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Shahrdad Sajjadi
Embry-Riddle Aeronautical University, sajja8b5@erau.edu

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By Shahrdad G. Sajjadi†

Department of Mathematics, Embry-Riddle Aeronautical University, 600 S. Clyde Morris Boulevard, Daytona Beach, FL 32114-3900, USA.

Abstract

The work described here shows that the known variational principle for the Navier-Stokes equations and the adjoint system can be modified to produce a set of Euler-Lagrange variational equations which have the same order and same solution as the Navier-Stokes equations provided the adjoint system has a unique solution, and provided in the steady state case, that the Reynolds number remains finite.

1. Introduction

Given a differential equation subject to some boundary conditions, it is an important question to decide if a variational principle exists, especially if this variational principle is to be used for providing an approximate solution to the original problem. For some problems minimum and maximum principles may hold, leading to reciprocal variational principles which provide a means for obtaining upper and lower bounds on a variational integral. In other problems, however, the variational principle, if at all exists, may only be a stationary principle and no minimum or maximum can be achieved.

For nonlinear and non self-adjoint operators the construction of variational principle is not well understood. Millikan [1] and Finlayson [2] have shown that there is no variational principle for the Navier-Stokes equations involving soley the velocity vector $\boldsymbol{u}$ and the pressure p unless either $\boldsymbol{u}\times\boldsymbol{\nabla}\times\boldsymbol{u} = 0$ or $\boldsymbol{u}.\boldsymbol{\nabla}\boldsymbol{u} = 0$, when the equations effectively becomes linear.

Under such circumstances we can express the velocity vector

$$\boldsymbol{u} = \boldsymbol{\nabla}\psi + \lambda\boldsymbol{\nabla}\mu$$

in which ψ, λ and μ are scalar functions of $\boldsymbol{x} \in \Omega$ and t, $\Omega \subseteq \Re^3$. The scalar functions λ and μ satisfy

$$\frac{D\lambda}{Dt} = \frac{D\mu}{Dt} = 0 \tag{1.1}$$

where $D/Dt \equiv \partial_t + \boldsymbol{u}.\boldsymbol{\nabla}$. Except for an arbitrary function of t alone, ψ is then determined uniquely by λ and μ *qua* functions of the space variables x_j for each t.

†Also visiting scholar at Trinity College, Cambridge.

With λ and μ given in the domain Ω, the residual equation of mass conservation

$$\begin{array}{lll} \boldsymbol{\nabla}.\boldsymbol{u} = \Delta\psi + \lambda\Delta\mu + \boldsymbol{\nabla}\lambda.\boldsymbol{\nabla}\mu = 0 & \text{in} & \Omega \\ \boldsymbol{n}.(\boldsymbol{\nabla}\psi + \lambda\boldsymbol{\nabla}\mu) = 0 & \text{on} & \partial\Omega \end{array}$$

(if Ω is taken to be bounded rigidly or

$$|\boldsymbol{\nabla}\psi + \lambda\boldsymbol{\nabla}\mu| \to 0 \quad \text{as} \quad |\boldsymbol{x}| \to \infty$$

if Ω is unbounded), comprise a Neumann problem for ψ. The problem is soluble in at least a weak form if merely $\lambda\boldsymbol{\nabla}\mu \in L^2(\Omega)$, where $\lambda \in L^\infty(\Omega)$ and $\mu \in H^1(\Omega)$. The respective variational principle of ψ is given by

$$J(\psi, \lambda, \mu) = \min_{f \in H^1(\Omega)} J(f, \lambda, \mu) \tag{1.2}$$

where

$$J(f, \lambda, \mu) = \int_\Omega \left(\tfrac{1}{2}|\boldsymbol{\nabla} f|^2 + \lambda\boldsymbol{\nabla}\mu.\boldsymbol{\nabla} f\right) d\boldsymbol{x} + \int_{\partial\Omega} f\lambda\boldsymbol{\nabla}\mu.\boldsymbol{n}\, dS$$

With ψ thus determined as a functional transformation of λ and μ, the Cauchy problem (1.1) with solution $(\lambda, \mu)(t)$ appears complete, and the following conclusion can be drawn. If potentials $(\lambda, \mu)(0)$ match the given initial data for the inviscid incompressible fluid equations, namely the Euler equations, then the evolution of (λ, μ) according to (1.1) with ψ fixed by (1.2) at each $t \geq 0$, fully reproduces the solution of the hydrodynamic problem.

