On Doubly Periodic Solutions of Quasilinear Hyperbolic Equations of the Fourth Order

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The problem on doubly periodic solutions is considered for a class of quasilinear hyperbolic equations. Effective sufficient conditions of solvability and unique solvability of this problem are established.

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The problem on periodic solutions for second-order partial differential equations of hyperbolic type has been studied rather intensively by various authors [1–9, 11–14]. Analogous problem for higher-order hyperbolic equations is little investigated. In the present paper for the quasilinear hyperbolic equations

\[ u^{(2,2)} = f_0(x, y, u) + f_1(x, u)u^{(2,0)} + f_2(x, u)u^{(0,2)} + f(x, y, u, u^{(1,0)}, u^{(0,1)}, u^{(1,1)}), \]

\[ u^{(2,2)} = f_0(x, y, u) + (f_1(x, y, u)u^{(1,0)})^{(1,0)} + (f_2(x, y, u)u^{(0,1)})^{(0,1)} \]

\[ + f(x, y, u, u^{(1,0)}, u^{(0,1)}, u^{(1,1)}) \]

we consider the problem on doubly periodic solutions

\[ u(x + \omega_1, y) = u(x, y), \quad u(x, y + \omega_2) = u(x, y) \quad \text{for} \ (x, y) \in \mathbb{R}^2. \]

Here \( \omega_1 \) and \( \omega_2 \) are prescribed positive numbers,

\[ u^{(j,k)}(x, y) = \frac{\partial^{j+k}u(x, y)}{\partial x^j \partial y^k}, \]

\( f_0(x, y, z), \ f_1(y, z), \ f_2(x, z), \ f_1(x, y, z), \ f_2(x, y, z), \) and \( f(x, y, z, z_1, z_2, z_3) \) are continuous functions, \( \omega_1 \)-periodic in \( x \), and \( \omega_2 \)-periodic in \( y \).
This problem was studied thoroughly for the linear equation

\[ u^{(2,2)} = p_0(x, y)u + p_1(x, y)u^{(2,0)} + p_2(x, y)u^{(0,2)} + q(x, y) \tag{5} \]

in [10]. The goal of the present paper is on the basis of the methods developed in [10] to obtain effective sufficient conditions of solvability, unique solvability, and well-posedness of problems (1), (3) and (2), (3).

Throughout the paper, we will use the following notation:

\[
\text{sgn}(z) = \begin{cases} 
1, & z > 1 \\
0, & z = 0 \\
-1, & z < 0 
\end{cases} \tag{6}
\]

\[ C^{m,n}_{\omega_1 \omega_2} (\mathbb{R}^2) \] is the space of continuous functions \( z : \mathbb{R}^2 \to \mathbb{R} \) \( \omega_1 \)-periodic in the first and \( \omega_2 \)-periodic in the second arguments, having the continuous partial derivatives \( u^{(j,k)} \) \( j \in \{0, \ldots, m\}, \ k \in \{0, \ldots, n\} \), with the norm

\[ \| z \|_{C^{m,n}_{\omega_1 \omega_2}} = \sup \left\{ \sum_{j=0}^{m} \sum_{k=0}^{n} |z^{(j,k)}(x,y)|^2 : (x,y) \in \mathbb{R}^2 \right\}^{1/2}. \tag{7} \]

\[ L^2_{\omega_1 \omega_2} (\mathbb{R}^2) \] is the space of locally square-integrable functions \( z : \mathbb{R}^2 \to \mathbb{R} \), \( \omega_1 \)-periodic in the first and \( \omega_2 \)-periodic in the second arguments, with the norm

\[ \| z \|_{L^2_{\omega_1 \omega_2}} = \left( \int_0^{\omega_1} \int_0^{\omega_2} |z(s,t)|^2 \, ds \, dt \right)^{1/2}. \tag{8} \]

\[ H^{m,n}_{\omega_1 \omega_2} (\mathbb{R}^2) \] is the space of functions \( z \in L^2_{\omega_1 \omega_2} (\mathbb{R}^2) \), having the generalized partial derivatives \( u^{(j,k)} \in L^2_{\omega_1 \omega_2} (\mathbb{R}^2) \), \( j \in \{0, \ldots, m\}, \ k \in \{0, \ldots, n\} \), with the norm

\[ \| z \|_{H^{m,n}_{\omega_1 \omega_2}} = \sum_{j=0}^{m} \sum_{k=0}^{n} \| u^{(j,k)} \|_{L^2_{\omega_1 \omega_2}}. \tag{9} \]

By a solution of problem (1), (3) (problem (2), (3)), we understand a classical solution, that is, a function \( u \in C^{2,2}_{\omega_1 \omega_2} (\mathbb{R}^2) \) satisfying (1) (equation (2)) everywhere in \( \mathbb{R}^2 \).

