacecraft has to thrust to maintain the desired orbit</td>
<td></td>
</tr>
<tr>
<td>4.6</td>
<td>Sun-Mercury $L_4$: Hamiltonian evolution</td>
<td>76</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>4.7</td>
<td>Sun-Mercury $L_3$: 3D solution</td>
<td>77</td>
</tr>
<tr>
<td>4.8</td>
<td>Sun-Mercury $L_3$: Separations between spacecraft throughout time</td>
<td>78</td>
</tr>
<tr>
<td>4.9</td>
<td>Sun-Mercury $L_3$: Velocity profiles in $x, y, z$</td>
<td>78</td>
</tr>
<tr>
<td>4.10</td>
<td>Sun-Mercury $L_3$: Control profiles in $x, y, z$</td>
<td>79</td>
</tr>
<tr>
<td>4.11</td>
<td>Sun-Mercury $L_3$: Thrust directions. The arrows represent the direction</td>
<td>79</td>
</tr>
<tr>
<td></td>
<td>where the spacecraft has to thrust to maintain the desired orbit</td>
<td></td>
</tr>
<tr>
<td>4.12</td>
<td>Sun-Mercury $L_3$: Hamiltonian evolution</td>
<td>80</td>
</tr>
<tr>
<td>4.13</td>
<td>Sun-Mercury $L_4$: 3D solution</td>
<td>81</td>
</tr>
<tr>
<td>4.14</td>
<td>Sun-Mercury $L_4$: Separations between spacecraft throughout time</td>
<td>82</td>
</tr>
<tr>
<td>4.15</td>
<td>Sun-Mercury $L_4$: Velocity profiles in $x, y, z$</td>
<td>82</td>
</tr>
<tr>
<td>4.16</td>
<td>Sun-Mercury $L_4$: Control profiles in $x, y, z$</td>
<td>83</td>
</tr>
<tr>
<td>4.17</td>
<td>Sun-Mercury $L_4$: Thrust directions. The arrows represent the direction</td>
<td>83</td>
</tr>
<tr>
<td></td>
<td>where the spacecraft has to thrust to maintain the desired orbit</td>
<td></td>
</tr>
<tr>
<td>4.18</td>
<td>Sun-Mercury $L_4$: Hamiltonian evolution</td>
<td>84</td>
</tr>
<tr>
<td>4.19</td>
<td>Sun-Mercury $L_5$: 3D solution</td>
<td>85</td>
</tr>
<tr>
<td>4.20</td>
<td>Sun-Mercury $L_5$: Separations between spacecraft throughout time</td>
<td>86</td>
</tr>
<tr>
<td>4.21</td>
<td>Sun-Mercury $L_5$: Velocity profiles in $x, y, z$</td>
<td>86</td>
</tr>
<tr>
<td>4.22</td>
<td>Sun-Mercury $L_5$: Control profiles in $x, y, z$</td>
<td>87</td>
</tr>
<tr>
<td>4.23</td>
<td>Sun-Mercury $L_5$: Thrust directions. The arrows represent the direction</td>
<td>87</td>
</tr>
<tr>
<td></td>
<td>where the spacecraft has to thrust to maintain the desired orbit</td>
<td></td>
</tr>
<tr>
<td>4.24</td>
<td>Sun-Mercury $L_5$: Hamiltonian evolution</td>
<td>88</td>
</tr>
<tr>
<td>4.25</td>
<td>Sun-Venus $L_1$: 3D solution</td>
<td>89</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>4.26</td>
<td>Sun-Venus $L_1$: Separations between spacecraft throughout time</td>
<td>90</td>
</tr>
<tr>
<td>4.27</td>
<td>Sun-Venus $L_1$: Velocity profiles in $x, y, z$</td>
<td>90</td>
</tr>
<tr>
<td>4.28</td>
<td>Sun-Venus $L_1$: Control profiles in $x, y, z$</td>
<td>91</td>
</tr>
<tr>
<td>4.29</td>
<td>Sun-Venus $L_1$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit</td>
<td>91</td>
</tr>
<tr>
<td>4.30</td>
<td>Sun-Venus $L_1$: Hamiltonian evolution</td>
<td>92</td>
</tr>
<tr>
<td>4.31</td>
<td>Sun-Venus $L_3$: 3D solution</td>
<td>93</td>
</tr>
<tr>
<td>4.32</td>
<td>Sun-Venus $L_3$: Separations between spacecraft throughout time</td>
<td>94</td>
</tr>
<tr>
<td>4.33</td>
<td>Sun-Venus $L_3$: Velocity profiles in $x, y, z$</td>
<td>94</td>
</tr>
<tr>
<td>4.34</td>
<td>Sun-Venus $L_3$: Control profiles in $x, y, z$</td>
<td>95</td>
</tr>
<tr>
<td>4.35</td>
<td>Sun-Venus $L_3$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit</td>
<td>95</td>
</tr>
<tr>
<td>4.36</td>
<td>Sun-Venus $L_3$: Hamiltonian evolution</td>
<td>96</td>
</tr>
<tr>
<td>4.37</td>
<td>Sun-Venus $L_4$: 3D solution</td>
<td>97</td>
</tr>
<tr>
<td>4.38</td>
<td>Sun-Venus $L_4$: Separations between spacecraft throughout time</td>
<td>98</td>
</tr>
<tr>
<td>4.39</td>
<td>Sun-Venus $L_4$: Velocity profiles in $x, y, z$</td>
<td>98</td>
</tr>
<tr>
<td>4.40</td>
<td>Sun-Venus $L_4$: Control profiles in $x, y, z$</td>
<td>99</td>
</tr>
<tr>
<td>4.41</td>
<td>Sun-Venus $L_4$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit</td>
<td>99</td>
</tr>
<tr>
<td>4.42</td>
<td>Sun-Venus $L_4$: Hamiltonian evolution</td>
<td>100</td>
</tr>
<tr>
<td>4.43</td>
<td>Sun-Venus $L_5$: 3D solution</td>
<td>101</td>
</tr>
<tr>
<td>4.44</td>
<td>Sun-Venus $L_5$: Separations between spacecraft throughout time</td>
<td>102</td>
</tr>
</tbody>
</table>
Figure | Page
---|---
4.45 Sun-Venus $L_5$: Velocity profiles in $x, y, z$ | 102
4.46 Sun-Venus $L_5$: Control profiles in $x, y, z$ | 103
4.47 Sun-Venus $L_5$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit | 103
4.48 Sun-Venus $L_5$: Hamiltonian evolution | 104
4.49 Sun-Earth $L_1$: 3D solution | 105
4.50 Sun-Earth $L_1$: Separations between spacecraft throughout time | 106
4.51 Sun-Earth $L_1$: Velocity profiles in $x, y, z$ | 106
4.52 Sun-Earth $L_1$: Control profiles in $x, y, z$ | 107
4.53 Sun-Earth $L_1$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit | 107
4.54 Sun-Earth $L_1$: Hamiltonian evolution | 108
4.55 Sun-Earth $L_4$: 3D solution | 109
4.56 Sun-Earth $L_4$: Separations between spacecraft throughout time | 110
4.57 Sun-Earth $L_4$: Velocity profiles in $x, y, z$ | 110
4.58 Sun-Earth $L_4$: Control profiles in $x, y, z$ | 111
4.59 Sun-Earth $L_4$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit | 111
4.60 Sun-Earth $L_4$: Hamiltonian evolution | 112
4.61 Sun-Mars $L_1$: 3D solution | 113
4.62 Sun-Mars $L_1$: Separations between spacecraft throughout time | 114
4.63 Sun-Mars $L_1$: Velocity profiles in $x, y, z$ | 114
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.64</td>
<td>Sun-Mars $L_1$: Control profiles in $x, y, z$</td>
<td>115</td>
</tr>
<tr>
<td>4.65</td>
<td>Sun-Mars $L_1$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit</td>
<td>115</td>
</tr>
<tr>
<td>4.66</td>
<td>Sun-Mars $L_1$: Hamiltonian evolution</td>
<td>116</td>
</tr>
<tr>
<td>4.67</td>
<td>Sun-Mars $L_3$: 3D solution</td>
<td>117</td>
</tr>
<tr>
<td>4.68</td>
<td>Sun-Mars $L_3$: Separations between spacecraft throughout time</td>
<td>118</td>
</tr>
<tr>
<td>4.69</td>
<td>Sun-Mars $L_3$: Velocity profiles in $x, y, z$</td>
<td>118</td>
</tr>
<tr>
<td>4.70</td>
<td>Sun-Mars $L_3$: Control profiles in $x, y, z$</td>
<td>119</td>
</tr>
<tr>
<td>4.71</td>
<td>Sun-Mars $L_3$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit</td>
<td>119</td>
</tr>
<tr>
<td>4.72</td>
<td>Sun-Mars $L_3$: Hamiltonian evolution</td>
<td>120</td>
</tr>
<tr>
<td>4.73</td>
<td>Sun-Mars $L_4$: 3D solution</td>
<td>121</td>
</tr>
<tr>
<td>4.74</td>
<td>Sun-Mars $L_4$: Separations between spacecraft throughout time</td>
<td>122</td>
</tr>
<tr>
<td>4.75</td>
<td>Sun-Mars $L_4$: Velocity profiles in $x, y, z$</td>
<td>122</td>
</tr>
<tr>
<td>4.76</td>
<td>Sun-Mars $L_4$: Control profiles in $x, y, z$</td>
<td>123</td>
</tr>
<tr>
<td>4.77</td>
<td>Sun-Mars $L_4$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit</td>
<td>123</td>
</tr>
<tr>
<td>4.78</td>
<td>Sun-Mars $L_4$: Hamiltonian evolution</td>
<td>124</td>
</tr>
<tr>
<td>4.79</td>
<td>Sun-Mars $L_5$: 3D solution</td>
<td>125</td>
</tr>
<tr>
<td>4.80</td>
<td>Sun-Mars $L_5$: Separations between spacecraft throughout time</td>
<td>126</td>
</tr>
<tr>
<td>4.81</td>
<td>Sun-Mars $L_5$: Velocity profiles in $x, y, z$</td>
<td>126</td>
</tr>
<tr>
<td>4.82</td>
<td>Sun-Mars $L_5$: Control profiles in $x, y, z$</td>
<td>127</td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>A.14</td>
<td>127</td>
<td></td>
</tr>
<tr>
<td>A.7</td>
<td>128</td>
<td></td>
</tr>
<tr>
<td>A.2</td>
<td>136</td>
<td></td>
</tr>
<tr>
<td>A.3</td>
<td>137</td>
<td></td>
</tr>
<tr>
<td>A.4</td>
<td>138</td>
<td></td>
</tr>
<tr>
<td>A.5</td>
<td>139</td>
<td></td>
</tr>
<tr>
<td>A.6</td>
<td>140</td>
<td></td>
</tr>
<tr>
<td>A.7</td>
<td>140</td>
<td></td>
</tr>
<tr>
<td>A.8</td>
<td>141</td>
<td></td>
</tr>
<tr>
<td>A.9</td>
<td>142</td>
<td></td>
</tr>
<tr>
<td>A.10</td>
<td>143</td>
<td></td>
</tr>
<tr>
<td>A.11</td>
<td>143</td>
<td></td>
</tr>
<tr>
<td>A.12</td>
<td>144</td>
<td></td>
</tr>
<tr>
<td>A.13</td>
<td>145</td>
<td></td>
</tr>
<tr>
<td>A.14</td>
<td>146</td>
<td></td>
</tr>
<tr>
<td>A.15</td>
<td>146</td>
<td></td>
</tr>
<tr>
<td>A.16</td>
<td>147</td>
<td></td>
</tr>
<tr>
<td>Figure</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>A.17 Sun-Venus $L_4$ ($L^2$-norm): 3D solution</td>
<td>148</td>
<td></td>
</tr>
<tr>
<td>A.18 Sun-Venus $L_4$ ($L^2$-norm): Separations between spacecraft</td>
<td>149</td>
<td></td>
</tr>
<tr>
<td>A.19 Sun-Venus $L_4$ ($L^2$-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit</td>
<td>149</td>
<td></td>
</tr>
<tr>
<td>A.20 Sun-Venus $L_4$ ($L^2$-norm): Hamiltonian evolution</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>A.21 Sun-Venus $L_5$ ($L^2$-norm): 3D solution</td>
<td>151</td>
<td></td>
</tr>
<tr>
<td>A.22 Sun-Venus $L_5$ ($L^2$-norm): Separations between spacecraft</td>
<td>152</td>
<td></td>
</tr>
<tr>
<td>A.23 Sun-Venus $L_5$ ($L^2$-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit</td>
<td>152</td>
<td></td>
</tr>
<tr>
<td>A.24 Sun-Venus $L_5$ ($L^2$-norm): Hamiltonian evolution</td>
<td>153</td>
<td></td>
</tr>
<tr>
<td>A.25 Sun-Mars $L_3$ ($L^2$-norm): 3D solution</td>
<td>154</td>
<td></td>
</tr>
<tr>
<td>A.26 Sun-Mars $L_3$ ($L^2$-norm): Separations between spacecraft</td>
<td>155</td>
<td></td>
</tr>
<tr>
<td>A.27 Sun-Mars $L_3$ ($L^2$-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit</td>
<td>155</td>
<td></td>
</tr>
<tr>
<td>A.28 Sun-Mars $L_3$ ($L^2$-norm): Hamiltonian evolution</td>
<td>156</td>
<td></td>
</tr>
<tr>
<td>A.29 Sun-Mars $L_5$ ($L^2$-norm): 3D solution</td>
<td>157</td>
<td></td>
</tr>
<tr>
<td>A.30 Sun-Mars $L_5$ ($L^2$-norm): Separations between spacecraft</td>
<td>158</td>
<td></td>
</tr>
<tr>
<td>A.31 Sun-Mars $L_5$ ($L^2$-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit</td>
<td>158</td>
<td></td>
</tr>
<tr>
<td>A.32 Sun-Mars $L_5$ ($L^2$-norm): Hamiltonian evolution</td>
<td>159</td>
<td></td>
</tr>
</tbody>
</table>
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Scaled units and $\mu^r$ ratio used in this work</td>
<td>25</td>
</tr>
<tr>
<td>2.2</td>
<td>Mercury and Venus libration points location using scaled units</td>
<td>27</td>
</tr>
<tr>
<td>2.3</td>
<td>Earth and Mars libration points location using scaled units</td>
<td>28</td>
</tr>
<tr>
<td>2.4</td>
<td>Optimal $\Delta V$’s and $C_3L$ found for Earth-Mars trajectories for 2030-2040</td>
<td>46</td>
</tr>
<tr>
<td>2.5</td>
<td>Optimal $\Delta V$’s and $C_3L$ found for Earth-Venus trajectories for 2030-2040</td>
<td>46</td>
</tr>
<tr>
<td>3.1</td>
<td>Monte Carlo approach for finding I.C.s</td>
<td>59</td>
</tr>
<tr>
<td>4.1</td>
<td>Sun-Mercury $L_1$: Initial conditions in scaled units</td>
<td>73</td>
</tr>
<tr>
<td>4.2</td>
<td>Sun-Mercury $L_1$: Results</td>
<td>76</td>
</tr>
<tr>
<td>4.3</td>
<td>Sun-Mercury $L_3$: Initial conditions in scaled units</td>
<td>77</td>
</tr>
<tr>
<td>4.4</td>
<td>Sun-Mercury $L_3$: Results</td>
<td>80</td>
</tr>
<tr>
<td>4.5</td>
<td>Sun-Mercury $L_4$: Initial conditions in scaled units</td>
<td>81</td>
</tr>
<tr>
<td>4.6</td>
<td>Sun-Mercury $L_4$: Results</td>
<td>84</td>
</tr>
<tr>
<td>4.7</td>
<td>Sun-Mercury $L_5$: Initial conditions in scaled units</td>
<td>85</td>
</tr>
<tr>
<td>4.8</td>
<td>Sun-Mercury $L_5$: Results</td>
<td>88</td>
</tr>
<tr>
<td>4.9</td>
<td>Sun-Venus $L_1$: Initial conditions in scaled units</td>
<td>89</td>
</tr>
<tr>
<td>4.10</td>
<td>Sun-Venus $L_1$: Results</td>
<td>92</td>
</tr>
<tr>
<td>4.11</td>
<td>Sun-Venus $L_3$: Initial conditions in scaled units</td>
<td>93</td>
</tr>
<tr>
<td>4.12</td>
<td>Sun-Venus $L_3$: Results</td>
<td>96</td>
</tr>
<tr>
<td>4.13</td>
<td>Sun-Venus $L_4$: Initial conditions in scaled units</td>
<td>97</td>
</tr>
<tr>
<td>4.14</td>
<td>Sun-Venus $L_4$: Results</td>
<td>100</td>
</tr>
<tr>
<td>Table</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>4.15  Sun-Venus $L_5$: Initial conditions in scaled units</td>
<td>101</td>
<td></td>
</tr>
<tr>
<td>4.16  Sun-Venus $L_5$: Results</td>
<td>104</td>
<td></td>
</tr>
<tr>
<td>4.17  Sun-Earth $L_1$: Initial conditions in scaled units</td>
<td>105</td>
<td></td>
</tr>
<tr>
<td>4.18  Sun-Earth $L_1$: Results</td>
<td>108</td>
<td></td>
</tr>
<tr>
<td>4.19  Sun-Earth $L_4$: Initial conditions in scaled units</td>
<td>109</td>
<td></td>
</tr>
<tr>
<td>4.20  Sun-Earth $L_4$: Results</td>
<td>112</td>
<td></td>
</tr>
<tr>
<td>4.21  Sun-Mars $L_1$: Initial conditions in scaled units</td>
<td>113</td>
<td></td>
</tr>
<tr>
<td>4.22  Sun-Mars $L_1$: Results</td>
<td>116</td>
<td></td>
</tr>
<tr>
<td>4.23  Sun-Mars $L_3$: Initial conditions in scaled units</td>
<td>117</td>
<td></td>
</tr>
<tr>
<td>4.24  Sun-Mars $L_3$: Results</td>
<td>120</td>
<td></td>
</tr>
<tr>
<td>4.25  Sun-Mars $L_4$: Initial conditions in scaled units</td>
<td>121</td>
<td></td>
</tr>
<tr>
<td>4.26  Sun-Mars $L_4$: Results</td>
<td>124</td>
<td></td>
</tr>
<tr>
<td>4.27  Sun-Mars $L_5$: Initial conditions in scaled units</td>
<td>125</td>
<td></td>
</tr>
<tr>
<td>4.28  Sun-Mars $L_5$: Results</td>
<td>128</td>
<td></td>
</tr>
<tr>
<td>A.1   Sun-Mercury $L_3$: Initial conditions in scaled units</td>
<td>136</td>
<td></td>
</tr>
<tr>
<td>A.2   Sun-Mercury $L_3$: Results</td>
<td>138</td>
<td></td>
</tr>
<tr>
<td>A.3   Sun-Mercury $L_4$: Initial conditions in scaled units</td>
<td>139</td>
<td></td>
</tr>
<tr>
<td>A.4   Sun-Mercury $L_4$: Results</td>
<td>141</td>
<td></td>
</tr>
<tr>
<td>A.5   Sun-Mercury $L_5$: Initial conditions in scaled units</td>
<td>142</td>
<td></td>
</tr>
<tr>
<td>A.6   Sun-Mercury $L_5$: Results</td>
<td>144</td>
<td></td>
</tr>
<tr>
<td>A.7   Sun-Venus $L_3$: Initial conditions in scaled units</td>
<td>145</td>
<td></td>
</tr>
<tr>
<td>Table</td>
<td>Page</td>
<td></td>
</tr>
<tr>
<td>-------</td>
<td>------</td>
<td></td>
</tr>
<tr>
<td>A.8   Sun-Venus $L_3$: Results</td>
<td>147</td>
<td></td>
</tr>
<tr>
<td>A.9   Sun-Venus $L_4$: Initial conditions in scaled units</td>
<td>148</td>
<td></td>
</tr>
<tr>
<td>A.10  Sun-Venus $L_4$: Results</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>A.11  Sun-Venus $L_5$: Initial conditions in scaled units</td>
<td>151</td>
<td></td>
</tr>
<tr>
<td>A.12  Sun-Venus $L_5$: Results</td>
<td>153</td>
<td></td>
</tr>
<tr>
<td>A.13  Sun-Mars $L_3$: Initial conditions in scaled units</td>
<td>154</td>
<td></td>
</tr>
<tr>
<td>A.14  Sun-Mars $L_3$: Results</td>
<td>156</td>
<td></td>
</tr>
<tr>
<td>A.15  Sun-Mars $L_5$: Initial conditions in scaled units</td>
<td>157</td>
<td></td>
</tr>
<tr>
<td>A.16  Sun-Mars $L_5$: Results</td>
<td>159</td>
<td></td>
</tr>
</tbody>
</table>
NOMENCLATURE

$x, y, z$ Components of the position vector of the spacecraft relative to the equilibrium point in the nondimensional and nonlinear form

$X_{eq}, Y_{eq}, Z_{eq}$ Coordinates of the equilibrium points

$x$ State vector

$u$ Control vector

$\dot{x}$ Derivative of the state vector with respect to time

$T_x, T_y, T_z$ Thrust components

$a_x, a_y, a_z$ Acceleration components

$J$ Cost function

$E$ Endpoint cost

$F$ Running cost

$m$ Mass

$\dot{m}$ Mass flow rate

$v_e$ Exhaust speed

$t_0$ Initial time

$t_f$ Final time

$H$ Control Hamiltonian

$\mathcal{H}$ Minimized Hamiltonian

$\lambda$ Costate vector

$\bar{E}$ Endpoint Lagrangian

$e$ Endpoint function

$\nu$ Endpoint covector
1. Introduction

Imagine waking up one day and realizing that there is no power in your house, that you have no cellphone service, and that basically anything that connects to a power outlet does not work anywhere in town. This could happen one day if a strong magnetic storm hits Earth.