In this work we present a variational principle for the Navier-Stokes equations and the adjoint system. This variational principle is then modified to produce a set of Euler-Lagrange variational equations which have the same order and solution as the Navier-Stokes equations provided the adjoint system has a unique solution.

2. A variational principle for a non self-adjoint problem

One of the simplest non self-adjoint boundary value problems is that for a damped harmonic oscillator with fixed end conditions,

$$y'' + 2ay' + by = 0, \qquad 0 < x < 1, \qquad (\ )' \equiv \frac{d}{dx} \tag{2.1}$$

where a and b are constant and $y(0) = \alpha$, $y(1) = \beta$. This problem has a unique solution provided $\sqrt{(b - a^2)}$ is not an integral multiple of π. Finding a variational principle for this problem, involving soley y and its derivatives without first introducing a transformation of variables of some type, is not possible. It is easy to show that (2.1) is exactly equivalent to the variational problem of finding stationary values of a functional of two independent functions of x. The appropriate problem is the following:

Find stationary values for the functional

$$J(y_1, y_2) = \frac{1}{2}\int_0^1 \left[(y_1'^2 - 2ay_2'y_1 - by_1^2) - (y_2'^2 - 2ay_1'y_2 - by_2^2)\right] dx \tag{2.2}$$

$$b - a^2 \neq m^2\pi^2, \qquad m \text{ an integer} \tag{2.3}$$

among the class of functions $y_1(x)$, $y_2(x)$ which have continuous second derivatives and satisfy the following boundary condtions as y,

$$y_1(0) = y_2(0) = \alpha, \qquad y_1(1) = y_2(1) = \beta \tag{2.4}$$

The Euler-Lagrange equations for the functional (2.2) are

$$\left.\begin{array}{ll} y_1: & y_1'' + 2ay_2' + by_1 = 0 \\ y_2: & y_2'' + 2ay_1' + by_2 = 0 \end{array}\right\} \tag{2.5}$$

so that the difference function $\overline{y} = (y_1 - y_2)/2$ satisfies the homogeneous problem

$$\overline{y}'' - 2a\overline{y}' + b\overline{y} = 0, \qquad \overline{y}(0) = \overline{y}(1) = 0$$

When the condition (2.3) is imposed, which is precisely the condition that the original problem should have a unique solution, it is easy to see that $\overline{y} = 0$, and that equations (2.5) with boundary conditions (2.4) become

$$y'' + 2ay' + by = 0, \quad y(0) = \alpha, \quad y(1) = \beta$$

the original problem.

The Lagrangian in the functional (2.2) may be written in term of

$$y = \frac{y_1 + y_2}{2}, \qquad \overline{y} = \frac{y_1 - y_2}{2}$$

and the functional then has the form

$$\begin{aligned} J(y_1, y_2) &= 2\int_0^1 (y'\overline{y}' + 2ay\overline{y}' - by\overline{y})\, dx - 2a\, [y\overline{y}]_0^1 \\ &= -2\int_0^1 \overline{y}\, (y'' + 2ay' + by)\, dx \end{aligned}$$

The variational problem is thus equivalent to a Galerkin method applied to the original problem with a weighting function which vanishes at the boundaries.

3. A variational principle for Navier-Stokes equations

We denote by $L^2(\Omega)$ the space of square integrable functions on Ω and by $H^1(\Omega)$ the Sobolev space made up of the functions which are in $L^2(\Omega)$. Let $H^{-1}(\Omega) = (H^1(\Omega))'$ be the topological dual space of $H^1(\Omega)$ such that $H^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. The domain Ω is assumed to be simply-connected with a C^∞ boundary $\partial\Omega$.

The Navier-Stokes equations for an incompressible flow, normalized to unit density, may be written in the form

$$\left.\begin{array}{l} \dfrac{\partial \boldsymbol{u}}{\partial t} - \nu\Delta\boldsymbol{u} + \displaystyle\sum_{j=1}^{3} u_j \frac{\partial \boldsymbol{u}}{\partial x_j} + \boldsymbol{\nabla} p = \boldsymbol{f} \\ \boldsymbol{\nabla}.\boldsymbol{u} = 0 \end{array}\right\} \quad \text{in } \Omega \cup \partial\Omega \tag{3.1}$$

with $\boldsymbol{f}$ given in $H^{-1}(\Omega)$.

