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Lorentz-Violating Electrostatics and Magnetostatics

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The static limit of Lorentz-violating electrodynamics in vacuum and in media is investigated. Features of the general solutions include the need for unconventional boundary conditions and the mixing of electrostatic and magnetostatic effects. Explicit solutions are provided for some simple cases. Electromagnetostatics experiments show promise for improving existing sensitivities to parity-odd coefficients for Lorentz violation in the photon sector.

I. INTRODUCTION

Since its inception, relativity and its underlying Lorentz symmetry have been intimately linked to classical electrodynamics. A century after Einstein, high-sensitivity experiments based on electromagnetic phenomena remain popular as tests of relativity. At present, many of these experiments are focused on the ongoing search for minuscule violations of Lorentz invariance that might arise in the context of an underlying unified theory at the Planck scale [1].

Much of the work involving relativity tests with electrodynamics has focused on the properties of electromagnetic waves, either in the form of radiation or in resonant cavities. Modern versions of the Michelson-Morley and Kennedy-Thorndike experiments using resonant cavities are among the best laboratory tests for relativity violations [2, 3, 4], while spectropolarimetric studies of cosmological birefringence currently offer the most sensitive measures of Lorentz symmetry in any system [5, 6]. However, the presence of Lorentz violation in nature would also affect other aspects of electrodynamics. Our primary goal in this work is to initiate the study of Lorentz-violating effects in electrostatics and magnetostatics. We find a variety of intriguing effects, the more striking of which may make feasible novel experimental tests attaining exceptional sensitivities to certain types of relativity violations.

The analysis in this work is performed within the framework of the Standard-Model Extension (SME) [7, 8], which enlarges general relativity and the Standard Model (SM) to include small arbitrary violations of Lorentz and CPT symmetry. The full lagrangian of the SME can be viewed as an effective field theory for gravitational and SM fields that incorporates all terms invariant under observer general coordinate and local Lorentz transformations. Terms having coupling coefficients with Lorentz indices control the Lorentz violation, and they could emerge as low-energy remnants of the underlying physics at the Planck scale [9]. Experimental tests of the SME performed to date include ones with photons [2, 3, 4, 5, 6], electrons [10, 11, 12], protons and neutrons [13, 14], mesons [15], and muons [16], while interesting possibilities exists for neutrinos [17] and the Higgs [18].

In the present work, we limit attention to the sector of the minimal SME comprising classical Lorentz-violating electrodynamics in Minkowski spacetime, coupled to an arbitrary 4-current source. There exists a substantial theoretical literature discussing the electrodynamics limit of the SME [19], but the stationary limit remains unexplored to date. We begin by providing some general information about this theory, including some aspects associated with macroscopic media. We then adapt Green-function techniques to obtain the general solution for the 4-potential in the electrostatics and magnetostatics limit. Among the associated unconventional effects is a mixing of electric and magnetic phenomena that is characteristic of Lorentz violation. For field configurations in Lorentz-violating electromagnetostatics, we show that the usual Dirichlet or Neumann boundary conditions are replaced by four natural classes of boundary conditions, a result also reflecting this mixing.

As one application, we obtain the Lorentz-violating 4-potential due to a stationary point charge. The usual radial electrostatic field is corrected by small Lorentz-violating terms, and a small nonzero magnetostatic field also emerges. Another solution presented here describes a nonzero scalar potential arising inside a conducting shell due to a purely magnetostatic source placed within the shell. This configuration appears well suited as the basis for a high-sensitivity experiment that would seek certain nonzero parity-breaking effects in Lorentz-violating electrodynamics. We discuss some aspects of an idealized experiment of this type, including the use of rotations and boosts to extract the signal. Finally, the appendix resolves some basic issues associated with coordinate redefinitions in the illustrative case of a classical charged point particle. Throughout this work, we adopt the conventions of Refs. [5, 7].

II. FRAMEWORK

A. Vacuum electrodynamics

The lagrangian density for the photon sector of the minimal SME can be written as

$$\begin{aligned}\mathcal{L} \;=\; & -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{4}(k_F)_{\kappa\lambda\mu\nu}F^{\kappa\lambda}F^{\mu\nu} \\ & +\tfrac{1}{2}(k_{AF})^{\kappa}\epsilon_{\kappa\lambda\mu\nu}A^{\lambda}F^{\mu\nu} - j^{\mu}A_{\mu}. \end{aligned} \qquad (1)$$

In this equation, $j^\mu = (\rho, \vec{J})$ is the 4-vector current source that couples to the electromagnetic 4-potential A_μ, and

$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, which satisfies the homogeneous equations

$$\epsilon^{\mu\nu\kappa\lambda}\partial_\mu F_{\kappa\lambda} = 0 \tag{2}$$

ensuring the U(1) gauge invariance of the action. The coefficients $(k_F)_{\kappa\lambda\mu\nu}$ and $(k_{AF})^\kappa$ control the Lorentz violation and are expected to be small.

To focus the analysis, some simplifying assumptions are adopted in what follows. Since our primary interest here is electromagnetostatics, we take the current j_μ to be conventional. Lorentz violation is then present only in the photon sector, and the Lorentz force is conventional. This limit is less restrictive than might first appear, since suitable coordinate redefinitions can move some of the Lorentz-violating effects into the matter sector without changing the physics. An explicit discussion of this issue for the case of a point charge is given in the appendix.

In the simplest scenarios for Lorentz violation the coefficients $(k_F)_{\kappa\lambda\mu\nu}$ and $(k_{AF})^\kappa$ are constant, so that energy and momentum are conserved. We adopt this assumption here. Variation of the lagrangian (1) then yields the inhomogeneous equations of motion

$$\partial_\alpha F_\mu{}^\alpha + (k_F)_{\mu\alpha\beta\gamma}\partial^\alpha F^{\beta\gamma} + (k_{AF})^\alpha \epsilon_{\mu\alpha\beta\gamma}F^{\beta\gamma} + j_\mu = 0, \tag{3}$$

which extend the usual covariant Maxwell equations to incorporate Lorentz violation. Although outside our present scope, a treatment allowing nonconservation of energy-momentum would be of interest.

