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On Nonlinear Generalizations of the KdV and BBM Equations from Long Range Water Wave Theory

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Abstract

A generalization of the famous KdV and BBM equation are considered with a new nonlinear term. Sufficient conditions of solvability, existence and uniqueness are established.

Mathematics Subject Classification: 35G25, 35Q53

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1 Introduction

It is well known [3-4] that the equations

$$u_{xxx} = (u+1)u_x + u_t \tag{1}$$

and

$$u_{xxt} = (u+1)u_x + u_t \tag{2}$$

perform extremely well as mathematical models for long time evolution of wave phenomenon. However, in the mathematical modeling leading up to these equations many of the physical conditions are either overlooked or drastically simplified to be taken as constant. If these physical conditions are not simplified then the above equations may not be valid. If this is the case then the modeling process must be redone, most likely leading to a completely new partial different equation.

For a simple illustration of how this process occurs one can recall the well known development of the standard partial differential equation for heat conduction. In the mathematical modeling of this phenomena if the physical assumptions allow the density, μ, specific heat, ρ, and the thermal conductivity, K, of the medium under consideration to all be taken as constant, then one obtains the well know partial differential for heat conduction $u_t = c^2 u_{xx}$, where $c^2 = \frac{K}{\mu\rho}$ = constant. However, if any one of the assumptions are violated then the above heat equation is not going to suffice as a mathematical model for the heat conduction within the medium. For example, if density and specific heat are allowed to be constant but the coefficient of thermal conductivity is taken as a function K of the spacial variable, then the equation for the heat distribution under consideration takes the form $\mu\rho u_t = K u_{xx} + K_x u_x$.

The modeling process of longtime evolution of wave phenomenon follows a very similar process, but it is a little more in depth on the physical side, thus, it will not be discussed in detail here. However, in short one can see that if certain physical conditions are either not simplified to be constant or if additional forces are added into the modeling picture then mathematical model for longtime evolution of wave phenomenon will generally take the general form of the above equation (1) or (2), but it will most likely have some additional terms. It is expected that the new equation will be an equation of the form

$$u_{xxt} = (u+1)u_x + u_t + f(x, t, u, u_x, u_{xx}) \tag{3}$$

and

$$u_{xxx} = (u+1)u_x + u_t + f(x, t, u, u_x, u_{xx}) \tag{4}$$

where the term $f(x, t, u, u_x, u_{xx})$ is left arbitrary, but is understood to be a direct result of the change in the physical modeling as detailed above. Various examples of this equation have been investigated in [5] but not necessarily solved nor has existence of solutions been guaranteed. For example in [6] recent work the equation

$$u_{xxt} = (u+1)u_x + u_t + c u_{xx} \tag{5}$$

was investigated as a mathematical model for long time water wave theory when viscosity, c, was considered in the mathematical modeling of long range water waves. In the following pages we will consider several generalizations of both the BBM (3)and KdV (4) type equations and develop conditions for existence, uniqueness and the like.

2 statement of results

Theorem 2.1 *Let g(x) be a continuous function such that*

$$\sup_{x \in R} |g(x)| \leq b < \infty$$

then there exists a $t_0(b)$ *such that the initial value problem*

$$u_{xxt} = (u+1)u_x + u_t + f(x,t,u,u_x,u_{xx}), u(x,0) = g(x) \tag{3}$$

has a unique solution defined in $0 < t < t_0$ *provided that the nonlinear function* $F(x,t,z_1,z_2,z_3)$ *satisfies the conditions*

$$|F(z_i) - F(\bar{z}_i)| \le L_i |z_i - \bar{z}_i|$$

for $i = 1,2,3$ *where* $F = \frac{df}{dx}$.

Example 2.2 *Let g(x) be a continuous function such that*

$$\sup_{x \in R} |g(x)| \le b < \infty$$

then there exists a $t_0(b)$ *such that the initial value problem*

$$u_{xxt} = (u+1)u_x + u_t + cu_{xx}, u(x,0) = g(x) \tag{5}$$

has a unique solution defined in $0 < t \le t_0, 0 < x \le R$ *provided that the value of c satisfies the inequality*

$$(1 + c + R)t_0 \le 1.$$

Theorem 2.3 *Let* $g(\xi)$ *be a continuous function such that*

$$\sup_{\xi \in R} |g(x)| \le b < \infty$$

then there exists a $\tau_0(b)$ *such that the initial value problem*

$$u_{xxx} = (u+1)u_x + u_t + f(x,t,u,u_x,u_{xx}), u(x,0) = g(x) \tag{4}$$

$$u(x,0) = g(x)$$

has a unique solution defined in the transformed coordinate system $x = \epsilon^{\frac{1}{2}}\tau$ *and* $t = \epsilon^{\frac{3}{2}}\xi + \epsilon^{\frac{1}{2}}\tau$. *This solution is defined for* $0 < \tau < \tau_0$ *provided that the nonlinear function* $F(\xi,\tau,z_1,z_2,z_3)$ *satisfies the conditions*

$$|F(z_i) - F(\bar{z}_i)| \le L_i |z_i - \bar{z}_i|$$

for $i = 1,2,3$ *where* $F = \frac{df}{d\xi}$.

