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Self-Adjoint Extension and Spectral Theory of a Linear Relation in a Hilbert Space

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Received 21 December 2013; Accepted 28 January 2014; Published 5 March 2014

Academic Editors: B. Franchi and G. L. Karakostas

The aim of this paper is to develop the conditions for a symmetric relation in a Hilbert space to have self-adjoint extensions in terms of defect indices and discuss some spectral theory of such linear relation.

1. Introduction

In this paper, we discuss the theory of linear relations in a Hilbert space. These linear relations were first studied by Arnes, Coddington, Dijksma, de Snoo, and Hassi et al in [1–4]. It has also been studied extensively more recently in [5]. The theory has particular interest because, in some of the application problems, a linear operator can have multivalued part; for example, see [6, 7]. Here, we concentrate on establishing the conditions for symmetric relations to have self-adjoint extensions in terms of defect indices. Moreover, we discuss the spectral theory of such self-adjoint relations. The analogous treatment on operator theory of some of the theorems on this paper can be found in [8].

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and denote by $\mathcal{H}^2$ the Hilbert space $\mathcal{H} \oplus \mathcal{H}$. A linear relation $\mathcal{R} = \{(f, g) : f, g \in \mathcal{H}\}$ on $\mathcal{H}$ is a subspace of $\mathcal{H}^2$. The graph of an operator is an example of a linear relation but note that a relation can have multivalued part.

These relations have been used in some of the eigenvalue problems in ordinary differential equations. For example, the canonical systems

$$Ju' = zH(x)u(x), \quad z \in \mathbb{C},$$

(1)

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $H(x)$ is a $2 \times 2$ positive semidefinite matrix whose entries are locally integrable, induce a multivalued linear relation. For instance, we may think of writing the systems in the form

$$H(x)^{-1}Ju' = zu$$

(2)

and consider it as an operator on a Hilbert space. But $H(x)$ is not invertible in general therefore can not be considered as an eigenvalue equation of an operator. Instead, the system induces a linear relation that may have a multivalued part. This is one of the main motivations for our work in this paper.

The boundary value problem of such canonical systems has been studied by using linear relations; see [6, 7, 9].

$$D(\mathcal{R}) = \{f \in \mathcal{H} : (f, g) \in \mathcal{R}\} \quad \text{and} \quad R(\mathcal{R}) = \{g \in \mathcal{H} : (f, g) \in \mathcal{R}\}$$

are respectively defined as the domain and range of the relation $\mathcal{R}$.

$$\mathcal{R}^{-1} = \{(g, f) : (f, g) \in \mathcal{R}\}$$

denotes the inverse relation. The adjoint of $\mathcal{R}$ on $\mathcal{H}$ is a closed linear relation defined by

$$\mathcal{R}^* = \{(h, k) \in \mathcal{H}^2 : \langle g, h \rangle = \langle f, k \rangle, \forall (f, g) \in \mathcal{R}\}.$$ (3)

A linear relation $\mathcal{S}$ is called symmetric if $\mathcal{S} \subset \mathcal{S}^*$ and self-adjoint if $\mathcal{S} = \mathcal{S}^*$. From now on, we write relation to mean linear relation.

A relation $\mathcal{R}$ is called isometry if

$$\langle f_1, f_2 \rangle = \langle g_1, g_2 \rangle, \quad \forall (f_1, g_1) \in \mathcal{R}$$

(4)

and $\mathcal{R}$ is unitary if it is isometry and $D(\mathcal{R}) = R(\mathcal{R}) = \mathcal{H}$.

Let $(z - \mathcal{R}) = \{(f, zf - g) : (f, g) \in \mathcal{R}\}$, $N(\mathcal{R}, z) = \{f : (f, zf) \in \mathcal{R}\}$ and $R_z = R(z - \mathcal{R})$.

It is clear to see that $N(\mathcal{R}^*, z) = R_z^\perp$.