We assume the boundary and initial conditions

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{U}, \quad \boldsymbol{u}|_{t=0} = \boldsymbol{a}$$

where Δ is the Laplacian operator which is an isomorphism from $H^1(\Omega)$ onto $H^{-1}(\Omega)$ and ν is a positive parameter; the kinematic viscosity.

We will now present a variational statement for the Navier-Stokes equations in the absence of any external forces ($\boldsymbol{f} = 0$); being a straightforward generalization of that for the one-dimensional damped harmonic oscillator, with fixed end conditions, of the previous section.

Consider the problem of finding stationary values for the functional

$$J(u_i, p, w_i, r) = \int_0^\tau \int_\Omega \mathscr{L}(u_i, p, w_i, r)\, d\boldsymbol{x}\, dt \tag{3.2}$$

where $\mathscr{L}$ is the Lagrangian

$$\begin{aligned}\mathscr{L} &= \frac{\nu}{2}\left(\frac{\partial u_i}{\partial x_j}\right)^2 + \frac{1}{2}(w_i + u_i)u_j\frac{\partial w_i}{\partial x_j} + u_i\frac{\partial p}{\partial x_i} + \frac{u_i}{2}\frac{\partial w_i}{\partial t} \\ &- \frac{\nu}{2}\left(\frac{\partial w_i}{\partial x_j}\right)^2 - \frac{1}{2}(u_i + w_i)w_j\frac{\partial u_i}{\partial x_j} - w_i\frac{\partial r}{\partial x_i} - \frac{w_i}{2}\frac{\partial u_i}{\partial t}\end{aligned}$$

of the functions $u_i, p, w_i, r (i = 1, 2, 3)$ dependent on the cartesian spatial variables $\boldsymbol{x} \in \Omega$ and the time interval $t \geq 0$. The double summation convention (summation over the reapeated indicies) is being used and the interval is over the domain Ω with smooth boundary $\partial\Omega$ and the time interval $[0, \tau]$. The class of admissible functions is such that

(i) u_i, w_i have second order continuous spatial derivatives and first order continuous time derivative, p, r, have first order continuous spatial derivatives,

(ii) $u_i = w_i = f_i \;\forall\; \boldsymbol{x} \in \partial\Omega$ for $t \in [0, \tau]$ where $\int_{\partial\Omega} f_i n_i = 0$,

(iii) $u_i = w_i = g_i \;\forall\; \boldsymbol{x} \in \Omega$ at $t = 0$ where g_i are the components of a solenoidal vector field with which f_i is compatiable. Hence g_i is treated as an element of the space H such that $g_i \in H^\infty(\Omega) \times H^\infty(\Omega) \subset f_i(\Omega \to \Re^3)$,

(iv) $u_i = w_i \;\forall\; \boldsymbol{x} \in \Omega$ at $t = \tau$,

(v) $p = r \;\forall\; \boldsymbol{x} \in \partial\Omega$ for $t \in [0, \tau]$.

It will become clear that certain additional constraints are required when the flow is steady and these will be stated at the appropriate time.

The variation of J due to variations δu_i etc. in the class of admissible functions is

$$\begin{aligned}\delta J(u_i, p, w_i, r) &= \int_0^\tau \int_\Omega \left[A_i(u, w, p)\delta u_i - A_i(w, u, r)\delta w_i\right]\, d\boldsymbol{x}\, dt \\ &+ \int_0^\tau \int_\Omega \left[B(u, w, p, \delta u, \delta w, \delta p) - B(w, u, r, \delta w, \delta u, \delta r)\right]\, d\boldsymbol{x}\, dt\end{aligned} \tag{3.3}$$

where

$$
\begin{aligned}
A_i(u,w,p) &= \frac{1}{2}\left[u_j\frac{\partial w_i}{\partial x_j} + (w_j+u_j)\frac{\partial w_j}{\partial x_i}\right] + \frac{\partial p}{\partial x_i} + \frac{1}{2}\frac{\partial w_i}{\partial t} - \frac{1}{2}w_j\frac{\partial u_i}{\partial x_j} \\
B(u,w,p,\delta u,\delta w,\delta p) &= \nu\frac{\partial u_i}{\partial x_j}\frac{\partial \delta u_i}{\partial x_j} + \frac{1}{2}(u_i+w_i)u_j\frac{\partial \delta w_i}{\partial x_j} + u_i\frac{\partial \delta p}{\partial x_i} + \frac{u_i}{2}\frac{\partial \delta w_i}{\partial t}
\end{aligned}
$$