3 Proofs and auxiliary statements

In the following Theorems we will be considering solutions to partial differential equations of the from

$$u_{xxt} = (u+1)u_x + u_t + f(x,t,u,u_x,u_{xx}) \tag{3}$$

and

$$u_{xxx} = (u+1)u_x + u_t + f(x,t,u,u_x,u_{xx}). \tag{4}$$

Both of the above equations will be taken with the initial conditions $u(x,0) = g(x)$, considered for a class of real nonperiodic functions $u(x,t)$ defined for $-\infty < x < +\infty, t \geq 0$.

Lemma 3.1 *The partial differential equation (3) can be as*

$$[(1-\partial_x^2)]u^{(0,1)} = -\partial_x[u + \frac{1}{2}u^2 - F(x,t,u(x,t),u_x,u_{xx})],$$

or as an integral equation

$$u(x,t) =$$

$$\int_0^t \int_{-\infty}^{+\infty} K(x-\xi)(u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F(\xi,\tau,u(\xi,\tau),u_\xi(\xi,\tau))d\xi d\tau,$$

where F is a function such that $\frac{\partial F}{\partial x} = f$.

proof of Lemma 3.1:
To begin we note that (3) can be rewritten as

$$u^{(0,1)} - u^{(2,1)} = f - u^{(1,0)} - u^{(1,0)}u.$$

We then see that equation (3) can be rewritten as

$$[(1-\partial_x^2)]u^{(0,1)} = -\partial_x[u + \frac{1}{2}u^2 - F],$$

where F is a functions such that $F = \frac{df}{dx}$. Following a similar argument as in [3] one views the above as a differential equation for $u^{(0,1)}$; hence, one obtains that

$$u^{(0,1)} = \int_{-\infty}^{+\infty} K(x-\xi)(u(\xi,t) + \frac{1}{2}u^2(\xi,t) - F(\xi,t))d\xi,$$

where the Kernel is defined as $K(x) = \frac{1}{2}(sgnx)e^{-|x|}$.

Now, the above pseudo differential equation can easily be rewritten as an integral equation as

$$u(x,y) = g(x) + \int_0^t \int_{-\infty}^{+\infty} K(x-\xi)(u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F(\xi,\tau))d\xi d\tau,$$

where $g(x) = u(x,0)$. This complete the proof of Lemma 1.

proof of Theorem 2.1:

We denote ξ_{t_0} as the set of functions that continuous and uniformly bounded on the strip $[0,R] \times t_0$, that have the norm $||u|| = \sup_{x\in R, 0\leq t\leq t_0} |u^{(i,0)}|$ with $i = 0,1,2$.

Now, let us define A as the integral operator, as in Lemma 1,

$$A[u] =$$

$$\int_0^t \int_{-\infty}^{+\infty} K(x-\xi)(u(\xi,\tau) + \frac{1}{2}u^2(\xi,\tau) - F(\xi,\tau,u(\xi,\tau),u_\xi(\xi,\tau),u_{\xi\xi}(\xi,\tau))d\xi d\tau$$

and view our partial differential equation (3) as an operator equation $u = g(x) + A[u]$. Prior to proceeding with the usual fixed point argument we must observe that

$$F(\xi,\tau,v_1,\frac{\partial v_1}{\partial \xi},\frac{\partial^2 v_1}{\partial \xi^2}) - F(\xi,\tau,v_2,\frac{\partial v_2}{\partial \xi},\frac{\partial^2 v_2}{\partial \xi^2})$$

$$= [F(\xi,\tau,v_1,\frac{\partial v_1}{\partial \xi},\frac{\partial^2 v_1}{\partial \xi^2}) - F(\xi,\tau,v_2,\frac{\partial v_1}{\partial \xi},\frac{\partial^2 v_1}{\partial \xi^2})]+$$

$$+[F(\xi,\tau,v_2,\frac{\partial v_1}{\partial \xi},\frac{\partial^2 v_1}{\partial \xi^2}) - F(\xi,\tau,v_2,\frac{\partial v_2}{\partial \xi},\frac{\partial^2 v_1}{\partial \xi^2})]$$

$$+[F(\xi,\tau,v_2,\frac{\partial v_2}{\partial \xi},\frac{\partial^2 v_1}{\partial \xi^2}) - F(\xi,\tau,v_2,\frac{\partial v_2}{\partial \xi},\frac{\partial^2 v_2}{\partial \xi^2})]$$