In Section 2, we establish the condition for a symmetric relation to have self-adjoint extensions in terms of defect indices and in Section 3 we discuss spectral theory.
2. Defect Indices and Self-Adjoint Extension

Let $\mathcal{R}$ be a relation on a Hilbert space $\mathcal{H}$. The set

$$\Gamma(\mathcal{R}) = \{ z \in \mathbb{C} : \text{there exists } C \text{ such that } \|zf - g\| \geq C(z) \|f\|, \forall (f,g) \in \mathcal{R} \}$$

is defined as the regularity domain of $\mathcal{R}$ and $S(\mathcal{R}) = \mathbb{C} - \Gamma(\mathcal{R})$ is defined as the Spectral Kernel of $\mathcal{R}$. Note that

(i) $z \in \Gamma(\mathcal{R})$ if and only if $(z - \mathcal{R})^{-1}$ is a bounded linear operator on $\mathcal{H}$;
(ii) if $\mathcal{R}$ is symmetric, then $\mathbb{C} - \mathbb{R} \subset \Gamma(\mathcal{R})$;
(iii) $\Gamma(\mathcal{R})$ is open.

The subspace $\mathcal{R}_z^\perp$ is called the defect space of $\mathcal{R}$ and $z$. The cardinal number $\beta(\mathcal{R}, z) = \dim \mathcal{R}_z^\perp$ is called the defect index of $\mathcal{R}$ and $z$.

**Theorem 1.** The defect index $\beta(\mathcal{R}, z)$ is constant on each connected subset of $\Gamma(\mathcal{R})$. If $\mathcal{R}$ is symmetric, then the defect index is constant on the upper and lower half-planes.

**Proof.** Let $Q_z$ denote the orthogonal projection onto $\mathcal{R}_z^\perp$. We first show that $\|Q_z - Q_{z_0}\| \to 0$ as $z \to z_0$, for any $z_0 \in \Gamma(\mathcal{R})$. Let $z_0 \in \Gamma(\mathcal{R})$; then there is a constant $C(z_0) > 0$ such that

$$\|f\| \leq C(z_0) \|z_0 f - g\|,$$

for all $(f, g) \in \mathcal{R}$. For $|z - z_0| < 1/(2C(z_0))$ and all $(f, g) \in \mathcal{R}$, we have

$$\|f\| \leq C(z_0) \|z_0 f - g\| \leq (\|zf - g\| + |z - z_0| \|f\|) \leq C(z_0) \|zf - g\| + \frac{1}{2} \|f\| \leq C(z_0) \|zf - g\|.$$

For $h \in R(Q_{z_0})^\perp = \mathcal{R}_z^\perp$,

$$\|Q_z h\| = \sup \{ \|h, zf - g\| : zf - g \in \mathcal{R}_z^\perp, \|zf - g\| \leq 1 \} = \sup \{ \|h, (z - z_0)f\| : zf - g \in \mathcal{R}_z^\perp, \|zf - g\| \leq 1 \} \leq \|h\| \|z - z_0\| C(z_0).$$

Similarly, for $h \in R(Q_{z_0})^\perp = \mathcal{R}_z^\perp$,

$$\|Q_z h\| \leq \|h\| \|z - z_0\| C(z_0).$$

It follows that

$$\|Q_z - Q_{z_0}\| \leq 2|z - z_0| C(z_0).$$

Let $P_z$ denote an orthogonal projection onto $\mathcal{R}_z^\perp$; then

$$\|P_z - P_{z_0}\| = \|Q_z - Q_{z_0}\| \leq 2|z - z_0| C(z_0) \to 0$$

as $z \to z_0$.

Hence, if we choose $0 < \epsilon < 1/(2C(z_0))$, then $\|P_z - P_{z_0}\| < 1$, for $|z - z_0| < \epsilon$. Therefore, $\dim \mathcal{R}_z^\perp = \dim \mathcal{R}_z^\perp$. It follows that

$$\beta(\mathcal{R}, z) = \beta(\mathcal{R}, z_0) \text{ for } |z - z_0| < \epsilon.$$  (12)

If $\mathcal{R}$ is symmetric, then the upper and lower half-planes are connected subsets of $\Gamma(\mathcal{R})$; therefore, the defect index is constant there.