The first integral involving B may be integrated by parts to give

$$
\begin{aligned}
&\int_0^\tau\int_\Omega B(u,w,p,\delta u,\delta w,\delta p)\,d\boldsymbol{x}\,dt = \\
-&\int_0^\tau\int_\Omega \left\{\nu\frac{\partial^2 u_i}{\partial x_j^2}\delta u_i + \frac{1}{2}\left[\frac{\partial(u_i+w_i)u_j}{\partial x_j} + \frac{\partial u_i}{\partial t}\right]\delta w_i + \frac{\partial u_i}{\partial x_i}\delta p\right\} d\boldsymbol{x}\,dt \\
+&\int_0^\tau\int_{\partial\Omega}\left\{\nu\frac{\partial u_i}{\partial x_j}\delta u_i + \frac{1}{2}(u_i+w_i)u_j\delta w_i + u_j\delta p\right\} n_j\,dS\,dt + \int_\Omega\left[\frac{1}{2}u_i\delta w_i\right]_0^\tau d\boldsymbol{x}
\end{aligned}
\qquad (3.4)
$$

Conditions (ii) and (v) ensure that the boundary integral vanishes and conditions (iii) and (iv) that the time independent integral vanishes. The resulting Euler-Lagrange equations are

$$
\left.
\begin{aligned}
p:&\quad \frac{\partial u_i}{\partial x_i} = 0 \\
r:&\quad \frac{\partial w_i}{\partial x_i} = 0 \\
u_i:&\quad \nu\frac{\partial^2 u_i}{\partial x_j^2} - \frac{\partial p}{\partial x_i} = \frac{\partial w_i}{\partial t} + \frac{u_j+w_i}{2}\left(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}\right) \\
w_i:&\quad \nu\frac{\partial^2 w_i}{\partial x_j^2} - \frac{\partial r}{\partial x_i} = \frac{\partial u_i}{\partial t} + \frac{w_j+u_i}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)
\end{aligned}
\right\}
\qquad (3.5)
$$

At this point it should be noted that if $u_i = w_i$ then $\boldsymbol{\nabla}p$ and $\boldsymbol{\nabla}r$ are identical and that there are only four independent equation; these are identical in form to equations (3.1) provided $p + u_i^2/2$ is interpreted as the pressure.

The difference functions

$$
\overline{v}_i = \frac{u_i - w_i}{2}, \qquad \overline{q} = \frac{p-r}{2} isnottoolarge \qquad (3.6)
$$

satisfy

$$
\begin{aligned}
\frac{\partial \overline{v}_i}{\partial x_i} &= 0 \\
\nu\frac{\partial^2 \overline{v}_i}{\partial x_j^2} - \frac{\partial \overline{q}}{\partial x_i} &= -\frac{\partial \overline{v}_i}{\partial t} - \frac{u_j+w_j}{2}\left(\frac{\partial \overline{v}_i}{\partial x_j} + \frac{\partial \overline{v}_j}{\partial x_i}\right)
\end{aligned}
\qquad (3.7)
$$

subject to the boundary and initial conditions

$$
\begin{aligned}
&\overline{v}_i = 0 \quad \forall \boldsymbol{x}\in\partial\Omega \quad \text{for} \quad t\in[0,\tau] \\
\text{and}\quad &\overline{v}_i = 0 \quad \forall \boldsymbol{x}\in\Omega \quad \text{at} \quad t = 0,\tau
\end{aligned}
$$

A problem essentially identical to that posed by equation (3.7) arises in providing uniqueness for weak solutions of the Navier-Stokes equations, see for example, Ladyzhenskaya [3]. However, a more direct approach proceeds as follows.