**Theorem 1.** Let there exists a positive constant \( \delta \) such that

\[ f_1(y, z) \geq \delta, \quad f_2(x, z) \geq \delta \quad \text{for} \ (x, y, z) \in \mathbb{R}^3. \tag{10} \]
Moreover let the functions \( f_1, f_2, f_0 \), and \( f \) satisfy the conditions

\[
(f_1(y, z) - f_1(y, z)) \sgn(z - \zeta) \sgn(z) \geq 0 \quad \text{for } y \in \mathbb{R}, z \zeta \geq 0,
\]

\[
(f_2(x, z) - f_2(x, z)) \sgn(z - \zeta) \sgn(z) \geq 0 \quad \text{for } x \in \mathbb{R}, z \zeta \geq 0,
\]

\[
f_0(x, y, z) \sgn(z) < 0 \quad \text{for } (x, y) \in \mathbb{R}^2, z \neq 0,
\]

\[
\lim_{z \to -\infty} \frac{\int_0^{\omega_1} \int_0^{\omega_1} f_0(x, y, z) \, dx \, dy}{\omega_1} = -\infty,
\]

\[
\lim_{z \to -\infty} \frac{f(x, y, z_1, z_2, z_3)}{f_0(x, y, z)} = 0 \quad \text{uniformly on } \mathbb{R}^2 \times \mathbb{R}^4.
\]

Then problem (1), (3) is solvable.

**Theorem 2.** Let \( f_1 \) and \( f_2 \) be continuously differentiable functions such that

\[
f_1(x, y, z) \geq \delta, \quad f_2(x, y, z) \geq \delta \quad \text{for } (x, y, z) \in \mathbb{R}^3
\]

for some positive \( \delta \). Moreover, let the functions \( f_0 \) and \( f \) satisfy the conditions of Theorem 1. Then problem (2), (3) is solvable.

**Remark 1.** Note that conditions (10) and (15) are optimal in a sense that we cannot take \( \delta = 0 \). Indeed, consider the problems

\[
u^{(2,2)} = -F(u) + \left( F'(u)u^{(1,0)} \right)^{(1,0)} + u^{(0,2)} + \pi \sin x,
\]

\[
u(x + 2\pi, y) = u(x, y), \quad u(x, y + 2\pi) = u(x, y),
\]

where \( F(z) = z^3 \), or \( F(z) = \arctan(z) \). Problem (16), (17) satisfies all of the conditions of Theorem 2 except condition (15). Instead of (15), we have that \( F'(z) \) is nonnegative and vanishes at 0, or at \( \infty \) only.

Let us show that problem (16), (17) has no solution. Assume the contrary: let \( u \) be a solution of (16), (17), and set \( \nu(x, y) = u^{(0,2)}(x, y) - F(u(x, y)) \). Then for every \( y \in \mathbb{R} \), the function \( \nu(\cdot, y) \) is a solution to the periodic problem

\[
\nu'' = v + \pi \sin x, \quad \nu(x + 2\pi) = \nu(x).
\]

This problem has a unique solution \( \nu(x) = -\pi/2 \sin x \). Therefore, problem (16), (17) is equivalent to the problem

\[
u^{(0,2)} = F_1(u) - \frac{\pi}{2} \sin x, \quad u(x, y + 2\pi) = u(x, y).
\]

However, problem (19) has no more than one solution. Indeed, let \( u_1 \) and \( u_2 \) be arbitrary solutions to problem (19). Then one easily gets the identity

\[
\int_0^{\omega_2} \left( \left( u_1^{(0,1)}(x, t) - u_2^{(0,1)}(x, t) \right)^2 + \left( F(u_1(x, t)) - F(u_2(x, t)) \right)(u_1(x, t) - u_2(x, t)) \right) dt \equiv 0,
\]

whence it follows that \( u_1(x, y) \equiv u_2(x, y) \).
Due to uniqueness, a solution of problem \((19)\) should be independent of \(y\). So finally we arrive to the functional equation

\[
F(u) = \frac{\pi}{2} \sin x,
\]

whence we get

\[
u(x, y) = \frac{3\sqrt{\pi} \sin x}{2} \quad \text{for } F(z) = z^3, \tag{21}\]

\[
u(x, y) = \tan \left( \frac{\pi}{2} \sin x \right) \quad \text{for } F(z) = \arctan(z). \tag{22}\]

In the first case \(u\) is not differentiable at \(\pi k, k \in \mathbb{Z}\), while in the second case \(u\) itself is a discontinuous function, because it blows up at points \(\pi/2 + \pi k, k \in \mathbb{Z}\).

Thus, it is clear that of problem \((16), (17)\) has no solutions in the both cases.

**Remark 2.** The conditions of Theorem 1 (as well as Theorem 2) do not guarantee the uniqueness of a solution. Indeed, for the equation

\[
u^{(2,2)} = -u^n + \nu^{(2,0)} + \nu^{(0,2)} - \left( \prod_{k=1}^{n} (u - k) - u^n \right), \tag{23}\]

all of the conditions of Theorem 1 (and Theorem 2) are fulfilled. Nevertheless, it has at least \(n\) solutions \(u_k(x, y) \equiv k (k = 1, 2, \ldots, n)\) satisfying conditions \((3)\).