Letting $\mathcal{R}$ be a symmetric relation on a Hilbert space, for $z \in \mathbb{C}^+$, the defect index $m = \beta(\mathcal{R}, z)$ and, for $w \in \mathbb{C}^-$, the defect index $n = \beta(\mathcal{R}, w)$ are written as a pair $(m, n)$ and are called the defect indices of $\mathcal{R}$.

The Cayley transform of a symmetric relation $\mathcal{R}$ on $\mathcal{H}$ is defined by the relation

$$\mathcal{V} = \{(g + if, g - if) : (f, g) \in \mathcal{R}\}.$$  (13)

Then clearly $D(\mathcal{V}) = R(\mathcal{R} + i)$ and $R(\mathcal{V}) = R(\mathcal{R} - i)$.

**Theorem 2.** If $\mathcal{R}$ is a symmetric relation on $\mathcal{H}$ and $\mathcal{V}$ is the Cayley transform of $\mathcal{R}$, then

(1) $\mathcal{V}$ is an isometry;
(2) $R(I - \mathcal{V}) = D(\mathcal{R})$ and $\mathcal{R} = \{(f - g, i(f + g)) : (f, g) \in \mathcal{V}\}$;
(3) $\mathcal{R}$ is multi-valued if and only if $N(I - \mathcal{V}) \neq \emptyset$.

**Proof.** (1) Let $(u_1, v_1), (u_2, v_2) \in \mathcal{V}$; then $u_i = g_i + if_i$ and $v_i = g_i - if_i$, for $(f_i, g_i) \in \mathcal{R}$, $i = 1, 2$; then

$$\langle u_1, u_2 \rangle = \langle g_1 + if_1, g_2 + if_2 \rangle = \langle g_1, g_2 \rangle + \langle g_1, f_2 \rangle + \langle if_1, g_2 \rangle + \langle if_1, f_2 \rangle = \langle g_1, g_2 \rangle + i\langle f_1, f_2 \rangle - i^2 \langle f_1, g_2 \rangle = \langle g_1 - if_1, g_2 - if_2 \rangle = \langle v_1, v_2 \rangle.$$

(2) It follows from the definition of Cayley transform.
(3) Suppose that $\mathcal{R}$ is multivalued; then there is $g \in \mathcal{H}, g \neq 0$ such that $(0, g) \in \mathcal{R}$. It follows by definition of $\mathcal{V}$ that $(g, g) \in \mathcal{V}$. Hence, $g \in N(I - \mathcal{V})$. On the other hand, let $g \in N(I - \mathcal{V}), g \neq 0$; then $(g, g) \in \mathcal{V}$; then, by (2), $(0, 2ig) \in \mathcal{R}$. Hence, $\mathcal{R}$ is multivalued.

**Theorem 3.** A relation $\mathcal{V}$ on $\mathcal{H}$ is the Cayley transform of a symmetric relation $\mathcal{R}$ if and only if $\mathcal{V}$ has the following properties.

(1) $\mathcal{V}$ is an isometric relation.
(2) $R(I - \mathcal{V}) = D(\mathcal{R})$.

The relation $\mathcal{R}$ is given by $\mathcal{R} = \{(f - g, i(f + g)) : (f, g) \in \mathcal{V}\}$.

**Proof.** If $\mathcal{V}$ is the Cayley transform of $\mathcal{R}$, then, by Theorem 2, $\mathcal{V}$ satisfies the properties (1) and (2). Conversely, suppose
that \( \mathcal{V} \) has properties (1) and (2), we show that \( \mathcal{R} = \{(f - g, if + g) : (f, g) \in \mathcal{V}\} \) is a symmetric relation.