Multiplying (3.7) by $\overline{v}_i$ and using the incompressibility conditions, we obtain

$$\frac{1}{2}\frac{\partial \overline{v}_i^2}{\partial t} = \nu\left(\frac{\partial \overline{v}_i}{\partial x_j}\right)^2 + \frac{\partial}{\partial x_j}\left\{\overline{v}_i\overline{q} - \nu\overline{v}_j\frac{\partial \overline{v}_j}{\partial x_i} - \frac{u_i + w_i}{4}\overline{v}_j^2\right\} - \frac{u_j + w_j}{2}\overline{v}_i\frac{\partial \overline{v}_j}{\partial x_i} \tag{3.8}$$

Integration of (3.8) over the domain Ω and use of the boundary condition $\overline{v}_i = 0$ on $\partial\Omega$ gives

$$\frac{1}{2}\frac{dE}{dt} = \frac{1}{2}\frac{d}{dt}\int_\Omega \overline{v}_i^2\, d\boldsymbol{x} = \int_\Omega\left[\nu\left(\frac{\partial \overline{v}_i}{\partial x_j}\right)^2 - \frac{\overline{v}_i\overline{v}_j}{2}\frac{\partial(u_j + w_j)}{\partial x_i}\right] d\boldsymbol{x} \tag{3.9}$$

As both u_i and w_i are components for solenoidal velocity fields the last term exeeds $-m\int_\Omega \overline{v}_i^2\, d\boldsymbol{x}$, where $m > 0$ is the maximum eigenvalue of the deformation tensor for the forcing flow,

$$\frac{1}{2}\left[\frac{\partial(u_j + w_j)}{\partial x_i} + \frac{\partial(u_i + w_i)}{\partial x_j}\right]$$

Integration of (3.9) over the range $t_0 < t < \tau$ where $0 \le t_0 \le \tau$ gives $E(t_0)e^{2mt_0} < E(\tau)e^{2m\tau}$. Condition (iii) has $u_i = w_i$ at $t = 0\ \forall\ \boldsymbol{x} \in \Omega$ thus $E(0) = 0$ and consequently $\overline{v}_i = 0$ and $\boldsymbol{\nabla} q = 0\ \forall\ t \in [0, \tau]\ \forall\ \boldsymbol{x} \in \Omega$.

It is of some interest to discuss the precise nature of the stationary point for the functional. The following argument shows that the stationary point is indeed unique. Suppose, otherwise, that there are two stationary points corresponding to (u_i^1, p^1, w_i^1, r^1) and (u_i^2, p^2, w_i^2, r^2). Each of these sets of functions satisfy the system (3.5). The previous method may be applied to the differences

$$u_i^* = \frac{u_i^1 - u_i^2}{2}, \qquad p^* = \frac{p^1 - p^2}{2}$$

or

$$w_i^* = \frac{w_i^1 - w_i^2}{2}, \qquad r^* = \frac{r^1 - r^2}{2}$$

with the results that $u_i^* = w_i^* = 0$ and $\boldsymbol{\nabla} p^* = \boldsymbol{\nabla} r^* = 0$. The functional is zero at the stationary point and this value is also taken at any point where the test functions are such that $u_i = w_i$ and $p = r$. Now the interchange of (u, p) with (w, r) changes the sign of J and thus is clear that the stationary point is neither a maximum nor a minimum.

4. A variational principle for the steady flow

When the flow is time independent the appropriate functional is a suitably modified version of (3.2), i.e.,

$$J^s(u_i, p, w_i, r) = \int_\Omega \mathscr{L}(u_i, p, w_i, r)\, d\boldsymbol{x} \tag{4.1}$$

Now the functions u_i, p, w_i, r are required to be sufficiently smooth as in condition (i) and such that the pairs $(u_i, p), (w_i, r)$ take the same values on the boundary $\partial\Omega$. A similar derivation for the Euler-Lagrange equations leads to a time independent version of (3.5) and the difference functions satisfy the steady versions of (3.7) and the same boundary conditions. To show the uniquness of the stationary point in non-steady problem, we set equation (3.9) to zero. We can then see that