On September 1, 1859, British astronomer Richard C. Carrington spotted a cluster of enormous dark spots (sunspots) on the Sun, and two patches of intensely bright and white light (solar flares) erupted from the sunspots (Balan et al., 2017). This event, known as the Carrington event, caused telegraph communications to fail and brilliant auroras occurred. The Carrington event has been the most extreme geomagnetic storm ever occurred in known history.

Other severe Space Weather events have also occurred. For instance, in Canada, on March 13, 1989, the Hydro-Quebec electric power grid collapsed in less than 2 minutes due to an extreme geomagnetic storm, which resulted in the loss of electric power to more than six million people for 9 hours at an economical cost estimated of around 13.2 billion Canadian dollars (Medford, Lanzerotti, Kraus, & Maclennan, 1989; Boteler, Pirjola, & Nevanlinna, 1998; Bolduc, 2002). More Space Weather events include the ones that occurred in 2001 and 2003 in New Zealand and Sweden, respectively, which caused power outages (Balan et al., 2017).

More recently, on July 23, 2012, a large and strong coronal mass ejection (CME) penetrated through Earth’s orbit. Fortunately, Earth was not there (it missed it with a margin of approximately nine days). A direct hit by this CME would have caused
widespread power blackouts, complete world-wide air-traffic shutdown, and disabling everything that plugs into a wall socket (Baker et al., 2013). It is estimated that if the Carrington event occurred today, it would produce about $2.6 trillion in damage in the US alone (Lloyd’s of London, 2013).

Although there are many institutions and centers trying to forecast Space Weather events, like NOAA’s Space Weather Prediction Center (SWPC) and the U.S. Air Force’s Weather Agency (AFWA), current prediction models cannot predict the dynamical evolution of CMEs due to various fluid and kinetic instabilities that can evolve at the boundary between CME ejecta and sheath, hence, leaving significant room for improvement (Nykyri & Udrea, 2016). One challenge is the enormous computational resources required to self-consistently model the required physics from the Sun to the Earth. Even the best supercomputers cannot do that. However, scientists rely on plasma approximations such as Magneto Hydro Dynamics (MHD), which treats plasma as a magnetized fluid but neglects the kinetic physics of individual particles. The second source of uncertainty relates to the initial conditions ingested into models that rely on remote sensing measurements. The present thesis is motivated by the need to develop a next-generation space weather prediction model that will utilize and synthesize both in-situ spacecraft observations from multiple locations.

1.1. Space Weather

The term Space Weather became popular in the 1990’s and it generally refers to conditions on the Sun, in the solar wind, and within Earth’s magnetosphere, ionosphere and thermosphere that can influence the performance and reliability of space-borne
and ground-based technological systems and can endanger human life or health (NASA, 2020b). The Sun continuously emits two main types of energy into space—electromagnetic (EM) radiation and corpuscular radiation. Visible light, radio waves, microwaves, infrared, ultraviolet, X-rays, and gamma rays are forms of EM radiation. On the other hand, charged atoms and sub-atomic particles (mostly protons and electrons), the so-called solar wind (SW), represent the corpuscular radiation. The SW expands out into the Solar System carrying with it the Sun’s magnetic field. SW speed typically varies between 350 km/s and 800 km/s. When SW reaches our planet, the Earth’s magnetic field acts as a shield. However, that shield can break, causing aurorae and large electrical currents that can disrupt power grids and satellite navigation systems. When high-speed SW overtakes slow-speed SW, it creates regions of very high densities and strong magnetic fields, called Corotating Interaction Regions (CIRs) (Alves, Echer, & Gonzalez, 2006).

The exact mechanism that generates the magnetic field of the Sun is not fully understood, but it is well established that the total number of localized regions of intense magnetic fields, called sunspots, are known to vary with roughly an 11-year period known as the solar cycle. A higher number of sunspots corresponds to increased solar activity. The solar cycle peak is known as solar maximum and the trough as solar minimum. During a solar minimum, most coronal holes exist at north and south heliographic poles, while at other times more transient holes appear at all latitudes (Wang & Sheeley Jr., 1990). The equatorial coronal holes are responsible for the source of fast plasma, which can be directed to Earth to drive geomagnetic storms effectively. Currently, Solar Cycle
Figure 1.1 Monthly average of sunspots count from solar cycle 12 to 24, spanning the years 1880 to 2020 (Hathaway, 2015).

24 is ending. The amplitude of Cycle 25 is most likely to lie within ±20% of Cycle 24, meaning no major change in the level of solar activity is expected. The remaining controversy that causes more concern is where in this range the cycle will peak (Petrovay, 2020). Figure (1.1) shows monthly averages of daily sunspot numbers since Cycle 12.

Occasionally, reconfigurations of the solar magnetic field cause a large portion of the corona to blast away from the Sun and out into the heliosphere, these phenomena are known as coronal mass ejections (CMEs). CMEs speeds can vary between 100 and 3,000 km/s. They also expand in size when they travel towards Earth.

The part of the Sun’s magnetic field pulled out into the heliosphere by the solar wind is called the interplanetary magnetic field (IMF). Its characteristic spiral configuration (an Archimedian spiral pattern), when viewed from above or below the equatorial plane, is due to the Sun’s rotation. This phenomenon is known as the Parker spiral, as Figure (1.2) shows. Besides the Archimedian spiral pattern, the IMF has a structure in the north-south direction because the magnetic equator of the Sun is not
perfectly aligned with the Sun’s spin axis. This 3D complex spiral shape is known as the heliospheric current sheet and is shown in Figure (1.3). The IMF orientation at Earth’s location determines where the SW energy can most effectively access and penetrate through the Earth’s magnetic shield; the magnetopause, as seen in Figure (1.4) (Moldwin, 2008; Nykyri & Udrea, 2016). The main physical processes at the Earth’s magnetopause that lead to solar wind mass and energy transport into the Earth’s magnetosphere are the magnetic reconnection (see e.g., (Burch et al., 2016) and references therein) and Kelvin-Helmholtz Instability (Fairfield et al., 2000; Otto & Fairfield, 2000; Nykyri & Otto, 2001).

1.1.1. Consequences

Space weather consequences are numerous, including damages on equipment on Earth all the way to the equipment in space such as satellites. Some of the consequences, as pointed by (Feynman & Gabriel, 2000) are:

1. Effects on Near Earth’s Surface
Figure 1.4 When the solar magnetic field lines (yellow) are anti-parallel to Earth’s magnetic field (blue), the fields can reconnect, and Earth’s magnetic shield (magnetopause) gets penetrated by the SW plasma (JAXA, 2016).

(a) Surges in power lines: During geomagnetic storms, large currents flow in the ionosphere and are induced in Earth, which can cause surges in power lines triggering expensive power system shutdowns. Degradation of pipeline materials also occurs due to the currents over periods of years (R. Pirjola, Amm, & Pulkkinen, 1998).

(b) Radiation hazards in the polar regions: The Earth’s magnetic field does not adequately protect the environment from high-energy particles over the poles. Hence, these particles have the potential to harm astronauts in polar orbits and may even have harmful effects on passengers and crew in high-altitude polar flights (Feynman & Gabriel, 2000).

2. Effects in the Outer Magnetosphere

(a) Surface charging: High-energy electrons can give rise to surface charging. Exposed spacecraft surface can become charged, giving rise to false electronic signals and spacecraft anomalies (Feynman & Gabriel, 2000).
3. High Energy Charged Particles (Electrons, Protons, and Heavy Ions)

(a) Electronic part degradation: Two types of degradation occur due to energy deposition in semiconductors:
   i. Total ionizing dose (TID) effects: These low dose rates affect many types of devices, including p-n junction diodes, bipolar transistors, junction field-effect transistors, and integrated circuits causing performance degradation and eventual failure after a high integrated dose (Pease, 1996).
   ii. Displacement damage effects: This damage occurs because of silicon atoms’ movement from their usual lattice positions to interstitial sites leaving behind a vacancy. Damages due to this include changes in the leakage properties of p-n junctions (Stephen, 1993).

(b) Materials degradation: The effects on materials are a function of the total absorbed dose (TID) and include swelling of mirror coatings and damage to thermal control coatings (Bourrieau, 1993). Moreover, lattice defects produce a darkening of glasses, which is responsible for changes in the optical properties, and hence thermal performance, of thermal control coatings (Feynman & Gabriel, 2000).

(c) CCD and sensor effects: When a high-energy charged particle passes through a charge-couple device (CCD), three harmful effects occur; ionization damage, displacement damage, and transient damage (G. R. Hopkinson & Marshall, 1996).
(d) Solar cell degradation: Displacement damage causes solar cell performance to be reduced. Power reductions can be as high as 30%, leading to the oversizing of arrays to ensure that the end-of-life requirement is met (Feynman & Gabriel, 2000).

(e) Single event effects: Deposition of a sufficient amount of energy or charge in a sensitive volume of an electronic device by a single ion can cause single-event effects, which can turn in devices to be completely burnt out and destroyed (McNulty, 1996).

(f) Sensor interference effects: High-energy charged particles can produce electron-hole pairs in the same way that photons of desired signals of some detectors such as those used in astronomy use. Hence, such detectors are prone to interference from the radiation environment (Feynman & Gabriel, 2000).

(g) Internal electrostatic discharge: High-energy electrons can penetrate through the outer thermal blanket of spacecraft or even through electronic boxes and deposit charge in insulators. This can give rise to high electric fields inside and outside the insulator, causing fake signals and anomalies (Frederickson, 1996).

(h) Man in space: Astronauts are exposed to large fluxes of high-energy protons and ions, which pose a threat to them due to high risks of cancer and possible lethal effects. Apollo 16 was launched in April of 1972. Apollo17 (the last of the six lunar landings) was launched in December. In August of 1972, one
of the largest solar proton events ever measured occurred. If the astronauts had been walking on the Moon during this event, there is an estimated 50–50 chance that one of them would have received a lethal dose (Moldwin, 2008; Wilson et al., 1997).

1.1.2. Current Status of Space Weather Prediction

Coronagraph images are used to estimate the size, speed, direction, and density of a CME, and whether CME might hit the Earth. In order to predict the strength of the resulting geomagnetic storm, estimates of the magnetic field strength and direction are crucial. At the present time, the magnetic field cannot be determined until it is measured as the CME passes over a monitoring satellite.

The NOAA's Wang-Sheeley-Arge (WSA)-Enlil model consists of:
1. A semiempirical near-Sun module constructed by averaging solar surface magnetic field over 27-day solar rotation that approximates the outflow at the base of the SW.

2. A 3-D MHD numerical model that simulates the resulting flow evolution out to Earth. When an Earth-directed CME is detected, coronagraph images from NASA spacecraft are used to characterize the basic properties of the CME, including timing, location, direction, and speed. This input is injected into the pre-existing ambient conditions (the so-called cone model), and the subsequent transient evolution forms the basis for the prediction of the CME arrival time at Earth, its intensity, and its duration (SWPC, 2015).

Space Weather prediction models such as WSA rely on single-point spacecraft measurements at Sun-Earth $L_1$. However, they are not adequate for accurate Space Weather prediction and unravel the 3-D CME properties because plasma conditions and magnetic field direction vary at different locations. In order to accurately predict magnetic field orientation in association with CMEs, multiple spacecraft, some with separations less than the CME size, and some at similar scale are desperately needed to allow for better prediction of CME properties and their geoeffectiveness (Nykyri & Udrea, 2016; Nykyri & Foullon, 2013).

1.1.3. Previous Missions

There have been many space missions dedicated to the study of space weather. Some examples of previous missions are:

- ISEE-3: This was the first spacecraft ever to be placed in a Halo orbit at the Lagrange point 1 ($L_1$) of the Sun-Earth system in the late 1970s. Its main objective
was the continuous monitoring of SW. It had an average stationkeeping cost of 8.5 m/s per year (Farquhar, 1998).

• SOHO: This spacecraft was launched in 1995 and is still active. It is placed in a quasi-periodic Halo orbit at $L_1$ Sun-Earth and its main objective is to study the solar phenomena, atmosphere, and dynamics. From 1996 until 2011, its typical annual stationkeeping cost was less than 2 m/s per year (Roberts, 2012).

• GGS WIND: This spacecraft was also placed at $L_1$ Sun-Earth in 1994 but in a Lissajous orbit. This mission’s primary goal is to study radio waves and plasma that occur in the SW and in Earth’s magnetosphere. From 1996 until 2011, its typical annual stationkeeping cost was about 2 m/s per year (Roberts, 2012).

• STEREO: Launched in 2006, it consists of two nearly identical space-based observatories, one ahead of Earth in its orbit, the other trailing behind. Its main scientific missions are the study of CMEs, and improvement of the determination of the structure of the ambient SW (NASA, 2013a).

• Parker Solar Probe: It was launched in 2018. Its primary goal is to trace the flow of energy and understand the heating of the solar corona and to explore what accelerates the solar wind (APL, 2019).

• Solar Orbiter: Developed by the European Space Agency’s (ESA), this satellite was launched at the beginning of 2020. Its primary scientific goals are to study the drivers of the solar wind and the origin of the coronal magnetic field, to determine how solar transients drive heliospheric variability, learn how solar
eruptions produce the energetic particles that fill the heliosphere, and study how the solar dynamo works and drives connections between the Sun and the heliosphere (NASA, 2020a).

Moreover, there have only been two 4-spacecraft missions to the Earth’s magnetosphere:

- Cluster: Composed of four identical spacecraft orbiting in a tetrahedral formation. Its mission is to study the Earth’s magnetosphere over the course of nearly two solar cycles (ESA, n.d.).

- MMS: Consists of four identical spacecraft that orbit around Earth. Its mission is to study a phenomenon from Earth’s magnetosphere called magnetic reconnection (NASA, 2020a).

Although there are many active missions dedicated to studying the space environment, there has never been a multi-spacecraft constellation mission in the SW upstream of the Sun-Earth $L_1$ point.

### 1.2. The Next Generation Space Weather Prediction Mission

A multi-spacecraft mission is proposed at various libration points to forecast Space Weather hazards with a 1-2 day warning time. At each Lagrange point, a constellation of 4 spacecraft will be placed in a tetrahedral formation. The required separations between each spacecraft vary depending on the physical scale size one needs to resolve. Each spacecraft will carry the necessary instruments for the scientific data needed. Spacecraft at each Lagrange point will be placed by a carrier spacecraft.
1.2.1. Mission Overview

The carrier spacecraft is expected to be launched by 2030-2031, and the mission is expected to start by 2033. The PI of the mission is professor Katariina Nykyri, who has been working on the scientific proof of concept for the mission with professor Xuanye Ma since 2017. The trajectories that the carrier spacecraft will follow have been designed by Mark Herring. For details on those trajectories, see (Herring, 2019). The restricted three-body problem is used to design the Halo orbits that the spacecraft will be placed at. The proposed libration points for placing the spacecraft are:

- Sun-Mercury: $L_1, L_3, L_4, L_5$
- Sun-Venus: $L_1, L_3, L_4, L_5$
- Sun-Earth: $L_1, L_4$

Figure 1.6 Heliophysics System Observatory (NASA, 2020a).
1.2.2. Goals and Objectives

The mission’s primary scientific goal is to predict the $B_z$ component of the IMF at the bow-shock nose in about 1-2 days in advance. The multiple spacecraft with appropriate separations and locations can be used to determine the k-spectrum and phase of the magnetic field fluctuations and plasma wave modes (the exact method will be published elsewhere). The computed k-spectra will then be continuously fed on a background SW model currently under development by X. Ma and K. Nykyri to predict the orientation of the field at Earth (Nykyri & Udrea, 2016).

In order to measure the properties and dynamics of the relevant wave modes, knowledge on the magnetic field’s strength, plasma density, and temperature is required. Hence, the spacecraft will have to be equipped with a magnetometer, and either an electrostatic analyzer or a Faraday cup.

1.2.3. Banana Region

This mission aims to improve the current space weather forecast capability by providing more upstream solar wind measurements. Hence, sending satellites on the path of SW to the Earth’s magnetosphere is the ideal plan from the science perspective. Regardless of the solar rotation and the Earth’s motion, SW is mostly straight towards Earth, because its azimuthal speed (i.e., few km/s) is much smaller than its radial speed, which is about 400 km/s. Therefore, Sun-Earth $L_1$ is highly important for SW. However, Sun-Earth $L_1$ is still too close to Earth, resulting in only about 20-40 minutes for prediction. Moreover, Sun-Earth $L_4$ and $L_5$ hardly provide significant improvement.
Figure 1.7 Banana Region: Traces of 9 years of the SW magnetic field, which eventually reaches the Sun-Earth $L_1$ point. The color bar represents the logarithmic number of the counts (X. Ma, personal communication, July 2, 2020).

Nevertheless, the typical size of the CMEs is $\sim 50^\circ$ (Yashiro et al., 2004).

Therefore, if there is always a satellite within $25^\circ$ apart from the Sun-Earth alignment, it will catch most of CMEs going towards Earth. As such, the use of Venus and Mercury’s libration points is the obvious choice. Besides, the Park SW model suggests that SW magnetic field is an Archimedean spiral. With the frozen-in condition’s help, it is still useful to measure the plasma in the magnetic field that eventually reaches Earth. By using the simple Weber-Davis SW model (Weber & Davis, 1967), the magnetic field line can easily be traced back by using Sun-Earth’s $L_1$ measurements. Figure (1.7) is obtained by tracing nine years of the SW magnetic field by using one-minute resolution OMIN data with 20 minutes average.
Figure (1.7) is known as the Banana Region, and it shows a statistical survey of SW magnetic field lines that eventually reach the Sun-Earth $L_1$ point. The color bar represents the logarithmic number of the counts, suggesting that the SW magnetic field passes through Mercury and Venus’ orbits within about $34^\circ$ and $17^\circ$, respectively. Therefore, if Mercury and Venus’ libration points are used, much better coverage of the upstream SW monitor can be achieved.

1.3. **The Circular Restricted Three Body Problem**

The main work of this thesis is the design and optimization of orbits in the three-body problem. Hence, it behooves to provide some historical background about it.

1.3.1. **Problem History**

As with most of today’s work in mathematics, the three-body problem exists due to Newton. The three-body problem holds due to Newton’s contribution to the gravitational force’s work between any two-point masses in 1687. Newton tried studying the Sun-Earth-Moon system; however, he did not accomplish any significant progress.

Years after, in 1772, Leonhard Euler formulated the circular restricted body-problem or CRTBP to study the Sun-Earth-Moon system. He was interested in studying the motion of the Moon about the Earth but perturbed by the Sun. Euler’s simplification of the general three-body problem consisted of considering one of the three masses to be negligible with respect to the other two. In that same year, Joseph Louis Lagrange showed the first solutions for the three-body problem in *Essai sur le problème des trois corps*. Moreover, Lagrange’s contribution to the general three-body problem
consisted of reducing the problem from a system of differential equations of order 18 to a system of order 7.

Carl Gustav Jacob Jacobi was another mathematician that made important contributions to the three-body problem. He was able to reduce the general problem to a sixth-order system and the restricted problem to a fourth-order one (Vallado, 2007). In 1836 he found an integration constant known as the *Jacobi constant*, which can be interpreted as the total energy of the negligible mass relative to the rotating frame, which is used to describe the regions of possible motion of the negligible mass as shown in Figure (2.3) (Curtis, 2013). Finally, Henri Poncaire showed in 1899 that Jacobi’s integral is the only exact integral for the three-body problem, which even earned him a prize from the king of Sweden. To this day, the problem is not solvable in closed form; however, particular solutions do exist for various cases (Worthington, 2012).
2. Background

In this chapter, the mathematics behind the CRTBP are introduced. The dimensional and nondimensional equations of motion are formulated. Also, the Jacobi’s constant is introduced, and its effects on the different Hill’s regions are shown. Moreover, it is shown how to find the libration points for any system, and stability analyses on those points are conducted. Furthermore, a brief introduction to pseudospectral methods is presented. Finally, a quick overview of Lambert’s problem is presented, and Gooding’s method is used to generate porkchop plots.

2.1. CRTBP

The CRTBP consists of three masses, two primaries (for instance, Earth and Sun) and one secondary (usually a spacecraft), as depicted in Figure (2.1). The CRTBP makes three major assumptions:

- The third body is assumed to possess infinitesimal mass compared to the other two bodies (primaries).
- Distance between primaries remains constant.
- Primaries rotate in circular orbits about their barycenter.