Suppose \((f_1 - g_1, (if_1 + g_1)), (f_2 - g_2, (if_2 + g_2)) \in \mathcal{R} \); then

\[
\langle i (f_1 + g_1), (f_2 - g_2) \rangle
= -i (\langle f_1, f_2 \rangle - \langle f_1, g_2 \rangle + \langle g_1, f_2 \rangle - \langle g_1, g_2 \rangle).
\]

(15)

Since \( \mathcal{V} \) is an isometry, for any \((f_1, g_1), (f_2, g_2) \in \mathcal{V}, (f_1, f_2) = (g_1, g_2) \), this implies that

\[
\langle i (f_1 + g_1), (f_2 - g_2) \rangle
= -i \langle g_1, f_2 \rangle + i \langle f_1, g_2 \rangle - i \langle g_1, f_2 \rangle + i \langle f_1, f_2 \rangle
= -i (\langle g_1 - f_1, g_2 \rangle + \langle g_1 - f_1, f_2 \rangle)
= \langle f_1 - g_1, i (f_2 + g_2) \rangle.
\]

Hence, \( \mathcal{R} \) is symmetric.

**Theorem 4.** A symmetric relation \( \mathcal{R} \) is self-adjoint if and only if \( \mathcal{V} \) is unitary.

**Proof.** We show that the \( \mathcal{R} \) is self-adjoint if and only if

\[
R(\mathcal{R} + i) = R(\mathcal{R} - i) = \mathcal{H}.
\]

(17)

Since \( \mathcal{R} \) is symmetric, we always have \( \mathcal{R} \subseteq \mathcal{R}^* \). Let \((f, g) \in \mathcal{R}^* \); then \( if - g \in \mathcal{H} \) and \( R(\mathcal{R} - i) = \mathcal{H} \) imply that there is \((h, k) \in \mathcal{R} \) such that \( k - ih = if - g \). So \( if + h = k + g \), so that \( f + h, if + h \in \mathcal{R}^* \). That is

\[
(f + h) \in N(\mathcal{R}^*, i) = R((\mathcal{R} + i)^+) = \{0\}.
\]

(18)

This implies \( f = -h \in D(\mathcal{R}) \). Hence, \( \mathcal{R} \) is self-adjoint.

Conversely, suppose that \( \mathcal{R} \) is self-adjoint. Let

\[
h \in R(\mathcal{R} - i)^+ = N(\mathcal{R}^*, -i) = N(\mathcal{R}, -i).
\]

(19)

So \((h, -ih) \in \mathcal{R} \). But

\[
(-ih, h) = (h, ih) \implies i(h, h) = -i(h, h).
\]

(20)

Hence, we must have \( h = 0 \). So \( R(\mathcal{R} - i) = \mathcal{H} \). Similarly, \( R(\mathcal{R} + i) = \mathcal{H} \).

**Theorem 5.** Let \( \mathcal{R} \) be a closed symmetric relation on a Hilbert space \( \mathcal{H} \) and let \( \mathcal{V} \) denote its Cayley transform. One has the following.

\( \mathcal{V}' \) is the Cayley transform of a closed symmetric extension \( \mathcal{R}' \) of \( \mathcal{R} \) if and only if the following holds.

There exist closed subspaces \( F_+ \) of \( R(\mathcal{R} - i)^+ \) and \( F_- \) of \( R(\mathcal{R} + i)^+ \) and an isometric relation \( \mathcal{V} \) on \( F_+ \oplus F_- \) for which

\[
\mathcal{V}' = \{(f + h, g + k) : (f, g) \in \mathcal{V}, (h, k) \in \mathcal{V}\},
\]

\[
D(\mathcal{V}') = R(\mathcal{R}' + i) = R(\mathcal{R}' + i) \oplus F_+,
\]

(21)

\[
R(\mathcal{V}') = R(\mathcal{R}' - i) = R(\mathcal{R}' - i) \oplus F_-.
\]

The spaces \( F_+ \) and \( F_- \) have the same dimension.

(2) The relation \( \mathcal{V}' \) in part (1) is unitary if and only if \( F_- = R(\mathcal{R}' - i)^+ \) and \( F_+ = R(\mathcal{R}' + i)^+ \).

(3) \( \mathcal{R} \) possess self-adjoint extension if and only if its defect indices are equal.