Some of the analysis of this theory is simplified by introducing certain convenient linear combinations of the coefficients $(k_F)_{\kappa\lambda\mu\nu}$ for Lorentz violation. One useful set is given by [5]

$$\begin{aligned}
(\kappa_{DE})^{jk} &= -2(k_F)^{0j0k}, \\
(\kappa_{HB})^{jk} &= \tfrac{1}{2}\epsilon^{jpq}\epsilon^{krs}(k_F)^{pqrs}, \\
(\kappa_{DB})^{jk} &= -(\kappa_{HE})^{kj} = (k_F)^{0jpq}\epsilon^{kpq}.
\end{aligned} \tag{4}$$

As an immediate application of these definitions, the microscopic equations (3) for Lorentz-violating electrodynamics *in vacuo* with $(k_{AF})^\kappa = 0$ can be cast in the form of the Maxwell equations for macroscopic media,

$$\begin{aligned}
\vec{\nabla}\cdot\vec{D} = \rho, \qquad & \vec{\nabla}\times\vec{H} - \partial_0\vec{D} = \vec{J}, \\
\vec{\nabla}\cdot\vec{B} = 0, \qquad & \vec{\nabla}\times\vec{E} + \partial_0\vec{B} = 0,
\end{aligned} \tag{5}$$

by adopting the definitions

$$\begin{aligned}
\vec{D} &\equiv (1+\kappa_{DE})\cdot\vec{E} + \kappa_{DB}\cdot\vec{B}, \\
\vec{H} &\equiv (1+\kappa_{HB})\cdot\vec{B} + \kappa_{HE}\cdot\vec{E},
\end{aligned} \tag{6}$$

which hold in vacuum.

Another useful set of combinations is [5]

$$\begin{aligned}
(\tilde{\kappa}_{e+})^{jk} &= \tfrac{1}{2}(\kappa_{DE}+\kappa_{HB})^{jk}, \\
(\tilde{\kappa}_{e-})^{jk} &= \tfrac{1}{2}(\kappa_{DE}-\kappa_{HB})^{jk} - \tfrac{1}{3}\delta^{jk}(\kappa_{DE})^{ll}, \\
(\tilde{\kappa}_{o+})^{jk} &= \tfrac{1}{2}(\kappa_{DB}+\kappa_{HE})^{jk}, \\
(\tilde{\kappa}_{o-})^{jk} &= \tfrac{1}{2}(\kappa_{DB}-\kappa_{HE})^{jk}, \\
\tilde{\kappa}_{\rm tr} &= \tfrac{1}{3}(\kappa_{DE})^{ll}.
\end{aligned} \tag{7}$$

This set is of particular relevance for certain experimental considerations. For example, if $(k_{AF})^\kappa = 0$ then birefringence induced by Lorentz violation is controlled entirely by the 10 coefficients $(\tilde{\kappa}_{e+})^{jk}$ and $(\tilde{\kappa}_{o-})^{jk}$.

Experiments constrain part of the space of coefficients for Lorentz violation [5, 6]. The CPT-odd coefficients $(k_{AF})^\kappa$ are stringently bounded by cosmological observations and are set to zero throughout this work. Spectropolarimetry of cosmologically distant sources limits the 10 combinations $(\tilde{\kappa}_{e+})^{jk}$ and $(\tilde{\kappa}_{o-})^{jk}$ to values below parts in 10^{32}. The remaining 9 linearly independent components of $(k_F)_{\kappa\lambda\mu\nu}$ are accessible in laboratory experiments [2, 3, 4], with the best sensitivity achieved to date on some components of $(\tilde{\kappa}_{e-})^{jk}$ at the level of about 10^{-15}.

For some of what follows, it is useful to consider the limit of the theory (1) in which the 10 coefficients $(\tilde{\kappa}_{e+})^{jk}$ and $(\tilde{\kappa}_{o-})^{jk}$ are set identically to zero. The pure-photon part of the lagrangian (1) then can be written as

$$\mathcal{L} = -\tfrac{1}{4}F^{\kappa\lambda}F^{\mu\nu}[\eta_{\kappa\mu}\eta_{\lambda\nu} + \eta_{\kappa\mu}(k_F)^\alpha{}_{\lambda\alpha\nu} + \eta_{\lambda\nu}(k_F)^\alpha{}_{\kappa\alpha\mu}]. \tag{8}$$

In this limit, corresponding simplifications occur in the combinations (4). For example, $(\kappa_{DB})^{jk}$ becomes an antisymmetric matrix.

B. Electrodynamics in media

The analysis of electrodynamics in ponderable media could in principle proceed by supplying a four-current density that describes the microscopic charge and current distributions in detail. However, as usual, it is more practical to adopt an averaging process.

Following standard techniques [21], we average the exact Maxwell equations (5) with the microscopic charge and current densities over elemental spatial regions. We introduce the usual notions of polarization $\vec{P}$ and magnetization $\vec{M}$ in terms of averaged molecular electric and magnetic dipole moments,

$$\begin{aligned}
\vec{P} &= \left\langle \sum_n \vec{p}_n\, \delta^3(\vec{x}-\vec{x}_n) \right\rangle, \\
\vec{M} &= \left\langle \sum_n \vec{m}_n\, \delta^3(\vec{x}-\vec{x}_n) \right\rangle.
\end{aligned} \tag{9}$$

The braces here denote a smooth spatial average over many molecules, and the sums are over each type n of molecule in the given material. For present purposes these multipoles suffice, and the higher-order microscopic moments arising from the averaging process can be ignored.

With these definitions, the analogue displacement field $\vec{D}$ and analogue magnetic field $\vec{H}$ appearing in the vacuum equations (5) are replaced with the macroscopic displacement field $\vec{D}_{\rm matter}$ and macroscopic magnetic field

$\vec{H}_{\rm matter}$, defined by

$$\begin{aligned}\vec{D}_{\rm matter} &= (1+\kappa_{DE})\cdot\vec{E}+\kappa_{DB}\cdot\vec{B}+\vec{P},\\ \vec{H}_{\rm matter} &= (1+\kappa_{HB})\cdot\vec{B}+\kappa_{HE}\cdot\vec{E}-\vec{M}.\end{aligned}\tag{10}$$

The inhomogeneous Maxwell equations in macroscopic media in the presence of Lorentz violation then still take the form (5).