Two coordinate systems are used in the development of the CRTBP; synodic and barycentric. Figure (2.1) shows the geometry of the CRTBP in the barycentric frame, which is the inertial reference with respect to $M_1$ and is fixed at the barycenter of the system.

Equations of motion can be found as described in Wie’s book *Space Vehicle Dynamics and Control* (Wie, 2008). Since the two primaries are assumed to rotate in
Figure 2.1 Geometry of the CRTBP.

circular orbits about their barycenter, that makes the system to rotate with constant angular velocity $\omega$. As shown in Figure (2.1), the position vector of the spacecraft relative to the barycenter is:

$$\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k} \quad (2.1)$$

Moreover, the inertial acceleration of the spacecraft is:

$$\ddot{\vec{R}} = (\dddot{X} - 2\omega\dot{Y} - \omega^2 X)\hat{i} + (\dddot{Y} + 2\omega\dot{X} - \omega^2 Y)\hat{j} + \dddot{Z}\hat{k} \quad (2.2)$$

since $\dot{i} = \omega\hat{j}$, $\dot{j} = \omega\hat{i}$, and $\dot{k} = 0$.

The generalized barycentric equation of motion for n-bodies is given as (Vallado, 2007):

$$\ddot{\vec{R}} = -G \sum_{j=1}^{n} \frac{M_j\vec{r}_j}{r_j^3} \quad (2.3)$$

where $G$ is the gravitational constant and $r_j = |\vec{r}_j|$. So for the CRTBP, the equation of
motion is:

$$\ddot{\vec{R}} = -\frac{GM_1}{r_1^3}\vec{r}_1 - \frac{GM_2}{r_2^3}\vec{r}_2$$  \hspace{1cm} (2.4)$$

Moreover, from Figure (2.1) it can be seen that the position vectors $\vec{r}_1$ and $\vec{r}_2$ are:

$$\vec{r}_1 = (X - D_1)\vec{i} + Y\vec{j} + Z\vec{k}$$

$$\vec{r}_2 = (X + D_2)\vec{i} + Y\vec{j} + Z\vec{k}$$

At this point, the standard gravitational parameter $\mu$ can be defined as the product of the gravitational constant and mass of the $j$th body: $\mu_j = GM_j$

Equation 2.4 can now then be rewritten as:

$$\ddot{\vec{R}} = -\frac{\mu_1}{r_1^3}\vec{r}_1 - \frac{\mu_2}{r_2^3}\vec{r}_2$$  \hspace{1cm} (2.5)$$

Equating Equations 2.2 and 2.5 and breaking them into components yield the dimensional equations of motion:

$$\ddot{X} - 2\omega\dot{Y} - \omega^2X = -\frac{\mu_1(X - D_1)}{r_1^3} - \frac{\mu_2(X + D_2)}{r_2^3}$$  \hspace{1cm} (2.6a)$$

$$\ddot{Y} + 2\omega\dot{X} - \omega^2Y = -\frac{\mu_1Y}{r_1^3} - \frac{\mu_2Y}{r_2^3}$$  \hspace{1cm} (2.6b)$$

$$\ddot{Z} = -\frac{\mu_1Z}{r_1^3} - \frac{\mu_2Z}{r_2^3}$$  \hspace{1cm} (2.6c)$$

To get the nondimensional equations of motion a ratio relating both the mass of the primaries and the distance between them is needed. If the distance between the primaries is set to 1, then the following ratio can be used $\mu^* = \frac{M_2}{M_1 + M_2}$, meaning $\mu^* = D_1 = M_2$ as shown in Figure (2.2).

If the angular velocity $\omega$ is set to unity and the $\mu^*$ ratio is used, then Equation 2.6 can be rewritten in the nondimensional, nonlinear form as:
\[ \ddot{x} = 2\dot{y} + x - \frac{(1 - \mu^*)(x - \mu^*)}{r_1^3} - \frac{\mu^*(x + 1 - \mu^*)}{r_2^3} \]
\[ \ddot{y} = -2\dot{x} + y - \frac{(1 - \mu^*)y}{r_1^3} - \frac{\mu^*y}{r_2^3} \]
\[ \ddot{z} = -\frac{(1 - \mu^*)z}{r_1^3} - \frac{\mu^*z}{r_2^3} \]

where:
\[ r_1 = \sqrt{(x + \mu^*)^2 + y^2 + z^2} \]
\[ r_2 = \sqrt{(x + \mu^* - 1)^2 + y^2 + z^2} \]

Another way of expressing the equations of motion is by means of the pseudopotential:
\[ U(x,y,z) = \frac{x^2 + y^2}{2} + 1 - \frac{\mu^*}{r_1} + \frac{\mu^*}{r_2} \]
Equations of motion are then expressed as:

\[ \ddot{x} - 2\dot{y} = \frac{\partial U}{\partial x} \] (2.9a)

\[ \ddot{y} + 2\dot{x} = \frac{\partial U}{\partial y} \] (2.9b)

\[ \ddot{z} = \frac{\partial U}{\partial z} \] (2.9c)

The equations of motion of the CRTBP allow an energy integral called *Jacobi’s integral*, which is related to the energy of the system $E$. Jacobi’s integral is given by (Koon, Lo, Marsden, & Ross, 2011) as:

\[ C = -2E \] (2.10)

where the energy of the system $E$ is defined as:

\[ E(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} - U(x, y, z) \] (2.11)

and where $U$ is the pseudopotential given in Equation 2.8.

$C$ is a constant of the motion of the secondary mass, just like the energy and angular momentum are constants of the relative motion in the two-body problem (Curtis, 2013). Most of the literature, such as (Murray & Dermott, 1999), derives Jacobi’s integral directly from the equations of motion of the system. First, Equation 2.9 is multiplied by $\dot{x}$, $\dot{y}$, $\dot{z}$, respectively. Adding them gives:

\[ \dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial y}\dot{y} + \frac{\partial U}{\partial z}\dot{z} = \frac{dU}{dt} \] (2.12)

which after integration yields Jacobi’s integral:

\[ C = 2U - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \] (2.13)

The projection of the four-dimensional phase space onto the position space ($x, y$) is called the Hill’s region and is divided into three domains; the interior region, the exterior region, and the forbidden regions. The boundaries of these Hill’s regions are known as
Zero Velocity Curves (Zotos, 2017). The various cases regarding the Hill’s regions are shown in Figure (2.3).

*Figure 2.3* Different Hill’s regions configurations for various values of the Jacobi constant $C$ of a system with a mass ratio of 0.1. The white domains correspond to the Hill’s regions, the gray shaded domains indicate the energetically forbidden regions, and the thick black lines depict the Zero Velocity Curves. The yellow and green points represent the primary bodies, while the crosses represent the locations of the five libration points.
2.2. Scaling Units

Scaling units is a common engineering technique used to increase the efficiency of computational problems. In this work, scaled (also called canonical) units are used to solve the three-body problem. Distance and time units are chosen as the scaled variables and are the ones used to represent the entire problem. One distance unit \( L \) is the distance between the primary bodies, while time unit \( T \) is chosen based on the orbital period of the secondary body as \( T = \frac{\text{Period}}{2\pi} \). Hence, the entire problem is represented using the following scaled units.

\[
\tilde{x} = \frac{x}{L} \rightarrow x = \tilde{x} L \quad (2.14a)
\]

\[
\tilde{y} = \frac{y}{L} \rightarrow y = \tilde{y} L \quad (2.14b)
\]

\[
\tilde{z} = \frac{z}{L} \rightarrow z = \tilde{z} L \quad (2.14c)
\]

\[
\tilde{t} = \frac{t}{T} \rightarrow t = \tilde{t} T \quad (2.14d)
\]

\[
\dot{x} = \frac{dx}{dt} = \frac{L}{T} \frac{d\tilde{x}}{d\tilde{t}} \quad (2.15a)
\]

\[
\dot{y} = \frac{dy}{dt} = \frac{L}{T} \frac{d\tilde{y}}{d\tilde{t}} \quad (2.15b)
\]

\[
\dot{z} = \frac{dz}{dt} = \frac{L}{T} \frac{d\tilde{z}}{d\tilde{t}} \quad (2.15c)
\]

\[
\ddot{x} = \frac{d}{dt} \left( \frac{dx}{dt} \right) = \frac{L}{T^2} \frac{d^2\tilde{x}}{d\tilde{t}^2} \quad (2.16a)
\]

\[
\ddot{y} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{L}{T^2} \frac{d^2\tilde{y}}{d\tilde{t}^2} \quad (2.16b)
\]

\[
\ddot{z} = \frac{d}{dt} \left( \frac{dz}{dt} \right) = \frac{L}{T^2} \frac{d^2\tilde{z}}{d\tilde{t}^2} \quad (2.16c)
\]
Once the problem simulation has been carried out, unscaling the results is rather simple. For instance, to unscale acceleration, the scaled acceleration term should simply be multiplied by $L/T^2$ as Equation 2.16 suggests. Table 2.1 shows all the distance and time units used in this work, as well as the various $\mu^*$ ratios.

Table 2.1
Scaled units and $\mu^*$ ratio used in this work.

<table>
<thead>
<tr>
<th></th>
<th>1 Distance Unit</th>
<th>1 Time Unit</th>
<th>$\mu^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sun-Mercury</td>
<td>57.91e6 km</td>
<td>14.00 days</td>
<td>1.6601e-7</td>
</tr>
<tr>
<td>Sun-Venus</td>
<td>108.2e6 km</td>
<td>38.67 days</td>
<td>2.4478e-6</td>
</tr>
<tr>
<td>Sun-Earth</td>
<td>149.6e6 km</td>
<td>58.13 days</td>
<td>3.0395e-6</td>
</tr>
<tr>
<td>Sun-Mars</td>
<td>227.9e6 km</td>
<td>109.3 days</td>
<td>3.2261e-7</td>
</tr>
</tbody>
</table>

2.3. Equilibrium Points

Equilibrium points (also called Lagrange or libration points), which are exclusive of the rotating coordinate system, are locations in space where the secondary body is at rest relative to the primaries. By using the rotating, nondimensional equations of motion (Equation 2.7), locations of all libration points can be found. First, all velocities and accelerations in Equation 2.7 are set equal to zero. Moreover, $z = 0$, or an out-of-plane acceleration would induce oscillatory motion, which does not represent an equilibrium
state (Vallado, 2007). After applying these conditions in Equation 2.7, expressions for \(x\) and \(y\) become:

\[
x - \frac{(1 - \mu^*)(x - \mu^*)}{r_1^3} - \frac{\mu^*(x + 1 - \mu^*)}{r_2^3} = 0
\]

(2.17)

\[
y \left(1 - \frac{1 - \mu^*}{r_1^3} - \frac{\mu^*}{r_2^3}\right) = 0
\]

(2.18)

The equilateral points can be found by simply letting \(r_1 = r_2 = 1\). Hence,

\[
L_4 = \frac{1}{2} - \mu^*, \frac{\sqrt{3}}{2}
\]

\[
L_5 = \frac{1}{2} - \mu^*, -\frac{\sqrt{3}}{2}
\]

The location of the collinear points can be found by first setting \(y = 0\) as Figure (2.4) shows. Solution to Equation 2.17 yields the location of \(L_1, L_2\) and \(L_3\). To accomplish that, three different quintic equations are used, and solving for \(x\) in each yield the location of the equilibrium points (Szebehely, 1967).

\[
x^5 + (3 - \mu^*)x^4 + (3 - 2\mu^*)x^3 - \mu^*x^2 - 2\mu^*x - \mu^* = 0
\]

(2.19)

\[
x^5 - (3 - \mu^*)x^4 + (3 - 2\mu^*)x^3 - \mu^*x^2 + 2\mu^*x - \mu^* = 0
\]

(2.20)

\[
x^5 + (2 + \mu^*)x^4 + (1 + 2\mu^*)x^3 - (1 - \mu^*)x^2 - 2(1 - \mu^*)x - (1 - \mu^*) = 0
\]

(2.21)

Notice that per Descarte’s sign rule, there exists only one positive root for each of the quintic equations above, which of course is the desired one. Another remark is that \(L_3, L_1\) and \(L_2\) exist respectively in the following intervals along the \(x\)-axis (Koon et al., 2011): \((-\infty, -\mu^*), (-\mu^*, 1 - \mu^*), (1 - \mu^*, \infty)\). Using scaled units, the libration points for each system studied in this thesis can be found in Tables 2.2 and 2.3. Note that more
decimals than the ones showed in Tables 2.2 and 2.3 are required for the actual simulation of the problem.

Table 2.2
Mercury and Venus libration points location using scaled units.

<table>
<thead>
<tr>
<th>Sun-Mercury</th>
<th>Sun-Venus</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Point</strong></td>
<td><strong>Point</strong></td>
</tr>
<tr>
<td>Barycenter</td>
<td>Barycenter</td>
</tr>
<tr>
<td>L1</td>
<td>L1</td>
</tr>
<tr>
<td>L2</td>
<td>L2</td>
</tr>
<tr>
<td>L3</td>
<td>L3</td>
</tr>
<tr>
<td>L4</td>
<td>L4</td>
</tr>
<tr>
<td>L5</td>
<td>L5</td>
</tr>
</tbody>
</table>
Table 2.3

Earth and Mars libration points location using scaled units.

<table>
<thead>
<tr>
<th>Point</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barycenter</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L_1$</td>
<td>0.9900</td>
<td>0</td>
</tr>
<tr>
<td>$L_2$</td>
<td>1.0100</td>
<td>0</td>
</tr>
<tr>
<td>$L_3$</td>
<td>-1.000</td>
<td>0</td>
</tr>
<tr>
<td>$L_4$</td>
<td>0.5000</td>
<td>0.8660</td>
</tr>
<tr>
<td>$L_5$</td>
<td>0.5000</td>
<td>-0.8660</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Point</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barycenter</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L_1$</td>
<td>0.9953</td>
<td>0</td>
</tr>
<tr>
<td>$L_2$</td>
<td>1.0048</td>
<td>0</td>
</tr>
<tr>
<td>$L_3$</td>
<td>-1.000</td>
<td>0</td>
</tr>
<tr>
<td>$L_4$</td>
<td>0.5000</td>
<td>0.8660</td>
</tr>
<tr>
<td>$L_5$</td>
<td>0.5000</td>
<td>-0.8660</td>
</tr>
</tbody>
</table>

2.3.1. Stability of Libration Points

Linearization of Equation 2.7 is necessary for the stability analysis of all libration points. Equation 2.7 can be linearized by doing a Taylor expansion about the equilibrium points $(X_{eq}, Y_{eq}, Z_{eq})$:

$$\ddot{x} - 2\dot{y} = \left. \frac{\partial^2 U}{\partial x^2} \right|_{Eq} x + \left. \frac{\partial^2 U}{\partial y \partial x} \right|_{Eq} y + \left. \frac{\partial^2 U}{\partial z \partial x} \right|_{Eq} z$$  (2.22a)

$$\ddot{y} + 2\dot{x} = \left. \frac{\partial^2 U}{\partial x \partial y} \right|_{Eq} x + \left. \frac{\partial^2 U}{\partial y^2} \right|_{Eq} y + \left. \frac{\partial^2 U}{\partial z \partial y} \right|_{Eq} z$$  (2.22b)

$$\ddot{z} = \left. \frac{\partial^2 U}{\partial x \partial z} \right|_{Eq} x + \left. \frac{\partial^2 U}{\partial y \partial z} \right|_{Eq} y + \left. \frac{\partial^2 U}{\partial z^2} \right|_{Eq} z$$  (2.22c)
where $U$ is the pseudopotential in Equation 2.8 and

$$x = x - X_{\text{Eq}}$$

$$y = y - Y_{\text{Eq}}$$

$$z = z - Z_{\text{Eq}}$$

By letting $U_{XX} \equiv \left. \frac{\partial^2 U}{\partial x^2} \right|_{\text{Eq}},$ $U_{YY} \equiv \left. \frac{\partial^2 U}{\partial y^2} \right|_{\text{Eq}},$ $U_{XY} \equiv \left. \frac{\partial^2 U}{\partial x \partial y} \right|_{\text{Eq}},$ and $U_{ZZ} \equiv \left. \frac{\partial^2 U}{\partial z^2} \right|_{\text{Eq}},$ the above equations then become:

$$\ddot{x} - 2\dot{y} = U_{XX} x + U_{XY} y$$  \hspace{1cm} (2.23a)

$$\ddot{y} + 2\dot{y} = U_{XY} x + U_{YY} y$$  \hspace{1cm} (2.23b)

$$\ddot{z} = U_{ZZ} z$$  \hspace{1cm} (2.23c)

The partial derivatives evaluated at the equilibrium point are:

$$U_{XX} = \frac{\mu^* - 1}{R_1^3} - \frac{\mu^*}{R_2^3} + 3\mu^* \left(2\mu^* + 2X_{\text{Eq}} - 2\right)^2 - \frac{3(2\mu^* + 2X_{\text{Eq}})^2(\mu^* - 1)}{4R_1^3} + 1$$  \hspace{1cm} (2.24a)

$$U_{YY} = \frac{\mu^* - 1}{R_1^3} - \frac{\mu^*}{R_2^3} - \frac{3Y_{\text{Eq}}^2(\mu^* - 1)}{R_1^3} + \frac{3\mu^* Y_{\text{Eq}}^2}{R_2^3} + 1$$  \hspace{1cm} (2.24b)

$$U_{ZZ} = \frac{\mu^* - 1}{R_1^3} - \frac{\mu^*}{R_2^3} - \frac{3Z_{\text{Eq}}^2(\mu^* - 1)}{R_1^3} + \frac{3\mu^* Z_{\text{Eq}}^2}{R_2^3}$$  \hspace{1cm} (2.24c)

$$U_{XY} = U_{YX} = \frac{3\mu^* Y_{\text{Eq}}(2\mu^* + 2X_{\text{Eq}} - 2)}{2R_2^3} - \frac{3Y_{\text{Eq}}(2\mu^* + 2X_{\text{Eq}})(\mu^* - 1)}{2R_1^3}$$  \hspace{1cm} (2.24d)

where:

$$R_1 = \sqrt{(\mu^* + X_{\text{Eq}})^2 + Y_{\text{Eq}}^2 + Z_{\text{Eq}}^2}$$

$$R_2 = \sqrt{(\mu^* + X_{\text{Eq}} - 1)^2 + Y_{\text{Eq}}^2 + Z_{\text{Eq}}^2}$$
Equation (2.23) can be expressed in state-space form as:

\[ \dot{x} = Ax \]  

(2.25)

where:

\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
U_{XX} & U_{XY} & 0 & 2 & 0 \\
U_{YX} & U_{YY} & 0 & -2 & 0 & 0 \\
0 & 0 & U_{ZZ} & 0 & 0 & 0
\end{bmatrix}
\]

For the equilateral points \((L_4 \text{ and } L_5)\), the characteristic equation is:

\[ \lambda^4 + \lambda^2 + \frac{27}{4} \mu^*(1 - \mu^*) = 0 \]  

(2.26)

Furthermore, the characteristic equation for the collinear points \((L_1, L_2, \text{ and } L_3)\) is

\[ \lambda^4 - (\Upsilon - 2)\lambda^2 - (2\Upsilon + 1)(\Upsilon - 1) = 0 \]  

(2.27)

where:

\[ \Upsilon = \frac{(1 - \mu^*)}{|X_{Eq} - \mu^*|^3} + \frac{\mu^*}{|X_{Eq} + 1 - \mu^*|^3} \]

Eigenvalues of each libration point at any system can then be calculated, and a stability analysis can be conducted. For demonstration purposes, \(L_4 \text{ and } L_1\) of the Sun-Venus system are studied below.

**Sun-Venus \(L_4\)**

The mass ratio of the Sun-Venus system is \(\mu^* = 2.448 \times 10^{-6}\), and \(L_4\) has coordinates \([0.499997552042192, 0.866025403784439, 0]^T\) as shown in Table 2.2. After plugging in the previous values in Equation 24, the following A matrix is obtained:
\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0.7500 & 1.299 & 0 & 0 & 2 & 0 \\
1.299 & 2.250 & 0 & -2 & 0 & 0 \\
0 & 0 & -1.000 & 0 & 0 & 0
\end{bmatrix}
\]

Eigenvalues of this matrix are:

\[\lambda_{1,2} = \pm j0.999991737786278\]

\[\lambda_{3,4} = \pm j0.004065024289030\]

\[\lambda_{5,6} = \pm j0.9999999968355\]

Since none of the eigenvalues has a positive real part and there are no repeated eigenvalues on the imaginary axis, it can be concluded that \(L_4\) is a stable equilibrium point. \(L_5\) is also stable due to the symmetry of the system.

**Sun-Venus \(L_1\)**

Now, \(L_1\) that has coordinates \([0.990682140685907, 0, 0]^T\) is considered. After plugging in the values in Equations 2.24a-2.24d, the following A matrix is obtained:
\[
A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
9.1136 & 0 & 0 & 0 & 0 & 2 \\
0 & -3.0568 & 0 & -2 & 0 & 0 \\
0 & 0 & -4.0568 & 0 & 0 & 0
\end{bmatrix}
\]

Eigenvalues of this matrix are:

\[
\lambda_{1,2} = \pm 2.5310
\]

\[
\lambda_{3,4} = \pm j2.0854
\]

\[
\lambda_{5,6} = \pm j2.0142
\]

Since one of the eigenvalues has a positive real part, the system is an unstable equilibrium point. This would generally be the same for the other two collinear points \(L_2\) and \(L_3\).