**Proof.** (1) Suppose that \( \mathcal{V}' \) has the given form. Then \( \mathcal{V}' \) is isometric relation, since for any \((f + h, g + k) \in \mathcal{V}' \), we have

\[
\|g + k\|^2 = \|f\|^2 + \|h\|^2 = \|f + h\|^2.
\]

(22)

Hence, we can define a symmetric extension \( \mathcal{R}' \) such that \( \mathcal{V}' \) is its Cayley transform. Conversely, if \( \mathcal{V}' \) is the Cayley transform of a symmetric extension \( \mathcal{R}' \) of \( \mathcal{R} \), then put \( F_- = R(\mathcal{R}' - i) \oplus R(\mathcal{R}' - i), F_+ = R(\mathcal{R}' + i) \oplus R(\mathcal{R}' + i) \) and \( \mathcal{V} = \mathcal{V}' \mid_{F_+ \oplus F_-} \). Then we have the desired properties.

(2) Here we have that \( \mathcal{V}' \) is unitary if and only if

\[
D(\mathcal{V}') = R(\mathcal{V}') = \mathcal{H}.
\]

(23)

That is, if and only if \( F_- = R(\mathcal{R}' - i)^+ \) and \( F_+ = R(\mathcal{R}' + i)^+ \).

(3) By (1) and (2), \( \mathcal{V} \) possess unitary extension if and only if there exists an isometry relation \( \mathcal{V} \) onto \( R(\mathcal{R}' + i)^+ \oplus R(\mathcal{R}' - i)^+ \). This happens if and only if

\[
\dim(R(\mathcal{R}' + i)^+) = \dim(R(\mathcal{R}' - i)^+)\]

(24)

By definition of Cayley transform, it is clear that if \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are the Cayley transforms of any two symmetric relations \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \), then

\[
\mathcal{R}_1 \subset \mathcal{R}_2 \quad \text{if and only if} \quad \mathcal{V}_1 \subset \mathcal{V}_2.
\]

(25)

**Theorem 6.** Let \( \mathcal{R} \) be a closed symmetric relation on a Hilbert space \( \mathcal{H} \) with defect indices \( (m, m) \). One has the following.

(1) \( \mathcal{R}' \) is a symmetric extension of \( \mathcal{R} \) if and only if the following holds. There are closed subspaces \( F_+ \) of \( R(\mathcal{R} + i)^2 \) and \( F_- \) of \( R(\mathcal{R} - i)^2 \) and an isometric mapping \( \mathcal{V} \) of \( F_- \) onto \( F_+ \) such that

\[
D(\mathcal{R}') = D(\mathcal{R}) + \{g + \mathcal{V} g : g \in F_+\}.
\]

(26)

(2) \( \mathcal{R}' \) is self-adjoint if and only if \( \mathcal{R}' \) is an \( m \)-dimensional extension of \( \mathcal{R} \).

**Proof.** (1) Let \( \mathcal{V}' \) and \( \mathcal{V}' \) be the Cayley transforms of the closed symmetric relation \( \mathcal{R} \) and its symmetric extension \( \mathcal{R}' \), respectively. By Theorem 5, there exist closed subspaces \( F_+ \) of \( R(\mathcal{R} - i)^+ \) and \( F_- \) of \( R(\mathcal{R} + i)^+ \) and an isometric relation \( \mathcal{V} \) on \( F_+ \oplus F_- \) for which

\[
\mathcal{V}' = \{(f + h, g + k) : (f, g) \in \mathcal{V}, (h, k) \in \mathcal{V}\},
\]

\[
D(\mathcal{V}') = R(\mathcal{R}' + i) = R(\mathcal{R}' + i) \oplus F_+,
\]

(27)

\[
R(\mathcal{V}') = R(\mathcal{R}' - i) = R(\mathcal{R}' - i) \oplus F_-.
\]
Then by definition of the Cayley transform, we see that
\[
D(\mathcal{R}') = R(1 - \mathcal{V}') = (1 - \mathcal{V}') D(\mathcal{V}') = (1 - \mathcal{V}') R(i + \mathcal{R}')
\]
\[
= (1 - \mathcal{V}') (R(i + \mathcal{R}) \oplus F_1)
\]
\[
= (1 - \mathcal{V}') (D(\mathcal{V}) \oplus F_1)
\]
\[
= (1 - \mathcal{V}) D(\mathcal{V}) + (1 - \mathcal{V}') F_1
\]
\[
= D(\mathcal{R}) + \{g - \overline{T}g : g \in F_1\}.
\]

The converse is similar.

