An Asymptotic Analysis for Generation of Unsteady Surface Waves on Deep Water by Turbulence

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An asymptotic analysis for generation of unsteady surface waves on deep water by turbulence

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The detailed mathematical study of the recent paper by Sajjadi, Hunt and Drullion (2014) is presented. The mathematical development considered by them, for unsteady growing monochromatic waves is also extended to Stokes waves. The present contribution also demonstrates agreement with the pioneering work of Belcher and Hunt (1993) which is valid in the limit of the complex part of the wave phase speed \( c_i \downarrow 0 \). It is further shown that the energy-transfer parameter and the surface shear stress for a Stokes wave reverts to a monochromatic wave when the second harmonic is excluded. Furthermore, the present theory can be used to estimate the amount of energy transferred to each component of nonlinear surface waves on deep water from a turbulent shear flow blowing over it. Finally, it is demonstrated that in the presence of turbulent eddy viscosity the Miles (1957) critical layer does not play an important role. Thus, it is concluded that in the limit of zero growth rate the effect of the wave growth arises from the elevated critical layer by finite turbulent diffusivity, so that the perturbed flow and the drag force is determined by the asymmetric and sheltering flow in the surface shear layer and its matched interaction with the upper region.

1. Introduction

It is well known that a surface wave travelling along a water surface can force a couple motion in the air and water, both propagating at the same speed, namely the eigenvalue \( c_r \), being the real part of the wave speed \( c \). Hence the surface wave could force an unstable shear mode in the air, which then grows and induces growth of the water wave. In a pioneering work, Miles (1957) constructed a model for generation of waves by shear flows by assuming that the critical height is sufficiently high that the turbulent stresses could be neglected. Given this assumption, he argued that the airflow perturbations are described by Rayleigh equation

\[
(U - c)(\phi'' - k^2 \phi) - U''\phi = 0 \tag{1.1}
\]

for the non-dimensional perturbation stream function \( \phi \). In (1.1), \( U(z) \) is the undisturbed velocity profile for the wind, blowing over the waves, and \( k \) is the wavenumber. Clearly, unless the wave amplitude varies with time, i.e. \( c_i \neq 0 \), the equation (1.1) is singular at critical point \( z_c \). By solving (1.1) above and below the air-water interface and by matching the vertical velocity and pressure at \( z_c \), Miles (1957) calculated \( c_i \) in the limit as \( c_i/U_1 \downarrow 0 \) from the resulting eigenvalue relationship.

Miles (1957) showed

\[
c^2 = c_w^2 + s(\alpha + i\beta)U_1^2 \tag{1.2}
\]

where \( c_w = \sqrt{g/k} \) is the free surface wave speed, \( s = \rho_a/\rho_w \ll 1 \) and \( U_1 = U_a/\kappa \), with \( U_a \) representing the wind friction velocity, \( \rho_a \) and \( \rho_w \) are air and water densities, respectively, and \( \kappa \)
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is the von Kármán’s constant. He then deduced the growth of a monochromatic wave is given by the following expression

\[ \zeta_a = 2 \mathcal{F}\{c\}/\mathcal{R}\{c\} = s \beta (U_1/c) \]  

(1.3)

The very important aspect of (1.2) and (1.3) is that, for a steady wave (in which the wave amplitude \( a \) remains constant), \( c \) must have a non-zero imaginary part, i.e. such a wave will only grow if \( c_i \neq 0 \). This is quite evident from equations (1.2) and (1.3).