The above formulation shows that, unlike the Lorentz-symmetric case, a scalar dielectric permittivity and a scalar magnetic permeability alone may be insufficient to describe the linear response of a simple material to applied electric and magnetic fields. Instead, we must allow for the existence of components of the induced moments that are orthogonal to the applied field, induced by atomic-structure modifications from the Lorentz violation. The nature of the response may vary for different materials. For homogeneous materials, the unconventional response can be described by the constituency relations

$$\begin{aligned}\vec{D}_{\rm matter} &= (\epsilon+\kappa_{DE}^{\rm vacuum}+\kappa_{DE}^{\rm matter})\cdot\vec{E}\\ &\quad +(\kappa_{DB}^{\rm vacuum}+\kappa_{DB}^{\rm matter})\cdot\vec{B},\\ \vec{H}_{\rm matter} &= (\tfrac{1}{\mu}+\kappa_{HB}^{\rm vacuum}+\kappa_{HB}^{\rm matter})\cdot\vec{B}\\ &\quad +(\kappa_{HE}^{\rm vacuum}+\kappa_{HE}^{\rm matter})\cdot\vec{E}.\end{aligned}\tag{11}$$

Here, the permittivity ϵ and the permeability μ are understood to be those in the absence of Lorentz violation, the coefficients $\kappa^{\rm vacuum}$ are those discussed in the previous subsection, and the coefficients $\kappa^{\rm matter}$ contain the pieces of the induced moments $\vec{P}$ and $\vec{M}$ that are leading-order in $(k_F)_{\kappa\lambda\mu\nu}$ and that may be partially or wholly orthogonal to the applied fields.

The explicit form of the coefficients $\kappa^{\rm matter}$ depends on the macroscopic medium. Indeed, unless the matter is isotropic, their values can depend on the orientation of the material. Applying the averaging process to an appropriate atomic or molecular model based on the SME could establish their form and represents an interesting open problem. Note that, in analyzing an experiment, it may be insufficient merely to replace expressions involving vacuum coefficients with ones involving the sum of vacuum and matter coefficients because the boundary conditions in the presence of matter may induce further modifications. An explicit example of this is given in subsection IV B.

In the remainder of the paper, coefficients without labels are understood to be those in the vacuum. The relevant matter coefficients are explicitly labeled as $\kappa^{\rm matter}$.

III. ELECTROMAGNETOSTATICS

In this section, we consider stationary solutions of the theory (3). The stationary fields *in vacuo* satisfy the time-independent equation of motion

$$\tilde{k}^{j\mu k\nu}\partial_j\partial_k A_\nu(\vec{x}) = j^\mu(\vec{x}).\tag{12}$$

In this equation, the coefficients $\tilde{k}^{j\mu k\nu}$ are defined by

$$\tilde{k}^{j\mu k\nu} = \eta^{jk}\eta^{\mu\nu}-\eta^{\mu k}\eta^{\nu j}+2(k_F)^{j\mu k\nu}.\tag{13}$$

From Eq. (2), the electrostatic and magnetostatic fields can be written in terms of a 4-potential $A^\lambda=(\Phi,A^j)$ according to $\vec{E}=-\vec{\nabla}\Phi$ and $\vec{B}=\vec{\nabla}\times\vec{A}$, as usual.

The appearance of both $\vec{E}$ and $\vec{B}$ in $\vec{D}$ and in $\vec{H}$, which occurs even in the vacuum (cf. Eq. (6)), means that in the presence of Lorentz violation a static charge density generates both an electrostatic and a (suppressed) magnetostatic field, and similarly a steady-state current density generates both a magnetostatic and a (suppressed) electrostatic field. We show below that the subjects of electrostatics and magnetostatics, which are distinct in the usual case, become convoluted in the presence of Lorentz violation. A satisfactory discussion therefore requires the simultaneous treatment of both electric and magnetic phenomena, even in the static limit [20].

A. Green functions and boundary conditions

To obtain a general solution for the potentials Φ and $\vec{A}$, we introduce indexed Green functions $G_{\mu\alpha}(\vec{x},\vec{x}')$ solving Eq. (12) for a point source,

$$\tilde{k}^{j\mu k\nu}\partial_j\partial_k G_{\mu\alpha}(\vec{x},\vec{x}') = \delta^\nu{}_\alpha\delta^3(\vec{x}-\vec{x}').\tag{14}$$

Once a suitable Green theorem incorporating the differential operator in Eq. (12) is found, the formal solution of Eq. (12) can be constructed using standard methods [21, 22].

The relevant Green theorem can be given in terms of arbitrary functions $X_\mu(\vec{x})$ and $Y_\mu(\vec{x})$. We obtain

$$\begin{aligned}&\int_V d^3x(X_\mu\tilde{k}^{j\mu k\nu}\partial_j\partial_k Y_\nu - Y_\nu\tilde{k}^{j\mu k\nu}\partial_j\partial_k X_\mu)\\ &= -\int_S d^2S\ \hat{n}^j(Y_\mu\tilde{k}^{j\mu k\nu}\partial_k X_\nu - X_\mu\tilde{k}^{j\mu k\nu}\partial_k Y_\nu),\end{aligned}\tag{15}$$

where $\hat{n}^j$ is the outward normal of the surface S bounding the region of interest.

Using Eqs. (12) and (14) in Eq. (15), we find that the general solution for the 4-potential A^λ is

$$\begin{aligned}A_\lambda(\vec{x}) &= \int_V d^3x' G_{\mu\lambda}(\vec{x}',\vec{x})j^\mu(\vec{x}')\\ &\quad -\int_S d^2S'\hat{n}'^j[G_{\mu\lambda}(\vec{x}',\vec{x})\tilde{k}^{j\mu k\nu}\partial'_k A_\nu(\vec{x}')\\ &\qquad -A_\mu(\vec{x}')\tilde{k}^{j\mu k\nu}\partial'_k G_{\nu\lambda}(\vec{x}',\vec{x})]\end{aligned}\tag{16}$$

up to the gradient of an arbitrary scalar. The first term contains the contribution from the sources within the volume V, while the remainder represents the effects of the bounding surface S.

Consider next the issue of appropriate boundary conditions. Inspection of Eq. (16) reveals that there are four natural classes of boundary condition that specify a solution. We summarize these in Table 1.