Now, applying the Lipschitz condition on the function $F(x,y,u,u_x,u_{xx})$ in u for the first square parentheses, then in u_x for the second and in u_{xx} for the third we obtain

$$|F(\xi,\tau,v_1,\frac{\partial v_1}{\partial \xi},\frac{\partial^2 v_1}{\partial \xi^2}) - F(\xi,\tau,v_2,\frac{\partial v_2}{\partial \xi},\frac{\partial^2 v_2}{\partial \xi^2})|$$

$$\leq L_1|v_1 - v_2| + L_2|\frac{\partial v_1}{\partial \xi} - \frac{\partial v_2}{\partial \xi}| + L_3|\frac{\partial^2 v_1}{\partial \xi^2} - \frac{\partial^2 v_2}{\partial \xi^2}|. \tag{6}$$

Recalling the integral operator A and considering the difference of two solutions v_1 and v_2 of our partial differential equation (3) we obtain that:

$$|A[v_1]-A[v_2]| = \int_0^t \int_{-\infty}^{+\infty} e^{-|x-\xi|}(|v_1(\xi,\tau)|-v_2(\xi,\tau)+\frac{1}{2}|v_1(\xi,\tau)^2-v_2(\xi,\tau)^2|)d\xi d\tau$$

$$+\int_0^t \int_{-\infty}^{+\infty} \frac{1}{2}e^{|x-\xi|}|f(\xi,\tau,v_1,\frac{\partial v_1}{\partial \xi},\frac{\partial^2 v_1}{\partial \xi^2}) - f(\xi,\tau,v_2,\frac{\partial v_2}{\partial \xi},\frac{\partial^2 v_2}{\partial \xi^2})|d\xi d\tau$$

$$= \int_0^t \int_{-\infty}^{+\infty} e^{-|x-\xi|}(|v_1(\xi,\tau)| - v_2(\xi,\tau) + \frac{1}{2}|v_1(\xi,\tau)^2 - v_2(\xi,\tau)^2|)d\xi d\tau$$

$$+\int_0^t \int_{-\infty}^{+\infty} \frac{1}{2}e^{|x-\xi|}|f(\xi,\tau,v_1,\frac{\partial v_1}{\partial \xi},\frac{\partial^2 v_1}{\partial \xi^2}) - f(\xi,\tau,v_2,\frac{\partial v_2}{\partial \xi},\frac{\partial^2 v_2}{\partial \xi^2})|d\xi d\tau.$$

Now, applying (6) we obtain that

$$|A[v_1] - A[v_2]| =$$

$$\leq \int_0^t \int_{-\infty}^{+\infty} e^{-|x-\xi|}(|v_1 - v_2| + \frac{1}{2}(|v_1| + |v_2||v_1 - v_2|)d\xi d\tau$$

$$+\int_0^t \int_{-\infty}^{+\infty} \frac{1}{2}e^{|x-\xi|}(L_1|v_1 - v_2| + L_2|\frac{\partial v_1}{\partial \xi} - \frac{\partial v_2}{\partial \xi}| + L_3|\frac{\partial^2 v_1}{\partial \xi^2} - \frac{\partial^2 v_2}{\partial \xi^2}|)d\xi d\tau$$

$$= (1+R)t|v_1 - v_2| + Lt||v_1 - v_2||,$$

where $||f|| = \sum |f^i|$, L= max L_i with $i = 1,2,3$ and R is a previously defined. Hence, we have obtained that

$$|A[v_1] - A[v_2]| \leq (1+R)t|v_1 - v_2| + L||v_1 - v_2||. \tag{7}$$

Now, we compute that

$$A[\frac{\partial v_1}{\partial x}] = \frac{\partial}{\partial x}[\int_0^t \int_{-\infty}^{+\infty} e^{-|x-\xi|}(v_1(\xi,\tau)+\frac{1}{2}v_1(\xi,\tau)^2)+\frac{1}{2}e^{|x-\xi|}f(\xi,\tau,v_1,\frac{\partial v_1}{\partial \xi},\frac{\partial^2 v_1}{\partial \xi^2})d\xi d\tau]$$

$$= \int_0^t \int_{-\infty}^{+\infty} sgn(x)(-e^{-|x-\xi|}(v_1(\xi,\tau)+\frac{1}{2}v_1(\xi,\tau)^2)+\frac{1}{2}e^{|x-\xi|}f(\xi,\tau,v_1,\frac{\partial v_1}{\partial \xi},\frac{\partial^2 v_1}{\partial \xi^2}))d\xi d\tau.$$

