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Abstract

The problem of classical solutions for the regularized long-wave equation is considered where various additional forcing terms are introduced which are often required for physical modifications in the wave theory. Sufficient conditions of solvability and existence are established and then these conditions are related to the structure of the forcing terms under consideration.

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1 Introduction

It is well known [3] that the equation

$$u_{xxt} = (u(x,t) + 1) \cdot u_x + u_t$$

(BBM equation henceforth) performs well as a mathematical model for long-time evolution of wave phenomenon. Moreover, when investigating the mathematical modeling leading up to the BBM equation in applications of wave phenomenon it becomes apparent that many physical conditions are either overlooked, or drastically simplified to be taken as constant. For example, if one adds the condition of viscosity to model water waves then the corresponding equation takes the form

$$(u(x,t) + 1) \cdot u_x + u_t - u_{xxt} = cu_{xx},$$

where is c a constant that represents the viscosity. Various results for the above equation were discussed in our recent work [6]. In addition, various other forcing terms have been investigated [5] and the corresponding equations often take the form

$$(u(x,t)+1)\cdot u_x + u_t - u_{xxt} = f(x,t,u(x,t)).$$

In the following pages we investigate the above equation and develop conditions on the forcing term f that are required so that one can assure existence of a solution of the corresponding partial differential equation.

Henceforth, we denote by $u^{(m,n)}(x,t)$ as the m+n partial derivative of a function $u(x,t)$ with respect to x m times and t n times, i.e. $u^{(m,n)}(x,t) = \frac{\partial^{m+n}u}{\partial x^m \partial t^n}$. Also, we denote ζ_T as the class of functions $u(x,t)$ that are continuous and uniformly bounded of $R \times [0,T]$. Moreover, we denote $\zeta_T^{(m,n)}$ as the class of functions such that $u^{(i,j)}(x,t) \in \zeta_T$ for $0 \leq i \leq m, 0 \leq j, \leq n$. And, we use the norm $||u|||_{\zeta_T} = \sup_{x\in R, 0\leq t\leq T} |u(x,t)|$.

2 statement of results

In the following Theorems we will be considering solutions to the partial differential equation

$$(u(x,t)+1)u^{(1,0)}(x,t) + u^{(0,1)}(x,t) - u^{(2,1)}(x,t) = f(x,t,u(x,t)) \tag{1}$$

with the initial conditions $u(x,0) = g(x)$, considered for a class of real non-periodic functions $u(x,t)$ defined for $-\infty < x < +\infty, t \geq 0$. Moreover, we will also be referring to the following integral equation for the same class of functions

$$u(x,t) = g(x) + \int_0^t \int_{-\infty}^{+\infty} K(x-\xi)\left\{u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F(\xi,\tau,u(\xi,\tau))\right\} d\xi d\tau. \tag{2}$$

Theorem 2.1 *Let g(x) be a continuous function such that*

$$\sup_{x\in R} |g(x)| \leq b < \infty$$

then there exists a $t_0(b)$ *such that the integral equation, (2.2) has a solution that is bounded and continuous for* $x \in R$ *and* $0 \leq b < 1$. *The kernel is* $K(x) = \frac{1}{2}sgn(x)e^{-x}$, *and* F *is a function that satisfies*

$$F(x,y,u) - F(x,y,\bar{u}) \leq L|u-\bar{u}|.$$

where L *is a positive constant.*

Theorem 2.2 *Let $g(x)$ be a continuous function such that*

$$\sup_{x\in R}|g(x)| \leq b < \infty$$

then there exists a $t_0(b)$ such that the integral equation, (2.2) has a solution that is bounded and continuous for $x \in R$ and $0 \leq b < 1$. The kernel is $K(x) = \frac{1}{2}sgn(x)e^{-x}$, and F is a function that satisfies

$$F(x,y,u) - F(x,y,\bar{u}) \leq G(|u-\bar{u}|).$$

where G is a nondecreasing function, such that $G(z) > 0$ for $z > 0$ and $\int_\varepsilon \frac{1}{G(z)}dz \to \infty$ as $\varepsilon \to 0$.

Theorem 2.3 *If $g(x) \in C^2$ then any solution of integral equation*

$$u(x,t) = g(x) + \int_0^t \int_{-\infty}^{+\infty} K(x-\xi)\left\{u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F\right\}d\xi d\tau$$

which is an element of $\zeta_{(0,1)}$ for a given $T > 0$ is also an element of $\zeta_{(0,1)}^{(2,\infty)}$ and is a classical solution of the partial differential equation (1) with the initial condition $g(x) = u(x,0)$ provided that the forcing term $f(u)$ satisfies either the condition of Theorem 2.1 or Theorem 2.2, and $F(x,t,u(x,t)$ is such that $\frac{dF}{dx} = f$.