	Fields on S	Green function on S
I	Φ, $\hat{n}\times\vec{A}$	$\epsilon^{jkl}\hat{n}^j G^l{}_\lambda = 0$, $G_{0\lambda}=0$
II	Φ, $\hat{n}\times\vec{H}$	$\hat{n}^j\tilde{k}^{jlk\nu}\partial_k G_{\nu\lambda} = \frac{1}{S}\delta^l{}_\lambda$, $G_{0\lambda}=0$
III	$\hat{n}\cdot\vec{D}$, $\hat{n}\times\vec{A}$	$\hat{n}^j\tilde{k}^{j0k\nu}\partial_k G_{\nu\lambda} = \frac{1}{S}\delta^0{}_\lambda$, $\epsilon^{jkl}\hat{n}^j G^l{}_\lambda = 0$
IV	$\hat{n}\cdot\vec{D}$, $\hat{n}\times\vec{H}$	$\hat{n}^j\tilde{k}^{j\mu k\nu}\partial_k G_{\nu\lambda} = \frac{1}{S}\delta^\mu{}_\lambda$

Table 1. Natural classes of boundary conditions.

Class I boundary conditions are expressed entirely in terms of the potentials, being specified by $\Phi \equiv A^0$ and the tangential component of $\vec{A}$. This class is the only one for which the explicit Green function is independent of the area of the surface. It is most closely analogous to Dirichlet boundary conditions in conventional electrostatics. Class II boundary conditions involve Φ and the tangential component of $\vec{H}$. The explicit boundary condition on the corresponding Green function incorporates a factor of the inverse surface area on the right-hand side, generating a term in the solution involving the average contribution of the potential over the bounding surface. A similar feature occurs in conventional electrostatics for Neumann boundary conditions. Class III boundary conditions involve the normal component of $\vec{D}$ and the tangential component of $\vec{A}$, while class IV boundary conditions involve the normal component of $\vec{D}$ and the tangential component of $\vec{H}$. The gauge freedom in specifying $\vec{A}$ in class I and III boundary conditions has no effect on the solutions for $\vec{E}$ and $\vec{B}$.

The uniqueness of the solutions for the fields $\vec{E}$ and $\vec{B}$ associated to each of the four classes above can be shown by using the general solution (16) to examine the difference $\Delta A_\mu = A^1_\mu - A^2_\mu$ of two solutions obtained for a specified choice of boundary conditions. Direct calculation verifies that ΔA_μ is either a constant or zero. It follows that the electric and magnetic fields from both solutions are identical. This result can also be verified without the use of Green functions by considering the first Green identity and the field boundary conditions.

We remark in passing that reciprocity relations for the Green functions can be obtained by combining the Green theorem (15) with the above results. For example, provided the bounding surface is at infinity, the Green functions satisfy $G_{\mu\nu}(\vec{x},\vec{x}') = G_{\nu\mu}(\vec{x}',\vec{x})$.

Note also that a given problem in Lorentz-violating electromagnetostatics effectively requires the simultaneous solution of all four potentials A_ρ from the corresponding boundary conditions. This differs from the usual case, in which electrostatics and magnetostatics can be regarded as distinct subjects, even though Eq. (16) reduces to standard results in the limit of zero k_F.

The solution (16) can be generalized to include regions of matter. In such cases the modified equations (11) apply. For simplicity, we limit attention to cases where the material is isotropic and the matter coefficients are constant everywhere inside. The equations of motion for the electromagnetic field are then

$$\tilde{k}^{j\mu k\nu}_{\rm matter}\partial_j\partial_k A_\nu(\vec{x}) = j^\mu(\vec{x}), \qquad (17)$$

where the coefficients $\tilde{k}^{j\mu k\nu}_{\rm matter}$ are given by

$$\begin{aligned}\tilde{k}^{j\mu k\nu}_{\rm matter} &= \epsilon\eta^{jk}\eta^{\mu 0}\eta^{\nu 0} - \tfrac{1}{\mu}\eta^{jk}\eta^{\mu l}\eta^{\nu l} - \tfrac{1}{\mu}\eta^{j\nu}\eta^{k\mu} \\ &\quad + 2(k_F)^{j\mu k\nu}_{\rm vacuum} + 2(k_F)^{j\mu k\nu}_{\rm matter}. \qquad (18)\end{aligned}$$

In this equation, the coefficients $(k_F)_{\rm matter}$ are related to the quantities $\kappa^{\rm matter}$ in Eq. (11) by definitions of the form (4). Assuming that $(k_F)_{\rm matter}$ has the symmetries of $(k_F)_{\rm vacuum}$ and that the volume V is filled with matter to the surface S, the solution (16) and the above formalism can be applied directly by replacing $\tilde{k}^{j\mu k\nu}_{\rm vacuum} \to \tilde{k}^{j\mu k\nu}_{\rm matter}$.

B. Conductors

The presence of conducting surfaces influences the determination of appropriate boundary conditions. In conventional electrostatics, the potential Φ is constant on a conductor in equilibrium. However, it unclear *a priori* whether this result holds in the presence of Lorentz violation, when the potential $\Phi \equiv \Phi_\rho + \Phi_J$ becomes the sum of a part Φ_ρ arising from the charge density and a (suppressed) part Φ_J arising from the current density.

To investigate this issue, consider one or more conductors positioned in a region that may also contain static charges and steady-state currents. Assuming the region contains matter with the general constituency relations (11), the electromagnetic energy density u can be written

$$\begin{aligned}u &= \tfrac{1}{2}[\vec{E}\cdot(\epsilon + \kappa^{\rm vacuum}_{DE} + \kappa^{\rm matter}_{DE})\cdot\vec{E} \\ &\quad + \vec{B}\cdot(\tfrac{1}{\mu} + \kappa^{\rm vacuum}_{HB} + \kappa^{\rm matter}_{HB})\cdot\vec{B}], \qquad (19)\end{aligned}$$

where for simplicity $\kappa^{\rm matter}$ and $\kappa^{\rm vacuum}$ are taken to have the same symmetries. The total electromagnetic energy U of the configuration is the integral of Eq. (19) over all space, including the volume occupied by the conductors. If the fields fall off sufficiently rapidly at the boundary of the region, the total energy can alternatively be expressed as an integral over all space involving the potentials Φ and $\vec{A}$, the charge density ρ, and the current density $\vec{J}$.