(2) By Theorem 5, \(\mathcal{R}'\) is self-adjoint if and only if \(\mathcal{V}'\) is unitary. This happens if and only if \(F_1 = R(\mathcal{R} + i)^{-1}\). So, by (1), \(\mathcal{R}'\) is self-adjoint if and only if it is an \(m\)-dimensional extension of \(\mathcal{R}\).

**Theorem 7.** Suppose that \(\mathcal{T}\) is a self-adjoint relation and suppose that \(z \in \Gamma(\mathcal{T})\); then
\[
\mathcal{H} = \{zf - g : (f, g) \in \mathcal{T}\}.
\]

**Proof.** We will show that \(R(z - \mathcal{T}) = \{zf - g : (f, g) \in \mathcal{T}\}\) is a closed subspace of \(\mathcal{H}\). Since \(z \in \Gamma(\mathcal{T})\), there is a constant \(C(z)\) such that
\[
\|zf - g\| \geq C(z) \|f\|.
\]

Let \(v_n \in R(z - \mathcal{T})\) and \(v_n \rightarrow v\) in \(\mathcal{H}\). Suppose that \(f_n \in D(\mathcal{T})\) such that \(f_n, g_n \in \mathcal{T}\) and \(v_n = zf_n - g_n\), so that \((f_n, v_n) \in z - \mathcal{T}\). But from the above relation we have
\[
\|v_n - v_m\| = \|zf_n - zf_m - (g_n - g_m)\| \geq C \|f_n - f_m\|.
\]

It follows that \(f_n\) is a Cauchy sequence in \(\mathcal{H}\), and it converges to some \(f\) in \(\mathcal{H}\). Hence, \((f_n, v_n) \rightarrow (f, v)\). Since \(\mathcal{T}\) is closed, \(f \in D(\mathcal{T})\) and \((f, v) \in z - \mathcal{T}\) and \(v = R(z - \mathcal{T})\). Hence, \(R(z - \mathcal{T})\) is closed. So we have
\[
\mathcal{H} = R(z - \mathcal{T}) \oplus R(z - \mathcal{T})^{-1}.
\]

We next show that \(R(z - \mathcal{T})^{-1} = \{0\}\). Let \(h \in R(z - \mathcal{T})^{-1} = N(\mathcal{T}, z); Then (h, \overline{zh}) \in \mathcal{T}\). But \(0 = \|zh - \overline{zh}\| \geq C(\overline{zh})\|h\|\) implies \(h = 0\) a.e.

Let \(\mathcal{T}\) be a self-adjoint relation on \(\mathcal{H}\) and \(z \in \Gamma(\mathcal{T})\). Define \(T : \mathcal{H} \rightarrow \mathcal{H}\) by \(T(zf - g) = f\). That is \(T = (z - \mathcal{T})^{-1} = \{(zf - g, f) : (f, g) \in \mathcal{T}\}\). Then \(T\) is a bounded linear operator since
\[
\|T\| = \sup_{\|zf - g\| = 1} \|T(zf - g)\| = \sup_{\|zf - g\| = 1} \|f\| \leq \frac{1}{C(z)}
\]
and \(\mathcal{T}\) is given by
\[
\mathcal{T} = \{(Tf, zTf - f) : f \in \mathcal{H}\}.
\]

### 3. Spectral Theory of a Linear Relation

**Definition 8.** Let \(\mathcal{R}\) be a closed relation on a Hilbert space \(\mathcal{H}\). We define
\[
\rho(\mathcal{R}) = \{z \in \mathbb{C} : \exists f \in B(\mathcal{H}), \mathcal{R} = \{(Tf, zTf - f) : f \in \mathcal{H}\}\}
\]
to be the resolvent set and \(\sigma(\mathcal{R}) = \mathbb{C} - \rho(\mathcal{R})\) to be the spectrum of \(\mathcal{R}\).