3 Proofs and auxiliary statements

Lemma 3.1 *The partial differential equation (1) can be as*

$$[(1-\partial_x^2)]u^{(0,1)} = -\partial_x[u + \frac{1}{2}u^2 - F(x,t,u(x,t))], \tag{3}$$

or as an integral equation

$$u(x,t) = \int_0^t \int_{-\infty}^{+\infty} K(x-\xi)(u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F(\xi,\tau,u(\xi,\tau))d\xi d\tau, \tag{4}$$

where F is a function such that $\frac{\partial F}{\partial x} = f(x,t,u(x,t))$.

proof:

To begin we note that using routing algebra (1) can be rewritten as

$$u^{(0,1)}(x,t) - u^{(2,1)}(x,t) = f(x,t,u(x,t) - u^{(1,0)}(x,t) - u^{(1,0)}(x,t)u(x,t).$$

Now, the left hand side of the above equation can be rewritten as

$$[(1-\partial_x)(1+\partial_x)]u^{(0,1)} = [(1-\partial_x^2)]u^{(0,1)}.$$

Similarly,

$$-u^{(1,0)} - u^{(1,0)}u = -\partial_x[u + \frac{1}{2}u^2].$$

Combining this one can rewrite the partial differential equation (1) as

$$[(1-\partial_x^2)]u^{(0,1)} = f(x,t,u(x,t)) - \partial_x[u + \frac{1}{2}u^2].$$

Or,

$$[(1-\partial_x^2)]u^{(0,1)} = -\partial_x[u + \frac{1}{2}u^2 - F(x,t,u(x,t)],$$

where F is a function such that $\frac{\partial F}{\partial x} = f(x,t,u(x,t))$. Now, one views the above equation as a differential equation for $u^{(0,1)}$; hence, one obtains the formal solution as

$$u^{(0,1)} = -\frac{1}{2}\int_{-\infty}^{+\infty} e^{|x-\xi|}\partial_\xi[u(\xi,t) + \frac{1}{2}u^2(\xi,t) - F(\xi,t,u(\xi,t))]d\xi.$$

Now, using integration by parts this can be written as

$$u^{(0,1)} = \int_{-\infty}^{+\infty} K(x-\xi)(u(\xi,t) + \frac{1}{2}u^2(\xi,t) - F(\xi,t,u(\xi,t))d\xi$$

where the Kernel is defined as $K(x) = \frac{1}{2}sgn(x)e^{-|x|}$. And, the above pseudo differential equation can be rewritten as

$$u(x,t) = \int_0^t \int_{-\infty}^{+\infty} K(x-\xi)(u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F(\xi,\tau,u(\xi,\tau))d\xi d\tau$$

which completes the proof of Lemma 1.

proof of Theorem 3.1:

To begin we will write the integral equation (4) from Lemma 3.1 in operator notation as

$$u = \mathbf{A}[u]$$

where

$$\mathbf{A}[u] = \int_0^t \int_{-\infty}^{+\infty} K(x-\xi)(u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F(\xi,\tau,u(\xi,\tau))d\xi d\tau.$$

Now for our considerations let us denote ζ_{t_0} as the class of functions $v(x,t)$ which satisfy the integral equation from Lemma 3.1, which are continuous and uniformly bounded on the infinite strip $R\times[0,t_0]$. Now, we will be consider this set of functions $v\in\zeta_{t_o}$ that have the norm $||v|||_{\zeta_{t_0}}$ which is a Banach space of bounded continuous functions where t_0 is left as arbitrary.

Now let us consider two functions $v_1, v_2\in\zeta_{t_o}$ and we observe that

$$|Av_1 - Av_2| =$$

$$\int_0^t\int_{-\infty}^{+\infty}|K(x-\xi)|(|v_1(\xi,\tau)-v_2(\xi,\tau)|+\frac{1}{2}|v_1^2(\xi,\tau)-v_2^2(\xi,\tau)|$$

$$+|F(\xi,\tau,v_2\xi,\tau)-F(\xi,\tau,v_1(\xi,\tau))|)d\xi d\tau$$

$$\leq\int_0^t\int_{-\infty}^{+\infty}|K(x-\xi)|(|v_1(\xi,\tau)-v_2(\xi,\tau)|+\frac{1}{2}|v_1(\xi,\tau)-v_2(\xi,\tau)||v_1(\xi,\tau)+v_2(\xi,\tau)|)d\xi d\tau$$

$$+\int_0^t\int_{-\infty}^{+\infty}|K(x-\xi)||L|v_2\xi,\tau)-v_1(\xi,\tau)|d\xi d\tau.$$

$$\leq||v_1-v_2||_\zeta(1+\frac{1}{2}||v_1+v_2||_\zeta+L)\int_0^t\int_{-\infty}^{+\infty}|K(x-\xi)|d\xi d\tau$$

$$\leq||v_1-v_2||_\zeta(1+\frac{1}{2}||v_1+v_2||_\zeta+L)t$$

$$\leq||v_1-v_2||_\zeta(1+\frac{1}{2}||v_1||_\zeta+\frac{1}{2}||v_2||_\zeta+L)t.$$