In equilibrium, the free charge on the conductor is arranged to minimize U. Consider a variation $\delta\rho$ of the charge distribution on the conductors away from the equilibrium configuration. With some manipulation of the above expressions and suitable use of the modified Maxwell equations, the corresponding change ΔU in the electromagnetic energy can be written as

$$\begin{aligned}\Delta U &\equiv U(\rho+\delta\rho) - U(\rho) \\ &= \int_V d^3x[\Phi_\rho\delta\rho \\ &\quad + \tfrac{1}{2}\delta\vec{E}\cdot(\epsilon + \kappa^{\rm vacuum}_{DE} + \kappa^{\rm matter}_{DE})\cdot\delta\vec{E} \\ &\quad + \tfrac{1}{2}\delta\vec{B}\cdot(\tfrac{1}{\mu} + \kappa^{\rm vacuum}_{HB} + \kappa^{\rm matter}_{HB})\cdot\delta\vec{B}]. \qquad (20)\end{aligned}$$

The first term represents a first-order variation, so the energy is extremized when it vanishes. This condition is satisfied for constant Φ_ρ in the conductors because the variation $\delta\rho$ integrates to zero. The remaining terms represent the second-order variation, and the energy is minimized when they are positive. This condition is also satisfied because κ_{DE} and κ_{HB} are small. The potential Φ_ρ from the charge density is thus expected to be constant in a conductor in equilibrium. This result generalizes the Thomson theorem of conventional electrostatics, and it establishes a partial condition on Φ.

The energy-based variational approach of Eq. (20) gives no information about the portion Φ_J of the potential due to the current density. This is because the contribution of Φ_J to Φ in the expression for the energy cancels against the portion $\vec{A}_\rho$ of $\vec{A}$ arising from the charge density. Instead, a condition on Φ_J can be obtained by considering the rate of work done by the fields in a fixed volume of conductor.

Since the Lorentz force is conventional, the power P in the volume V of conductor is given by

$$P = -\int_V d^3x\ \vec{J}\cdot\vec{E}. \tag{21}$$

Using $\vec{E} = -\vec{\nabla}\Phi$, the steady-state assumption $\vec{\nabla}\cdot\vec{J} = 0$, and the vanishing of the normal component of the current at the surface of the conductor reveals that P vanishes. Substituting the steady-state Maxwell equation $\vec{\nabla}\times\vec{H}_{\rm matter} = \vec{J}$ in Eq. (21) and using $\vec{\nabla}\times\vec{E} = 0$ then gives the condition

$$\int_S d^2x\ \vec{E}\cdot\hat{n}\times\vec{H}_{\rm matter} = 0 \tag{22}$$

on the surface of the conductor. This is the statement that there is no net outward flow of field momentum: the integral of the normal component of the generalized Poynting vector $\vec{S} = \vec{E}\times\vec{H}_{\rm matter}$ vanishes over the surface. The point of interest is that Eq. (22) is satisfied for vanishing tangential electric field at the surface, which in turn implies that Φ_J is constant. The condition (22) is therefore consistent with requiring that the total potential Φ is constant on the surface of a conductor.

Note that Eq. (22) could in principle also be satisfied for more general field configurations. Suppose that for linear conductors we introduce a generalized Ohm law of the form

$$\vec{J} = (\sigma + \tilde{\sigma}_E)\cdot\vec{E} + \tilde{\sigma}_B\cdot\vec{B}, \tag{23}$$

where $\tilde{\sigma}_E$, $\tilde{\sigma}_B$ involve coefficients for Lorentz violation. It can be shown using the above assumptions that only the final term in this expression could produce leading-order deviations from constant Φ in equilibrium. A term of this type might conceivably be generated through Lorentz violation under suitable circumstances. Although outside our present scope, it would be of interest to investigate this issue within specific models of conductors.

As an aside, we remark that the above results can be used to show that certain precision Cavendish-type experiments searching for a nonzero photon mass μ are insensitive to Lorentz violation. For example, the value of μ in the Proca electrostatics equation [23] $(\vec{\nabla}^2 - \mu^2)\Phi = 0$ was bounded by Williams *et al.* [24]. This experiment basically sought a nonzero potential difference inside a metal shell held at fixed potential. However, at leading order, the corresponding equation with a nonzero k_F coefficient for Lorentz violation is the Laplace equation. Fixing the potential across the shell thus also fixes it inside the shell, and so the experiment is insensitive to k_F.

IV. APPLICATIONS

A. Point charge

As an application of the general solution using the Green function (16), consider the fields for the special case of boundary conditions at infinity. The potentials can then be obtained from

$$A_\lambda(\vec{x}) \ = \ \int d^3x\ G_{\mu\lambda}(\vec{x},\vec{x}')j^\mu(\vec{x}'). \tag{24}$$

Imposing the Coulomb gauge, this integral can be solved at leading order in $(k_F)_{\kappa\lambda\mu\nu}$ for an arbitrary source j^μ, using the symmetric Green function

$$\begin{aligned} G_{\mu\nu}(\vec{x}-\vec{x}') \ &= \ \frac{\eta_{\mu\nu} + (k_F)_{\mu j\nu j}}{4\pi|\vec{x}-\vec{x}'|} \\ &\qquad -\frac{(k_F)_{\mu j\nu k}(\vec{x}-\vec{x}')^j(\vec{x}-\vec{x}')^k}{4\pi|\vec{x}-\vec{x}'|^3}. \end{aligned} \tag{25}$$

This Green function can be extracted from the differential equations (12) and (14) by Fourier decomposition in momentum space.

As an example, consider a classical point charge as the source. The action for this case is discussed in the appendix, along with some of the subtleties associated with the freedom to redefine the choice of coordinates. In the rest frame of the charge, taken to be located at the origin, the source 4-current is $j^\mu(\vec{x}) = {\delta_0}^\mu q\delta^{(3)}(\vec{x})$. Substituting this source and the Green function (25) into the solution (24) and performing the integral, we find the potentials at leading order in Lorentz violation are given by

$$\begin{aligned} \Phi(\vec{x}) \ &= \ \frac{q}{4\pi|\vec{x}|}\Big(1 - (k_F)^{0j0k}\hat{x}^j\hat{x}^k\Big), \\ A^j(\vec{x}) \ &= \ \frac{q}{4\pi|\vec{x}|}\Big((k_F)^{0kjk} - (k_F)^{jk0l}\hat{x}^k\hat{x}^l\Big). \end{aligned} \tag{26}$$

We have defined the charge q so that the first term in Φ has the usual normalization. The solution for Φ agrees with that previously obtained in Ref. [5]. As discussed above, the appearance of a nonzero vector potential from a point charge at rest is to be expected and can be traced to the mixing of electrostatics and magnetostatics in the presence of Lorentz violation.

