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Stability of Solitary and Cnoidal Traveling Wave Solutions for a Fifth Order Korteweg-de Vries Equation

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Stability of solitary and cnoidal traveling wave solutions for a fifth order Korteweg-de Vries equation

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We establish the nonlinear stability of solitary waves (solitons) and periodic traveling wave solutions (cnoidal waves) for a Korteweg-de Vries (KdV) equation which includes a fifth order dispersive term. The traveling wave solutions which yield solitons for zero boundary conditions and wave-trains of cnoidal waves for nonzero boundary conditions are analyzed using stability theorems, which rely on the positivity properties of the Fourier transforms. We show that all families of solutions considered here are (orbitally) stable.

*Keywords***:** cnodial waves, solitary waves, fifth order KdV equation, stability of traveling waves.

I. INTRODUCTION

We investigate the stability of the traveling wave solutions Korteweg-de Vries (KdV) equation with a fifth order dispersive term [17, 22]. Numerically it was investigated in a study of magneto-acoustic waves in a cold collision-free plasma [19], and takes the form

$$u_t + \gamma u u_x + \alpha u_{xxx} = \beta u_{xxxxx}. \tag{1}$$

α, β are the third and fifth order dispersive terms, respectively, γ is a wave steepening parameter for a unidirectional traveling wave $u(\xi) = u(x - ct)$, of velocity c in the x direction at time t which satisfies the first conservation law

$$-cu + \frac{\gamma}{2}u^2 + \alpha u_{\xi\xi} - \beta u_{\xi\xi\xi\xi} = \mathcal{A}, \tag{2}$$

and by multiplying by u_ξ, integrating and using (2) again, leads to the second conservation law

$$-\frac{c}{2}u^2 + \frac{\gamma}{3}u^3 + \alpha\left(uu_{\xi\xi} - \frac{1}{2}{u_\xi}^2\right) - \beta\left(uu_{\xi\xi\xi\xi} - u_\xi u_{\xi\xi\xi} + \frac{1}{2}{u_{\xi\xi}}^2\right) = \mathcal{B}. \tag{3}$$

The constants $\mathcal{A}, \mathcal{B}$ can be regarded as the mass and energy flux, and should be both zero for solitary waves and nonzero for oscillatory tails [11].

We establish the orbital stability for traveling wave solutions of (1), which depend on the parameter γ, by applying an existing stability criterion found in [2] and [27]. Previously, the stability for solitary wave solutions to the initial value problem for the KdV equation was already established by Benjamin [6], but it did not include the fifth order dispersive term, though the literature is rich of papers [4, 7, 8, 32] that develop sufficient conditions which imply stability for long wave equations with a general linear dispersion term $u_t + u_x + u^p u + \mathcal{L}u_x = 0$. The case where $\gamma = \alpha = 1$ is addressed in [26], and the case where $\beta = \frac{1}{1680}$, $\gamma = 1$, and $\alpha = \frac{13}{420}$ is studied in [2]. In [4, 7], the computation of the spectrum of $\mathcal{L}$ and the ability to verify an inequality involving the eigenfunctions for $\mathcal{L}$ is required, whereas in [2] the positivity of the Fourier transform of the solitary wave is used in conjunction with the inner product $I = (\chi, \varphi)_{L^2(\mathbb{R})}$ to obtain stability. Here, the class PF(2) [3, 18] is used to determine the necessary spectral properties of $\mathcal{L}$. For the theory of instability of solitary waves we refer to the papers [5, 10], where the authors establish instability for solitary waves associated to a generalized fifth order KdV equation of the form