Thus, we obtain, by taking sup for $t\in[0,t_0]$ we obtain that

$$|Av_1-Av_2|\leq t_0(L+1+\frac{1}{2}||v_1||_\zeta+\frac{1}{2}||v_2||_\zeta))||v_1-v_2||_{\bar{\zeta}}$$

From this it follows that A is a continuous mapping; moreover, A is a contraction (i.e. satisfies Lipschitz condition) on the ball $||v||_\zeta<R$ if

$$t_0(L+1+R)<1.$$

If this condition is satisfied then by the Theory of fixed points [2] one can assure that A has a unique fixed point in the ball $||v||_\zeta<R$ which completes the proof.

proof of Theorem 3.2:

The proof of Theorem 3.2 is identical to the proof of theorem 3.1 with the exception of a slight change in the algebra due to the adjustment of $|u-\bar{u}|$

becoming $G(|u-\bar{u}|)$. Moreover, this results in the Lipschitz constant L in the proof becoming replaced by a new constant C which is obtained from the function G. One can see an analogously development for the classic Osgood's theorem from the theory of ODE [1] which can be extended to this PDE. Moreover, some generalizations of these conditions, such as Perron's or Krasnosel'skill-Krein's, are also discussed in [1] for the theory of ODE which again could be extended.

proof of Theorem 3.3:

To begin we will note that under the conditions posed $\mathbf{A}[u]$ is a continuously differentiable function of t; hence, $u^{(0,1)}(x,t)$ exist and is given by

$$u^{(0,1)}(x,t)=\partial_t([\mathbf{A}]u)^{(0,1)}=\int_{-\infty}^{+\infty}K(x-\xi)(u(\xi,t)+\frac{1}{2}u^2(\xi,t)-F(\xi,t),u(\xi,t)d\xi. \tag{5}$$

One can see that (5) is continuous in x, continuous in t and bounded on $R\times[0,T]$. Now, it can be argued that

$$u^{(0,2)}=\int_{-\infty}^{+\infty}K(x-\xi)\partial_t(u(\xi,t)+\frac{1}{2}u^2(\xi,t)-F(\xi,t),u(\xi,t))d\xi.$$

And,

$$u^{(0,m)}=\int_{-\infty}^{+\infty}K(x-\xi)\partial_t^{m-1}(u(\xi,t)+\frac{1}{2}u^2(\xi,t)-F(\xi,t,u(\xi,t)))d\xi.$$

From the above equation, using the fact that u^2 has the same degree of regularity as u if u is bounded and noting that the conditions of F also maintain this degree of regularity, one can conclude that $u^{(1,0)}$ also has equal regularity. Thus, one can obtain that $u^{(0,1)}$ is continuous and bounded on $R\times[0,T]$. Using induction one can see that this can be carried on for any value of m, hence the statement of the lemma concerning the t-dependence of u is verified.

Now, let complete the lemma by confirming the existence of $u^{(1,0)}$. To do this we recall that from (2) that

$$u(x,t)=g(x)+\int_0^t\int_{-\infty}^{+\infty}K(x-\xi)(u(\xi,\tau)+\frac{1}{2}u^2(\xi,\tau)-F(\xi,\tau,u(\xi,\tau))d\xi d\tau$$

and by dividing the range of integration at $\xi=x$ we obtain

$$u^{(1,0)}=g'(x)+\int_0^t(u(x,\tau)+\frac{1}{2}u^2(x,\tau)-F(x,\tau))d\tau$$

$$-\int_0^t\int_{-\infty}^{+\infty}\frac{1}{2}e^{-|x-\xi|}(u(\xi,\tau)+\frac{1}{2}u^2(\xi,\tau)-F(\xi,\tau,u(\xi,\tau))d\xi d\tau.$$

which shows that $u^{(1,0)}(x,t)$ is continuous and bounded. Now, we can continue in this fashion and obtain that

$$u^{(2,0)} = g''(x) + \int_0^t \partial_x (u(x,\tau) + \frac{1}{2}u^2(x,\tau) - F(x,\tau), u(x,\tau) d\tau$$

$$+ \int_0^t \int_{-\infty}^{+\infty} K(x-\xi)(u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F(\xi,\tau,u(\xi,\tau))d\xi d\tau$$

$$= g''(x) + \int_0^t \partial_x (u(x,\tau) + \frac{1}{2}u^2(x,\tau) - F(x,\tau), u(x,\tau))d\tau + u(x,t) - g(x)$$

which is also a continuous and bounded function. Now for the t derivatives of $u^{(1,0)}$ and $u^{(2,0)}$ they can be developed inductively as above for $u^{(0,m)}$, hence, their continuity and boundedness are obtained. Thus, it follows that the solution $u(x,t)$ of (2) does indeed satisfy (1) pointwise in $R \times [0,T]$ which completes the proof.

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