2.4. Spacecraft Formation Design and Control

David Richardson designed the Halo orbit for the ISEE-3 mission (Richardson, 1980), the first spacecraft ever to be placed at a Lagrange point. Up to date, most orbits at Lagrange points are designed using the same process that Richardson followed.

2.4.1. Standard Procedure

The standard procedure for spacecraft formation design about a Lagrange point consists of two main parts. First, a reference orbit is designed by some version of
Figure 2.5 Recreation of Richardson’s ISEE-3 halo reference orbit as described in *Halo Orbit Formulation for the ISEE-3 Mission* (Richardson, 1980).

Poincaré approximation. It’s important that the reference orbit is close to a periodic solution of the nonlinear equations of motion. Then, formation control techniques around the reference orbit are applied by adding control acceleration components to the equations of motion (this will be covered in section 3). As an example, consider the ISEE-3 mission, which was the first spacecraft to be placed in a Halo orbit at the $L_1$ Sun-Earth system. David Richardson designed the reference trajectory for the spacecraft in the late 1970s and early 1980s (Wie, 2008). He came up with a third-order analytical approximation by first setting a desired $z$ amplitude $A_z$ of 110,000 km. Then, successive
approximations in conjunction with a form of the Lindstedt-Poincaré method were
applied. The complete third-order periodic solution that he constructed is as follows:
\[ x_r = a_{21}A_x^2 + a_{22}A_x^2 - A_x \cos(\lambda \tau + \phi) + (a_{23}A_x^2 - a_{24}A_x^2) \cos 2(\lambda \tau + \phi) \]
\[ + (a_{31}A_x^3 - a_{32}A_xA_x^2) \cos 3(\lambda \tau + \phi) \]  
\[ (2.28a) \]
\[ y_r = A_y \sin(\lambda \tau + \phi) + (b_{21}A_x^2 - b_{22}A_x^2) \sin 2(\lambda \tau + \phi) \]
\[ + (b_{31}A_x^3 - b_{32}A_xA_x^2) \sin 3(\lambda \tau + \phi) \]  
\[ (2.28b) \]
\[ z_r = -3d_{21}A_xA_x + A_x \cos(\lambda \tau + \phi) + d_{21}A_xA_x \cos 2(\lambda \tau + \phi) \]
\[ + (d_{32}A_xA_x^2 - d_{31}A_x^3) \cos 3(\lambda \tau + \phi) \]  
\[ (2.28c) \]
where \( \lambda \) denotes the in-plane frequency, \( \phi \) is the phase angle which determines the
initial positions in the orbit, \( A_x, y, z \) are the amplitudes of the orbit, and \( \tau \) is an independent
variable with a frequency correction \( \omega \) such that:
\[ \tau = \omega x \]
\[ \omega = \sum_{n \geq 1} \omega_n \]
\[ s = n_E t \]
where \( t \) denotes time, and \( n_E \) is the orbital mean motion of the Earth (for the ISEE-3
mission). The remaining \( a, b, c, d \) terms can be found in Richardson’s paper, previously
cited. Figure (2.5) is a recreation of Richardson’s original ISEE-3 halo reference orbit.

The actual mission reference orbit was constructed numerically using a differential
correction procedure to adjust the halo orbits produced by the analytical approximation.
Once the reference orbit has been designed, controls can be applied to it. For instance,
disturbance accommodating control, which was first demonstrated by (Hoffman, 1993), could be used. (Wie, 2008) used this approach by assuming the control acceleration to be continuous to allow the design of a linear state-feedback controller. He showed that a linear state-feedback controller without disturbance accommodation resulted in $\Delta V$ of 146 (m/s)/year. Once disturbance accommodation is used after 10 iterations, the $\Delta V$ per year is estimated to be 8.3 (m/s)/year. The actual ISEE-3 required a $\Delta V$ of approximately 10-15 (m/s)/year.

2.5. Pseudospectral Methods in Optimal Control

Pseudospectral (PS) methods for solving nonlinear control problems were first used in the 1990s by (Elnagar, Kazemi, & Razzaghi, 1995; Elnagar & Kazemi, 1998). They developed this technique based on spectral collocation methods used in the solution of differential equations. The merit of this approach is that the optimal control problem is replaced by an algebraic nonlinear programming problem to which existing, well-developed optimization algorithms may be applied (Elnagar et al., 1995).

2.5.1. Pseudospectral Discretization

As described by Elnagar, Kazemi, and Razzaghi (1995), discretization of optimal control problems can be done as follows, by considering the following problem:

\[
\begin{align*}
\text{maximize} & \quad J[X, U] = H(X(T), T) + \int_0^T G(X(\tau), U(\tau), \tau) \, d\tau \\
\text{subject to} & \quad \dot{X}(\tau) = F(X(\tau), U(\tau), \tau), \quad 0 \leq \tau \leq T, \quad X(0) = x_0, \\
& \quad S_j(X(\tau), U(\tau), \tau) \leq 0, \quad (j = 1, \ldots, \nu), \quad \tau \in [0, T]
\end{align*}
\] (2.29)

In the problem statement from above, $F$, $H$, $G$, and $S$ are usually nonlinear. Also, it is assumed that the problem has a unique solution. The Legendre polynomial $L_N(t)$ of
order $N$ is introduced with $t_0$ and $t_N$ being -1 and 1, respectively. Additionally, $t_m, m = 1, 2, \ldots, N - 1$ is let to be the zeros of the $L_N(t)$ derivative.

For constructing a pseudospectral Legendre polynomial of a given function $F(t)$ defined over [-1,1], first, the Lagrange polynomials must be constructed:

$$
\phi(t) = \frac{1}{N(N+1)L_N(t_l)} \cdot \frac{(t^2 - 1)L_N(t)}{t - t_l}, \quad (l = 0, 1, \ldots, N) \tag{2.30}
$$

Now, the $N_{th}$ degree pseudospectral Legendre polynomial and its derivative are given by:

$$
F^N(t) = \sum_{l=0}^{N} F(t_l) \phi_l(t) \tag{2.31}
$$

$$
\dot{F}^N(t_m) = \sum_{l=0}^{N} D_{ml} F(t_l) \tag{2.32}
$$

where,

$$
D = (D_{ml}) = \begin{cases} 
\frac{L_N(t_m)}{L_N(t_l)} \cdot \frac{1}{(t_m - t_l)}, & m \neq l \\
\frac{N(N+1)}{4}, & m = l = 0 \\
\frac{N(N+1)}{4}, & m = l = N \\
0, & \text{otherwise}
\end{cases} \tag{2.33}
$$

In addition, the following transformation is introduced: $\tau = \frac{T}{2}(T + 1)$. The original problem statement can then be rewritten as:

**maximize** $J[x,u] = h(x(1), T) + \int_{-1}^{1} g(x(t), u(t), t, T) \, dt$

subject to $\dot{x}(t) = f(x(t), u(t), t, T), \quad -1 \leq t \leq 1, \quad x(-1) = x_0, \tag{2.34}$

$$
S_j(x(t), u(t), t, T) \leq 0, \quad (j = 1, \ldots, 0), \quad -1 \leq t \leq 1
$$
As in Equation 2.31, the pseudospectral Legendre polynomials for $x(t)$ and $u(t)$
are:

$$
\begin{align*}
x^N(t) &= \sum_{l=0}^{N} a_l \phi_l(t), \\
u^N(t) &= \sum_{l=0}^{N} b_l \phi_l(t)
\end{align*}
$$

(2.35)

where the vectors $a_l$ and $b_l$ are yet to be determined. Also notice the following properties:

$$
a_k = x^N(t_k) = x(t_k), \quad b_k = u^N(t_k) = u(t_k)
$$

(2.36)

With the previous information, the problem now becomes:

maximize  \[ J^N = h(x^N(1), T) + \int_{-1}^{1} g(x^N(t), u^N(t), t, T) \, dt \]

subject to  \[ \dot{x}^N(t_k) = f(x^N(t), u^N(t_k), t_k, T), \quad (k = 0, 1, \ldots, N), \quad x^N(t_0) = x_0, \]

(2.37)

$$
S_j(x^N(t_k), u^N(t_k), t_k, T) \leq 0, \quad (j = 1, \ldots, \nu), \quad (k = 0, 1, \ldots, N)
$$

For a general nonlinear function $g$, the cost function is approximated, and the
constraints are rewritten to give the final approximation form of the original problem 2.29
as:

maximize  \[ J^N(\alpha, \beta, T) = h(a_N, T) + \sum_{k=0}^{N} g(a_k, b_k, t_k, T) \, w_k \]

subject to  \[ A_k = f(a_k, b_k, t_k, T) - d_k = 0, \quad (k = 0, 1, \ldots, N), \]

(2.38)

$$
B_k = S(a_k, b_k, t_k, T) \leq 0, \quad (k = 0, 1, \ldots, N),
$$

$$
a_0 = x_0
$$

where $\alpha = (a_0, a_1, \ldots, a_N)$, and $\beta = (b_0, b_1, \ldots, b_N)$. Moreover, $w_k$ is given by

Equation 2.39, $d_k$ is the $k$th component of the vector $\hat{D}(\hat{a}_0^T, \hat{D}(\hat{a}_1^T, \ldots, \hat{D}(\hat{a}_{N-1}^T)^T, \hat{D}$ is
given by matrix 2.40, and the entries of $D$ are given by Equation 2.33.

$$
w_k = \frac{2}{N(N+1)} \cdot \frac{1}{(L_N(t_k))^2}, \quad k = 0, 1, \ldots, N
$$

(2.39)
\[
\dot{D} = \begin{bmatrix}
D & 0 \\
0 & D
\end{bmatrix}
\] (2.40)

### 2.6. PS Theory

PS theory is a theoretical-computational framework that was conceived in the early 2000’s by Ross for solving optimal control problems. The following four elements are the basis of PS theory: domain transformation, interpolation, differentiation, and integration.

The state trajectory \( x(\cdot) \) in spectral methods is expressed as:

\[
x(t) = \sum_{m=0}^{\infty} a_m P_m(t)
\] (2.41)

where \( P_m(t) \) is a polynomial in \( t \) of degree \( m \). Equation 2.41 is known as the modal representation of \( x(\cdot) \), which is a form of a Fourier expansion. However, it is more convenient to express Equation 2.41 as a nodal representation:

\[
x(t) = \sum_{j=0}^{\infty} x_j \phi_j(t)
\] (2.42)

where \( t_j, j = 0, 1, 2, \ldots \) are discrete points in time called nodes and \( \phi_j(t) \) is a Lagrange interpolating polynomial. On the other hand, the control function is written as:

\[
u(t) = \sum_{j=0}^{\infty} u_j \psi_j(t)
\] (2.43)

where \( \psi(t) \) is a special interpolating function that makes the pair \( t \mapsto (x, u) \) dynamically feasible. Notice that in Equation 2.43, \( \phi_j \) is not used since doing so can be limiting in applicability, instead \( \psi_j \) is used. This facilitates applications such as real-time optimal control, and has inspired an alternative PS method known as the Bellman pseudospectral method (Ross, Gong, & Sekhavat, 2007; Ross & Gong, 2013).
A great aspect of PS optimal control theory is the connection between Pontryagin’s principle and the actual computation of the optimal control problem. That is, when using PS methods, the discretization and dualization of the optimal control problem commute with respect to an appropriate transformation. This connection is known as the Covector Mapping Principle (CMP) and is very useful because it links theory and computation.

What the CMP says is that given a general optimal control Problem $B$, and a discrete approximation to $B$ denoted by Problem $B^N$, there exists an order-preserving map between the dual variables corresponding to the dualized Problem $B^\lambda_N$ and the discretized Problem $B^{\lambda_N}$ (Ross & Fahroo, 2002). This idea is illustrated in Figure (2.6).

Problem $B$ is the original optimal control problem, whereas Problem $B^N$ is the discretized problem with $N$ number of discrete points. Moreover, Problem $B^{\lambda_N}$ refers to the set of necessary conditions obtained by applying the Karush-Kuhn-Tucker (KKT) theorem. On the top part of the figure, Problem $B^\lambda$ is the boundary value problem (BVP) obtained

Figure 2.6 Schematic of dualization, discretization and the Covector Mapping Principle (Ross & Fahroo, 2002).
Figure 2.7 Complete Covector Mapping Principle (Ross, 2015).

by applying Pontryagin’s principle to Problem $B$ while Problem $B^{\lambda N}$ represents the
discretization of Problem $B^{\lambda}$. For a more in-depth and detailed explanation of the CMP
process see (Ross, 2006a).

Notice that in Figure (2.6) there’s a “gap” in between Problem $B^{N\lambda}$ and Problem $B^{\lambda}$.
This is because not all methods commute for the discretization and dualization. Methods
that satisfy closing the gap are Hager’s family of Runge-Kutta methods and the Legendre
pseudospectral method. Out of those two, the Legendre pseudospectral method provides
a simpler transformation in the sense that it is linear and symmetric. Hence, the map
showing the schematic of dualization, discretization, and the CMP with no gap is depicted
in Figure (2.7). Methods that do not satisfy closing the gap include the Hermite-Simpson
method and some Runge-Kutta methods (Ross & Fahroo, 2002).
2.7. Porckchop Plots

The orbital two-point boundary value problem or Lambert’s problem, as it is commonly known, refers to the determination of an orbit that passes between two positions within a specified time-of-flight (TOF). By solving Lambert’s problem one can then plot contours of interplanetary trajectory parameters such as $C_3$ and $\Delta V$ in a launch-date/arrival-date space. This results in bi-lobed characteristic shapes which look like pork chops and hence the name of these type of plots. There are two lobes for each porkchop plot since Lambert’s problem allows two solutions; type 1 and type 2. Type 1 trajectories are characterized by heliocentric transfer angles, which are less than 180 degrees and are shorter in duration, while type 2 trajectories have transfer angles greater than 180 degrees and are longer in duration, as shown in Figure (2.8). The use of these type of plots is beneficial for preliminary mission design because it aides engineers in selecting launch dates, and calculating launch energies as well as $\Delta V$ budgets. (Woolley & Whetsel, 2013; Eagle, 2012).

There exist various methods for solving Lambert’s problem. Perhaps, the most used and famous solutions are:

- Gauss’s solution
- Thorne’s solution
- Prussing and Conway’s solution
- Solution by universal variables $f$ and $g$
- Battin’s solution
Figure 2.8 Paths for solving Lambert’s problem. For the long way, the transfer angle is greater than 180°, and vice versa (Vallado, 2007).

- Gooding’s solution
- Sun’s solution

Although there are many methods for solving Lambert’s problem, all of them share some preliminary concepts. For instance, in all of them, the angle through which the transfer takes place should be known, as Figure (2.8) illustrates. Also, the gravity source during the flight is the focus of the orbit that connects the initial and final position. For instance, the gravity source in an Earth-Venus interplanetary flight is the Sun. Initial geometry is also the same for most methods and is shown in Figure (2.9), where \( \mathbf{r}_1 \) is the initial position vector and \( \mathbf{r}_2 \) the final position vector. Moreover, \( F \) is the focus, \( F^* \) the unoccupied focus, \( f_i \) the true anomalies measured relative to periapsis, \( \Delta f \) the angular change between the initial and final position vectors, and \( c \) the relative vector between the final and initial positions (Schaub & Junkins, 2009).
Also, most of the methods converge to the solution through iterations. For instance, Gooding’s method, which is an extension of Lancaster and Blanchard’s method, uses a high-order Halley root-finding algorithm. Gooding’s algorithm can be summarized as follows (Wagner, Wie, & Kaplinger, 2015):

1. The inputs are the TOF and the Lambert parameter $q$.

2. Evaluate $T_0$ when $x = 0$ and determine the initial guess as defined by (Gooding, 1988), and (Lancaster & Blanchard, 1968).

3. If $x$ is close to 1.0, calculate $E$ and $K$ to evaluate $T$, $T'$, and $T''$ from Lancaster’s transcendental equations or the Gooding’s series solution.

4. Otherwise, calculate $z$, $d$, $y$, and $E$ and evaluate $T$, $T'$, and $T''$ as given by Gooding.

5. Update $x$ using Halley’s method.
6. Go to step 2 and repeat until the desired tolerance for the change in $x$ is met or a maximum number of iterations is exceeded.

7. Output the initial and final radius vectors.

For the construction of the porckchop plots, a date span in days is provided for both the launch and arrival dates. Then, Lambert’s problem is solved for all pairs of dates, and the data is saved in matrices. With the recorded data, contours of the parameters found, such as total $\Delta V$ or $C_3 L$ (launch energy), are plotted in a launch-date/arrival-date space. Figures (2.10) and (2.11) are Earth-Mars and Earth-Venus porkchop plots for 2030-2040.

![Earth-to-Mars Porkchop Plots 2030-2040](image)

**Figure 2.10** Multiple Earth-Mars porkchop plots from 2030-2040 showing change in shape and structure.
Figure 2.11 Multiple Earth-Venus porkchop plots from 2030-2040 showing change in shape and structure.

Gooding’s solution to Lambert’s problem advantage over other methods is that it uses Halley’s iteration method, which converges faster than others, such as the Newton-Raphsen. New porkchop plots appear every 26 months for Mars trajectories and every 19 months for Venus. In the 2030-2040 period, the optimal transfer $\Delta V$ to Mars is 5.85 km/s with launch and arrival dates of 6/28/2035 and 1/15/2036, respectively. Furthermore, for Venus, the optimal transfer $\Delta V$ is 5.90 km/s with launch and arrival dates of 12/6/2032 and 5/1/2033, respectively. The porkchop plots also show that the optimal trajectories to Mars are both in the short and long ways, whereas the optimal trajectories to Venus are all in the long way (type 2). Tables 2.4 and 2.5 summarize the results of Figures (2.10) and (2.11).
Table 2.4
Optimal $\Delta V$’s and $C_3L$ found for Earth-Mars trajectories for 2030-2040.

<table>
<thead>
<tr>
<th>Year</th>
<th>$\Delta V_{\text{Tot}}$</th>
<th>$\Delta V_1$</th>
<th>$\Delta V_2$</th>
<th>$C_3L$</th>
<th>Transfer</th>
<th>Type</th>
<th>LD</th>
<th>AD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[km/s]</td>
<td>[km/s]</td>
<td>[km/s]</td>
<td>[km$^2$/s$^2$]</td>
<td>[Days]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2033</td>
<td>6.3345</td>
<td>3.0062</td>
<td>3.3283</td>
<td>9.0730</td>
<td>198</td>
<td>1</td>
<td>4/16/2033</td>
<td>10/31/2033</td>
</tr>
<tr>
<td>2035</td>
<td>5.8527</td>
<td>3.2219</td>
<td>2.6308</td>
<td>10.3803</td>
<td>201</td>
<td>1</td>
<td>6/28/2035</td>
<td>1/15/2036</td>
</tr>
<tr>
<td>2037</td>
<td>6.8493</td>
<td>4.0580</td>
<td>2.7913</td>
<td>16.4673</td>
<td>352</td>
<td>2</td>
<td>8/18/2037</td>
<td>8/5/2038</td>
</tr>
<tr>
<td>2039</td>
<td>6.0197</td>
<td>3.5408</td>
<td>2.4789</td>
<td>12.5371</td>
<td>341</td>
<td>2</td>
<td>9/20/2039</td>
<td>8/26/2040</td>
</tr>
</tbody>
</table>

Table 2.5
Optimal $\Delta V$’s and $C_3L$ found for Earth-Venus trajectories for 2030-2040.

<table>
<thead>
<tr>
<th>Year</th>
<th>$\Delta V_{\text{Tot}}$</th>
<th>$\Delta V_1$</th>
<th>$\Delta V_2$</th>
<th>$C_3L$</th>
<th>Transfer</th>
<th>Type</th>
<th>LD</th>
<th>AD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>[km/s]</td>
<td>[km/s]</td>
<td>[km/s]</td>
<td>[km$^2$/s$^2$]</td>
<td>[Days]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2031</td>
<td>6.3749</td>
<td>2.5757</td>
<td>3.7992</td>
<td>6.6344</td>
<td>156</td>
<td>2</td>
<td>5/24/2031</td>
<td>10/27/2031</td>
</tr>
<tr>
<td>2032</td>
<td>5.8964</td>
<td>3.1773</td>
<td>2.7191</td>
<td>10.0952</td>
<td>158</td>
<td>2</td>
<td>12/6/2032</td>
<td>5/13/2033</td>
</tr>
<tr>
<td>2036</td>
<td>8.0619</td>
<td>4.7244</td>
<td>3.3376</td>
<td>22.3198</td>
<td>196</td>
<td>2</td>
<td>1/5/2036</td>
<td>7/19/2036</td>
</tr>
<tr>
<td>2037</td>
<td>7.7281</td>
<td>2.8264</td>
<td>4.9017</td>
<td>7.9884</td>
<td>161</td>
<td>2</td>
<td>10/23/2037</td>
<td>4/2/2038</td>
</tr>
<tr>
<td>2039</td>
<td>6.4680</td>
<td>2.5337</td>
<td>3.9343</td>
<td>6.4198</td>
<td>159</td>
<td>2</td>
<td>5/19/2039</td>
<td>10/25/2039</td>
</tr>
</tbody>
</table>
For a detailed derivation of Gooding’s method see (Gooding, 1988), and Lancaster and Blanchard original method (Lancaster & Blanchard, 1968). For more information on solving Lambert’s problem and generating porkchop plots, the following are good resources: (Vallado, 2007; Conte, 2014).
3. Theory and Procedures

The approach used for the design and control of the spacecraft formations in this thesis is inspired by (Infeld, Josselyn, Murray, & Ross, 2007). This approach does not follow the same steps as the “standard procedure” that was discussed in the previous section. Instead, the orbits and the formation control strategies are designed concurrently using the framework of multi-agent optimal control theory. That is that the design problem is not separated from the control problem, and both the reference orbit design and the formation-keeping control are solved in one stroke (Infeld et al., 2007).