We will give a uniqueness theorem for the equations

\[
u^{(2,2)} = f_0(x, y, u) + \left( f_1(x, y)u^{(1,0)} \right)^{(1,0)} + \left( f_2(x, y)u^{(0,1)} \right)^{(0,1)}, \tag{24}\]

\[
u^{(2,2)} = f_0(x, y, u) + \left( f_1(x, y)u^{(1,0)} \right)^{(1,0)} + \left( f_2(x, y)u^{(0,1)} \right)^{(0,1)}
+ \varepsilon f \left( x, y, u, u^{(1,0)}, u^{(0,1)}, u^{(1,1)}, u^{(2,0)}, u^{(0,2)}, u^{(2,1)}, u^{(1,2)} \right). \tag{25}\]

**Theorem 3.** Let there exists \(\delta > 0\) such that

\[
f_1(x, y) \geq \delta, \quad f_2(x, y) \geq \delta \quad \text{for } (x, y) \in \mathbb{R}^2, \tag{26}\]

\[
\left( f_0(x, y, z) - f_0(x, y, z) \right) \operatorname{sgn}(z - z) \leq -\delta |z - z| \quad \text{for } (x, y) \in \mathbb{R}^2, z, z \in \mathbb{R}. \tag{27}\]

Then problem \((24), (3)\) is uniquely solvable. Moreover, for every \(f(x, y, z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8)\) that is Lipschitz continuous with respect to the last eight phase variables, there exists a positive \(\varepsilon_0\) such that problem \((25), (3)\) is uniquely solvable for every \(\varepsilon \in (-\varepsilon_0, \varepsilon_0)\).

To prove Theorems 1–3, we will need the following lemmas.
Lemma 1. Let \( p_0, p_1, p_2, \) and \( q \in C_{\omega_1 \omega_2}(\mathbb{R}^2) \), and let there exist a positive constant \( \delta \) and a nondecreasing continuous function \( \eta : [0, +\infty) \to [0, +\infty) \), \( \eta(0) = 0 \) such that

\[
\begin{align*}
p_1(x, y) &\geq \delta, \quad p_2(x, y) \geq \delta, \\
|p_1(x_1, y_1) - p_1(x_2, y_2)| + |p_2(x_1, y_1) - p_2(x_2, y_2)| &\leq \eta(|x_1 - x_2| + |y_1 - y_2|) \quad \text{for } (x_i, y_i) \in \mathbb{R}^2 \quad (i = 1, 2).
\end{align*}
\]

Then an arbitrary solution \( u \) of problem (5), (3) admits the estimate

\[
\begin{align*}
\int_0^{\omega_1} \int_0^{\omega_2} \left( |u^{(2,0)}(x, y)|^2 + |u^{(0,2)}(x, y)|^2 + |u^{(2,1)}(x, y)|^2 + |u^{(1,2)}(x, y)|^2 \right) dx \, dy \\
&\leq M \int_0^{\omega_1} \int_0^{\omega_2} \left( |u(x, y)|^2 + |u^{(1,0)}(x, y)|^2 + |u^{(0,1)}(x, y)|^2 + q^2(x, y) \right) dx \, dy,
\end{align*}
\]

where the constant \( M > 0 \) depends on \( \delta, \|p_0\|_{C_{\omega_1 \omega_2}}, \) and the function \( \eta. \)

**Proof.** Let \( u \) be a an arbitrary solution of problem (5), (3). For any \( h > 0 \), set

\[
\begin{align*}
p_{ih}(x, y) &= \frac{1}{h^2} \int_x^{x+h} \int_y^{y+h} p_i(s, t) \, ds \, dt \quad (i = 1, 2), \\
Q_h[u](x, y) &= (p_1(x, y) - p_{1h}(x, y)) u^{(2,0)}(x, y) + (p_2(x, y) - p_{2h}(x, y)) u^{(0,2)}(x, y).
\end{align*}
\]

Then \( u \) satisfies the equation

\[
u^{(2,2)} = p_0(x, y)u + p_{1h}(x, y)u^{(2,0)} + p_{2h}(x, y)u^{(0,2)} + Q_h[u](x, y) + q(x, y).
\]

Multiplying successively (32) by \( u(x, y), u^{(2,0)}, \) and \( u^{(0,2)} \), integrating over the rectangle \([0, \omega_1] \times [0, \omega_2]\), and using integration by parts, we observe that

\[
\begin{align*}
\int_0^{\omega_1} \int_0^{\omega_2} \left( p_{ih}(x, y) \right| u^{(1,0)}(x, y) \left| \right|^2 + p_{2h}(x, y) \right| u^{(0,1)}(x, y) \left| \right|^2 + \left| u^{(1,1)}(x, y) \right|^2 \right) dx \, dy \\
&= \int_0^{\omega_1} \int_0^{\omega_2} \left( Q_h[u](x, y) - p_{1h}^{(1,0)}(x, y)u^{(1,0)}(x, y) - p_{2h}^{(0,1)}(x, y)u^{(0,1)}(x, y) \right. \\
&\quad + p_0(x, y)u(x, y) + q(x, y) \left. \right) u(x, y) dx \, dy,
\end{align*}
\]
Doubly periodic solutions