And, performing a similar calculation for $A[\frac{\partial v_2}{\partial x}]$ and, after some algebra, we obtain

$$A[\frac{\partial v_1}{\partial x}] - A[\frac{\partial v_1}{\partial x}]| \leq sgn(x)(-1 - \frac{1}{2}|v_1| - \frac{1}{2}|v_2|)t|v_1 - v_2| + Lt||v_1 - v_2||.$$

Which yields, in the range $x \geq 0$

$$|A[\frac{\partial v_1}{\partial x}] - A[\frac{\partial v_1}{\partial x}]| \leq (-1 - R)|v_1 - v_2| + Lt||v_1 - v_2||,$$

or, in the range $x < 0$

$$|A[\frac{\partial v_1}{\partial x}] - A[\frac{\partial v_1}{\partial x}]| \leq (1 + R)|v_1 - v_2| + L||v_1 - v_2||.$$

Hence, calling $K = (-1 - R)$ if we are in $x \geq 0$ or $K = (1 + R)$ if we are in $x < 0$ we have obtained

$$|A[\frac{\partial v_1}{\partial x}] - A[\frac{\partial v_2}{\partial x}]| \leq (K)t|v_1 - v_2| + L||v_1 - v_2||. \tag{8}$$

We perform a similar calculation for $A[\frac{\partial^2 u}{\partial x^2}]$ and proceed in the same manner we obtain

$$|A[\frac{\partial^2 v_1}{\partial x^2}] - A[\frac{\partial^2 v_2}{\partial x^2}]| \leq (1 + R)t|v_1 - v_2| + L||v_1 - v_2||. \tag{9}$$

Thus, combining equations (7),(8) and (9) we obtain

$$|A[v_1] - A[v_2]| + |A[\frac{\partial v_1}{\partial x}] - A[\frac{\partial v_2}{\partial x}]| + |A[\frac{\partial^2 v_1}{\partial x^2}] - A[\frac{\partial^2 v_2}{\partial x^2}]|$$

$$\leq (1+R)t|v_1-v_2|+L||v_1-v_2||+(K)t|v_1-v_2|+L||v_1-v_2||+(1+R)t|v_1-v_2|+L||v_1-v_2||.$$

Which is the same as

$$||A[v_1 - v_2]||$$

$$\leq (2 + 2R + K)t|v_1 - v_2| + 3L||v_1 - v_2||.$$

And, noting that $|v_1 - v_2| \leq ||v_1 - v_2||$, we obtain the inequality

$$||v_1 - v_2|| \leq \bar{L}||v_1 - v_2|| \tag{10}$$

where $\bar{L} = (2 + 2R + K + 3L)$. Hence, we have obtained the necessary inequality for the standard Fixed point argument, thus, completed the proof of this theorem. However, it is very important to determine the exact value of the constant $\bar{L}$ due to the fact that in order to apply the standard Banach's contraction principle [2] there will be certain requirements, often $\bar{L} \leq 1$, required in the necessary equation $d(F(u), F(v)|| \leq \bar{L}d(u, v)$, with d being the metric begin considered. It will be illustrated in the next example of how the value of K can be obtained for particular partial differential equations under consideration and how this is interpreted for the intervals where the solution is defined.

proof of Example 2.2:

The partial differential equation

$$u_{xxt} = (u + 1)u_x + u_t + cu_{xx} \tag{5}$$

$$u(x, 0) = g(x)$$

is a special case of (3) with $f = cu_{xx}$ and $F = cu_x$. Thus, one can see that the function f does indeed satisfy the condition $|F(z_i) - F(\bar{z}_i)| \leq L|z_i - \bar{z}_i|$ for Theorem 1. Namely, $|F(z_i) - F(\bar{z}_i)| = 0$ for $i = 0$ and $i = 2$ and $|F(z_i) - F(\bar{z}_i)| = c|z_i) - \bar{z}_i|$ for $i = 1$. Thus, the results of Theorem 1 do establish that a the partial differential equation (5) does have a unique solution. However, in order to obtain further insight into the solution, it is necessary to find the exact value of the constant K applied in the standard fixed point argument, as noted at the end of the proof of Theorem 1. To do this we note that the equation yields the value that $\bar{L} = 1 + R + c$ for our example. Thus, in order for one to satisfy the conditions required for Banach's contraction principle it follows that the constant $K = (1 + R + c)t_0$ must be bounded by one. Hence, it follows that $t_0 \leq \frac{1}{1+R+c}$ which gives the local existence as expected and standard continuation arguments can be applied to gain global existence. Further details of this are discussed for (11) in a recent paper [6] which focused on the physical applications and interpretations of the partial differential equation (5).