A challenging part while designing the orbits is having “good” initial conditions. To ease that matter, a Monte Carlo approach is used. The optimal control problem is then solved using an optimization software, and the results are validated through Pontryagin’s principle.

3.1. Dynamics and Control of the Spacecraft

Controls of the spacecraft can be defined by thrust directions, as shown in Figure (3.1). The thrust terms represent the control function of the spacecraft based on

![Figure 3.1 Controls of the spacecraft.](image)
accelerations. Acceleration components rather than thrust components are used because by doing this, knowledge on the spacecraft’s mass is not required. Equation 3.1 represents the dynamics of the system with the added control components.

\[ \dot{x} = v_x \] (3.1a)

\[ \dot{y} = v_y \] (3.1b)

\[ \dot{z} = v_z \] (3.1c)

\[ \ddot{x} = 2\dot{y} + x - \frac{(1-\mu^s)(x - \mu^s)}{r_1^3} - \frac{\mu^s(x + 1 - \mu^s)}{r_2^3} + a_x \] (3.1d)

\[ \ddot{y} = -2\dot{x} + y - \frac{(1-\mu^s)y}{r_1^3} - \frac{\mu^s y}{r_2^3} + a_y \] (3.1e)

\[ \ddot{z} = -\frac{(1-\mu^s)z}{r_1^3} - \frac{\mu^s z}{r_2^3} + a_z \] (3.1f)

where \( \mu^s \) is the ratio introduced in section 2, and \( r_1 \) and \( r_2 \) are defined in Equation 2.7. State and control variables for one spacecraft can be defined as:

\[ x = (x, y, z, \dot{x}, \dot{y}, \dot{z}) \] (3.2)

\[ u = (u_x, u_y, u_z) \] (3.3)

Consequently, it follows that for a set of multiple spacecraft the state and control variables are given by:

\[ x = (x^1, \ldots, x^{N_s}) \] (3.4)

\[ u = (u^1, \ldots, u^{N_s}) \] (3.5)

where \( N_s \) denotes the number of spacecrafts of the formation. Hence, the dynamics of the spacecraft formation are represented as:

\[ \dot{x}(t) = f(x(t), u(t), t) \quad u \in \mathbb{U} \] (3.6)
where $U = U^1 \times \cdots \times U^N$. For all the formations designed in this work, functions $f^1, \ldots, f^N$ are all the same. Therefore, the dynamics of a four-spacecraft formation are:

$$
\dot{x} = \begin{bmatrix}
    v_{x}^i \\
    v_{y}^i \\
    v_{z}^i \\
    2v_{x}^i + x^i - \frac{(1-\mu^*)(x^i-\mu^*)}{r_1^2} - \frac{\mu^*(x^i+1-\mu^*)}{r_2^2} + a_x^i \\
    -2v_{x}^i + y^i - \frac{(1-\mu^*)y^i}{r_1^2} - \frac{\mu^*y^i}{r_2^2} + a_y^i \\
    -(1-\mu^*)z^i - \frac{\mu^*z^i}{r_1^2} + a_z^i \\
    \vdots \\
    v_{x}^{iv} \\
    v_{y}^{iv} \\
    v_{z}^{iv} \\
    2v_{y}^{iv} + x^{iv} - \frac{(1-\mu^*)(x^{iv}-\mu^*)}{r_1^2} - \frac{\mu^*(x^{iv}+1-\mu^*)}{r_2^2} + a_x^{iv} \\
    -2v_{x}^{iv} + y^{iv} - \frac{(1-\mu^*)y^{iv}}{r_1^2} - \frac{\mu^*y^{iv}}{r_2^2} + a_y^{iv} \\
    -(1-\mu^*)z^{iv} - \frac{\mu^*z^{iv}}{r_1^2} + a_z^{iv}
\end{bmatrix}
$$

### 3.2. Cost Function

Cost functions measure performance, and their objective in optimal control problems is to minimize (or maximize) a desired quantity. The standard cost function is known as the Bolza cost function and is defined as:

$$
J[\mathbf{x}(\cdot), \mathbf{u}(\cdot), t_f] = E(\mathbf{x}_f, t_f) + \int_{t_0}^{t_f} F(\mathbf{x}(t), \mathbf{u}(t), t) dt
$$

(3.7)
where \( x(\cdot) \) and \( u(\cdot) \) represent the state trajectory and control trajectory, respectively. Moreover, \( E \) is known as the *endpoint cost* or *Mayer cost*, and \( F \) as the *running cost* or *Lagrange cost* (Ross, 2015). The feasibility of space missions is dictated by the amount of propellant required by spacecraft to achieve their objectives. Therefore, the cost function used in this work is chosen to minimize propellant consumption.

Steering of spacecraft can be achieved by gimbaled, ungimbaled, or vernier engines. If a configuration of six thrusters (ungimbaled thrust) is chosen, the rocket equation is given as:

\[
\dot{m} = -\frac{|T_x| + |T_y| + |T_z|}{v_e} = -\frac{\|T\|_1}{v_e}
\]

where \( \dot{m} \) represents the mass flow rate, \( v_e \) exhaust speed, and \( T_x, T_y \) and \( T_z \) the thrust forces acting on the spacecraft as seen on Figure (3.1). Notice that propellant consumption is the change in mass of the spacecraft. Hence, if \( v_e \) is constant and Equation 3.8 is used, then propellant consumption is simply given by:

\[
m(t_0) - m(t_f) = -\int_{t_0}^{t_f} \dot{m} = \frac{1}{v_e} \int_{t_0}^{t_f} \|T(t)\|_1 \, dt
\]  

(3.9)

Recalling that acceleration terms are used as the control variables, then the cost function for any of the spacecraft \( i \) becomes:

\[
J_i = \int_{t_0}^{t_f} \|u^i(t)\|_1 \, dt
\]  

(3.10)

Consequently, cost function for a formation is:

\[
J = \sum_{i=1}^{N_s} J_i = \int_{t_0}^{t_f} \sum_{i=1}^{N_s} \|u^i(t)\|_1 \, dt
\]  

(3.11)

However, Equation 3.11 is not yet the cost function to be used. Instead, it is a better option to measure the propellant consumed by the spacecraft for one orbit period \( t_f - t_0 \) as Equation 3.12 suggests. Since the orbits are desired to be either periodic or almost
periodic, it can be assumed that the same amount of propellant would be needed for each orbit period during the mission lifetime. Furthermore, \( t_f \) is let to be free to get an optimal period, which consequently leads to an optimal orbit.

\[
J = \frac{1}{t_f - t_0} \int_{t_0}^{t_f} \sum_{i=1}^{N} \| u_i(t) \|_1 \, dt
\]

(3.12)

Notice that endpoint cost \( E \) is zero in Equation 3.12 and \( J \) only includes a running cost. Additionally, notice that cost functions in Equations 3.10-3.12 are all the \( L^1 \)-norm of control. Many places mistakenly suggest that quadratic cost functions should be used for measuring fuel consumption. However, it has been proved that the fuel penalty for using quadratic costs is at least 18% and could be as high as 50% (Ross, 2004, 2006b). In the results section, comparisons of using quadratic costs versus \( L^1 \)-norms are presented.

### 3.3. Constraints

The most important constraint is that the orbits must be periodic. Therefore, to represent full complete orbits, the following condition must be satisfied for all spacecraft of the formation:

\[
x^i(t_0) = x^i(t_f) \quad \forall \ i
\]

(3.13)

Constraints for keeping desired distances between the spacecraft throughout time can also be imposed. For instance, if a certain distance between the values \( a \) and \( b \) is desired to be maintained for two spacecraft, then the following constraint could be used:

\[
a < \sqrt{(x_{ii} - x_i)^2 + (y_{ii} - y_i)^2 + (z_{ii} - z_i)^2} < b
\]

(3.14)

This constraint works well for two spacecraft. However, when the same constraint is added to a formation with more spacecraft, the computational time is considerably increased, and \( \Delta V \) values for the orbit maintenance of all orbits greatly increases
(especially for orbits in non-stable libration points). Therefore, Monte Carlo simulations, which are discussed in section 3.5.1 are used instead. In addition, bounds are imposed for the search of the control values.

### 3.4. Solving the Optimal Control Problem

The proposed optimal control can be summarized as follows:

\[
\begin{align*}
\text{minimize} & \quad J[x(\cdot), u(\cdot), t_0, t_f] = \frac{1}{t_f - t_0} \int_{t_0}^{t_f} \sum_{i=1}^{N} \| u^i(t) \|_1 \, dt \\
\text{subject to} & \quad \dot{x}(t) = f[x(t), u(t), t], \\
& \quad u(t) \in \mathbb{U}[t, x(t)], \\
& \quad [x(t_0), x(t_f), t_0, t_f] \in \mathbb{E}
\end{align*}
\]

where \( \mathbb{U} \) and \( \mathbb{E} \) represent the control space and endpoint conditions space, respectively.

The nonsmooth, nonlinear, multi-agent control Problem 3.15 is solved using DIDO (Ross, 2015), a MATLAB toolbox for solving optimal control problems. DIDO is a great tool since the results can be verified through Pontryagin’s principle, as demonstrated in section 3.7.1. DIDO allows the user to choose the number of desired nodes. The higher the nodes, the higher the accuracy, but the longer the computational time. Nodes are the grid points \( \pi^N = t_0, t_1, ..., t_N \) of the discretization.

DIDO implements Ross and Fahroo’s pseudospectral knotting methods for solving optimal control problems (Ross & Fahroo, 2004). PS knots enable the practical implementation of discontinuous controls as well as jumps in the state variables (Ross & Karpenko, 2012). PS knotting methods require that both the integral of the cost function
and the problem’s dynamics be discretized. The integral is approximated by a sum and the dynamics by discrete differential operators. The main idea of knotting methods is to divide the time interval [-1,1] into smaller subintervals, renormalize each subinterval to [-1,1], and then apply PS discretization on each subinterval. By doing that, information can be exchanged between the subintervals through the double Lobatto points (called PS knots) at -1 and 1. Knots can be free or fixed. If free, knots are treated as an extra decision variable. Figures (3.2) and (3.3) are examples of an unknotted and a knotted optimal control for a given problem. As can be seen, the unknotted control lacks accuracy at the jump points while the knotted seems to be much more accurate. An interesting property of knotting methods is that the distribution of the nodes can be controlled. Hence, it is advantageous to place the knots near to where there is a great presence of nodes (Gong & Ross, 2008).

In a few words, what DIDO does is that it takes Problem B of Figures (2.7) and (3.4) as an input, then uses PS theory (hence knotting methods) and outputs a candidate...
solution to Problem $B^\lambda$. The candidate solution can then be tested for optimality through Pontryagin’s principle. Figure (3.4) depicts the preceding ideas. For a full overview of the algorithm implemented in DIDO and details on the CMP see (Gong & Ross, 2008; Ross, 2006a).

3.5. Initial Conditions

Initial conditions (I.C.s), which are the same as the final conditions, as stated by Equation 3.13, are required for solving the optimal control problem. The problem is very sensitive to I.C.s, especially in the unstable libration points. It results nearly impossible to choose the I.C.s manually, especially since the problem requires 24 different of them ($x, y, z, v_x, v_y, v_z$; 4 spacecraft). It also becomes challenging since the spacecraft are required to maintain specific separations throughout time, meaning that their I.C.s cannot be the same. The proposed solution for finding I.C.s is through Monte Carlo simulations.
Figure 3.4 DIDO takes Problem $B$ as an input and outputs a candidate solution to Problem $B^\lambda$. The candidate solution can then be tested for optimality through Pontryagin’s principle (Ross, 2015).

3.5.1. Monte Carlo

I.C.s are usually chosen near the libration point. Hence, the procedure used in this thesis is as follows. The desired distances between the spacecraft are known; therefore, their initial locations are selected by placing them in a tetrahedral formation taking the desired separation as a tetrahedron unit. A location near the libration point is selected as the origin of the formation. The regular tetrahedron of unit side has the following geometry (Paschmann & Daly, 2000):

\[
\begin{align*}
\mathbf{d}_1 &= (1, 0, 0) \\
\mathbf{d}_2 &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right) \\
\mathbf{d}_3 &= \left(\frac{1}{2}, \frac{\sqrt{3}}{6}, \frac{\sqrt{6}}{3}\right)
\end{align*}
\]
\[ \mathbf{d}_4 = (0, 0, 0) \]

Therefore, the initial locations for the spacecraft formation can be chosen as:

\[
[x_i^0, y_i^0, z_i^0] = [sep + (x_{L^4} + shft), y_{L^4}, 0] \quad (3.16a)
\]

\[
[x_i^{ii}, y_i^{ii}, z_i^{ii}] = \left[ \frac{1}{2} \cdot sep + (x_{L^4} + shft), \frac{\sqrt{3}}{2} \cdot sep + y_{L^4}, 0 \right] \quad (3.16b)
\]

\[
[x_i^{iii}, y_i^{iii}, z_i^{iii}] = \left[ \frac{1}{2} \cdot sep + (x_{L^4} + shft), \frac{\sqrt{3}}{6} \cdot sep + y_{L^4}, \frac{\sqrt{6}}{3} \cdot sep \right] \quad (3.16c)
\]

\[
[x_i^{iv}, y_i^{iv}, z_i^{iv}] = [x_{L^4} + shft, y_{L^4}, 0] \quad (3.16d)
\]

where \(sep\) is the desired separation between the spacecraft, \(shft\) is a value to shift the formation in the \(x\)-axis, so it is not initially located at the libration point, and \(x_{L^4}\) and \(y_{L^4}\) are the \(x\) and \(y\) coordinates of the libration point, respectively.

To illustrate the above ideas, consider \(L^4\) of the Sun-Mars system. If a desired separation between the spacecraft throughout the orbit is between 1,500 km and 4,500 km, then \(sep\) can be chosen as 3,000 km and \(shft\) as 2,000 km. Figure (3.5) depicts the initial geometry of such formation.

Values for the velocities are selected based on results of previous simulations. However, if the obtained results are not satisfactory, a Monte Carlo approach can be used by randomly selecting the initial velocity values.

If the previous strategy does not yield satisfactory results, then the following method can be used. First, minimum and maximum distances from each coordinate of the libration point are selected. For instance, if the separation between spacecraft through time is required to be between 4,000 km and 8,000 km, then those distances can
be used as a starting point. For example, if the libration point is the origin \([0, 0, 0]\), then a Monte Carlo simulation can be performed using \([-2000, 2000]\) km (see Table 3.1) as the range in where to look for the initial conditions. Notice that those values seem to be a good starting point for this specific example because some initial positions will start already having the desired separations. Velocities are held fixed to decrease the number of variables being randomly selected. Values for velocities are chosen based on previous simulation results.

In Table 3.1, the green boxes are the I.C.s that are being randomly generated by MATLAB. In this specific example, values between -2,000 and 2,000 km are being generated for \(x\) and \(y\), while for \(z\) they are between 0 and 2,000 km. Moreover, the red boxes are the I.C.s for all the velocity terms that remain unchanged to reduce the number of free variables.
Table 3.1

Monte Carlo approach for finding I.C.s.

<table>
<thead>
<tr>
<th>$x^i_0$</th>
<th>$y^i_0$</th>
<th>$z^i_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v^i_{x_0}$</td>
<td>$v^i_{y_0}$</td>
<td>$v^i_{z_0}$</td>
</tr>
<tr>
<td>$x^{ii}_0$</td>
<td>$y^{ii}_0$</td>
<td>$z^{ii}_0$</td>
</tr>
<tr>
<td>$v^{ii}_{x_0}$</td>
<td>$v^{ii}_{y_0}$</td>
<td>$v^{ii}_{z_0}$</td>
</tr>
<tr>
<td>$x^{iii}_0$</td>
<td>$y^{iii}_0$</td>
<td>$z^{iii}_0$</td>
</tr>
<tr>
<td>$v^{iii}_{x_0}$</td>
<td>$v^{iii}_{y_0}$</td>
<td>$v^{iii}_{z_0}$</td>
</tr>
<tr>
<td>$x^{iv}_0$</td>
<td>$y^{iv}_0$</td>
<td>$z^{iv}_0$</td>
</tr>
<tr>
<td>$v^{iv}_{x_0}$</td>
<td>$v^{iv}_{y_0}$</td>
<td>$v^{iv}_{z_0}$</td>
</tr>
</tbody>
</table>

Some recommendations to consider are:

- Keeping nodes low when first starting to look for I.C.s since this will ensure faster convergence times (if a solution exists).

- Keeping the number of runs low at first to ensure that the obtained results look good or promising.

- Keeping track of the time each run takes. This is helpful in estimating how long the code will take to complete a run with many iterations.

To sum up, the first strategy suggests placing the spacecraft in a tetrahedral formation and using known values for the velocities, and if that does not work, then velocities are chosen randomly. The second strategy is the opposite; velocities are held fixed while the spacecraft’s initial locations are chosen randomly within a range.
Both strategies were used for the design and formation keeping of the orbits in this thesis rather than imposing constraints as Equation 3.14 would otherwise suggest. The use of Equation 3.14 was tried in some of the libration points. However, the obtained \( \Delta V \)s were really high. On the other hand, by following the Monte Carlo approach, the spacecraft followed their “natural” (of course, thrust is being exerted throughout the orbit to maintain control) path without any extra constraints.

### 3.6. Delta-V Calculation

Delta-V (\( \Delta V \)) is used in orbital mechanics as a criterion to evaluate orbit control performance. \( \Delta V \) is commonly used as a measure of performance since the fuel needed for orbit control cannot be determined unless the mass of a spacecraft is known (Wie, 2008). \( \Delta V \) can be calculated by integrating the control acceleration inputs with respect to time as Equation 3.17 suggests.

\[
\Delta V = \int_{t_0}^{t_f} \sqrt{a_x^2 + a_y^2 + a_z^2} \, dt
\]  

(3.17)

### 3.7. Verification and Validation

Verification and validation (V&V) of the results must be done in order to verify the feasibility of the solutions. As it was stated before, DIDO outputs a candidate solution to the optimal control problem. One way of validating and verifying DIDO’s solution is by applying Pontryagin’s principle.

#### 3.7.1. Pontryagin’s Principle

Pontryagin’s principle is a powerful tool for testing the optimality of the candidate solution that DIDO outputs. What Pontryagin’s principle does is that it states the necessary conditions that a candidate optimal solution must satisfy. Pontryagin’s principle
states that given an optimal solution to Problem B, there exists an absolutely continuous
covector function $\lambda(\cdot)$ and a covector $\nu$ that satisfy the Hamiltonian minimization
condition, the Hamiltonian value condition, the Hamiltonian evolution equation, the
adjoint equations, and the transversality conditions (Ross, 2015).

Ross suggests carrying out the following steps for any given problem to develop its
necessary conditions for optimality:

1. Construct the Hamiltonian.
2. Develop the adjoint equations.
4. Evaluate the Hamiltonian value condition.
5. Integrate the Hamiltonian evolution equation.
6. Formulate the transversality conditions.

**Constructing the Hamiltonian**

The Hamiltonian of any given Problem B is found by adding the running cost $F$ to
the scalar $\lambda \cdot f$:

$$H(\lambda, x, u, t) = F(x, u, t) + \lambda^T f(x, u, t)$$  \hspace{1cm} (3.18)

In Equation 3.18, $\lambda$ is the costate (a covector for measuring the value of the cost
function $J$), and $f$ represents the dynamics of the problem. The size of the costate is the
same as the size of the state vector $x$. For instance, if $x = [x \; v]^T \in \mathbb{R}^2$, it follows that $\lambda =
\begin{bmatrix}
\lambda_x \\
\lambda_v
\end{bmatrix} \in \mathbb{R}^2$. Moreover, the entries of $f$ are the right-hand-sides of $\dot{x}$ and $\dot{v}$. Consider the
following example adapted from (Ross, 2015), where \( u = u \in \mathbb{R} \), the running cost \( F \) is \( \frac{u^2}{2} \), and the dynamics \( \dot{x} \) and \( \dot{v} \) are \( v \) and \( u \), respectively. The Hamiltonian then is:

\[
H = \frac{u^2}{2} + [\lambda_x \quad \lambda_v] \begin{bmatrix} v \\ u \end{bmatrix} = \frac{u^2}{2} + \lambda_x v + \lambda_v u
\]

**Developing the Adjoint Equations**

The adjoint equations are found as follows:

\[
-\dot{\lambda} = \frac{\partial H}{\partial x}
\] (3.19)

Following the previous example, the adjoint equations would be:

\[
-\dot{\lambda}_x = \frac{\partial H}{\partial x} = 0 \quad \Rightarrow \lambda_x = a
\]

\[
-\dot{\lambda}_v = \frac{\partial H}{\partial v} = \lambda_x \quad \Rightarrow \lambda_v = -at - b
\]

where \( a \) and \( b \) are constants of integration. This specific example would mean that in dual space, the costate trajectories (given by DIDO) are straight lines.