The electromagnetostatic fields due to the point charge at rest can be derived directly from the results (26). We find

$$\begin{aligned} E^j(\vec{x}) &= \frac{q}{4\pi|\vec{x}|^2}\Big(\hat{x}^j + 2(k_F)^{0j0k}\hat{x}^k \\ &\qquad -3(k_F)^{0k0l}\hat{x}^j\hat{x}^k\hat{x}^l\Big), \\ B^j(\vec{x}) &= \frac{q}{4\pi|\vec{x}|^2}\epsilon^{jkl}\Big((k_F)^{0mkm}\hat{x}^l \\ &\qquad +[(k_F)^{0kml} + (k_F)^{0mkl}]\hat{x}^m \\ &\qquad +3(k_F)^{0mnk}\hat{x}^m\hat{x}^n\hat{x}^l\Big). \end{aligned} \tag{27}$$

These fields display an inverse-square behavior modulated by anisotropic Lorentz-violating parts.

B. Magnet inside conducting shell

We consider next a more involved example, consisting of a localized magnetic source surrounded by a grounded conducting shell. This situation is designed to exploit the mixing between electrostatic and magnetostatic effects in a manner that has direct application to laboratory searches for Lorentz violation, as is discussed in the next subsection.

For definiteness, the magnetic source is taken to be a sphere of radius a and uniform magnetization $\vec{M}$. At zeroth order in Lorentz violation, the associated magnetic field is uniform inside the sphere and is dipolar outside, with dipole moment $\vec{m} = 4\pi a^3\vec{M}/3$. The grounded conductor is taken to be a concentric spherical shell of radius R. We seek the solution for the scalar potential Φ in the region $a < r < R$, where r is the radial coordinate from the center of the sphere. We first solve the problem treating the source as an idealized bound current density, and then present the modifications induced by the magnet permittivity and matter coefficients for Lorentz violation.

The idealized solution can be found using Eq. (16) with class I boundary conditions,

$$\begin{aligned} \Phi(\vec{x}) &= \int_V d^3x' G_{j0}(\vec{x}', \vec{x}) j^j(\vec{x}') \\ &\quad + \int_S d^2S'\ \hat{n}'^j A_\mu(\vec{x}')\tilde{k}^{j\mu k\nu}\partial'_k G_{\nu 0}(\vec{x}', \vec{x}). \end{aligned} \tag{28}$$

The charge density ρ vanishes by assumption, so the source consists of the bound current density $\vec{J} = \vec{\nabla}\times\vec{M}$ due to the magnetization of the sphere. At leading order in the coefficients for Lorentz violation, Eq. (28) can be manipulated into the form

$$\Phi^{(1)}(\vec{x}) = -\int_V d^3x' G^{(0)}(\vec{x}, \vec{x}')\vec{\nabla}'\cdot\kappa_{DB}\cdot\vec{B}^{(0)}(\vec{x}'). \tag{29}$$

The labels (0) and (1) indicate zeroth- and first-order contributions in the coefficients for Lorentz violation.

The structure of Eq. (29) implies that the potential can be viewed as arising from an effective charge density obtained from the derivatives of the conventional magnetic field. The proposed application of this problem lies in the laboratory, so in evaluating Eq. (29) it is an excellent approximation to take $\kappa_{DB} = \tilde{\kappa}_{o+}$ as antisymmetric, following the discussion in section II A. For the zeroth-order Green function $G^{(0)}(\vec{x}, \vec{x}')$, we can take the conventional Dirichlet Green function for a spherical grounded shell of radius R. Performing the integral, we find that the Lorentz-violating potential Φ is

$$\Phi(\vec{x}) = \frac{\hat{r}\cdot\tilde{\kappa}_{o+}\cdot\vec{m}}{4\pi}\left(\frac{1}{r^2} - \frac{r}{R^3}\right). \tag{30}$$

This solution is valid for $a < r < R$.

The electrostatic and magnetostatic fields can be obtained by direct calculation. The electrostatic field in the region $a < r < R$ is given by

$$E^j(\vec{x}) = \frac{(\tilde{\kappa}_{o+})^{jk}m^k}{4\pi}\left(\frac{1}{R^3} - \frac{1}{r^3}\right) + \frac{3(\tilde{\kappa}_{o+})^{kl}\hat{r}^j\hat{r}^k m^l}{4\pi r^3}. \tag{31}$$

The magnetostatic field is given by

$$B^j(\vec{x}) = \frac{3\hat{r}^j(\hat{r}\cdot\vec{m}) - m^j}{4\pi r^3} \tag{32}$$

at zeroth order.

The solution (30) becomes modified in the more realistic scenario with the magnet consisting of matter obeying the constituency relations (11). To obtain the modified result for the case of an isotropic material with constant matter coefficients, note that the leading-order potential $\Phi^{(1)}$ satisfies the Laplace equation everywhere except at $r = a$ and that the normal component of the electric field satisfies the boundary condition $\Delta(\epsilon E_n^{(1)}) = \sigma^{\rm eff}$, where the effective surface charge $\sigma^{\rm eff}$ is determined by the discontinuity of the magnetic field at $r = a$. The problem is then formally identical to a conventional electrostatics problem, and standard techniques [21] apply. We find that the solution (30) is adjusted by the replacements

$$\begin{aligned} \vec{m} &\to \frac{3\vec{m}}{[2 + \epsilon + (1-\epsilon)a^3/R^3]}, \\ \tilde{\kappa}_{o+} &\to \tilde{\kappa}_{o+}^{\rm vacuum} + \tfrac{2}{3}\tilde{\kappa}_{o+}^{\rm matter}, \end{aligned} \tag{33}$$

where the dielectric constant ϵ is taken to be a constant scalar.