$$\begin{aligned}\nu\int_\Omega\left(\frac{\partial\overline{v}_i}{\partial x_j}\right)^2 d\boldsymbol{x} &= \left|\int_\Omega \frac{\overline{v}_i\overline{v}_j}{2}\frac{\partial(u_j+w_j)}{\partial x_i}\,d\boldsymbol{x}\right| \\ &\leq \left\{\int_\Omega (\overline{v}_i\overline{v}_j)^2\,d\boldsymbol{x}\right\}^{\frac{1}{2}}\left\{\int_\Omega \frac{1}{4}\left(\frac{\partial(u_j+w_j)}{\partial x_i}\right)^2 d\boldsymbol{x}\right\}^{\frac{1}{2}}\end{aligned}$$

using the Schwarz's inequality, and in three-dimensions the above inequality is replaced by its generalized form given by Serrin [4]

$$\nu\int_\Omega\left(\frac{\partial\overline{v}_i}{\partial x_j}\right)^2 d\boldsymbol{x} \leq 3^{-3/4}\left\{\int_\Omega \overline{v}_i^2\,d\boldsymbol{x}\right\}^{\frac{1}{4}}\left\{\int_\Omega\left(\frac{\partial\overline{v}_i}{\partial x_j}\right)^2 d\boldsymbol{x}\right\}^{\frac{3}{4}}\left\{\frac{1}{4}\int_\Omega\left(\frac{\partial(u_j+w_j)}{\partial x_i}\right)^2 d\boldsymbol{x}\right\}^{\frac{1}{2}}$$

Payne and Weinberger [5] have provided a further inequality on the first integral on the right so that finally

$$\nu\int_\Omega\left(\frac{\partial\overline{v}_i}{\partial x_j}\right)^2 d\boldsymbol{x} \leq \frac{3^{-3/4}}{\lambda^{1/4}}\int_\Omega\left(\frac{\partial\overline{v}_i}{\partial x_j}\right)^2 d\boldsymbol{x}\left\{\frac{1}{4}\int_\Omega\left(\frac{\partial(u_j+w_j)}{\partial x_i}\right)^2 d\boldsymbol{x}\right\}^{\frac{1}{2}}$$

where λ is approximately $20R^{-2}$, and R is the radius of the smallest sphere enclosing Ω. The solution for $\overline{v}_i$ is zero when

$$\frac{R^{1/2}}{\nu}\left\{\frac{1}{4}\int_\Omega\left(\frac{\partial(u_j+w_j)}{\partial x_i}\right)^2 d\boldsymbol{x}\right\}^{\frac{1}{2}} \leq R^{1/2}\lambda^{1/4}3^{3/4} \approx 3 \tag{4.2}$$

Estimates for the left hand side of this expression in terms of boundary data can be provided in the following way.

The steady state versions of equations (3.5) imply that

$$\frac{\nu}{2}\frac{\partial^2(u_i+w_i)}{\partial x_j^2} - \frac{1}{2}\frac{\partial(p+r)}{\partial x_i} = \frac{1}{4}(u_j+w_j)\left\{\frac{\partial(u_i+w_i)}{\partial x_j}+\frac{\partial(u_j+w_j)}{\partial x_i}\right\} \tag{4.3}$$

If equations (4.3) are multiplied by $(u_i+w_i)/2$ and then integrated over the domain Ω, we obtain

$$\begin{aligned}&\frac{\nu}{4}\int_\Omega\left[\frac{\partial(u_i+w_i)}{\partial x_j}\right]^2 d\boldsymbol{x} + \frac{1}{4}\int_{\partial\Omega}\left[p+r+\frac{(u_j+w_j)^2}{4}\right](u_i+w_i)n_i\,dS \\ = &\frac{1}{4}\int_\Omega (u_i+w_i)(u_j+w_j)\frac{\partial(u_j+w_j)}{\partial x_i}\,d\boldsymbol{x}\end{aligned}$$

The integral on the right hand side can be shown to be smaller than

$$\frac{3^{-3/4}}{\lambda^{1/4}}\left\{\frac{1}{4}\int_{\Omega}\left(\frac{\partial(u_j+w_j)}{\partial x_i}\right)^2 d\boldsymbol{x}\right\}^{\frac{3}{2}}$$

by a similar manipulation to that used previously so that if condition (4.2) is satisfied then

$$\begin{aligned}&\frac{1}{4}\left\{\nu-\frac{3^{-3/4}}{4\lambda^{1/4}}\int_{\Omega}\left(\frac{\partial(u_j+w_j)}{\partial x_i}\right)^2 d\boldsymbol{x}\right\}^{\frac{1}{2}}\int_{\Omega}\left(\frac{\partial(u_i+w_i)}{\partial x_j}\right)^2 d\boldsymbol{x}\\ <\ &\left|\frac{1}{4}\int_{\partial\Omega}\left[p+r+\frac{(u_j+w_j)^2}{4}\right](u_i+w_i)n_i\,dS\right|\end{aligned}$$

As in the non-steady problem, the stationary point is unique and is neither a maximum or a minimum.