**Minimizing the Hamiltonian**

The minimized Hamiltonian can be expressed as:

\[
\mathcal{H}(\lambda, x, t) = \min_{u \in \mathcal{U}} H(\lambda, x, u, t)
\] (3.20)

Again, following the previous example and using the minimized Hamiltonian concept, the extremal control can be found by minimizing the Hamiltonian with respect to the control \( u \).

\[
\frac{\partial H}{\partial u} = 0 \quad \Rightarrow u + \lambda_v = 0 \quad \Rightarrow u = -\lambda_v
\]

**Evaluating the Hamiltonian Value Condition**

The Hamiltonian Value condition is given by:

\[
\mathcal{H}[@t_f] = -\frac{\partial E}{\partial t_f}
\] (3.21)
where $\bar{E}$ is the endpoint Lagrangian. To compute $\bar{E}$, first the endpoint function $e$ and endpoint covector $\nu$ must be determined. The endpoint function is constructed from the information given in the problem statement. For instance, consider that the following conditions are given as part of the example problem:

\[
\begin{align*}
    x_f - 1 &= 0 \\
    v_f &= 0 \\
    t_f - 1 &= 0
\end{align*}
\]

then, the endpoint function would be:

\[
e(x_f, t_f) = \begin{bmatrix} x_f - 1 \\ v_f \\ t_f - 1 \end{bmatrix}
\]

Since $e \in \mathbb{R}^3$ it follows that the endpoint covector should also be $\nu \in \mathbb{R}^3$.

\[
\nu = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}
\]

The endpoint Lagrangian can now be computed as:

\[
\bar{E}(x_f, t_f) = E + \nu^T e
\]

(3.22)

where $E$ is the endpoint cost which is also given as part of the problem statement.

Substituting the now known values of the example in Equation 3.22, and assuming the endpoint cost to be zero, the endpoint Lagrangian would be:
\[
\begin{bmatrix}
x_f - 1 \\
v_f \\
t_f - 1
\end{bmatrix}
\begin{bmatrix}
E = 0 + [v_1 \; v_2 \; v_3] \\
v_f \\
t_f - 1
\end{bmatrix} = v_1(x_f - 1) + v_2 v_f + v_3(t_f - 1)
\]

and hence, the Hamiltonian Value Condition would be:

\[
\mathcal{H}[@t_f] = -\frac{\partial E}{\partial t_f} = -v_3
\]

**Analyzing the Hamiltonian Evolution Equation**

The Hamiltonian Evolution Equation is given by:

\[
\frac{dH}{dt} = \frac{\partial H}{\partial t}
\]

Thus, following with the example:

\[
\dot{H} = \frac{\partial H}{\partial t} = 0
\]

This means that the minimized Hamiltonian is a constant with respect to time, which again is helpful because by plotting the minimized Hamiltonian obtained through DIDO, a straight line is expected.

**Determining the Transversality Conditions**

The transversality conditions are obtained by differentiating the endpoint Lagrangian:

\[
\lambda(t_f) = \frac{\partial E}{\partial x_f}
\]

Transversality conditions for the example would be:

\[
\lambda_x(t_f) = \frac{\partial E}{\partial x_f} = v_1 \\
\lambda_v(t_f) = \frac{\partial E}{\partial v_f} = v_2
\]
3.7.2. V & V on the Proposed Optimal Control Problem

The same steps from the previous section can be applied to more complex problems such as the optimal control problem proposed in this thesis. First, Pontryagin’s principle is applied to the optimal control problem, just considering one spacecraft for exemplification. Then, the same steps are applied to the full problem (four spacecraft).

One Spacecraft

By considering just one spacecraft, Problem 3.15 becomes:

\[
\begin{align*}
\text{minimize} \quad & J = \frac{1}{t_f - t_0} \int_{t_0}^{t_f} (|a_x| + |a_y| + |a_z|) \, dt \\
\text{subject to} \quad & \dot{x} = v_x, \\
& \dot{y} = v_y, \\
& \dot{z} = v_z, \\
& \ddot{x} = 2\dot{y} + x - \frac{(1 - \mu^*)(x - \mu^*)}{r_1^3} - \frac{\mu^*(x + 1 - \mu^*)}{r_2^3} + a_x, \\
& \ddot{y} = -2\dot{x} + y - \frac{(1 - \mu^*)y}{r_1^3} - \frac{\mu^*y}{r_2^3} + a_y, \\
& \ddot{z} = -\frac{(1 - \mu^*)z}{r_1^3} - \frac{\mu^*z}{r_2^3} + a_z, \\
& t_0 = 0, \\
& \left(x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0}\right) = \left(x^0, y^0, z^0, v_{x0}, v_{y0}, v_{z0}\right), \\
& \left(x_f, y_f, z_f, v_{x_f}, v_{y_f}, v_{z_f}\right) = \left(x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0}\right)
\end{align*}
\]
where the superscripts 0 represent given initial values. Moreover, \( x = [x, y, z, v_x, v_y, v_z]^T \) and \( u = [a_x, a_y, a_z]^T \). Additionally, \( r_1 \) and \( r_2 \) are as given in Equation 2.7. Therefore, the Hamiltonian of Problem 3.25 is:

\[
H = (|a_x| + |a_y| + |a_z|) + [\lambda_x \lambda_y \lambda_z \lambda_{vx} \lambda_{vy} \lambda_{vz}]
\]

\[
\begin{bmatrix}
v_x \\
v_y \\
v_z \\
2v_y + x - \frac{(1-\mu^*)(x-\mu^*)}{r_1^3} - \frac{\mu^*(x+1-\mu^*)}{r_2^3} + a_x \\
-2v_x + y - \frac{(1-\mu^*)y}{r_1^3} - \frac{\mu^*y}{r_2^3} + a_y \\
-\frac{(1-\mu^*)z}{r_1^3} - \frac{\mu^*z}{r_2^3} + a_z
\end{bmatrix}
\]

Multiplying the matrices yields:

\[
H = |a_x| + |a_y| + |a_z| + \lambda_x v_x + \lambda_y v_y + \lambda_z v_z + \lambda_{vx} s_1 + \lambda_{vy} s_2 + \lambda_{vz} s_3
\]

(3.26)

where:

\[
s_1 = 2v_y + x - \frac{(1-\mu^*)(x-\mu^*)}{r_1^3} - \frac{\mu^*(x+1-\mu^*)}{r_2^3} + a_x
\]

\[
s_2 = -2v_x + y - \frac{(1-\mu^*)y}{r_1^3} - \frac{\mu^*y}{r_2^3} + a_y
\]

\[
s_3 = -\frac{(1-\mu^*)z}{r_1^3} - \frac{\mu^*z}{r_2^3} + a_z
\]

The adjoint equations are computed using Equation 3.19:

\[
-\dot{\lambda}_{vx} = \frac{\partial H}{\partial v_x} = \lambda_x - 2\lambda_{vx}
\]

(3.27)

\[
-\dot{\lambda}_{vy} = \frac{\partial H}{\partial v_y} = \lambda_y + 2\lambda_{vx}
\]

(3.28)

\[
-\dot{\lambda}_{vz} = \frac{\partial H}{\partial v_z} = \lambda_z
\]

(3.29)
Notice that the adjoint equations $-\dot{\lambda}_x$, $-\dot{\lambda}_y$, and $-\dot{\lambda}_z$ are not presented here because they are long and do not provide any useful information.

Next, the Hamiltonian can be minimized using Equation 3.20:

$$\frac{\partial H}{\partial a_x} = \frac{a_x}{|a_x|} + \lambda_{v_x} = 0 \Rightarrow \lambda_{v_x} = -\frac{a_x}{|a_x|} = \pm 1 \quad (3.30)$$

$$\frac{\partial H}{\partial a_y} = \frac{a_y}{|a_y|} + \lambda_{v_y} = 0 \Rightarrow \lambda_{v_y} = -\frac{a_y}{|a_y|} = \pm 1 \quad (3.31)$$

$$\frac{\partial H}{\partial a_z} = \frac{a_z}{|a_z|} + \lambda_{v_z} = 0 \Rightarrow \lambda_{v_z} = -\frac{a_z}{|a_z|} = \pm 1 \quad (3.32)$$

As can be seen from the above Equations 3.30-3.32, if $\lambda_{v_x}$, $\lambda_{v_y}$, and $\lambda_{v_z}$ are plotted they should result in a straight line at either -1 or +1 starting at $t_0$ and finishing at $t_f$.

Finally, the Hamiltonian evolution equation can be analyzed. Since the Hamiltonian does not explicitly depend on time, Equation 3.23 yields:

$$\dot{\mathcal{H}} = \frac{d\mathcal{H}}{dt} = \frac{\partial H}{\partial t} = 0 \quad (3.33)$$

Hence, the Hamiltonian should remain constant throughout time: $\mathcal{H} \left[ @t \right] = \text{constant}$, as Figure (3.6) depicts. Note that the transversality conditions are not computed since they will not provide any useful information at this point.

*Figure 3.6* Example of how the Hamiltonian evolution should look for the optimal control problem of one spacecraft.
Four Spacecraft

Now, if four spacecraft are considered, Problem 3.15 becomes:

\[
\text{minimize } \quad J = \frac{1}{t_f - t_0} \int_{t_0}^{t_f} \left( |a_x^i| + |a_y^i| + |a_z^i| + \cdots + |a_x^iv| + |a_y^iv| + |a_z^iv| \right) \, dt
\]

subject to

\[
\begin{align*}
\dot{x}^i &= v_x^i, \\
\dot{y}^i &= v_y^i, \\
\dot{z}^i &= v_z^i, \\
\dot{x}^i &= 2\dot{y}^i + x^i - \left( 1 - \mu^* \right) \left( x^i - \mu^* \right) \frac{r_1^i}{r_1^{i3}} - \mu^* \left( x^i + 1 - \mu^* \right) \frac{r_2^i}{r_2^{i3}} + a_x^i, \\
\dot{y}^i &= -2\dot{x}^i + y^i - \left( 1 - \mu^* \right) y^i \frac{r_1^i}{r_1^{i3}} - \mu^* y^i \frac{r_2^i}{r_2^{i3}} + a_y^i, \\
\dot{z}^i &= -\left( 1 - \mu^* \right) z^i \frac{r_1^i}{r_1^{i3}} - \mu^* z^i \frac{r_2^i}{r_2^{i3}} + a_z^i, \\
\vdots, \\
\dot{x}^iv &= 2\dot{y}^iv + x^iv - \left( 1 - \mu^* \right) \left( x^iv - \mu^* \right) \frac{r_1^iv}{r_1^{iv3}} - \mu^* \left( x^iv + 1 - \mu^* \right) \frac{r_2^iv}{r_2^{iv3}} + a_x^iv, \\
\dot{y}^iv &= -2\dot{x}^iv + y^iv - \left( 1 - \mu^* \right) y^iv \frac{r_1^iv}{r_1^{iv3}} - \mu^* y^iv \frac{r_2^iv}{r_2^{iv3}} + a_y^iv, \\
\dot{z}^iv &= -\left( 1 - \mu^* \right) z^iv \frac{r_1^iv}{r_1^{iv3}} - \mu^* z^iv \frac{r_2^iv}{r_2^{iv3}} + a_z^iv, \\
t_0 &= 0,
\end{align*}
\]

\[
\begin{align*}
(x_0, x_0^i, x_0^ii, x_0^iii, x_0^iv) &= (x_0^0, x_0^0i, x_0^0ii, x_0^0iii, x_0^0iv), \\
(x_f, x_f^i, x_f^ii, x_f^iii, x_f^iv) &= (x_f^0, x_f^0i, x_f^0ii, x_f^0iii, x_f^0iv)
\end{align*}
\]
where the superscripts $i$, $ii$, $iii$, and $iv$ correspond to each specific spacecraft. Again, the superscripts 0 represent given initial values, and $r_{1,2}$ are given in Equation 2.7. The Hamiltonian of Problem 3.34 is then:

$$H = |a_x^i| + |a_y^i| + |a_z^i| + \cdots + |a_x^{iv}| + |a_y^{iv}| + |a_z^{iv}| +$$

$$
\begin{bmatrix}
\lambda_x^i \\
\lambda_y^i \\
\lambda_z^i \\
\vdots \\
\lambda_x^{iv} \\
\lambda_y^{iv} \\
\lambda_z^{iv}
\end{bmatrix}^T
\begin{bmatrix}
y_x^i \\
y_y^i \\
y_z^i \\
\vdots \\
y_x^{iv} \\
y_y^{iv} \\
y_z^{iv}
\end{bmatrix} +
\begin{bmatrix}
2v_y^i + x^i - \frac{(1-\mu^i)(x^i-\mu^i)}{r_1^i} - \frac{\mu^i(x^i+1-\mu^i)}{r_2^i} + a_x^i \\
-2v_x^i + y^i - \frac{(1-\mu^i)y^i}{r_1^i} - \frac{\mu^i y^i}{r_2^i} + a_y^i \\
-\frac{(1-\mu^i)z^i}{r_1^i} - \frac{\mu^i z^i}{r_2^i} + a_z^i \\
\vdots \\
2v_y^{iv} + x^{iv} - \frac{(1-\mu^{iv})(x^{iv}-\mu^{iv})}{r_1^{iv}} - \frac{\mu^{iv}(x^{iv}+1-\mu^{iv})}{r_2^{iv}} + a_x^{iv} \\
-2v_x^{iv} + y^{iv} - \frac{(1-\mu^{iv})y^{iv}}{r_1^{iv}} - \frac{\mu^{iv} y^{iv}}{r_2^{iv}} + a_y^{iv} \\
-\frac{(1-\mu^{iv})z^{iv}}{r_1^{iv}} - \frac{\mu^{iv} z^{iv}}{r_2^{iv}} + a_z^{iv}
\end{bmatrix}$$
Multiplying the matrices yields:

\[
H = |a_x^i| + |a_y^i| + |a_z^i| + |a_x^{iv}| + |a_y^{iv}| + |a_z^{iv}| + |a_x^{iii}| + |a_y^{iii}| + |a_z^{iii}|
\]

\[
|a_y^{iv}| + |a_z^{iv}| + \lambda_x^iv x^i + \lambda_y^iv y^i + \lambda_z^iv z^i + \lambda_x^{iv} s^i_1 + \lambda_y^{iv} s^i_2 + \lambda_z^{iv} s^i_3 +
\]

\[
\lambda_x^{ii} v_x^{ii} + \lambda_y^{ii} v_y^{ii} + \lambda_z^{ii} v_z^{ii} + \lambda_x^{ii} s^{ii}_1 + \lambda_y^{ii} s^{ii}_2 + \lambda_z^{ii} s^{ii}_3 +
\]

\[
\lambda_x^{iii} v_x^{iii} + \lambda_y^{iii} v_y^{iii} + \lambda_z^{iii} v_z^{iii} + \lambda_x^{iii} s^{iii}_1 + \lambda_y^{iii} s^{iii}_2 + \lambda_z^{iii} s^{iii}_3 +
\]

\[
\lambda_x^{iv} v_x^{iv} + \lambda_y^{iv} v_y^{iv} + \lambda_z^{iv} v_z^{iv} + \lambda_x^{iv} s^{iv}_1 + \lambda_y^{iv} s^{iv}_2 + \lambda_z^{iv} s^{iv}_3
\]

where:

\[
s^i_1 = 2v_x^i + x^i - \frac{(1 - \mu^*) (x^i - \mu^*)}{r_1^{i3}} - \frac{\mu^* (x^i + 1 - \mu^*)}{r_2^{i3}} + a_x^i
\]

\[
s^i_2 = -2v_y^i + y^i - \frac{(1 - \mu^*) y^i}{r_1^{i3}} - \frac{\mu^* y^i}{r_2^{i3}} + a_y^i
\]

\[
s^i_3 = -\frac{(1 - \mu^*) z^i}{r_1^{i3}} - \frac{\mu^* z^i}{r_2^{i3}} + a_z^i
\]

\[
s^{ii}_1 = 2v_y^{ii} + x^{ii} - \frac{(1 - \mu^*) (x^{ii} - \mu^*)}{r_1^{ii3}} - \frac{\mu^* (x^{ii} + 1 - \mu^*)}{r_2^{ii3}} + a_x^{ii}
\]

\[
s^{ii}_2 = -2v_x^{ii} + y^{ii} - \frac{(1 - \mu^*) y^{ii}}{r_1^{ii3}} - \frac{\mu^* y^{ii}}{r_2^{ii3}} + a_y^{ii}
\]

\[
s^{ii}_3 = -\frac{(1 - \mu^*) z^{ii}}{r_1^{ii3}} - \frac{\mu^* z^{ii}}{r_2^{ii3}} + a_z^{ii}
\]

\[
s^{iii}_1 = 2v_y^{iii} + x^{iii} - \frac{(1 - \mu^*) (x^{iii} - \mu^*)}{r_1^{iii3}} - \frac{\mu^* (x^{iii} + 1 - \mu^*)}{r_2^{iii3}} + a_x^{iii}
\]

\[
s^{iii}_2 = -2v_x^{iii} + y^{iii} - \frac{(1 - \mu^*) y^{iii}}{r_1^{iii3}} - \frac{\mu^* y^{iii}}{r_2^{iii3}} + a_y^{iii}
\]

\[
s^{iii}_3 = -\frac{(1 - \mu^*) z^{iii}}{r_1^{iii3}} - \frac{\mu^* z^{iii}}{r_2^{iii3}} + a_z^{iii}
\]

\[
s^{iv}_1 = 2v_y^{iv} + x^{iv} - \frac{(1 - \mu^*) (x^{iv} - \mu^*)}{r_1^{iv3}} - \frac{\mu^* (x^{iv} + 1 - \mu^*)}{r_2^{iv3}} + a_x^{iv}
\]
\[ s_2^{iv} = -2y_x^{iv} + y^{iv} - \frac{(1 - \mu^*)y^{iv}}{r_1^{iv3}} - \frac{\mu^* y^{iv}}{r_2^{iv3}} + a_y^{iv} \]

\[ s_3^{iv} = - \frac{(1 - \mu^*)z^{iv}}{r_1^{iv3}} - \frac{\mu^* z^{iv}}{r_2^{iv3}} + a_z^{iv} \]

Notice again that since the Hamiltonian does not explicitly depend on time, Equation 3.23 yields:

\[ \dot{\mathcal{H}} = \frac{d\mathcal{H}}{dt} = \frac{\partial H}{\partial t} = 0 \]  \hspace{1cm} (3.36)

That means that the Hamiltonian should remain constant throughout time: \( \mathcal{H} \) \([@t]\) = constant, as Figure (3.6) shows.
4. Simulations and Results

Simulations and results of the Sun-Mercury, Sun-Venus, Sun-Earth, and Sun-Mars systems are presented here. Most of the units used in this section are scaled. To unscale them, refer to Section 2.2. Most of the simulations are performed using 40 nodes since smooth results are obtained, and computation times are still relatively low. For the comparison between the $L^1$ and $L^2$ norms of control, the problem is first solved using the $L^1$-norm and time is left “free”. The optimal time is then used (meaning it is now fixed) for the $L^2$-norm scenario; that way, both orbits have the same period. As it will be evident in the following pages, there is an outlier in the optimal period found in Table 4.22. In the Sun-Mars $L_1$, if the optimal period when using the $L^1$-norm is then used in the $L^2$-norm case, the resulting $\Delta V$ is really high. Therefore, for this specific case, the optimal period that is obtained when using the $L^2$-norm is also shown along with the spacecraft total $\Delta V$.