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$u_t + u_{xxxxx} + bu_{xxx} = (G(u, u_x, u_{xx}))_x$ for $b \neq 0$, where $G(q, r, s) = F_q(q, r) - rF_{qr}(q, r) - sF_{rr}(q, r)$ and $F(q, r)$ a homogeneous function of degree $p + 1$ for $p > 1$. When considering periodic traveling waves (of cnoidal type) we note that the literature is not as extensive as is the case for solitary waves. Pomeau et al. [29] compute the amplitudes of continuous-wave tails in the fifth-order Korteweg-de Vries equation in order to discuss the structural stability of the KdV equation under singular perturbation, while Pava and Natali [27] provide a program well suited for addressing the issue of stability for the periodic waves considered herein. There, the explicit expression for the periodic wave is based on the Jacobi elliptic functions and the Fourier series representation thereof. The stability theory is established under the conditions that $\widehat{\varphi_c} > 0$, $\widehat{\varphi_c^p} \in PF(2)$, and $I = (\chi, \varphi)_{L^2_{2L}} < 0$. The first case of proving stability of periodic traveling waves of the KdV equation appears in [24] where the integrability of the KdV equation is exploited. In [28] the authors adapt the modern theory of stability of solitary waves [9, 32] to the periodic context. They show that periodic waves corresponding to the KdV equation are stable with respect to small, periodic perturbations in the context of the initial-value problem. In [16] the authors consider an abstract Hamiltonian system in the presence of symmetry and which, relying on a sharp identification of the lowest eigenvalues in the spectrum of the linearized problem, reduces the stability issue to verifying the convexity of a specific functional dependent on the wave speed.

II. STABILITY OF ANALYTICAL SOLUTIONS

In this section we discuss the traveling wave solutions along with their stability properties. First, we consider the case where the traveling wave is a solitary wave, whereas the case of periodic traveling waves are dealt with in the subsequent subsection.

A. Solitary waves

Assuming α, $\beta > 0$, and using zero boundary conditions, the solution to (1) which was obtained previously by Hereman et al. [17] using the direct algebraic method, and more recently by Mancas [23] using the elliptic function method, takes the form

$$u(x, t) = \frac{105\alpha^2}{169\gamma\beta}\text{sech}^4\left[\frac{1}{2}\sqrt{\frac{\alpha}{13\beta}}\left(x - \frac{36\alpha^2}{169\beta}t\right)\right] \equiv \varphi_c(\xi) \tag{4}$$

and represents a solitary wave that translates to the right with velocity $c = \frac{36\alpha^2}{169\beta}$ that is fixed by both dispersion coefficients.

For the special case of $\beta = 0$, (1) reduces to the well-known KdV equation which describes the motion of small amplitude and large wavelength shallow waves in dispersive systems [21]

$$u_t + \gamma u u_x + \alpha u_{xxx} = 0, \tag{5}$$

with mass and energy flux given by

$$\begin{aligned} &-cu + \tfrac{\gamma}{2}u^2 + \alpha u_{\xi\xi} = \mathcal{A}, \\ &-\tfrac{c}{2}u^2 + \tfrac{\gamma}{3}u^3 + \alpha\left(u u_{\xi\xi} - \tfrac{1}{2}{u_\xi}^2\right) = \mathcal{B}. \end{aligned} \tag{6}$$

It is worth noting that for (5) a more general case can be adopted, that is by inclusion of the linear term Cu_x

$$u_t + Cu_x + \gamma u u_x + \alpha u_{xxx} = 0 \tag{7}$$

which appears in the work of [14], and more recently in [12, 13]. In [12] a Hamiltonian formulation is given for the governing equations describing the two-dimensional nonlinear interaction between coupled surface waves, internal waves, and an underlying current with piecewise constant vorticity in a two-layered fluid overlying a flat bed. In [14] the authors develop a Hamiltonian perturbation theory for the long-wave limits, and carry out analysis of the principal long-wave scaling regimes for irrotational flows (with zero vorticity). Note that in the presence of non-uniform underlying currents there is non-zero vorticity, and the most wide ocean motion with coherent travelling waves occurs in the equatorial Pacific (over more than $12,000$ km.), where underlying currents are of great significance [13]. The revised model (7) can be reduced to the original Eq. (5) by looking at a frame moving at a suitable constant speed, that is, after a change of variables (x,t) to $(x-Ct,t)$.