In his paper Miles derived an integral expression for, what is commonly known as, the ‘energy-transfer parameter’ \( \beta \),

\[ \beta = -\mathcal{F}\left\{ \int_{\eta_c}^{\infty} |\phi|^2 (w''/w) d\eta \right\} \]  

(1.4)

where \( w \) is the dimensionless wind velocity profile and the suffix \( c \) indicating evaluation at the critical point \( \eta = \eta_c \) where the wind velocity equals the wave speed. However, in evaluation the integral in (1.4) and hence arriving at his ‘well known’ inviscid expression for \( \beta \) at \( \eta = \eta_c \), for \( w_c = 0 \) and \( c_i \neq 0 \),

\[ \beta = -\pi |\phi_c|^2 (w''_c/w'_c), \]  

(1.5)

he states “The path of integration in (1.5) must be indented either over or under the singularity at \( \eta = \eta_c \), where \( w(\eta_c) = 0 \), and on this choice depends the sign of \( \beta \)...”. At the bottom of the same paragraph he concludes that “... the path must be indented under the singularity.” Note the asterisks attached to the word ‘singularity’ refers to a crucial footnote which holds the key to Miles’ (1957) critical-layer theory. In this footnote he states “This assumes \( c = c_w \) is real. In the next approximation \( \mathcal{F}\{c\} > 0 \), so that the singularity lies slightly above the real axis (assuming \( w''_c/w'_c < 0 \)), and the path of integration in (1.4) passes under the singularity without the necessity of indentation.”

This important footnote is generally overlooked by many who refer to Miles’ (1957) critical-layer mechanism. This footnote is very significant in that (a) has a physical consequence which contradicts the results in (1.2) and (1.3) which shows clearly if \( c_i = 0 \) then \( \beta = 0 \) and hence waves cannot grow; and (b) has a mathematical consequence which indicates that equation (1.5) is valid if \( c_i \neq 0 \).

1.1. Physical mechanisms

We now present two alternative physical arguments that not only their results agree with each other but proves rigorously that Miles (1957) critical-layer mechanism is valid only for slowly-growing waves which does not apply to growth of surface waves by strong or turbulent shear flows in open ocean.

Belcher & Hunt (1993) (referred to as BH therein) considered a fully developed boundary layer over a two-dimensional monochromatic wave of small steepness \( ak \) propagating with small wave speed \( c \) and calculated the perturbations in the asymptotic limit \( \mathcal{W} = (U_c + c)/U \downarrow 0 \). In this limit, the critical height \( z_c \) lies within the inner surface layer, where the perturbation Reynolds shear stress varies slowly. Then, by considering the equation for the shear stress, they constructed solutions across the critical layer and demonstrated that the shear stress perturbation plays an important role at the critical height which in turn implies Miles’ (1957) inviscid theory is not the dominating mechanism for the wave growth in this parameter range. Note that the perturbations above the inner surface layer are not directly influenced by the critical height and the region below \( z_c \) where the flow reversal occurs (see figure 1). In fact this is very similar to the perturbations due to a static undulation, but with the difference that the effective roughness length, that determines the shape of the unperturbed velocity profile, is modified according to

\[ z_c = z_0 \exp(\kappa c/U_4). \]
F I G U R E 1. Schematic diagram for flow geometry and asymptotic multi-layer structure for analysing turbulent shear flow over steady and unsteady monochromatic waves. Taken from Sajjadi et al. (2014).

BH then used the solutions for the perturbations to the boundary layer and calculated the wave growth, which is determined, in the leading order of perturbation, by the asymmetric pressure perturbation induced by the thickening of the perturbed boundary layer on the leeside of the wave crest. To the first order in $\mathcal{W}$, BH discovered that there are new effects that contribute significantly to the rate of growth: (a) the asymmetries in both the normal and shear Reynolds shear stresses associated with the leeside thickening of the boundary layer, this they termed the non-separated sheltering (cf. Jeffreys 1925); (b) asymmetrical perturbations which are induced by the varying surface velocity associated with the fluid motion in the wave; and, (c) asymmetries induced by the variation in the surface roughness along the wave. The theoretical value, predicted by their theory, for the shear stress perturbation at the crest of the wave on the wave surface as well as on the top of the inner region is in good agreement with laboratory measurements. Hence, despite the restriction that $\mathcal{W} \ll 1$, their theory describes a large portion of the experimental observations of the wave growth rate made at sea and in the laboratory.