\[ \int_{0}^{\omega_1} \int_{0}^{\omega_2} \left( p_{1h}(x,y) \left| u^{(2,0)}(x,y) \right|^2 + p_{2h}(x,y) \left| u^{(1,1)}(x,y) \right|^2 + \left| u^{(2,1)}(x,y) \right|^2 \right) dx \, dy \]

\[ = \int_{0}^{\omega_1} \int_{0}^{\omega_2} \left( p_{2h}^{(0,1)}(x,y)u^{(2,0)}(x,y)u^{(0,1)}(x,y) - p_{2h}^{(1,0)}(x,y)u^{(1,1)}(x,y)u^{(0,1)}(x,y) \right) dx \, dy \]

\[ - \int_{0}^{\omega_1} \int_{0}^{\omega_2} \left( Q_h[u](x,y) + p_0(x,y)u(x,y) + q(x,y) \right) u^{(2,0)}(x,y) dx \, dy, \]

(34)

However,

\[ \int_{0}^{\omega_1} \int_{0}^{\omega_2} \left| Q_h[u](x,y) \right| \left( \left| u(x,y) \right| + \left| u^{(2,0)}(x,y) \right| + \left| u^{(0,2)}(x,y) \right| \right) dx \, dy \]

\[ \leq 2\eta(h) \left( \left\| u \right\|_{L_{2w_1w_2}^2}^2 + \left\| u^{(2,0)} \right\|_{L_{2w_1w_2}^2}^2 + \left\| u^{(0,2)} \right\|_{L_{2w_1w_2}^2}^2 \right) , \]

(36)

\[ \int_{0}^{\omega_1} \int_{0}^{\omega_2} \left( \left| p_0(x,y) \right| \left| u(x,y) \right| + \left| q(x,y) \right| \right) \left( \left| u(x,y) \right| + \left| u^{(2,0)}(x,y) \right| + \left| u^{(0,2)}(x,y) \right| \right) dx \, dy \]

\[ \leq \left( \frac{2}{\varepsilon} \left\| p_0 \right\|_{C_{w_1w_2}}^2 + 2\varepsilon \right) \left\| u \right\|_{L_{2w_1w_2}^2}^2 + \frac{2}{\varepsilon} \left\| q \right\|_{L_{2w_1w_2}^2}^2 + 2\varepsilon \left( \left\| u^{(2,0)} \right\|_{L_{2w_1w_2}^2}^2 + \left\| u^{(0,2)} \right\|_{L_{2w_1w_2}^2}^2 \right) , \]

(37)

\[ \int_{0}^{\omega_1} \int_{0}^{\omega_2} \left( \left| p_{2h}^{(1,0)}(x,y) \right| \left| u^{(2,0)}(x,y) \right| \left| u^{(0,1)}(x,y) \right| \right) dx \, dy \]

\[ \leq \frac{2\eta(h)}{h} \varepsilon \left( \left\| u^{(2,0)} \right\|_{L_{2w_1w_2}^2}^2 + \left\| u^{(1,1)} \right\|_{L_{2w_1w_2}^2}^2 \right) + \frac{2\eta(h)}{h\varepsilon} \left\| u^{(0,1)} \right\|_{L_{2w_1w_2}^2}^2 , \]

(38)

\[ \int_{0}^{\omega_1} \int_{0}^{\omega_2} \left( \left| p_{lh}^{(1,0)}(x,y) \right| \left| u^{(0,2)}(x,y) \right| \left| u^{(1,0)}(x,y) \right| \right) dx \, dy \]

\[ \leq \frac{2\eta(h)}{h} \varepsilon \left( \left\| u^{(0,2)} \right\|_{L_{2w_1w_2}^2}^2 + \left\| u^{(1,1)} \right\|_{L_{2w_1w_2}^2}^2 \right) + \frac{2\eta(h)}{h\varepsilon} \left\| u^{(1,0)} \right\|_{L_{2w_1w_2}^2}^2 . \]

(39)

Now taking \( h > 0 \) and \( \varepsilon > 0 \) sufficiently small from (33)–(39), we immediately get estimate (30).

The following lemma immediately follows from [10, Lemma 2.7].
Lemma 2. Let $p_0, p_1, p_2,$ and $q \in C_{\omega_1,\omega_2}(\mathbb{R}^2),$ and let $p_1$ and $p_2$ satisfy conditions (28). Then an arbitrary solution $u$ of problem (5), (3) admits the estimate

$$\|u\|_{C_{\omega_1,\omega_2}^{2,2}} \leq r \left( \int_0^{\omega_1} \int_0^{\omega_2} \left( \|u(x,y)\| + \|u^{(2,0)}(x,y)\| + \|u^{(0,2)}(x,y)\| \right) \, dx \, dy + \|q\|_{C_{\omega_1,\omega_2}} \right), \tag{40}$$

where $r$ is a positive constant depending on $\delta$, $\|p_0\|_{C_{\omega_1,\omega_2}}$, $\|p_1\|_{C_{\omega_1,\omega_2}}$, and $\|p_2\|_{C_{\omega_1,\omega_2}}$ only.