The solution of (5) using zero boundary conditions is the solitary wave [21, 23, 30]

$$u(x,t) = \frac{3c}{\gamma}\mathrm{sech}^2\left[\frac{1}{2}\sqrt{\frac{c}{\alpha}}(x-ct)\right] \equiv \phi_c(\xi) \tag{8}$$

which propagates with arbitrary velocity to the right $c>0$ when $\alpha>0$ or to the left $c<0$ when $\alpha<0$.

We first establish the stability of solitary waves for (5) using (8) and then for (1) using (4), with suitable conditions on the parameters γ, α, β. Throughout we will make use of the results and definitions found in [2, 26]. In [2] the author remarks that the results therein can be extended to include more general nonlinearities, and with this in mind we consider a version of Eq. (1.1) in [2] with an appended factor of γ to the nonlinear term $u^p u_x$ that takes the form

$$u_t + \gamma u u_x - (Mu)_x = 0, \tag{9}$$

where M is a differential operator with positive symbol defined by $M = \beta\frac{d^4}{dx^4} - \alpha\frac{d^3}{dx^3}$.

Next, we establish the stability of the family of solutions given by (8) by applying Theorem 3.1 in [2]. Using the traveling wave ansatz and integrating once (9) assuming zero integration constant, we obtain

$$(M+c)u - \frac{\gamma}{p+1}u^{p+1} = 0. \tag{10}$$

We define a solitary wave φ as an even function which lies in the space $H^{\mu/2}$ and is a solution to (10). To study the stability of traveling waves for (1) we must consider the associated linear operator $\mathcal{L} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$

$$\mathcal{L}\zeta = (M+c)\,\zeta - \gamma u^p \zeta. \tag{11}$$

Proposition 2.1 in [2] establishes that $\mathcal{L}$ is a linear, closed, unbounded, self-adjoint operator defined on a dense subspace of $L^2(\mathbb{R})$. In particular this means $\mathcal{L}$ has the eigenvalue 0, with eigenfunction $\frac{du}{dx}$.

Definition II.1. Let φ be a solitary traveling-wave solution of (1) and consider $\tau_r\varphi(x) = \varphi(x+r)$, $x, r \in \mathbb{R}$. We define the set $\Omega_\varphi \subset H^2(\mathbb{R})$ the orbit generated by φ, as

$$\Omega_\varphi = \{g \mid g = \tau_r\varphi, \text{for some } r \in \mathbb{R}\}.$$

Moreover, for any $\eta > 0$, define $U_\eta \subset H^2(\mathbb{R})$ by

$$U_\eta = \left\{ f : \inf_{g\in\Omega_\varphi} \|f-g\|_{H^2} < \eta \right\}.$$

Using this terminology φ is said to be *(orbitally) stable* if

(i) the initial value problem associated with (1) is globally well-posed in $H^2(\mathbb{R})$.

(ii) For every $\epsilon > 0$, there exists $\delta > 0$ such that for all $u_0 \in U_\delta$, the solution u to (1) with initial condition $u(0,x) = u_0$ satisfies $u(t) \in U_\epsilon$ for all $t > 0$.

Theorem II.2. The smooth family of solutions (8) is orbitally stable in $H^2(\mathbb{R})$ under the flow of (5).

Proof. Recall the linear operator associated to (5)

$$\mathcal{L} = \left(-\alpha\frac{d^2}{dx^2} + c\right) - \gamma\phi_c, \tag{12}$$

with $M = M_{1,1} = -\alpha \frac{d^2}{dx^2}$, therefore by Theorem 4.6 in [2] it suffices to show that $I = (\phi_c, \psi)_2 < 0$. To do this we compute $\frac{d}{dc}\|\phi_c\|^2$, where

$$\begin{aligned}\|\phi_c\|^2 &= \left(\frac{3c}{\gamma}\right)^2 \int_{\mathbb{R}} \operatorname{sech}^4\left(\frac{1}{2}\sqrt{\frac{c}{\alpha}}\xi\right)\, d\xi = \left(\frac{3c}{\gamma}\right)^2 2\sqrt{\frac{\alpha}{c}} \int_{\mathbb{R}} \operatorname{sech}^4(\chi)\, d\chi = \left(\frac{3c}{\gamma}\right)^2 2\sqrt{\frac{\alpha}{c}}\left(\frac{4}{3}\right) \\ &= \frac{24\alpha^{1/2}}{\gamma^2} c^{3/2}. \end{aligned} \tag{13}$$