5. An extended variational principle

It is possible to remove a number of the restrictions on the class of admissible functions by adding a surface integral to the previous functional J given by (3.2). Consider the new functional

$$I = J + \int_0^{\mathcal{T}}\int_{\partial\Omega}\mathscr{M}_j(u_i,p,w_i,r)n_j\,dS\,dt \tag{5.1}$$

where n_j are the components of the outward normal to $\partial\Omega$ and

$$\begin{aligned}\mathscr{M}_j(u_i,p,w_i,r) &= -u_j^s p + w_j^s r + \frac{u_j u_i^2 + u_i^2 w_j}{4} - \frac{w_j w_i^2 + w_i^2 u_j}{4}\\ &+ \nu\left[-(u_i-u_i^s)\frac{\partial u_i}{\partial x_j} + (w_i-w_i^s)\frac{\partial w_i}{\partial x_j}\right.\\ &+ \left.\varepsilon_{ijk}(w_k-w_k^s-u_k+u_k^s)\frac{\partial}{\partial x_i}|\boldsymbol{u}-\boldsymbol{u}^s+\boldsymbol{w}-\boldsymbol{w}^s|\right]\end{aligned}$$

where ε_{ijk} is the permutation tensor and the superscript s indicates the surface values.

The derivation of this expression for $\mathscr{M}_j$ is complicated by the fact that when $\nu = 0$ the physical boundary condition on $\partial\Omega$ should relate only to the normal velocity component: accordingly, normal and tangential components require separate treatment.

We require I to be stationary ($\delta I = 0$) subject to the conditions (i), (ii) and (iv) where in addition $u_i = w_i$ at one point on $\partial\Omega$ and the variation are such that $\delta u_i + \delta w_i = 0$ on $\partial\Omega$. It is then readily verified that variations of u_i, p, w_i, r leads to the Navier-Stokes equations together with its adjoint system.

However, the boundary conditions on $\partial\Omega$ require a little care. Incorporating the additional integral of $\mathscr{M}_j$, the variations in w_j gives

$$n_i(u_i-u_i^s)(u_j-u_j^s) + \nu n_i\varepsilon_{ijk}\frac{\partial}{\partial x_k}|\boldsymbol{u}-\boldsymbol{u}^s| = 0 \tag{5.2}$$

Taking the scalar product of (5.2) with n_j, we obtain

$$n_i(u_i - u_i^s) = 0 \ \ \forall \ \ \boldsymbol{x} \in \partial\Omega$$

independently of whether ν is zero or not. This is, of course, the inviscid boundary conditions, which prescribes the normal velocity component at each point on $\partial\Omega$. Accordingly, the second term in (5.2) must itself vanish. In vector notation, this is just $\nu(\boldsymbol{n}\times\boldsymbol{\nabla}|\boldsymbol{u} - \boldsymbol{u}^s|) = 0$, which if $\nu \neq 0$, asserts that $|\boldsymbol{u} - \boldsymbol{u}^s|$ remains constant on $\partial\Omega$. If we now introduce the further requirement that $\boldsymbol{u} = \boldsymbol{u}^s$ at a *single point* on $\partial\Omega$, we have the boundary conditions for viscous flow that

$$u_i = u_i^s \ \ \forall \ \ \boldsymbol{x} \in \partial\Omega$$

Variations of p and u_i give the corresponding boundary conditions for w_i on $\partial\Omega$. Variation of p leads immediately to

$$n_i w_i = 0 \tag{5.3}$$

corresponding to the inviscid boundary condition for u_i. Variation of u_i, subject to the requirement that $\delta u_i = 0$ on $\partial\Omega$, leads to

$$\nu n_i w_j \left\{ \frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_j}{\partial x_i} - \varepsilon_{ijk} \frac{\partial |\delta \boldsymbol{u}|}{\partial x_k} \right\} = 0 \ \ \forall \ \ \boldsymbol{x} \in \partial\Omega$$