C. Experiment

Among the combinations of coefficients listed in Eq. (7), experiments to date are least sensitive to $\tilde{\kappa}_{o+}$ and $\tilde{\kappa}_{\rm tr}$. In the case of $\tilde{\kappa}_{o+}$, this reduced sensitivity can be attributed to the parity-odd nature of the corresponding Lorentz-violating effects, while high sensitivity to $\tilde{\kappa}_{\rm tr}$ is difficult to attain because it is a scalar. The configuration discussed in the previous subsection is constructed to be directly sensitive to parity-odd effects, as is reflected in the dominance of the combination $\tilde{\kappa}_{o+}$ in the solution (30).

In this section, we consider an idealized experiment that could attain high sensitivity to the three independent components of $\tilde{\kappa}_{o+}$ and indirectly also to $\tilde{\kappa}_{\rm tr}$. The idea is to measure the potential (30) inside the spherical cavity. For a conservative estimate of the sensitivity that might in principle be attainable in the ideal case, suppose for simplicity the spherical source is a hard ferromagnet with strength 10^{-1} T near its surface, and suppose the potential is measured with a voltmeter of nV sensitivity. Then, a null measurement in principle could achieve a bound of $\tilde{\kappa}_{o+} \lesssim 10^{-15}$, which would represent an improvement of four orders of magnitude over best existing sensitivities [3, 4]. Using SQUID-based devices, this might in principle be improved by another four orders of magnitude, suggesting Planck-scale sensitivity to this type of Lorentz violation is attainable in the laboratory.

The basic setup for the experiment would be to insert one or more voltage probes, referenced to each other or to ground, into the inner region $a < r < R$. The solution (30) shows that the maximum voltage sensitivity occurs when the probe is close to the magnetic source. The conducting shell surrounding the magnet serves to shield the apparatus in the interior from external electric fields.

In the presence of Lorentz violation, rotating the entire apparatus produces a signal with a definite time variation, which may increase sensitivity and reduce systematics. The expected time variation of the signal can be obtained by referring the laboratory coefficients to the standard Sun-centered celestial-equatorial frame [5], which is an approximately inertial frame appropriate for reporting results of arbitrary tests of Lorentz violation. By virtue of the orbital speed $|\vec{\beta}| \simeq 10^{-4}$ of the Earth in this frame, a measurement of the three components of $\tilde{\kappa}_{o+}$ in the laboratory achieving a sensitivity S then translates in the Sun-centered frame into a sensitivity of order S to the three components of $\tilde{\kappa}_{o+}$ and a sensitivity of order $10^4 S$ to $\tilde{\kappa}_{\rm tr}$.

To illustrate these points, consider the case of fixed probes recording a potential difference $\Delta\Phi$ between two points in the inner region. We seek to characterize the expected time dependence of the signal due to the rotation of the Earth and its orbital motion about the Sun. Other rotations, such as those induced by turntables in the laboratory, can also be treated by these methods.

In a frame fixed to the laboratory and within the idealized approximations of the previous subsection, $\Delta\Phi$ can be written as

$$\Delta\Phi = (\mathcal{M}_{DB})^{jk}_{\rm lab}(\kappa_{DB})^{jk}_{\rm lab} = (\mathcal{M}_{DB})^{jk}_{\rm lab}(\tilde{\kappa}_{o+})^{jk}_{\rm lab}, \quad (34)$$

where $(\mathcal{M}_{DB})_{\rm lab}$ is an experiment-specific constant matrix that is determined for the chosen probe configuration by applying Eq. (30). The time dependence of the signal $\Delta\Phi$ can be exhibited by transforming the laboratory-frame combinations $(\tilde{\kappa}_{o+})^{jk}_{\rm lab}$ to the standard Sun-centered frame. Following Ref. [5], with upper-case letters denoting Sun-centered coordinates, we find

$$\begin{aligned}(\tilde{\kappa}_{o+})^{jk}_{\rm lab} &= T_0^{jkJK}(\kappa_{DB})^{JK} \\ &\quad +(T_1^{kjJK} - T_1^{jkJK})(\kappa_{DE})^{JK} \\ &= T_0^{jkJK}(\tilde{\kappa}_{o+})^{JK} + 2T_1^{kjJJ}\tilde{\kappa}_{\rm tr} \\ &\quad +(T_1^{kjJK} - T_1^{jkJK})(\tilde{\kappa}_{e-})^{JK}, \quad (35)\end{aligned}$$

where $T_0^{jkJK} = R^{jJ}R^{kK}$ and $T_1^{jkJK} = R^{jP}R^{kJ}\epsilon^{KPQ}\beta^Q$ are tensors containing the time dependence induced by the action of the rotations R^{jJ} and boost β^J. Appendix C of Ref. [5] provides explicit expressions for these quantities and fixes the coordinate choices for the laboratory and Sun-centered frames.

In performing an experiment along the above lines, some attention should be given to possible unconventional effects induced by the apparatus, in addition to the usual variety of systematic effects such as surface patch charges. For example, certain voltmeters are based on devices that measure currents. The currents are determined by the dipole moment of a coiled wire, which is measured using an internal magnetic field to determine the torque. The presence of Lorentz violation implies this internal magnetic field could generate a corresponding electric field that could interfere with the signal from the magnetized sphere. In practice, devices of this type could still be used under appropriate conditions, such as an internal magnetic field significantly weaker than that of the magnetized sphere. As another caution, inspection of the solution (30) shows that if the magnetic material chosen has a large dielectric constant then the signal would be suppressed, so a magnetic source of small dielectric constant is preferable.

Related experiments that could provide interesting sensitivity to $\tilde{\kappa}_{o+}$ and $\tilde{\kappa}_{\rm tr}$ may also be possible. For example, a kind of converse of the above experiment could involve attempting to measure a magnetic field created from a source of charge, in analogy with Eq. (27). Modern SQUID measurements of the magnetic field at the level of 10^{-14} T from a large vacuum electric-field source of 10^{12} V/m could in principle yield comparable bounds to those above.

Another approach could be to take advantage of the strong electric fields in the vicinity of an atomic nucleus. Atomic spectroscopy might then reveal the small accompanying Lorentz-violating magnetic field through observable frequency shifts. For this case, we can adapt some existing theoretical and experimental studies of Lorentz-violating effects in atoms [13]. Suppose as before the 10 birefringence-inducing coefficients in the photon sector are negligible. A coordinate transformation of the type described in the appendix can then be used to move the remaining 9 coefficients in the photon sector to the matter sector, where they appear as symmetric components of a c-type coefficient for Lorentz violation, $c_{\mu\nu} \supset (k_F)^{\alpha}{}_{\mu\alpha\nu}$. For example, the parity-odd coefficients $(\tilde{\kappa}_{o+})^{jk}$ of interest above are contained in the three symmetric combinations $(c_{0j} + c_{j0})$. To date no clock-comparison experiment has measured parity-odd c-type coefficients, but future laboratory or space-based experiments could achieve Planck-scale sensitivities [14] by incorporating into the analysis the boost effects arising from the orbital motion of the Earth or a satellite.