Therefore $\frac{d}{dc}\|\phi_c\|^2 > 0$, for all $c > 0$. ■

Next, we establish the stability of the traveling wave solution (4). Since (4) does not define a family of solutions in c, we make use of Gegenbauer polynomials [2]. We can use these polynomials to help determine the sign of the inner product $I = (\varphi, \psi)_2$. Specifically, we use Theorem 4.10 in [2] which provides us with the proper expression for I in terms of the gamma function which takes the form

$$I = a \sum_{j=0}^{\infty} \left(\frac{\lambda_{2j}}{1-\lambda_{2j}}\right) \left\{\frac{\Gamma(2j+1)\cdot(2j+n+r-\frac{1}{2})}{\Gamma(2j+2n+2r-1)}\right\} \left\{\frac{\Gamma(j+n)\Gamma(j+n+r-\frac{1}{2})}{\Gamma(j+1)\Gamma(j+r+\frac{1}{2})}\right\}^2, \tag{14}$$

where $a = \left(\frac{\gamma 2^{n+r-1}\Gamma(r)}{\pi\Gamma(n)}\right)$, $\lambda_m = \frac{\Gamma(r+m)}{\Gamma(r+1)} \cdot \frac{\Gamma(r+2n+1)}{\Gamma(r+2n+m)}$, $r = 4$ and $n = 2$. Let b_j represent the jth term of the series, since $b_0 < 0$ it suffices to show that $\sum_{j=1}^{\infty} b_j < |b_0|$, where

$$b_j = \frac{1680\left(2j+\frac{11}{2}\right)(j+1)^2\left(j+\frac{9}{2}\right)^2(2j)!}{\left[(2j+4)(2j+5)(2j+6)(2j+7)-1680\right](2j+10)!},$$

and $|b_0| = \left(\frac{11}{10!}\right)\left(\frac{81}{4}\right) \approx 6.14 \times 10^{-5}$. By the use of Stirling's formula $b_j \sim j^{-2r-1}$ as $j \to \infty$, hence we have $\sum_{j=1}^{\infty} b_j \approx 5.05 \times 10^{-6} < |b_0|$.

Theorem II.3. The smooth family of solutions (4) is orbitally stable in $H^2(\mathbb{R})$ under the flow of (1).

First we note that the differential operator corresponding to (1) is given by $M_{2,1} = \beta\frac{d^4}{dx^4} - \alpha\frac{d^2}{dx^2}$. We again appeal to Theorem 4.6 in [2], hence it remains to show that $I = (\varphi, \psi) < 0$ which was already established above. ■

B. Periodic traveling waves

For the KdV Eq. (5) with nonzero boundary conditions the solution found in [23] is

$$u(x,t) = \frac{3c+\sqrt{\Delta}}{2\gamma} \operatorname{cn}^2\left[\frac{1}{2}\frac{\sqrt[4]{\Delta}}{\sqrt{3\alpha}}(x-ct); \sqrt{\frac{1}{2}\left(1+\frac{3c}{\sqrt{\Delta}}\right)}\right] \equiv \varphi_c(\xi), \tag{15}$$

which can be written compactly as

$$\varphi_c(\xi) = A\, \operatorname{cn}^2\left(\frac{1}{2}\frac{\sqrt[4]{\Delta}}{\sqrt{3\alpha}}\xi; k\right), \tag{16}$$

where $\operatorname{cn}(\theta; k)$ is the Jacobian elliptic function with amplitude $A = \frac{3c+\sqrt{\Delta}}{2\gamma}$, and modulus $k = \frac{\sqrt{A\gamma}}{\sqrt[4]{\Delta}}$. This solution represents a train of periodic cnoidal waves which propagate with arbitrary velocity and wavelength given by

$$\lambda = \frac{4\sqrt{3\alpha}}{\sqrt[4]{\Delta}} K(k), \tag{17}$$

where $\Delta = 9c^2 + 24\mathcal{A}\gamma > 0$, and $K(k)$ is the complete elliptic integral of the first kind [1] $K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$.