Lemma 3. Let $p_1, p_2 \in C_{\omega_1,\omega_2}(\mathbb{R}^2)$ satisfy the conditions of Lemma 1. Then there exist $\lambda > 0$ and $M_1 > 0$ depending on $\delta$, $\|p_1\|_{C_{\omega_1,\omega_2}}$, $\|p_2\|_{C_{\omega_1,\omega_2}}$, and the function $\eta$ such that for every $q \in C_{\omega_1,\omega_2}(\mathbb{R}^2)$, the equation

$$u^{(2,2)} = -\lambda u + p_1(x,y)u^{(2,0)} + p_2(x,y)u^{(0,2)} + q(x,y) \tag{41}$$

has a unique solution $u$ satisfying conditions (3), and

$$\|u\|_{C_{\omega_1,\omega_2}^{2,2}} \leq M_1 \|q\|_{C_{\omega_1,\omega_2}}. \tag{42}$$

Proof. This lemma easily follows from Lemmas 1 and 2. Indeed, let $u$ be an arbitrary solution of problems (41), (3). Multiplying successively (41) by $u(x,y)$, $u^{(2,0)}$, and $u^{(0,2)}$, integrating over the rectangle $[0,\omega_1] \times [0,\omega_2]$, and using integration by parts, we get

$$\lambda \left( \|u\|_{L_{\omega_1,\omega_2}^2}^2 + \|u^{(1,0)}\|_{L_{\omega_1,\omega_2}^2}^2 + \|u^{(0,1)}\|_{L_{\omega_1,\omega_2}^2}^2 \right) \leq \left( 1 + \|p_1\|_{L_{\omega_1,\omega_2}^2}^2 + \|p_2\|_{L_{\omega_1,\omega_2}^2}^2 \right) \left( \|u\|_{L_{\omega_1,\omega_2}^2}^2 + \|u^{(2,0)}\|_{L_{\omega_1,\omega_2}^2}^2 + \|u^{(0,2)}\|_{L_{\omega_1,\omega_2}^2}^2 \right) + \|q\|_{L_{\omega_1,\omega_2}^2}^2. \tag{43}$$

Validity of Lemma 3 immediately follows from estimates (30), (40), and (43).

Consider the linear equation

$$u^{(2,2)} = p_0(x,y)u + \left( p_1(x,y)u^{(1,0)} \right)^{(1,0)} + \left( p_2(x,y)u^{(0,1)} \right)^{(0,1)} + q(x,y). \tag{44}$$

If $p_1$ and $p_2$ satisfy (28), then by $g_1(\cdot,\cdot,x) : \mathbb{R}^2 \to \mathbb{R}$ and $g_2(\cdot,\cdot,y) : \mathbb{R}^2 \to \mathbb{R}$, respectively, denote Green’s functions of the problems

$$\frac{d^2z}{dy^2} = p_1(x,y)z, \quad z(y + \omega_2) = z(y), \tag{45}$$

$$\frac{d^2z}{dx^2} = p_2(x,y)z, \quad z(x + \omega_1) = z(x),$$

(see [10, Lemmas 2.1 and 2.2]).
Lemma 4. Let \( u \) be a solution of problem (44), (3). Then the following representation is valid

\[
\begin{align*}
    u^{(2,0)}(x,y) &= p_2(x,y)u \\
    &+ \int_y^{y+\omega_2} g_1(y,t,x) \left( (p_0(x,t) + p_1(x,t)p_2(x,t)) u(x,t) \\
    &+ p_1^{(1,0)}(x,t)u^{(1,0)}(x,t) + q(x,t) \right) dt, \\
    u^{(0,2)}(x,y) &= p_1(x,y)u \\
    &+ \int_x^{x+\omega_1} g_2(x,s,y) \left( (p_0(s,y) + p_1(s,y)p_2(s,y)) u(s,y) \\
    &+ p_2^{(0,1)}(s,y)u^{(0,1)}(s,y) + q(s,y) \right) ds, \\
    u(x,y) &= \int_y^{y+\omega_2} \int_x^{x+\omega_1} g_1(y,t,x)g_2(x,s,y) \left( (p_0(s,t) + p_1(s,t)p_2(s,t)) u(s,t) \\
    &+ p_2^{(0,1)}(s,t)u^{(0,1)}(s,t) + q(s,t) \right) ds dt.
\end{align*}
\]  

(46)

We omit the proof of Lemma 4, since it is similar to the proof of [10, Lemma 2.7].