**Remark 9.** When a relation \(\mathcal{R}\) is an operator on \(\mathcal{H}\), then
\[
\rho(\mathcal{R}) = \{z \in \mathbb{C} : (z - \mathcal{R})^{-1} \in B(\mathcal{H})\}\.
\]

A complex number \(z \in \mathbb{C}\) is called an **eigenvalue** of a relation \(\mathcal{R}\) if there exists \(f \in \mathcal{H}\), \(f \neq 0\) such that \((f, zf) \in \mathcal{R}\). The set of all eigenvalues of \(\mathcal{R}\) is called the **point spectrum** of \(\mathcal{R}\) and is denoted by \(\sigma_p(\mathcal{R})\).

**Remark 10.** For any closed relation \(\mathcal{R}\) on a Hilbert space \(\mathcal{H}\), \(\sigma_p(\mathcal{R}) \subset \sigma(\mathcal{R})\).

Let \(\mathcal{L} = \{(0, g) \in \mathcal{R}\} and \mathcal{Z} = \{g : (0, g) \in \mathcal{R}\} be the multivalued part of a relation \(\mathcal{R}\). Clearly \(\mathcal{Z}\) is a closed subspace of \(\mathcal{H}\). Note that \(D(\mathcal{R})\) is not dense if \(\mathcal{R}\) is multivalued. Now define the quotient space \(\mathcal{H}_z = \mathcal{H}/\mathcal{Z}\). We know that this quotient space is also a Hilbert space with the norm defined by
\[
\|f\| = \inf_{g \in \mathcal{Z}} \|f + g\|.
\]

Define a relation \(\mathcal{R}_z\) on \(\mathcal{H} \oplus \mathcal{H}\) by \(\mathcal{R}_z = \{(f, [g]) : (f, g) \in \mathcal{R}\}\). We consider the relation \(\mathcal{R}_z\) as the restriction of \(\mathcal{R}\) on \(\mathcal{H}_z\). By natural isomorphism, the space \(\mathcal{H}_z\) is identified to \(\mathcal{H} \oplus \mathcal{Z}\) and the relation \(\mathcal{R}_z\) as \(\mathcal{R} \oplus \mathcal{Z}\). Then clearly \(\mathcal{R}_z\) is an operator on \(\mathcal{H}_z\) with \(D(\mathcal{R}_z) = D(\mathcal{R})\).

**Theorem 11.** If \(\mathcal{T}\) is a self-adjoint relation on \(\mathcal{H}\), then
\[
S(\mathcal{T}) = \sigma(\mathcal{T}) = \sigma(\mathcal{T}_z).
\]

**Proof.** Letting \(z \in \Gamma(\mathcal{T})\), then there exists a constant \(C > 0\) such that
\[
\|zf - g\| \geq C \|f\|, \quad \forall (f, g) \in \mathcal{T}.
\]

For such \(z\), we can define \(T = (z - \mathcal{T})^{-1}\) as a bounded linear operator on \(\mathcal{H}\) such that \(\mathcal{T} = \{(Th, zTh - h) : h \in \mathcal{H}\}\). So \(z \in \rho(\mathcal{T})\). On the other hand, let \(z \in \rho(\mathcal{T})\); then there exists \(T \in B(\mathcal{H})\) such that \(\mathcal{T} = \{(Th, zTh - h) : h \in \mathcal{H}\}\). For any \((f, g) \in \mathcal{T}\), there is \(h \in \mathcal{H}\) such that \(Th = f\) and \(zTh - h = g\).

So
\[
\|zf - g\| = \|zTh - zTf + h\| \geq C \|Th\| = C \|f\|
\]
for some \(C > 0\) and hence \(z \in \Gamma(\mathcal{T})\). Hence, \(S(\mathcal{T}) = \sigma(\mathcal{T})\).