For the special case $\alpha = 0$, $\beta \neq 0$, (1) takes the form

$$u_t + \gamma u u_x = \beta u_{xxxxx}. \tag{18}$$

This equation was first studied by Hasimoto [15] for shallow water waves near some critical value of surface tension, while Nagashima [25] performed experiments, and observed solitary waves with small oscillating tails using an oscilloscope. Its conservation laws are

$$\begin{array}{l} -cu + \frac{\gamma}{2}u^2 - \beta u_{\xi\xi\xi\xi} = \mathcal{A}, \\ -\frac{c}{2}u^2 + \frac{\gamma}{3}u^3 - \beta\left(u u_{\xi\xi\xi\xi} - u_\xi u_{\xi\xi\xi} + \frac{1}{2}{u_{\xi\xi}}^2\right) = \mathcal{B}. \end{array} \tag{19}$$

The solution to (18) obtained also by [23, 33] is

$$u(x,t) = \frac{5c}{2\gamma}\,\mathrm{cn}^4\left[\frac{\sqrt{2}}{2}\sqrt[4]{\frac{c}{42\beta}}\,(x-ct); \frac{\sqrt{2}}{2}\right] \equiv \phi_c(\xi), \tag{20}$$

and represents a train of periodic cnoidal waves which only propagate to the right with shape preserved by the constant modulus, and wavelength given by $\lambda = 2\sqrt{2}\sqrt[4]{\frac{42\beta}{c}}K\left(\frac{\sqrt{2}}{2}\right)$.

Next, we establish the stability of the periodic traveling-wave solutions given by (15) and (20). For this we make use of the techniques developed in [27]. This requires the following adjustments to our current setup. That is, we consider traveling-wave solutions to (1) of the form $u(x,t) = \varphi_c(x-ct)$, where the profile φ_c is a smooth periodic function with fundamental period $\lambda = 2L$ given by (17) for $L > 0$.

The notion of stability carries over for periodic traveling waves in this context, that is we say the orbit generated by φ_c denoted $\Omega_{\varphi_c} = \{\varphi_c(\cdot + y) \mid y \in \mathbb{R}\}$ is stable in $H^{m_2}_{per}([-L,L])$ by the periodic flow generated by (1). If, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for $u_0 \in H^{m_2}_{per}([-L,L])$ with $d(u_0, \Omega_{\varphi_c}) \equiv \inf_{y\in\mathbb{R}} \|u_0 - \varphi_c(\cdot + y)\|_{H^{m_2}_{per}} < \delta$. The solution u of (1) with $u(x,0) = u_0$ is global in time and satisfies $d\left(u(\cdot,t), \Omega_{\varphi_c}\right) < \epsilon$ for all $t \in \mathbb{R}$. With this notion of stability the conditions found in the papers [6, 9, 16, 27, 31] can be used to imply stability:

(P_0) There is a nontrivial smooth curve of periodic solutions for (1) of the form $c \in I \subset \mathbb{R} \to \varphi_c \in H^{m_2}_{per}([-L,L])$,

(P_1) $\mathcal{L}$ has a unique negative eigenvalue λ and it is simple,

(P_2) the eigenvalue is 0,

(P_3) $\frac{d}{dc}\int_{-L}^{L}\varphi_c^2(x)dx > 0$.

By considering the periodic solutions (15) and (20) condition (P_0) will be satisfied. To check conditions (P_1) and (P_2) we use Theorem 4.1 in [27] which relies on the positivity properties of the Fourier transform of the solution. The main theorems used to verify conditions $(P_0)-(P_2)$ are Theorem 5.1 and Theorem 4.1 in [27].