Let

\[
\varphi_\rho(z) = \begin{cases} 
1 & \text{for } |z| \leq \rho, \\
\rho + 1 - |z| & \text{for } |z| \in [\rho, \rho + 1], \\
0 & \text{for } |z| \geq \rho + 2,
\end{cases}
\]

\[
\chi_\rho(z) = \int_0^z \varphi_\rho(s)ds,
\]  

(47)

and let \( \Phi_\rho : C_{\omega_1,\omega_2}^1 \rightarrow \mathbb{R} \) be a continuous nonlinear functional defined by the equality

\[
\Phi_\rho(u) = \varphi_\rho \left( \|u\|_{C_{\omega_1,\omega_2}^1} \right).
\]  

(48)

Consider the equation

\[
\begin{align*}
    u^{(2,2)} &= f_0(x,y,\chi_\rho(u)) + f_1(y,\Phi_\rho(u)u)u^{(2,0)} + f_2(x,\Phi_\rho(u))u^{(0,2)} \\
    &+ \Phi_\rho(u)f(x,y,u^{(1,0)},u^{(0,1)},u^{(1,1)}) - \lambda u + \lambda \chi_\rho(u).
\end{align*}
\]  

(49)

Lemma 5. Let \( \lambda > 0 \) and \( \rho > 0 \). Then every solution \( u \) of problem (49), (3) admits the estimates

\[
\begin{align*}
    \int_0^{\omega_1} \int_0^{\omega_1} \left( |f_0(x,y,\chi_\rho(u(x,y)))| |u(x,y)| + \left| u^{(1,0)}(x,y) \right|^2 \\
    + \left| u^{(0,1)}(x,y) \right|^2 + \left| u^{(1,1)} \right|^2 \right) dx dy \leq r_0,
\end{align*}
\]  

(50)

where \( r_0 \) is a positive constant independent of \( \rho, \lambda, \) and \( u \).
Proof. Let \(u\) be a solution of problems (49), (3). Multiplying (49) by \(u(x,y)\), integrating over the rectangle \([0, \omega_1] \times [0, \omega_2]\), and using integration by parts, we get

\[
\int_0^{\omega_1} \int_0^{\omega_2} \left( - f_0(x,y, \chi_\rho(u(x,y))) + \lambda u - \lambda \chi_\rho(u) \right) u(x,y) - f_1(y, u(x,y)) u^{(2,0)}(x,y) u(x,y) \\
- f_2(x,u(x,y)) u^{(0,2)}(x,y) u(x,y) + \left| u^{(1,1)}(x,y) \right|^2 \right) \, dx \, dy \\
= \int_0^{\omega_1} \int_0^{\omega_2} \Phi_\rho(u) f \left(x,y,u(x,y), u^{(1,0)}(x,y), u^{(0,1)}(x,y), u^{(1,1)}(x,y) \right) u(x,y) \, dx \, dy.
\]

(51)

By conditions (13) and (14), we have

\[
( - f_0(x,y, \chi_\rho(u(x,y))) + \lambda \left( u(x,y) - \chi_\rho(u(x,y)) \right) ) u(x,y) \\
\geq \left| f_0(x,y, \chi_\rho(u(x,y))) \right| u(x,y),
\]

(52)

\[
\Phi_\rho(u) \left| f \left(x,y,u(x,y), u^{(1,0)}(x,y), u^{(0,1)}(x,y), u^{(1,1)}(x,y) \right) \right| \left| u(x,y) \right| \\
\leq r_1 + \frac{1}{2} \left| f_0(x,y, \chi_\rho(u(x,y))) \right| \left| u(x,y) \right|,
\]

(53)

where \(r_1\) is a positive constant independent of \(\rho, \lambda,\) and \(u\).

For \(h > 0\), set

\[
f_{i_h}(y,z) = \frac{1}{h} \int_z^{z+h} f_i(y, \xi) \, d\xi.
\]

(54)

Then by condition (11), we have

\[
- \int_0^{\omega_1} \int_0^{\omega_2} f_{i_h}(y, \Phi_\rho(u) u(x,y)) u(x,y) u^{(2,0)}(x,y) \, dx \, dy \\
= \int_0^{\omega_1} \int_0^{\omega_2} f_{i_h}(y, \Phi_\rho(u) u(x,y)) \left| u^{(1,0)}(x,y) \right|^2 \, dx \, dy \\
+ \frac{\Phi_\rho(u)}{h} \int_0^{\omega_1} \int_0^{\omega_2} \left( f_i(y, \Phi_\rho(u)(u(x,y) + h) - f_i(y, \Phi_\rho(u) u(x,y)) \right) u(x,y) \left| u^{(1,0)}(x,y) \right|^2 \, dx \, dy \\
\geq \int_0^{\omega_1} \int_0^{\omega_2} f_{i_h}(y, \Phi_\rho(u) u(x,y)) \left| u^{(1,0)}(x,y) \right|^2 \, dx \, dy \\
- \Phi_\rho(u) \int_{D_h} \left| f_i(y, \Phi_\rho(u)(u(x,y) + h) - f_i(y, \Phi_\rho(u) u(x,y)) \right| \left| u^{(1,0)}(x,y) \right|^2 \, dx \, dy,
\]