However, since $\delta u_i = 0$ on $\partial\Omega$,

$$\begin{aligned} \frac{\partial \delta u_i}{\partial x_j} &= n_i \frac{\partial \delta u_i}{\partial \ell} \\ \frac{\partial |\delta \boldsymbol{u}|}{\partial x_k} &= n_k \frac{\partial |\delta \boldsymbol{u}|}{\partial \ell} \end{aligned}$$

where ℓ denotes distance along the outward normal on $\partial\Omega$. This yields

$$\nu(n_i n_j w_j + w_i) \frac{\partial \delta u_i}{\partial \ell} = 0$$

the third term vanishing identically. Using (5.3) we have, from variation in u_i,

$$w_i = 0 \ \ \forall \ \ \boldsymbol{x} \in \partial\Omega$$

whenever $\nu \neq 0$. There is of course a corresponding result for a steady state problem.

6. Summary

To summarise the results obtained in this work we note the following:

(a) the functional (3.2) of the eight functions $u_i, p, w_i, r (i = 1, 2, 3)$ which belong to the class of functions (i)-(v) has a unique stationary value of zero at the point (v_i, q, v_i, q) where the equations satisfied by v_i and q are the Navier-Stokes equations

$$\begin{aligned} \frac{\partial v_i}{\partial x_i} &= 0 \\ \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} &= -\frac{\partial}{\partial x_i}\left(q + \tfrac{1}{2} v_j^2\right) + \nu \frac{\partial^2 v_i}{\partial x_j^2} \end{aligned}$$

within a domain Ω with smooth boundary $\partial\Omega, v_i$ is given on Ω at $t = 0$ and on $\partial\Omega$ for $t \in [0, \tau]$.

(b) The functional (4.1) of the eight functions $u_i, p, w_i, r (i = 1, 2, 3)$ which belong to the class functions satisfying

(i) u_i, w_i have continuous second order spatial derivatives and p and r have continuous first order spatial derivatives within the domain Ω,

(ii) $u_i = w_i = g_i \ \forall \ \boldsymbol{x} \in \partial\Omega$ where $\int_{\partial\Omega} g_i n_i \, dS = 0$,

(iii) $p = r \ \forall \ \boldsymbol{x} \in \partial\Omega$, and

(iv) condition (4.2) is satisfied,

has a unique stationary value zero at the stationary point (v_i, q, v_i, q) where v_i, q satisfy the steady state Navier-Stokes equations

$$\begin{aligned} \frac{\partial v_i}{\partial x_i} &= 0 \\ v_j \frac{\partial v_i}{\partial x_j} &= -\frac{\partial}{\partial x_i}\left(q + \tfrac{1}{2} v_j^2\right) + \nu \frac{\partial^2 v_i}{\partial x_j^2} \end{aligned}$$

within a domain Ω with smooth boundary $\partial\Omega$ and where v_i is given on $\partial\Omega$.

It is to be noted that in the steady state case if the Reynolds number does not remain finite (4.2) will not be satisfied and so its use is somewhat restricted.

Although these variational integrals yield the correct equations in Ω, they do not incorporate the correct physical boundary conditions on $\partial\Omega$. To circumvent this difficulty, it is necessary to add a further surface integral, namely (5.1).

References

[1] Millikan, C.B. (1929) On the steady motion of viscous incompressible fluids, with particular reference to a variational principle. *Phil. Mag.*, **7**, 641.

[2] Finlayson, B.A. (1972) Existence of variational principles for the Navier-Stokes equations. *Phys. Fluids*, **15**, 963.

[3] Ladyzhenskaya, O.A. (1969) *The mathematical theory of viscous incompressible flow.* Gordon and Breach.

[4] Serrin, J. (1963) The initial value problem for the Navier-Stokes equations. In *Nonlinear problems* ed. R.E. Langer, The University of Wisconsin press, page 98.

[5] Payne, L.E and Weinberger, H.F. (1963) An exact stability bound for Navier-Stokes flow in a sphere. In *Nonlinear problems* ed. R.E. Langer, The University of Wisconsin press, page 311.