The first periodic traveling wave solution we consider is (15) which is a solution to (5) with condition $\mathcal{A}\gamma > 0$. Using the wavelength formula given by (17) with $L = \frac{\lambda}{2}$, we can write (15) in a simpler form

$$\varphi_c(\xi) = \frac{2\mathcal{M}(c)K^2(k)}{L^2}\mathrm{cn}^2\left[\frac{K(k)}{L}\xi; k\right], \tag{21}$$

where $\mathcal{M}(c) = \frac{3\alpha}{\gamma}\left(1 + \frac{3c}{\sqrt{\Delta}}\right) = \frac{6\alpha A}{\sqrt{\Delta}} > 0$.

Theorem II.4. *The periodic traveling wave solution* (21) *is stable in* $H^1_{per}([0,L])$ *by the flow of Eq.* (18).

Proof. We first consider the Fourier expansion of $\mathrm{cn}^2(\cdot,k)$

$$\mathrm{cn}^2\left(\frac{K}{L}\xi; k\right) = 1 - \frac{1}{k^2}\left(1 - \frac{E}{K}\right) + \frac{2\pi^2}{k^2K^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\cos\left(\frac{n\pi}{L}\xi\right), \tag{22}$$

where E is the complete integral of the second kind [1]

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2(\theta)}\, d\theta.$$

The series in (22) converges when the nome $q = e^{-\frac{\pi K}{K'}}$ satisfies $qe^{2\mathrm{Im}(\zeta)} < 1$ where $\zeta = \frac{\pi\xi}{L}$. Since $\mathrm{Im}(\zeta) = 0$ for $\xi \in \mathbb{R}$ and $q < 1$ we see that the series (22) converges. Furthermore

$$\frac{q^n}{1-q^{2n}} = \frac{1}{2}\,\mathrm{csch}\left(\frac{n\pi K'}{K}\right), \tag{23}$$

where $K'(k) = K(k')$ and $k' = \sqrt{1-k^2}$. Therefore

$$K^2\mathrm{cn}^2\left(\frac{K}{L}\xi; k\right) = K^2 - K\frac{(K-E)}{k^2} + \frac{\pi^2}{k^2}\sum_{n=1}^{\infty} n\ \mathrm{csch}\left(\frac{n\pi K'}{K}\right)\cos\left(\frac{n\pi}{L}\xi\right). \tag{24}$$

From this we obtain that the Fourier coefficients of φ_c are

$$\widehat{\varphi_c}(n) = \begin{cases} \frac{2\mathcal{M}K}{L^2}\left(K - \frac{K-E}{k^2}\right), & n = 0 \\ \frac{2\mathcal{M}\pi^2}{L^2k^2} n\ \mathrm{csch}\left(\frac{n\pi K'}{K}\right), & n \neq 0. \end{cases} \tag{25}$$

The expression $\frac{2\mathcal{M}K}{L^2}\left(K - \frac{K-E}{k^2}\right)$ is positive on $(0,1)$ (this is discussed below) therefore $\varphi_c > 0$. Moreover, we see that $\widehat{\varphi}_c > 0$ due to the Fourier coefficients of φ_c.

We consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{2\mathcal{M}\pi^2}{L^2k^2} x\ \mathrm{csch}\left(\frac{\pi x K'}{K}\right)$. To show that $\widehat{\varphi_c}$ belongs to $PF(2)$ in the discrete case we define the function $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = \frac{2\mathcal{M}K}{L^2}\left(K - \frac{K-E}{k^2}\right)$, and $h(x) = f(x)$ for $x \in (-\infty, -1] \cup [1, \infty)$. On $(-1, 1)$ we extend f in a differentiable manner such that $h(x)$ belongs to $PF(2)$ continuous case. Hence, in the discrete case $h(n) = \widehat{\varphi_c}(n)$ is in $PF(2)$ discrete. By Theorem 4.1 in [27] properties (P_1) and (P_2) in Definition 5.1 are satisfied. Next, we set $\chi = -\frac{d}{dc}\varphi_c$ since $\mathcal{L}\chi = \varphi_c$, by Parseval's theorem it follows that