(55)
where \( D_h = \{ (x, y) \in [0, \omega_1] \times [0, \omega_2] : |u(x, y)| \leq h \} \). Hence we immediately get that

\[
- \int_{0}^{\omega_1} \int_{0}^{\omega_2} f_1(y, \Phi_p(u(x, y))) u(x, y) u^{(2,0)}(x, y) \, dx \, dy \\
\geq \int_{0}^{\omega_1} \int_{0}^{\omega_2} f_1(y, \Phi_p(u(x, y))) \left| u^{(1,0)}(x, y) \right|^2 \, dx \, dy.
\] (56)

In the same way, we show that

\[
- \int_{0}^{\omega_1} \int_{0}^{\omega_2} f_2(x, \Phi_p(u(x, y))) u(x, y) u^{(0,2)}(x, y) \, dx \, dy \\
\geq \int_{0}^{\omega_1} \int_{0}^{\omega_2} f_2(y, \Phi_p(u(x, y))) \left| u^{(0,1)}(x, y) \right|^2 \, dx \, dy.
\] (57)

Taking into account (52)–(57), from (51), we immediately get (50) with \( r_0 = (2 + \delta^{-1})r_1 \).

**Proof of Theorem 1.** Let \( v \in C^{1,1}_{\omega_1 \omega_2} (\mathbb{R}^2) \) be an arbitrary function. Set

\[
p_1[v](x, y) = f_1(y, \Phi_p(v)v(x, y)), \quad p_2[v](x, y) = f_2(x, \Phi_p(v)v(x, y)),
\]

\[
q[v](x, y) = f_0(x, y, \chi_p(v(x, y))) + \Phi_p(v)f(x, y, v, v^{(1,0)}(x, y), v^{(0,1)}(x, y), v^{(1,1)}(x, y)).
\] (58)

Consider the equation

\[
u^{(2,2)} = -\lambda u + p_1[v](x, y) u^{(2,0)} + p_2[v](x, y) u^{(0,2)} + \lambda \chi_p(v(x, y)) + q[v](x, y).
\] (59)

Note that due to definitions of \( p_1 \) and \( p_2 \) for every \( \rho > 0 \), there exists a continuous function \( \eta_\rho : [0, +\infty) \rightarrow [0, +\infty) \), \( \eta_\rho(0) = 0 \) such that

\[
| p_1[v](x_1, y_1) - p_2[v](x_2, y_2) | + | p_2[v](x_1, y_1) - p_2[v](x_2, y_2) | \\
\leq \eta_\rho(| x_1 - x_2 | + | y_1 - y_2 |).
\] (60)

By Lemma 3, there exist \( \lambda > 0 \) and \( M_1 > 0 \) depending on \( \rho \), \( \delta \), and the function \( \eta_\rho \) only, such that for every \( v \in C^{1,1}_{\omega_1 \omega_2} (\mathbb{R}^2) \), problem (59), (3) has a unique solution \( u[v] \) admitting the estimate

\[
\| u[v] \|_{C^{2,2}_{\omega_1 \omega_2}} \leq M_1(\| q[v] \|_{C^{1,1}_{\omega_1 \omega_2}} + \lambda \rho).
\] (61)

It is easy to see that the operator \( \mathcal{A} : v \mapsto u[v] \) is a continuous operator from \( C^{1,1}_{\omega_1 \omega_2} (\mathbb{R}^2) \) into \( C^{2,2}_{\omega_1 \omega_2} (\mathbb{R}^2) \), and therefore it is a completely continuous operator from \( C^{1,1}_{\omega_1 \omega_2} (\mathbb{R}^2) \) into \( C^{1,1}_{\omega_1 \omega_2} (\mathbb{R}^2) \). Moreover,

\[
\| \mathcal{A}(v) \|_{C^{1,1}_{\omega_1 \omega_2}} \leq M_1(\| q[v] \|_{C^{1,1}_{\omega_1 \omega_2}} + \lambda \rho) \leq M_1 \epsilon_\rho,
\] (62)

where \( \epsilon_\rho \) is a positive constant independent of \( v \).
By Schauder’s fixed point theorem, the operator \( A \) has a fixed point \( u \in C_{\omega_1,\omega_2}^{2,2}(\mathbb{R}^2) \), which is a solution of the functional differential equation (49).

By Lemma 5, \( u \) admits estimate (50). Conditions (13) and (50) imply the estimate

\[
\|u\|_{H_{\omega_1,\omega_2}^{1,1}} \leq r_1, \tag{63}
\]

where \( r_1 \) is a positive constant independent of \( \rho, \lambda, \) and \( u \). On the other hand, one can easily establish the inequalities

\[
\|u\|_{C_{\omega_1,\omega_2}} \leq \Omega \|u\|_{H_{\omega_1,\omega_2}^{1,1}}, \tag{64}
\]

\[
|u(x_1,y_1) - u(x_2,y_2)| \leq \Omega \|u\|_{H_{\omega_1,\omega_2}^{1,1}} \left( \sqrt{|x_1-x_2|} + \sqrt{|y_1-y_2|} \right), \tag{65}
\]

where

\[
\Omega = \frac{1}{\sqrt{\omega_1}} + \frac{1}{\sqrt{\omega_2}} + \frac{1}{\sqrt{\omega_1 \omega_2}} + \sqrt{\omega_1} + \sqrt{\omega_2}. \tag{66}
\]