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APPENDIX A: CLASSICAL POINT CHARGE

This appendix discusses some issues associated with the choice of coordinate system in the presence of Lorentz violation [25]. For definiteness, we consider the theory of a single classical charged particle in Lorentz-violating electrodynamics.

The action is taken to be

$$S \;=\; S_0 + S_{\rm int} + S_{\rm em}, \tag{A1}$$

where S_0 is the action for the free classical particle, $S_{\rm int}$ contains the interaction, and $S_{\rm em}$ is the action containing the pure-photon part of the lagrangian (1). The free action S_0 is

$$S_0 = -m \int d\lambda \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\eta_{\mu\nu}}, \tag{A2}$$

and the interaction is assumed conventional,

$$S_{\rm int} = -\int d^4x \; j^\mu A_\mu = -q \int d\lambda \; A_\mu(x^\alpha)\frac{dx^\mu}{d\lambda}. \tag{A3}$$

As usual, the equations of motion for the charged particle are obtained by varying with respect to $x^\alpha(\lambda)$ and reparametrizing with the proper time $d\tau^2 = \eta_{\mu\nu}dx^\mu dx^\nu$. This shows that the conventional Lorentz-force law,

$$m\frac{d^2x^\mu}{d\tau^2} = qF^\mu{}_\alpha\frac{dx^\alpha}{d\tau}, \tag{A4}$$

holds despite the Lorentz violation in the photon sector.

A suitable coordinate transformation can move 9 of the 19 coefficients for Lorentz violation from the photon sector to the matter sector, while leaving unaffected the form $j^\mu A_\mu$ of the interaction. For simplicity in what follows, we keep only the relevant 9 coefficients, which corresponds to restricting the pure-photon lagrangian to Eq. (8), and we work at leading order in the coefficients $(k_F)_{\kappa\lambda\mu\nu}$ for Lorentz violation. The relevant coordinate transformation is $x^\mu \to x^{\mu'} = x^\mu - \frac{1}{2}(k_F)^{\alpha\mu}{}_{\alpha\nu}x^\nu$. Under this transformation, the total action becomes

$$\begin{aligned} S \;=\; & \int d^4x \; (-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu) \\ & -m\int d\lambda \sqrt{\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}(\eta_{\mu\nu} + (k_F)^\alpha{}_{\mu\alpha\nu})} \; . \end{aligned} \tag{A5}$$

In the new coordinates, the electromagnetic field has conventional kinetic and interaction terms and thus conventional dynamics. However, the charged particle now has Lorentz-violating dynamics controlled by the coefficients $(k_F)^\alpha{}_{\mu\alpha\nu}$. These are the classical analogue of the symmetric traceless coefficients $c_{\mu\nu}$ in the matter sector of Lorentz-violating quantum electrodynamics. The equations of motion for the classical charged particle are now expressed in terms of the proper time $d\tau^2 = (\eta_{\mu\nu} + (k_F)^\alpha{}_{\mu\alpha\nu})dx^\mu dx^\nu$, and they represent a modified Lorentz force,

$$m\frac{d^2x^\mu}{d\tau^2} + m(k_F)^{\alpha\mu}{}_{\alpha\beta}\frac{d^2x^\beta}{d\tau^2} = qF^\mu{}_\alpha\frac{dx^\alpha}{d\tau}. \tag{A6}$$

Phenomenological analysis could proceed with either of the two actions (A1) or (A5), or indeed with various other actions obtained by coordinate transformations of Eq. (A1). The physically observable effects are always equivalent, but care is required in matching calculations to physical situations. For example, it might seem tempting to conclude that the action (A1) cannot describe physical Lorentz violation involving the 9 coefficients considered above because there exists a coordinate system with conventional photon dynamics. However, in practice charged particles are used to measure properties of the electromagnetic field, and so in the new coordinate system the physical Lorentz violation appears because observables are affected by the modified force law (A6). Determining which set of coordinates is most appropriate for a given experiment involves establishing the underlying choice of standard rods and clocks to which the experimental observables are ultimately being referenced.

Comparisons between results obtained with the two different actions must include the appropriate coordinate transformation. For example, observers using the two different actions (A1) and (A5) disagree on the force between two charged particles. Consider an observer for whom two point charges q and q' are at rest and separated by a distance $\vec{x}$. If the observer uses the action (A1), then the force between the two point charges is obtained using the Lorentz-violating result (26) together with the conventional Lorentz force (A4). At leading order, the force on charge q' due to charge q is then

$$m\frac{du^0}{d\tau} \;=\; 0, \quad m\frac{du^j}{d\tau} = q'E^j(\vec{x}), \tag{A7}$$

where $E^j(\vec{x})$ is the electric field (27) of the charge q. A second observer using the action (A5) finds instead a force given by applying the corresponding coordinate transformation to Eq. (A7),

$$\begin{aligned} m\frac{du^{0'}}{d\tau} \;&=\; \frac{-qq'}{8\pi|\vec{x}'|^2}(k_F)^{0j'l'j'}\hat{x}^{l'}, \\ m\frac{du^{j'}}{d\tau} \;&=\; \frac{qq'}{4\pi|\vec{x}'|^2}\big(\hat{x}^{j'} + 2(k_F)^{0j'0k'}\hat{x}^{k'} - \tfrac{1}{2}(k_F)^{0'l'j'l'}\tilde{x}^{0'} \\ &\qquad + \tfrac{3}{2}(k_F)^{0'l'k'l'}\hat{x}^{j'}\hat{x}^{k'}\tilde{x}^{0'}\big), \end{aligned} \tag{A8}$$

where $\tilde{x}^{0'} = x^{0'}/|\vec{x}'|$. Note that the charges are moving in this second frame. The result (A8) can be derived directly from the modified Lorentz force (A6) with conventional photon dynamics, provided care is taken to account for the motion of the charges.

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