Next, assuming that \(z \in \Gamma(\mathcal{T})\), then for any \((f, [g]) \in \mathcal{T}_z\), there exists a constant \(C > 0\) such that
\[
\|z[f] - [g]\| \geq C \|f\|.
\]
For any \((f, g) \in \mathcal{T}\) we have
\[
\|zf - g\| \geq \|z[f] - [g]\| \geq C\|f\| = C\|f\|. \tag{42}
\]
Hence \(z \in \Gamma(\mathcal{T})\). On the other hand, suppose that \(z \in \Gamma(\mathcal{T})\), then there is a constant \(C > 0\) such that
\[
\|zf - g\| \geq C\|f\|. \tag{43}
\]
For any \(([f], [g]) \in \mathcal{T}_s\), we have
\[
\|z[f] - [g]\| = \inf_{u \in \mathcal{Z}} \|zf - g + u\| = \inf_{u \in \mathcal{Z}} (\|zf - g\| + \|u\|) \geq \|zf - g\| \geq C\|f\| = C\|f\|. \tag{44}
\]
This implies that \(z \in \Gamma(\mathcal{T}_s)\). Thus, \(S(\mathcal{T}_s) = S(\mathcal{T})\). Hence, \(S(\mathcal{T}) = \sigma(\mathcal{T}) = \sigma(\mathcal{T}_s)\).

Remark 12. If \(\mathcal{T}\) is a self-adjoint relation on \(\mathcal{H}\), then \(\sigma(\mathcal{T}) \subseteq \mathbb{R}\).

Theorem 13. Let \(z \in \Gamma(\mathcal{T})\) and \(T = z - 1/(1/(z - \lambda))\). One has the following.

1. If \(\lambda \in \Gamma(T),\) then \((z - (1/\lambda)) \in \Gamma(\mathcal{T})\).
2. If \(\lambda \in S(\mathcal{T}),\) then \((1/(z - \lambda)) \in S(T)\).
3. \(S(T) \subseteq \sigma(T)\).

Proof. (1) Let \(\lambda \in \Gamma(T);\) then by definition there exists \(C(\lambda) > 0\) such that
\[
\|\lambda(zf - g) - f\| \geq C(\lambda)\|zf - g\|, \quad \forall (f, g) \in \mathcal{T}. \tag{45}
\]
Note that \(\lambda \neq 0\). For any \((f, g) \in \mathcal{T},\) we have
\[
\left\|\left(z - \frac{1}{\lambda}\right)f - g\right\| = \frac{1}{|\lambda|}\|z\lambda f - f - \lambda g\|
\]
\[
= \frac{1}{|\lambda|}\|\lambda(zf - g) - f\|
\]
\[
\geq \frac{C(\lambda)}{|\lambda|}\|zf - g\| \geq \frac{C(\lambda)(\lambda)}{|\lambda|}\|f\|. \tag{46}
\]
So \((z - (1/\lambda)) \in \Gamma(\mathcal{T})\).

(2) Let \(\lambda \in S(\mathcal{T})\) and suppose that \((1/(z - \lambda)) \notin S(T).\) Then \((1/(z - \lambda)) \in \Gamma(T).\) But by (1), \((z - 1/(1/(z - \lambda))) \notin \Gamma(\mathcal{T}).\) This implies that \(\lambda \in \Gamma(\mathcal{T})\) which is a contradiction.

(3) Let \(\lambda \in \rho(T)\); then \((\lambda - T)^{-1}\) is bounded and is defined on all of \(\mathcal{H}\); then for any \((f, g) \in \mathcal{T}\)
\[
\|zf - g\| = \|(\lambda - T)^{-1}(\lambda - T)(zf - g)\|
\]
\[
\leq \|(\lambda - T)^{-1}\|\|\lambda(zf - g) - T(zf - g)\|
\]
\[
\implies \|\lambda(zf - g) - T(zf - g)\|
\]
\[
\geq \frac{1}{\|(\lambda - T)^{-1}\|}\|zf - g\|. \tag{47}
\]
\(\implies \lambda \in \Gamma(T).\) This shows that \(S(T) \subseteq \sigma(T)\). \(\Box\)