Boundary constraints to the events, states, and controls of the problem are imposed. These lower and upper constraints are imposed to ensure that feasible solutions are obtained. Using improper boundary constraints could lead to unnecessary thrusting maneuvers. By not choosing the correct boundary constraints, the program could look for solutions far from the libration point location. This is important, especially when working with the $L_1$ point of any system, due to the closeness of the $L_2$ point. All the figures in this section show the simulations’ results when using the $L^1$-norm. Results of some simulations using the $L^2$-norm and the same I.C.s as in here, can be found in the Appendix section.
4.1. Sun-Mercury

4.1.1. $L_1$

Table 4.1

Sun-Mercury $L_1$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9961289</td>
<td>-0.0001243</td>
<td>0.0004073</td>
<td>0.000300</td>
<td>0.000100</td>
<td>0.000100</td>
</tr>
<tr>
<td>2</td>
<td>0.9961230</td>
<td>-0.0002498</td>
<td>0.0004755</td>
<td>0.000300</td>
<td>0.000100</td>
<td>0.000100</td>
</tr>
<tr>
<td>3</td>
<td>0.9962059</td>
<td>0.0000051</td>
<td>0.0003383</td>
<td>0.000300</td>
<td>0.000100</td>
<td>0.000100</td>
</tr>
<tr>
<td>4</td>
<td>0.9959953</td>
<td>0.0002381</td>
<td>0.0003263</td>
<td>0.000300</td>
<td>0.000100</td>
<td>0.000100</td>
</tr>
</tbody>
</table>

Figure 4.1 Sun-Mercury $L_1$: 3D solution.
Figure 4.2 Sun-Mercury $L_1$: Separations between spacecraft throughout time.

Figure 4.3 Sun-Mercury $L_1$: Velocity profiles in $x, y, z$. 
Figure 4.4 Sun-Mercury $L_1$: Control profiles in $x$, $y$, $z$.

Figure 4.5 Sun-Mercury $L_1$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Table 4.2

Sun-Mercury $L_1$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L^2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.5564</td>
<td>18.352</td>
<td>14.326</td>
</tr>
<tr>
<td>2</td>
<td>7.5564</td>
<td>18.134</td>
<td>13.630</td>
</tr>
<tr>
<td>3</td>
<td>7.5564</td>
<td>17.840</td>
<td>13.180</td>
</tr>
<tr>
<td>4</td>
<td>7.5564</td>
<td>17.955</td>
<td>13.821</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table 4.1 are found using the second strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 5,000 km and 45,000 km, respectively, as shown by Figure (4.2). The optimal period using the $L^1$-norm is 7.5564 TU or 106 days. Figure (4.5) shows that most of the thrusting occurs at the beginning/end of the orbit. Furthermore, the overall $\Delta V$ is lower when using the $L^2$-norm, as Table 4.2 shows. Figure (4.6) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
### 4.1.2. $L_3$

Table 4.3

Sun-Mercury $L_3$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.9999930</td>
<td>0.0000068</td>
<td>0.0001517</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>2</td>
<td>-1.0000089</td>
<td>-0.0000116</td>
<td>0.0000539</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>3</td>
<td>-1.0000048</td>
<td>0.0000106</td>
<td>0.0004328</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>4</td>
<td>-1.0000052</td>
<td>0.0000073</td>
<td>0.0003565</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
</tbody>
</table>

Figure 4.7 Sun-Mercury $L_3$: 3D solution.
Figure 4.8 Sun-Mercury $L_3$: Separations between spacecraft throughout time.

Figure 4.9 Sun-Mercury $L_3$: Velocity profiles in $x$, $y$, $z$. 
Figure 4.10 Sun-Mercury $L_3$: Control profiles in $x$, $y$, $z$.

Figure 4.11 Sun-Mercury $L_3$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
The initial conditions shown in Table 4.3 are found using the second strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 4,500 km and 24,000 km, respectively, as shown by Figure (4.8). The optimal period using the $L^1$-norm is 1.2203 TU or 17 days. Figure (4.11) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L^1$-norm, as Table 4.4 shows. Figure (4.12) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
4.1.3. $L_4$

Table 4.5

Sun-Mercury $L_4$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4998864</td>
<td>0.8660935</td>
<td>0.0002568</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>2</td>
<td>0.5001526</td>
<td>0.8657955</td>
<td>0.0002827</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>3</td>
<td>0.4999960</td>
<td>0.8661451</td>
<td>0.0002858</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>4</td>
<td>0.5002786</td>
<td>0.8662033</td>
<td>0.0002667</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
</tbody>
</table>

Figure 4.13  Sun-Mercury $L_4$: 3D solution.
**Figure 4.14** Sun-Mercury $L_4$: Separations between spacecraft throughout time.

**Figure 4.15** Sun-Mercury $L_4$: Velocity profiles in $x, y, z$. 
Figure 4.16 Sun-Mercury $L_4$: Control profiles in $x$, $y$, $z$.

Figure 4.17 Sun-Mercury $L_4$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Table 4.6

Sun-Mercury $L_4$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L_1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L_2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.3440</td>
<td>0.51713</td>
<td>0.50346</td>
</tr>
<tr>
<td>2</td>
<td>1.3440</td>
<td>0.61898</td>
<td>0.57239</td>
</tr>
<tr>
<td>3</td>
<td>1.3440</td>
<td>0.50824</td>
<td>0.53304</td>
</tr>
<tr>
<td>4</td>
<td>1.3440</td>
<td>0.67612</td>
<td>0.71649</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table 4.5 are found using the second strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 4,000 km and 25,000 km, respectively, as shown by Figure (4.14). The optimal period using the $L_1$-norm is 1.3440 TU or 19 days. Figure (4.17) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L_1$-norm, as Table 4.6 shows. Figure (4.18) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
4.1.4. \( L_5 \)

Table 4.7

Sun-Mercury \( L_5 \): Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( z_0 )</th>
<th>( v_x )</th>
<th>( v_y )</th>
<th>( v_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5001656</td>
<td>-0.8661379</td>
<td>0.0002867</td>
<td>0.0001000</td>
<td>0.0001000</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.5003247</td>
<td>-0.8658092</td>
<td>0.0002543</td>
<td>0.0001000</td>
<td>0.0001000</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.4999078</td>
<td>-0.8660680</td>
<td>0.0002843</td>
<td>0.0001000</td>
<td>0.0001000</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.5000676</td>
<td>-0.8658495</td>
<td>0.0002684</td>
<td>0.0001000</td>
<td>0.0001000</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.19 Sun-Mercury \( L_5 \): 3D solution.
Figure 4.20 Sun-Mercury $L_5$: Separations between spacecraft throughout time.

Figure 4.21 Sun-Mercury $L_5$: Velocity profiles in $x$, $y$, $z$. 
Figure 4.22 Sun-Mercury $L_5$: Control profiles in $x$, $y$, $z$.

Figure 4.23 Sun-Mercury $L_5$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Table 4.8
Sun-Mercury $L_5$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [m/s]/1TU</th>
<th>$\Delta V$ ($L^2$-norm) [m/s]/1TU</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.81189</td>
<td>0.59956</td>
<td>0.68310</td>
</tr>
<tr>
<td>2</td>
<td>0.81189</td>
<td>0.46315</td>
<td>0.54788</td>
</tr>
<tr>
<td>3</td>
<td>0.81189</td>
<td>0.46753</td>
<td>0.52620</td>
</tr>
<tr>
<td>4</td>
<td>0.81189</td>
<td>0.45442</td>
<td>0.46556</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table 4.7 are found using the second strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 14,000 km and 29,000 km, respectively, as shown by Figure (4.20). The optimal period using the $L^1$-norm is 0.81189 TU or 11 days. Figure (4.23) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L^1$-norm, as Table 4.8 shows. Figure (4.24) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
4.2. Sun-Venus

4.2.1. $L_1$

Table 4.9

Sun-Venus $L_1$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9906188</td>
<td>-0.0002082</td>
<td>0.0002990</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>2</td>
<td>0.9905402</td>
<td>-0.0001742</td>
<td>0.0002705</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>3</td>
<td>0.9905190</td>
<td>-0.0001437</td>
<td>0.0002543</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>4</td>
<td>0.9907448</td>
<td>-0.0001008</td>
<td>0.0003039</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
</tbody>
</table>

*Figure 4.25* Sun-Venus $L_1$: 3D solution.
Figure 4.26  Sun-Venus $L_1$: Separations between spacecraft throughout time.

Figure 4.27  Sun-Venus $L_1$: Velocity profiles in $x$, $y$, $z$. 
Figure 4.28  Sun-Venus $L_1$: Control profiles in $x$, $y$, $z$.

Figure 4.29  Sun-Venus $L_1$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Table 4.10

Sun-Venus $L_1$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L^2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.2800</td>
<td>28.61684</td>
<td>39.52761</td>
</tr>
<tr>
<td>2</td>
<td>6.2800</td>
<td>28.82315</td>
<td>36.47854</td>
</tr>
<tr>
<td>3</td>
<td>6.2800</td>
<td>28.76790</td>
<td>41.52295</td>
</tr>
<tr>
<td>4</td>
<td>6.2800</td>
<td>28.41486</td>
<td>39.24179</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table 4.9 are found using the second strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 5,000 km and 60,000 km, respectively, as shown by Figure (4.26). The optimal period using the $L^1$-norm is 6.28 TU or 243 days. Figure (4.29) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L^1$-norm, as Table 4.10 shows. Figure (4.30) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
4.2.2. $L_3$

Table 4.11

Sun-Venus $L_3$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.9996591</td>
<td>0</td>
<td>0</td>
<td>0.0002134</td>
<td>0.0000879</td>
<td>0.0000747</td>
</tr>
<tr>
<td>2</td>
<td>-0.9998208</td>
<td>0.0002801</td>
<td>0</td>
<td>0.0002142</td>
<td>0.0001776</td>
<td>0.0000532</td>
</tr>
<tr>
<td>3</td>
<td>-0.9998208</td>
<td>0.0000934</td>
<td>0.0002641</td>
<td>0.0002292</td>
<td>0.0001531</td>
<td>0.0001589</td>
</tr>
<tr>
<td>4</td>
<td>-0.9999825</td>
<td>0</td>
<td>0</td>
<td>0.0001713</td>
<td>0.0002021</td>
<td>0.0002211</td>
</tr>
</tbody>
</table>

*Figure 4.31* Sun-Venus $L_3$: 3D solution.
Figure 4.32  Sun-Venus $L_3$: Separations between spacecraft throughout time.

Figure 4.33  Sun-Venus $L_3$: Velocity profiles in $x$, $y$, $z$. 
**Figure 4.34** Sun-Venus $L_3$: Control profiles in $x$, $y$, $z$.

**Figure 4.35** Sun-Venus $L_3$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
The initial conditions shown in Table 4.11 are found using the first strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 33,000 km and 40,500 km, respectively, as shown by Figure (4.32). The optimal period using the $L_1$-norm is 0.9632 TU or 37 days. Figure (4.35) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L_1$-norm, as Table 4.12 shows.

Figure (4.36) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
4.2.3. \( L_4 \)

Table 4.13

Sun-Venus \( L_4 \): Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( z_0 )</th>
<th>( v_x )</th>
<th>( v_y )</th>
<th>( v_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4999900</td>
<td>0.8660001</td>
<td>0.0000805</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>2</td>
<td>0.4999900</td>
<td>0.8660001</td>
<td>0.0000616</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>3</td>
<td>0.5004000</td>
<td>0.8660002</td>
<td>0.0006200</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>4</td>
<td>0.5004000</td>
<td>0.8660002</td>
<td>0.0002500</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
</tbody>
</table>

*Figure 4.37* Sun-Venus \( L_4 \): 3D solution.
Figure 4.38  Sun-Venus $L_4$: Separations between spacecraft throughout time.

Figure 4.39  Sun-Venus $L_4$: Velocity profiles in x, y, z.
**Figure 4.40** Sun-Venus $L_4$: Control profiles in $x$, $y$, $z$.

**Figure 4.41** Sun-Venus $L_4$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Figure 4.42 Sun-Venus $L_4$: Hamiltonian evolution.

Table 4.14

Sun-Venus $L_4$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L^2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.18911</td>
<td>0.38287</td>
<td>0.33621</td>
</tr>
<tr>
<td>2</td>
<td>2.18911</td>
<td>0.56416</td>
<td>0.52072</td>
</tr>
<tr>
<td>3</td>
<td>2.18911</td>
<td>0.64477</td>
<td>0.61738</td>
</tr>
<tr>
<td>4</td>
<td>2.18911</td>
<td>0.40291</td>
<td>0.40136</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table 4.13 are found using the second strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 29,000 km and 60,000 km, respectively, as shown by Figure (4.38). The optimal period using the $L^1$-norm is 2.18911 TU or 85 days. Figure (4.41) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L^2$-norm, as Table 4.14 shows. Figure (4.42) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
4.2.4. $L_5$

Table 4.15

Sun-Venus $L_5$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5000225</td>
<td>-0.8659479</td>
<td>0.0002947</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>2</td>
<td>0.4997774</td>
<td>-0.8660550</td>
<td>0.0002523</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>3</td>
<td>0.4998477</td>
<td>-0.8659384</td>
<td>0.0002861</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>4</td>
<td>0.5001889</td>
<td>-0.8659769</td>
<td>0.0002535</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
</tbody>
</table>

Figure 4.43  Sun-Venus $L_5$: 3D solution.
**Figure 4.44** Sun-Venus $L_5$: Separations between spacecraft throughout time.

**Figure 4.45** Sun-Venus $L_5$: Velocity profiles in $x$, $y$, $z$. 
Figure 4.46  Sun-Venus $L_5$: Control profiles in $x$, $y$, $z$.

Figure 4.47  Sun-Venus $L_5$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Figure 4.48 Sun-Venus $L_5$: Hamiltonian evolution.

Table 4.16

Sun-Venus $L_5$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L^2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.4050</td>
<td>0.38263</td>
<td>0.36515</td>
</tr>
<tr>
<td>2</td>
<td>2.4050</td>
<td>0.31940</td>
<td>0.31907</td>
</tr>
<tr>
<td>3</td>
<td>2.4050</td>
<td>0.36834</td>
<td>0.35780</td>
</tr>
<tr>
<td>4</td>
<td>2.4050</td>
<td>0.47099</td>
<td>0.44104</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table 4.15 are found using the second strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 12,500 km and 45,000 km, respectively, as shown by Figure (4.44). The optimal period using the $L^1$-norm is 2.4050 TU or 93 days. Figure (4.47) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L^2$-norm, as Table 4.16 shows. Figure (4.48) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
4.3. Sun-Earth

4.3.1. $L_1$

Table 4.17

Sun-Earth $L_1$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9893509</td>
<td>0</td>
<td>0</td>
<td>0.0000891</td>
<td>0.0000796</td>
<td>0.0000794</td>
</tr>
<tr>
<td>2</td>
<td>0.9893342</td>
<td>0.0000289</td>
<td>0</td>
<td>0.0000798</td>
<td>0.0000812</td>
<td>0.0000805</td>
</tr>
<tr>
<td>3</td>
<td>0.9893342</td>
<td>0.0000096</td>
<td>0.0000273</td>
<td>0.0001326</td>
<td>0.0000992</td>
<td>0.0001182</td>
</tr>
<tr>
<td>4</td>
<td>0.9893175</td>
<td>0</td>
<td>0</td>
<td>0.0000850</td>
<td>0.0000796</td>
<td>0.0000677</td>
</tr>
</tbody>
</table>

Figure 4.49  Sun-Earth $L_1$: 3D solution.
Figure 4.50 Sun-Earth $L_1$: Separations between spacecraft throughout time.

Figure 4.51 Sun-Earth $L_1$: Velocity profiles in $x$, $y$, $z$. 
Figure 4.52 Sun-Earth $L_1$: Control profiles in $x$, $y$, $z$.

Figure 4.53 Sun-Earth $L_1$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Figure 4.54 Sun-Earth $L_1$: Hamiltonian evolution.

Table 4.18

Sun-Earth $L_1$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L^2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8.9792</td>
<td>30.127</td>
<td>51.005</td>
</tr>
<tr>
<td>2</td>
<td>8.9792</td>
<td>30.870</td>
<td>46.273</td>
</tr>
<tr>
<td>3</td>
<td>8.9792</td>
<td>30.267</td>
<td>46.445</td>
</tr>
<tr>
<td>4</td>
<td>8.9792</td>
<td>30.019</td>
<td>45.484</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table 4.17 are found using the first strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 10,000 km and 250,000 km, respectively, as shown by Figure (4.50). The optimal period using the $L^1$-norm is 8.9792 TU or 522 days. Figure (4.53) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L^2$-norm, as Table 4.18 shows. Figure (4.54) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
4.3.2. \( L_4 \)

Table 4.19

Sun-Earth \( L_4 \): Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( z_0 )</th>
<th>( v_x )</th>
<th>( v_y )</th>
<th>( v_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4998861</td>
<td>0.8659619</td>
<td>0.0002500</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>2</td>
<td>0.4998711</td>
<td>0.8661401</td>
<td>0.0006200</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>3</td>
<td>0.4999939</td>
<td>0.8660464</td>
<td>-0.0000702</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>4</td>
<td>0.4999860</td>
<td>0.8661492</td>
<td>0.0000125</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
</tbody>
</table>

Figure 4.55  Sun-Earth \( L_4 \): 3D solution.
Figure 4.56 Sun-Earth $L_4$: Separations between spacecraft throughout time.

Figure 4.57 Sun-Earth $L_4$: Velocity profiles in $x$, $y$, $z$. 
**Figure 4.58** Sun-Earth $L_4$: Control profiles in $x$, $y$, $z$.

**Figure 4.59** Sun-Earth $L_4$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
The initial conditions shown in Table 4.19 are found using the second strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 6,800 km and 13,000 km, respectively, as shown by Figure (4.56). The optimal period using the $L^1$-norm is 1 TU or 58 days. Figure (4.59) shows that most of the thrusting occurs at the beginning/end of the orbit. Furthermore, the overall $\Delta V$ is lower when using the $L^2$-norm, as Table 4.20 shows. Figure (4.60) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
4.4. Sun-Mars

4.4.1. \( L_1 \)

Table 4.21

Sun-Mars \( L_1 \): Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( z_0 )</th>
<th>( v_x )</th>
<th>( v_y )</th>
<th>( v_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9952794</td>
<td>0</td>
<td>0.0000132</td>
<td>0.0001354</td>
<td>0.0000938</td>
<td>0.0000809</td>
</tr>
<tr>
<td>2</td>
<td>0.9952720</td>
<td>0.0000129</td>
<td>0.0000132</td>
<td>0.0000716</td>
<td>0.0000787</td>
<td>0.0001109</td>
</tr>
<tr>
<td>3</td>
<td>0.9952720</td>
<td>0.0000043</td>
<td>0.0000253</td>
<td>0.0000923</td>
<td>0.0000611</td>
<td>0.0001358</td>
</tr>
<tr>
<td>4</td>
<td>0.9952646</td>
<td>0</td>
<td>0.0000132</td>
<td>0.0000591</td>
<td>0.0000562</td>
<td>0.0000705</td>
</tr>
</tbody>
</table>

Figure 4.61  Sun-Mars \( L_1 \): 3D solution.
Figure 4.62 Sun-Mars $L_1$: Separations between spacecraft throughout time.

Figure 4.63 Sun-Mars $L_1$: Velocity profiles in $x$, $y$, $z$. 
Figure 4.64  Sun-Mars $L_1$: Control profiles in $x$, $y$, $z$.

Figure 4.65  Sun-Mars $L_1$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Table 4.22
Sun-Mars $L_1$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [m/s]/1TU</th>
<th>$\Delta V$ ($L^2$-norm) [m/s]/1TU</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.9817</td>
<td>21.680</td>
<td>$&gt;1000$ (10.377; $t_f=9.4248$)</td>
</tr>
<tr>
<td>2</td>
<td>3.9817</td>
<td>21.680</td>
<td>$&gt;1000$ (12.136; $t_f=9.4248$)</td>
</tr>
<tr>
<td>3</td>
<td>3.9817</td>
<td>21.680</td>
<td>$&gt;1000$ (12.828; $t_f=9.4248$)</td>
</tr>
<tr>
<td>4</td>
<td>3.9817</td>
<td>21.680</td>
<td>$&gt;1000$ (9.7206; $t_f=9.4248$)</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table 4.21 are found using the first strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 4,000 km and 97,000 km, respectively, as shown by Figure (4.62). The optimal period using the $L^1$-norm is 3.9817 TU or 435 days. Figure (4.65) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L^1$-norm, as Table 4.22 shows. Figure (4.66) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
### 4.4.2. \( L_3 \)

Table 4.23

Sun-Mars \( L_3 \): Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( z_0 )</th>
<th>( v_x )</th>
<th>( v_y )</th>
<th>( v_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.9999787</td>
<td>0</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>2</td>
<td>-0.9999861</td>
<td>0.0000129</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>3</td>
<td>-0.9999861</td>
<td>0.0000043</td>
<td>0.0000121</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>4</td>
<td>-0.9999936</td>
<td>0</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
</tbody>
</table>

*Figure 4.67* Sun-Mars \( L_3 \): 3D solution.
Figure 4.68 Sun-Mars $L_3$: Separations between spacecraft throughout time.

Figure 4.69 Sun-Mars $L_3$: Velocity profiles in $x$, $y$, $z$. 
Figure 4.70 Sun-Mars $L_3$: Control profiles in $x$, $y$, $z$.