Lemma 3.2 *The partial differential equation*

$$u^{(0,1)} + u^{(1,0)}u + u^{(3,0)} - \epsilon u^{(2,1)} = u^{(2,0)} \tag{11}$$

can be rewritten as

$$u_\xi + u_\tau + u_\xi u - u_{\xi\xi\tau} = 0 \tag{12}$$

if the change of variables $\xi = \epsilon^{-\frac{1}{2}}x + \epsilon^{-\frac{3}{2}}t$ *and* $\tau = \epsilon^{-\frac{1}{2}}x =$ *are applied.*

proof of Lemma 3.2

To begin we introduce the change the independent variables $\xi = \epsilon^k x + \epsilon^R t$ and $\tau = \epsilon^k t$. And, applying the standard chain rule we observe that

$$u^{(0,1)} = \frac{\partial u}{\partial \xi}\epsilon^R + \frac{\partial u}{\partial \tau}\epsilon^k, u^{(2,0)} = \frac{\partial^2 u}{\partial \xi^2}\epsilon^{2k}, u^{(3,0)} = \frac{\partial^3 u}{\partial \xi^3}\epsilon^{3k}, u^{(2,1)} = \frac{\partial^3 u}{\partial \xi^3}\cdot\epsilon^{2K+R} + \frac{\partial^3 u}{\partial \xi^2 \partial \tau}\cdot\epsilon^{3k}.$$

Plugging these values into (11) we observe that our partial differential equation becomes

$$\epsilon^R u_\xi + \epsilon^k u_\tau + \epsilon^k u_\xi u + \epsilon^{3k} u_{\xi\xi\xi} - \epsilon^{2k+R+1} u_{\xi\xi\xi} - \epsilon^{3k+1} u_{\xi\xi\tau} = 0.$$

Now, if we force $k = R + 1$ the partial differential equation takes the desired form

$$\epsilon^k(\epsilon u_\xi + u_\tau + u_\xi u - \epsilon^{2k+1} u_{\xi\xi\tau}) = 0.$$

Finally, multiplying by $\epsilon^{\frac{1}{2}}$ and selecting $k = -\frac{1}{2}$ we see that our partial differential equation has become exactly equation (12) as stated in the Lemma, hence, this complete the proof of Lemma 3.2.

proof of Theorem 2.3

As it was shown in Theorem 2.1 the partial differential equation

$$u_{xxt} = (u+1)u_x + u_t + f(x, t, u, u_x, u_{xx}), u(x, 0) = g(x)$$

has a unique solution expressed by the integral equation in Lemma 3.1 that is defined in $0 \le t \le t_0$ where t_0 is ultimately determined by the Lipshitz constant from F with $\frac{dF}{dx} = f$. Moreover, as it was shown in Lemma 3.2 the partial differential equation (11) can be transformed into

$$u_\xi + u_\tau + u_\xi u - u_{\xi\xi\tau} = 0 \tag{12}$$

where $\xi = \epsilon^{-\frac{1}{2}} x + \epsilon^{-\frac{3}{2}} t$ and $\tau = \epsilon^{-\frac{1}{2}}$.

In order to obtain insight of the solution to our desired patrial differential equation

$$u_{xxx} = (u+1)u_x + u_t + f(x, t, u, u_x, u_{xx}), u(x, 0) = g(x) \tag{5}$$

we first investigate the equation

$$u_{xxx} - \epsilon u_{xxt} = (u+1)u_x + u_t + f(x, t, u, u_x, u_{xx}), u(x, 0) = g(x).$$

Now, if we apply the transformation outlined in Lemma 3.2 to the above partial differential equation it will become.

$$u_{\xi\xi\tau} = \epsilon u_{\xi} + u_{\xi} + u_{\tau} + \bar{f}(x, t, u, u_{\xi}, u_{\xi\xi}),$$

where $\bar{f}$ is the function f from (4) after the ξ, τ transformation has been applied. Now, the logic outlined in the proof of theorem 2.1 shows that the above partial differential equation does have a unique solution. Thus, by inverting the transformation we can assure that our equation (12) does have a unique solution. Then, by taking the limit as $\epsilon \to 0$ we obtain that the partial differential equation (5) has a unique solution and this completes the proof of Theorem 2.3.

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