In a subsequent study, Belcher et al. (1999) (hereafter will be referred to as BHC) considered turbulent flow over growing waves, using triple-deck boundary-layer theory originally developed by Lighthill (1957) and Stewartson (1974), to analyse the sheltering mechanism described above. They suggested that Miles’ (1957) critical-layer theory generates growth of waves but this was not demonstrated in data. In this model they assumed the atmospheric mean flow is neutrally stable and is logarithmic and the surface wave, moving in the positive $x$-direction, is monochromatic whose profile is given by

$$z' = \text{Re}\{ae^{ik(x'-cr)}\}$$

where $c = c_r + ic_i$ is a complex wave speed, such that $c_r$ is the phase speed and the wave amplitude, $a$, grows exponentially at the rate $kc_i$. They considered a frame of reference moving with the wave at a speed $c_r$ so that the mean velocity profile can be expressed as

$$U(z) = U_1 \log(z/z_0) - c_r$$

where $U_1 = U_*/\kappa$, and that this wave speed vanishes at the critical height $z_c$. In this frame of
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reference the surface wave is described by

\[ z = \text{Re}\{ae^{ik(x-ict)}\} . \]

The boundary conditions imposed at the wave surface is that the wind velocity is equal to the
surface velocity of the water flow, being approximated by the surface value of the orbital velocity
of an irrotational wave on deep water. The other boundary condition is that perturbations in the
basic flow vanish as \( k z \uparrow \infty \).

BHC modelled the turbulent shear stress in the inner region, adjacent to the wave surface,
using a mixing length model, and in the region above this, the outer region, they invoked rapid
distortion theory to describe the turbulence. They showed that the depth of the inner region, \( \ell_i \),
may be obtained from the following implicit relation

\[ k \ell_i = \frac{2 \kappa^2}{|\ln(\ell_i/z_0) - \kappa c_r/U^*|} \]

where the variation of solution to this equation for \( \ell_i \) as a function of \( c_r/U^* \) for
\( k z_0 = 10^{-4} \) is shown in figure 2.

They further assumed the perturbations to the air flow are governed by equation (1.7) where the
turbulent stresses on the right-hand side is modelled by an eddy viscosity. The vertical component
of the velocity perturbation \( \vec{U} \equiv \vec{U} = (U, 0, W) \)
is expanded in the normal form

\[ W(x, z, t) = \hat{W}(k, z)e^{i(kx - ic_i t)} , \]

where the amplitude of the perturbation, \( \hat{W} \), satisfies the inhomogeneous Rayleigh equation

\[ \frac{\partial^2 \hat{W}}{\partial z^2} - \left( k^2 + \frac{U''}{U - ic_i} \right) \hat{W} = \frac{i}{U - ic_i} \frac{\partial^2 \hat{W}}{\partial z^2} \left( \nu_e \frac{\partial^2 \hat{W}}{\partial z^2} \right) \] \tag{1.6}

where \( \nu_e \) is an eddy viscosity.

BHC showed that in the middle layer, \( \ell_i \ll z \ll \ell_m \), the advection term is negligible compared
with the curvature term and the thus (1.6) reduces to

\[ \frac{\partial^2 \hat{W}}{\partial z^2} - \frac{U''}{U - ic_i} \hat{W} \sim 0 \] \tag{1.7}

whose leading-order solution may be expressed as

\[ \hat{W} \sim \{U(z) - ic_i\} \left\{ A + B \int \frac{d\xi}{U(\xi) - ic_i} \right\} \] \tag{1.8}

In (1.8) \( A \) and \( B \) are constants which may be determined by matching the inner and upper layer
solutions. Now, for slow waves, the critical layer lies close to the surface in the inner region.
Hence, the solution (1.8) is regular, since \( U(z) > 0 \) and does not vanish there. Therefore, the
integrand (1.8) is regular throughout the middle layer. Note, the same argument applies to the
moderate waves, but there are differences in its application.