Figure 4.71 Sun-Mars $L_3$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Table 4.24

Sun-Mars $L_3$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L^2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.7163</td>
<td>0.30414</td>
<td>0.43776</td>
</tr>
<tr>
<td>2</td>
<td>0.7163</td>
<td>0.19769</td>
<td>0.27653</td>
</tr>
<tr>
<td>3</td>
<td>0.7163</td>
<td>0.213301</td>
<td>0.26749</td>
</tr>
<tr>
<td>4</td>
<td>0.7163</td>
<td>0.19838</td>
<td>0.28786</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table 4.23 are found using the first strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 2,750 km and 4,100 km, respectively, as shown by Figure (4.68). The optimal period using the $L^1$-norm is 0.7163 TU or 78 days. Figure (4.71) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L^1$-norm, as Table 4.24 shows. Figure (4.72) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
### 4.4.3. $L_4$

Table 4.25

Sun-Mars $L_4$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5000216</td>
<td>0.8660254</td>
<td>0</td>
<td>0.0000665</td>
<td>0.0000951</td>
<td>0.0000500</td>
</tr>
<tr>
<td>2</td>
<td>0.5000150</td>
<td>0.8660368</td>
<td>0</td>
<td>0.0000742</td>
<td>0.0000574</td>
<td>0.0000657</td>
</tr>
<tr>
<td>3</td>
<td>0.5000150</td>
<td>0.8660292</td>
<td>0.000107</td>
<td>0.0000586</td>
<td>0.0000796</td>
<td>0.0000829</td>
</tr>
<tr>
<td>4</td>
<td>0.5000085</td>
<td>0.8660254</td>
<td>0</td>
<td>0.0000674</td>
<td>0.0000527</td>
<td>0.0000524</td>
</tr>
</tbody>
</table>

**Figure 4.73** Sun-Mars $L_4$: 3D solution.
Figure 4.74 Sun-Mars $L_4$: Separations between spacecraft throughout time.

Figure 4.75 Sun-Mars $L_4$: Velocity profiles in $x$, $y$, $z$. 
Figure 4.76 Sun-Mars $L_4$: Control profiles in $x$, $y$, $z$.

Figure 4.77 Sun-Mars $L_4$: Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Table 4.26

Sun-Mars $L_4$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L^2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.2464</td>
<td>0.35457</td>
<td>0.41036</td>
</tr>
<tr>
<td>2</td>
<td>1.2464</td>
<td>0.22148</td>
<td>0.24068</td>
</tr>
<tr>
<td>3</td>
<td>1.2464</td>
<td>0.22531</td>
<td>0.24314</td>
</tr>
<tr>
<td>4</td>
<td>1.2464</td>
<td>0.21540</td>
<td>0.25114</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table 4.25 are found using the first strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 1,800 km and 4,300 km, respectively, as shown by Figure (4.74). The optimal period using the $L^1$-norm is 1.2464 TU or 137 days. Figure (4.77) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L^1$-norm, as Table 4.26 shows. Figure (4.78) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
4.4.4. \( L_5 \)

Table 4.27

Sun-Mars \( L_5 \): Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( z_0 )</th>
<th>( v_x )</th>
<th>( v_y )</th>
<th>( v_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5000058</td>
<td>-0.8660254</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>2</td>
<td>0.4999983</td>
<td>-0.8660125</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>3</td>
<td>0.4999983</td>
<td>-0.8660211</td>
<td>0.0000121</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>4</td>
<td>0.4999909</td>
<td>-0.8660254</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
</tbody>
</table>

*Figure 4.79* Sun-Mars \( L_5 \): 3D solution.
Figure 4.80 Sun-Mars $L_5$: Separations between spacecraft throughout time.

Figure 4.81 Sun-Mars $L_5$: Velocity profiles in $x$, $y$, $z$. 
Figure 4.82 Sun-Mars \( L_5 \): Control profiles in \( x, y, z \).

Figure 4.83 Sun-Mars \( L_5 \): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Table 4.28

Sun-Mars $L_5$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [m/s]/1TU</th>
<th>$\Delta V$ ($L^2$-norm) [m/s]/1TU</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5183</td>
<td>0.33203</td>
<td>0.44430</td>
</tr>
<tr>
<td>2</td>
<td>0.5183</td>
<td>0.21699</td>
<td>0.28375</td>
</tr>
<tr>
<td>3</td>
<td>0.5183</td>
<td>0.22258</td>
<td>0.28388</td>
</tr>
<tr>
<td>4</td>
<td>0.5183</td>
<td>0.22278</td>
<td>0.28446</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table 4.27 are found using the first strategy, as suggested in Section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 2,950 km and 3,690 km, respectively, as shown by Figure (4.80). The optimal period using the $L^1$-norm is 0.5183 TU or 57 days. Figure (4.83) shows that most of the thrusting occurs at the beginning/end of the orbit. The overall $\Delta V$ is lower when using the $L^1$-norm, as Table 4.28 shows. Figure (4.84) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
5. Conclusion

This thesis’s primary purpose was to show that it is possible to have spacecraft constellations with relatively low station-keeping costs at various libration points of different planets of the Solar System for the Next Generation Space Weather Prediction Mission. As expected, the $L_4$ and $L_5$ points of all the systems studied were the easiest to work with due to their natural stability. Although $L_3$ points are unstable, not many Monte Carlo simulations were needed to get satisfactory results. $L_1$ points, on the other hand, were the most challenging ones.

It was found that $L_1$ points are really sensitive to I.C.s. Therefore, a lot of Monte Carlos simulations had to be performed to get solutions that converged. In most cases, the $L^1$-norm of control yielded the lowest station-keeping costs, as some literature suggests. However, there were a few exceptions. For the Sun-Mercury $L_1$ point, the $L^2$-norm happened to be 27% more efficient than the $L^1$-norm. The other two exceptions were the Sun-Venus $L_4$ and $L_5$ points, with percentage differences of 6% and 4%, respectively, between the $L^1$-norm and the $L^2$-norm. It is also worth noting that when the $L^1$-norm vs. the $L^2$-norm were compared, the optimal period that was obtained from the $L^1$-norm simulation is the one that was used in the $L^2$-norm simulation. That means that maybe the actual $L^2$-norm cost is lower than what the results presented suggest. For instance, for the Sun-Mars $L_1$ point, the $\Delta V$ for each spacecraft resulted in more than 1,000 (m/s)/TU. However, by letting the period free rather than using the optimal period from $L_1$, $\Delta V$ values were about 11 (m/s)/TU. Based on the previous observations, it is advised to try both cost functions when designing these orbits to secure the lowest station-keeping cost.
It is also worth pointing out that by using different cost functions, the shape and hence the separations between the spacecraft throughout the orbit change as well. Some orbits included in the appendix have the same I.C.s and period times as their equivalent in the results section so that the reader can compare both of them. Furthermore, in all the cases, the Hamiltonian is not exactly 0. This is most likely due to numerical errors and the precision of the I.C.s.

Future work could be done in all $L_1$ points to reduce station-keeping costs. Perhaps the standard procedure of first designing a reference orbit and then applying controls to it could be tried. The culminating remark is that the Next Generation Space Weather Prediction Mission is feasible from an astrodynamics perspective.
REFERENCES


Nykyri, K., & Udrea, B. (2016). *The next generation space weather prediction mission* [Internal Accelerate Research Initiative Project Description]. Embry-Riddle Aeronautical University, Daytona Beach, Florida.


A. APPENDIX - More Results

A.1. Sun-Mercury

A.1.1. $L_3$

Table A.1

Sun-Mercury $L_3$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.9997238</td>
<td>0</td>
<td>0</td>
<td>0.0001399</td>
<td>0.0002427</td>
<td>0.0000585</td>
</tr>
<tr>
<td>2</td>
<td>-0.9998749</td>
<td>0.0002617</td>
<td>0</td>
<td>0.0002446</td>
<td>0.0000878</td>
<td>0.0001834</td>
</tr>
<tr>
<td>3</td>
<td>-0.9998749</td>
<td>0.0000872</td>
<td>0.0002467</td>
<td>0.0001673</td>
<td>0.0001850</td>
<td>0.0001222</td>
</tr>
<tr>
<td>4</td>
<td>-1.0000260</td>
<td>0</td>
<td>0</td>
<td>0.0001741</td>
<td>0.0002122</td>
<td>0.0000539</td>
</tr>
</tbody>
</table>

Figure A.1  Sun-Mercury $L_3$ ($L^2$-norm): 3D solution.
Figure A.2 Sun-Mercury $L_3$ ($L^2$-norm): Separations between spacecraft throughout time.

Figure A.3 Sun-Mercury $L_3$ ($L^2$-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
The initial conditions shown in Table A.1 are found using the first strategy, as suggested in section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 17,000 km and 19,200 km, respectively, as shown by Figure (A.2). The optimal period using the $L^2$-norm is 0.8702 TU or 12 days. Figure (A.4) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
A.1.2. \( L_4 \)

Table A.3

Sun-Mercury \( L_4 \): Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( z_0 )</th>
<th>( v_x )</th>
<th>( v_y )</th>
<th>( v_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5003279</td>
<td>0.8660254</td>
<td>0</td>
<td>0.0001842</td>
<td>0.0001930</td>
<td>0.0001784</td>
</tr>
<tr>
<td>2</td>
<td>0.5001768</td>
<td>0.8662871</td>
<td>0</td>
<td>0.0001338</td>
<td>0.0001282</td>
<td>0.0002132</td>
</tr>
<tr>
<td>3</td>
<td>0.5001768</td>
<td>0.8661126</td>
<td>0.0002467</td>
<td>0.0001135</td>
<td>0.0002129</td>
<td>0.0002078</td>
</tr>
<tr>
<td>4</td>
<td>0.5000257</td>
<td>0.8660254</td>
<td>0</td>
<td>0.0002205</td>
<td>0.0001511</td>
<td>0.0001771</td>
</tr>
</tbody>
</table>

Figure A.5 Sun-Mercury \( L_4 \) (\( L^2 \)-norm): 3D solution.
Figure A.6 Sun-Mercury $L_4$ ($L^2$-norm): Separations between spacecraft throughout time.

Figure A.7 Sun-Mercury $L_4$ ($L^2$-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Figure A.8  Sun-Mercury $L_4$ ($L^2$-norm): Hamiltonian evolution.

Table A.4

Sun-Mercury $L_4$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L^2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3662</td>
<td>0.40812</td>
<td>0.60338</td>
</tr>
<tr>
<td>2</td>
<td>0.3662</td>
<td>0.36279</td>
<td>0.47228</td>
</tr>
<tr>
<td>3</td>
<td>0.3662</td>
<td>0.41205</td>
<td>0.59911</td>
</tr>
<tr>
<td>4</td>
<td>0.3662</td>
<td>0.39271</td>
<td>0.64219</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table A.3 are found using the first strategy, as suggested in section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 17,100 km and 18,100 km, respectively, as shown by Figure (A.6). The optimal period using the $L^2$-norm is 0.3662 TU or 5 days. Figure (A.8) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
A.1.3. $L_5$

Table A.5

Sun-Mercury $L_5$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5003279</td>
<td>-0.8660254</td>
<td>0</td>
<td>0.0001650</td>
<td>0.0000620</td>
<td>0.0000970</td>
</tr>
<tr>
<td>2</td>
<td>0.5001768</td>
<td>-0.8657637</td>
<td>0</td>
<td>0.0001206</td>
<td>0.0002142</td>
<td>0.0000531</td>
</tr>
<tr>
<td>3</td>
<td>0.5001768</td>
<td>-0.8659382</td>
<td>0.0002467</td>
<td>0.0000586</td>
<td>0.0000838</td>
<td>0.0001798</td>
</tr>
<tr>
<td>4</td>
<td>0.5000257</td>
<td>-0.8660254</td>
<td>0</td>
<td>0.0001963</td>
<td>0.0001795</td>
<td>0.0001402</td>
</tr>
</tbody>
</table>

*Figure A.9* Sun-Mercury $L_5$ ($L^2$-norm): 3D solution.
Figure A.10  Sun-Mercury $L_5$ ($L^2$-norm): Separations between spacecraft throughout time.

Figure A.11  Sun-Mercury $L_5$ ($L^2$-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
The initial conditions shown in Table A.5 are found using the first strategy, as suggested in section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 15,500 km and 19,250 km, respectively, as shown by Figure (A.10). The optimal period using the $L^2$-norm is 1.0613 TU or 15 days. Figure (A.12) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.

Table A.6

Sun-Mercury $L_5$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L^2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0613</td>
<td>0.44067</td>
<td>0.51047</td>
</tr>
<tr>
<td>2</td>
<td>1.0613</td>
<td>0.33460</td>
<td>0.33376</td>
</tr>
<tr>
<td>3</td>
<td>1.0613</td>
<td>0.30261</td>
<td>0.34572</td>
</tr>
<tr>
<td>4</td>
<td>1.0613</td>
<td>0.41306</td>
<td>0.51912</td>
</tr>
</tbody>
</table>

Figure A.12 Sun-Mercury $L_5$ ($L^2$-norm): Hamiltonian evolution.
A.2. Sun-Venus

A.2.1. \( L_3 \)

Table A.7

Sun-Venus \( L_3 \): Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( z_0 )</th>
<th>( v_x )</th>
<th>( v_y )</th>
<th>( v_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.9996591</td>
<td>0</td>
<td>0</td>
<td>0.0002134</td>
<td>0.0000879</td>
<td>0.0000747</td>
</tr>
<tr>
<td>2</td>
<td>-0.9998208</td>
<td>0.0002801</td>
<td>0</td>
<td>0.0002142</td>
<td>0.0001776</td>
<td>0.0000532</td>
</tr>
<tr>
<td>3</td>
<td>-0.9998208</td>
<td>0.0000934</td>
<td>0.0002641</td>
<td>0.0002292</td>
<td>0.0001531</td>
<td>0.0001589</td>
</tr>
<tr>
<td>4</td>
<td>-0.9999825</td>
<td>0</td>
<td>0</td>
<td>0.0001713</td>
<td>0.0002021</td>
<td>0.0002211</td>
</tr>
</tbody>
</table>

Figure A.13  Sun-Venus \( L_3 \) (\( L^2 \)-norm): 3D solution.
Figure A.14 Sun-Venus $L_3$ ($L^2$-norm): Separations between spacecraft throughout time.

Figure A.15 Sun-Venus $L_3$ ($L^2$-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Table A.8

Sun-Venus $L_3$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L_1$-norm) [m/s]/1TU</th>
<th>$\Delta V$ ($L_2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9632</td>
<td>0.41136</td>
<td>0.46987</td>
</tr>
<tr>
<td>2</td>
<td>0.9632</td>
<td>0.31194</td>
<td>0.39139</td>
</tr>
<tr>
<td>3</td>
<td>0.9632</td>
<td>0.38086</td>
<td>0.42844</td>
</tr>
<tr>
<td>4</td>
<td>0.9632</td>
<td>0.32836</td>
<td>0.33317</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table A.7 are found using the first strategy, as suggested in section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 33,000 km and 36,500 km, respectively, as shown by Figure (A.14). The optimal period using the $L_2$-norm is 0.9632 TU or 37 days. Figure (A.16) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
A.2.2.  \( L_4 \)

Table A.9

Sun-Venus \( L_4 \): Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( z_0 )</th>
<th>( v_x )</th>
<th>( v_y )</th>
<th>( v_z )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5003534</td>
<td>0.8660254</td>
<td>0</td>
<td>0.0000526</td>
<td>0.0000878</td>
<td>0.0000801</td>
</tr>
<tr>
<td>2</td>
<td>0.5001870</td>
<td>0.8663135</td>
<td>0</td>
<td>0.0000929</td>
<td>0.0000994</td>
<td>0.0000965</td>
</tr>
<tr>
<td>3</td>
<td>0.5001870</td>
<td>0.8661215</td>
<td>0.0002717</td>
<td>0.0000705</td>
<td>0.0000500</td>
<td>0.0000770</td>
</tr>
<tr>
<td>4</td>
<td>0.5000207</td>
<td>0.8660254</td>
<td>0</td>
<td>0.0000604</td>
<td>0.0000610</td>
<td>0.0000663</td>
</tr>
</tbody>
</table>

Figure A.17  Sun-Venus \( L_4 \) (\( L^2 \)-norm): 3D solution.
Figure A.18  Sun-Venus \( L_4 \) (\( L^2 \)-norm): Separations between spacecraft throughout time.

Figure A.19  Sun-Venus \( L_4 \) (\( L^2 \)-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
The initial conditions shown in Table A.9 are found using the first strategy, as suggested in section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 35,500 km and 37,500 km, respectively, as shown by Figure (A.18). The optimal period using the $L^2$-norm is 0.69538 TU or 27 days. Figure (A.20) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
A.2.3. $L_5$

Table A.11

Sun-Venus $L_5$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5003072</td>
<td>-0.8660254</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>2</td>
<td>0.5001408</td>
<td>-0.8657373</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>3</td>
<td>0.5001408</td>
<td>-0.8659294</td>
<td>0.0002717</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>4</td>
<td>0.4999744</td>
<td>-0.8660254</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
</tbody>
</table>

![L5 Solution](image)

*Figure A.21* Sun-Venus $L_5$ ($L^2$-norm): 3D solution.
Figure A.22 Sun-Venus $L_5$ ($L^2$-norm): Separations between spacecraft throughout time.

Figure A.23 Sun-Venus $L_5$ ($L^2$-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Table A.12

Sun-Venus $L_5$ ($L^2$-norm): Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L^2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4635</td>
<td>0.32345</td>
<td>0.42017</td>
</tr>
<tr>
<td>2</td>
<td>0.4635</td>
<td>0.30101</td>
<td>0.33559</td>
</tr>
<tr>
<td>3</td>
<td>0.4635</td>
<td>0.30250</td>
<td>0.39021</td>
</tr>
<tr>
<td>4</td>
<td>0.4635</td>
<td>0.29741</td>
<td>0.38108</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table A.11 are found using the first strategy, as suggested in section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 33,500 km and 36,500 km, respectively, as shown by Figure (A.22). The optimal period using the $L^2$-norm is 0.4635 TU or 18 days. Figure (A.24) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.

Figure A.24  Sun-Venus $L_5$ ($L^2$-norm): Hamiltonian evolution.
A.3. Sun-Mars

A.3.1. $L_3$

Table A.13

Sun-Mars $L_3$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.9999787</td>
<td>0</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>2</td>
<td>-0.9999861</td>
<td>0.0000129</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>3</td>
<td>-0.9999861</td>
<td>0.0000043</td>
<td>0.0000121</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>4</td>
<td>-0.9999936</td>
<td>0</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
</tbody>
</table>

Figure A.25  Sun-Mars $L_3$ ($L^2$-norm): 3D solution.
Figure A.26  Sun-Mars $L_3$ ($L^2$-norm): Separations between spacecraft throughout time.

Figure A.27  Sun-Mars $L_3$ ($L^2$-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
Figure A.28 Sun-Mars $L_3$ ($L^2$-norm): Hamiltonian evolution.

Table A.14

Sun-Mars $L_3$: Results.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$t_f$ [TU]</th>
<th>$\Delta V$ ($L^1$-norm) [(m/s)/1TU]</th>
<th>$\Delta V$ ($L^2$-norm) [(m/s)/1TU]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.7163</td>
<td>0.30414</td>
<td>0.43776</td>
</tr>
<tr>
<td>2</td>
<td>0.7163</td>
<td>0.19769</td>
<td>0.27653</td>
</tr>
<tr>
<td>3</td>
<td>0.7163</td>
<td>0.213301</td>
<td>0.26749</td>
</tr>
<tr>
<td>4</td>
<td>0.7163</td>
<td>0.19838</td>
<td>0.28786</td>
</tr>
</tbody>
</table>

The initial conditions shown in Table A.13 are found using the first strategy, as suggested in section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 700 km and 4,400 km, respectively, as shown by Figure (A.26). The optimal period using the $L^2$-norm is 0.7163 TU or 78 days. Figure (A.28) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.
Table A.15

Sun-Mars $L_5$: Initial conditions in scaled units.

<table>
<thead>
<tr>
<th>Spacecraft</th>
<th>$x_0$</th>
<th>$y_0$</th>
<th>$z_0$</th>
<th>$v_x$</th>
<th>$v_y$</th>
<th>$v_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spacecraft 1</td>
<td>0.5000058</td>
<td>-0.8660254</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>Spacecraft 2</td>
<td>0.4999983</td>
<td>-0.8660125</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>Spacecraft 3</td>
<td>0.4999983</td>
<td>-0.8660211</td>
<td>0.0000121</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
<tr>
<td>Spacecraft 4</td>
<td>0.4999909</td>
<td>-0.8660254</td>
<td>0</td>
<td>0.0003000</td>
<td>0.0001000</td>
<td>0.0001000</td>
</tr>
</tbody>
</table>

Figure A.29 Sun-Mars $L_5$ ($L^2$-norm): 3D solution.
Figure A.30  Sun-Mars $L_5$ ($L^2$-norm): Separations between spacecraft throughout time.

Figure A.31  Sun-Mars $L_5$ ($L^2$-norm): Thrust directions. The arrows represent the direction where the spacecraft has to thrust to maintain the desired orbit.
The initial conditions shown in Table A.15 are found using the first strategy, as suggested in section 3.5.1. The minimum and maximum separations between the spacecraft in this simulation are approximately 3,100 km and 3,475 km, respectively, as shown by Figure (A.30). The optimal period using the $L^2$-norm is 0.5183 TU or 57 days. Figure (A.32) depicts the Hamiltonian evolution of the simulation, as described by Equation 3.23.