$$I = -\frac{L}{2}\frac{d}{dc}\|\varphi_c\|^2_{L^2_{per}} = -\frac{L}{2}\frac{d}{dc}\|\widehat{\varphi_c}\|^2_{\ell^2},$$

where

$$\|\widehat{\varphi_c}\|^2_{\ell^2} = \frac{4\mathcal{M}^2K^2}{L^4}\left(K - \frac{K-E}{k^2}\right)^2 + \frac{4\mathcal{M}^2\pi^4}{L^4k^4}\sum_{n\neq 0} n^2\ \mathrm{csch}^2\left(\frac{n\pi K'}{K}\right),$$

and $D(k) = \frac{K(k)-E(k)}{k^2}$ is the Legendre integral given by $D(k) = \int_0^{\pi/2}\frac{\sin^2(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}d\theta$.

Therefore,

$$\begin{aligned} \frac{d}{dc}\|\widehat{\varphi_c}\|^2_{\ell^2} &= \frac{4\mathcal{M}K}{L^4}(K-D)^2\frac{d}{dc}(\mathcal{M}K) + \frac{8\mathcal{M}^2K^2}{L^4}(K-D)\frac{d}{dc}(K-D) \\ &+ \frac{4\pi^4}{L^4}\left(\frac{k\frac{d\mathcal{M}}{dc} - 2\mathcal{M}\frac{dk}{dc}}{k^3}\right)\sum_{n\neq 0} n^2\ \mathrm{csch}^2\left(\frac{n\pi K'}{K}\right) \\ &+ \frac{8\pi^5}{L^4}\left(\frac{\mathcal{M}}{k^2}\right)^2\left(\frac{K'\frac{dK}{dk} - K\frac{dK'}{dk}}{K^2}\right)\frac{dk}{dc}\sum_{n\neq 0} n^3\ \mathrm{csch}^2\left(\frac{n\pi K'}{K}\right)\coth\left(\frac{n\pi K'}{K}\right). \end{aligned} \tag{26}$$

To determine the sign of $\frac{d}{dc}\|\widehat{\varphi_c}\|^2_{\ell^2}$ we consider the following terms:

(i) $\frac{4\mathcal{M}K}{L^4}(K-D)^2\frac{d}{dc}(\mathcal{M}K)$,

(ii) $\frac{8\mathcal{M}^2K^2}{L^4}(K-D)\frac{d}{dc}(K-D)$,

(iii) $\frac{4\pi^4}{L^4k^3}\left(k\frac{d\mathcal{M}}{dc}-2\mathcal{M}\frac{dk}{dc}\right)$,

(iv) $\frac{8\pi^5}{L^4}\left(\frac{\mathcal{M}}{k^2}\right)^2\left(\frac{K'\frac{dK}{dk}-K\frac{dK'}{dk}}{K^2}\right)\frac{dk}{dc}$.

For (i), note that $\mathcal{M}(c)$, $K(c)\geq 0$, furthermore $\mathcal{M}(c)'=\frac{216|\alpha|\mathcal{A}\gamma}{[9c^2+24\mathcal{A}\gamma]^{3/2}}\geq 0$, $\frac{dK}{dc}=\frac{dK}{dk}\frac{dk}{dc}>0$.
For (ii), note that $K(k)-D(k)=\int_0^{\pi/2}\frac{1-\sin(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}d\theta>0$ and also $\frac{d}{dc}(K-D)>0$.
For (iii), note that $\mathcal{M}=\frac{6|\alpha|k^2}{\gamma}$ hence $k\frac{d\mathcal{M}}{dc}-2\mathcal{M}\frac{dk}{dc}=\frac{3|\alpha|}{\gamma}\left(4k^2\frac{dk}{dc}-4k^2\frac{dk}{dc}\right)=0$.
Lastly, for (iv), since K', $\frac{dK}{dk}>0$ and $\frac{dK'}{dk}<0$ we have $K'\frac{dK}{dk}-K\frac{dK'}{dk}>0$. Therefore, $I=-\frac{d}{dc}\|\widehat{\varphi_c}\|^2_{\ell^2}<0$ hence, by Theorem 5.1 in [27] the positive cnoidal waves φ_c are stable in $H^1_{per}([0,L])$. ■

Finally, we turn to the question of stability for the periodic traveling wave solution (20) of (18).