Choosing \( \rho > \Omega r_1 \), we observe that \( u \) is a solution of the equation

\[
u^{(2,2)} = f_0(x,y,u) + f_1(y,\Phi_\rho(u)u)^{(2,0)} + f_2(x,\Phi_\rho(u))^{(0,2)} + \Phi_\rho(u)f(x,y,u)^{(1,0)}, \tag{70}\]

Due to (63) and (65), there exists a nondecreasing continuous function \( \eta : [0, +\infty) \rightarrow [0, +\infty) \), \( \eta(0) = 0 \) independent of \( \rho \) such that

\[
|f_1(y_1,\Phi_\rho(u)u(x_1,y_1)) - f_1(y_2,\Phi_\rho(u)u(x_2,y_2))| + |f_2(x_1,\Phi_\rho(u)u(x_1,y_1)) - f_2(x_2,\Phi_\rho(u)u(x_2,y_2))| \leq \eta(|x_1-x_2| + |y_1-y_2|). \tag{68}\]

By Lemma 1 and inequality (68), there exists a positive constant \( M \) independent of \( \rho \) such that \( u \) admits the estimate (30). Choosing \( \rho > \Omega(r_1 + M) \), we get that an arbitrary solution of problems (67), (3) satisfies the inequality

\[
\|u\|_{C_{[0,\omega_1,\omega_2]}^{1,1}} < \rho. \tag{69}\]

Consequently \( u \) is a solution of problem (1), (3) too. \( \square \)

We omit the proof of Theorem 2, since it can be proved in much the same way. The only difference is that instead of Lemmas 1–3 one should use Lemma 4 to get necessary a priori estimates.

**Proof of Theorem 3.** Let \( q \in C_{\omega_1,\omega_2}(\mathbb{R}^2) \). Consider the equation

\[
u^{(2,2)} = f_0(x,y,u) + (f_1(x,y)u^{(1,0)})^{(1,0)} + (f_2(x,y)u^{(0,1)})^{(0,1)} + q(x,y). \tag{70}\]

By Theorem 2, problems (70), (3) are solvable. Let \( u_1 \) and \( u_2 \) be two arbitrary solutions of problems (70), (3), and let \( v(x,y) = u_1(x,y) - u_2(x,y) \). Then applying (27), we easily
get the inequality
\[
\int_0^{\omega_1} \int_0^{\omega_2} \left( \delta v^2(x, y) + f_1(x, y) \left| v^{(1,0)}(x, y) \right|^2 + f_2(x, y) \left| v^{(0,1)}(x, y) \right|^2 \right) dx dy \leq 0. 
\] (71)

Hence it follows that \( u_1(x, y) \equiv u_2(x, y) \).

Thus for every \( q \in C_{\omega_1, \omega_2}(\mathbb{R}^2) \), problem (70), (3) has a unique solution \( u[q] \). Applying Lemmas 1 and 2, one can easily show that the operator \( \mathcal{A} : q \rightarrow u[q] \) is a continuous operator from \( C_{\omega_1, \omega_2}(\mathbb{R}^2) \) into \( C_{\omega_1, \omega_2}^{2,2}(\mathbb{R}^2) \) and that
\[
\| \mathcal{A}(q_1) - \mathcal{A}(q_2) \|_{C_{\omega_1, \omega_2}^{2,2}} \leq a \| q_1 - q_2 \|_{C_{\omega_1, \omega_2}}, \quad (72)
\]
where \( a \) is a positive constant independent of \( q_1 \) and \( q_2 \). Therefore problem (25), (3) is equivalent to the operator equation
\[
u(x, y) = \mathcal{A} \left( \varepsilon f \left( x, y, u, u^{(1,0)}, u^{(0,1)}, u^{(1,1)}, u^{(2,0)}, u^{(0,2)}, u^{(1,2)} \right) \right)(x, y) = \mathcal{B}_\varepsilon(u)(x, y). \quad (73)
\]

Due to Lipschitz continuity of the function \( f \), there exists a positive constant \( b \) such that
\[
\left| f(x, y, z_1, \ldots, z_8) - f(x, y, \bar{z}_1, \ldots, \bar{z}_8) \right| \leq b \sum_{i=1}^{8} | z_i - \bar{z}_i|. \quad (74)
\]

From (72) and (74), it is clear that for \( \varepsilon \in (-1/ab, 1/ab) \), the operator \( \mathcal{B}_\varepsilon \) is a contractive operator from \( C_{\omega_1, \omega_2}^{2,2}(\mathbb{R}^2) \) into \( C_{\omega_1, \omega_2}^{2,2}(\mathbb{R}^2) \). Hence (73), and consequently, problem (25), (3) is uniquely solvable. 

\[\square\]

References


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