Theorem II.5. *The periodic traveling wave solution* (20) *is stable in* $H^1_{per}([0,L])$ *by the flow of Eq.* (18).

Proof. In order to proceed, we first consider the Fourier series expansion for cn^4, see [20].

$$k^4\mathrm{cn}^4(z,k)=\frac{1}{3}\left[2\left(k^2-k'^2\right)\left(\frac{E}{K}-k'^2\right)+k^2k'^2\right]+\frac{2\pi^2}{K^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\frac{1}{3}\left(2(k^2-k'^2)+\frac{n^2\pi^2}{2K^2}\right)\cos\left(\frac{n\pi}{K}z\right),\tag{27}$$

where $z=\frac{\sqrt{2}}{2}\sqrt[4]{\frac{c}{42\beta}}\,\xi$, the series converges since $0=\mathrm{Im}(z/K)<\mathrm{Im}(iK'/K)$. Moreover, for $k=\frac{\sqrt{2}}{2}$ (27) reduces to

$$\mathrm{cn}^4(z,\sqrt{2}/2)=\frac{1}{3}+\frac{2\pi^4}{3K^4}\sum_{n=1}^{\infty}n^3\ \mathrm{csch}(n\pi)\cos\left(\frac{n\pi}{K}z\right).\tag{28}$$

From this we compute the Fourier coefficients of ϕ_c which are

$$\widehat{\phi_c}(n)=\begin{cases}\frac{5c}{6\gamma}, & n=0\\ \frac{5c\pi^4}{3\gamma K^4}n^3\ \mathrm{csch}(n\pi), & n\neq 0.\end{cases}\tag{29}$$

Since $\widehat{\phi_c}(n)>0$, following the same argument as in the previous proof we can conclude that $\widehat{\phi_c}(n)$ is in $PF(2)$ discrete. Let $\chi=-\frac{d}{dc}\phi_c$ then $\mathcal{L}\chi=\phi_c$, hence, by Parseval's theorem it follows that $I=-\frac{L}{2}\frac{d}{dc}\|\phi_c\|^2_{L^2_{per}}=-\frac{L}{2}\frac{d}{dc}\left\|\widehat{\phi_c}\right\|^2_{\ell^2}$, thus

$$\left\|\widehat{\phi_c}\right\|^2_{\ell^2}=\frac{25c^2}{36\gamma^2}+\frac{25c^2\pi^8}{9\gamma^2K^8}\sum_{n\neq 0}n^6\ \mathrm{csch}^2(n\pi).$$

Hence, it is immediate that $I=-\frac{d}{dc}\left\|\widehat{\phi_c}\right\|^2_{\ell^2}<0$ therefore, by Theorem 5.1 in [27] the cnoidal waves ϕ_c are also stable in $H^1_{per}([0,L])$. ■

III. CONCLUSION

In this paper, we established the *(orbital) stability* of traveling wave solutions in the case of solitary waves and periodic waves for a KdV equation which includes a fifth order dispersive term. We demonstrate that the sufficient conditions for stability in the current literature are satisfied for the set of traveling waves considered herein. When the solution is given in terms of a differentiable family, the inner product $I=(\psi,\varphi)_2$ can be computed with the help of Parseval's theorem. The inner product represents a functional constructed from conserved quantities and is used in the proof of stability theorems. In the case where the solution does not present itself as a differentiable family, the method of Gegenbauer polynomials was used to determine the sign of the inner product I. The stability is determined by applying existing results in the current literature which exploit the positivity properties of the Fourier transform of the solutions. For periodic solutions which are given as powers of a Jacobian elliptic function we use the recurrence formula for the coefficients of the Fourier series found in [20]. To our knowledge the stability for the traveling waves considered here have not been previously established.

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