Suppose now for a range of intermediate waves the critical layer lies in the outer region.
Thus, we can expect, for a particular range of \( c_r/U_* \), the critical layer lies within \( \ell_i \ll z \ll \ell_m \).
Neglecting the Reynolds stresses in the vicinity of the critical layer, \( \hat{W} \) satisfies (1.7) with the
solution given by (1.8).

However, in this scenario, the critical layer lies within the range of the integral in (1.8) and if
\( c_i = 0 \) then there is a singularity in this integrand at the critical layer, where \( U(z_c) = 0 \). This
singularity may be resolved by inertial effects, i.e. inviscid processes that control its dynamics.

If we now suppose the wave grows so that \( c_i > 0 \), then the integral (1.8) is regular at \( z = z_c \). If
we further assume the waves grows slowly, such that \( c_i \ll U_c^2 / U'' \), where the suffix \( c \) indicates
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Figure 2. Variation with $c_r/U_1$ of solutions for the normalized inner-region height, $k\ell_i$, and critical height, $kz_c$, when $kz_0 = 10^{-4}$. For given $c_r/U_1$, an inner, local equilibrium region lies between $kz = 0$ and the smallest value of $k\ell_i$, and an outer, rapid-distortion, region lies above. Solid lines: $k\ell_i$; dotted lines: $kz_c$.

evaluation at $z = z_c$, then the integral in (1.8) can be evaluated approximately. To do this, $U(z)$ is Taylor expanded in the vicinity of the critical layer, i.e. $U(z) \sim \xi U_c' + \frac{1}{2} \xi^2 U''_c$, where $\xi = z - z_c$ and thus the integral becomes

$$I = \int_{z_c}^{z} \frac{d\xi}{[U(\xi) - ic_i]^2} \sim \int \frac{d\xi}{\{\xi U_c' + \frac{1}{2} \xi^2 U''_c - ic_i\}^2}$$

$$= \frac{1}{c_i U_c'} \int \frac{d\xi}{(\xi U_c' - i)^2} \left\{1 + \frac{\varepsilon \left(\frac{\xi}{2} + i\right)^2}{2 \left(\frac{\xi}{2} + i\right)}\right\}^{-2}$$

(1.9)

where $\xi = \xi U_c'/c_i$ and $\varepsilon = c_i U''_c/U_c'^2 \ll 1$. Note that, if $U(z)$ has a logarithmic profile then $\varepsilon = c_i/U_s \ll 1$ which confirms that these are slowly-growing waves. In the limit of slow-growing waves the factor in braces can be expanded for $\varepsilon \ll 1$ to give

$$I \sim \frac{1}{c_i U_c'} \int \frac{d\xi}{(\xi U_c' - i)^2} \left\{1 - \varepsilon \left(\frac{\xi}{\xi - i}\right)^2 + O(\varepsilon^2)\right\}$$

$$= \frac{1}{c_i U_c'} \left[\frac{1}{\xi - i} + \varepsilon \left\{\frac{1}{2(\xi - i)^2} - \frac{2i}{\xi - i} + \ln(\xi - i)\right\} + O(\varepsilon^2)\right]$$

(1.10)

From (1.10) we can deduce that far from the critical height $I$ is dominated by the logarithmic term and hence (1.10) reduces to

$$I \sim \frac{\varepsilon}{c_i U_c'} \ln(\xi - i) \quad \text{as} \quad \xi \to \pm \infty$$

$$= \frac{\varepsilon}{2c_i U_c'} \ln(\xi^2 + 1) + i\theta$$

(1.11)
where $\theta$ is given by

$$
\tan \theta = -\xi^{-1} = -c_i/U_*'(z - z_c)
$$

(1.12)

For a logarithmic velocity profile $\tan \theta = \varepsilon z_c/(z - z_c)$ and hence $\theta$ varies between

$$
\theta \rightarrow 0 \quad \text{as} \quad (z - z_c)/\ell_c \rightarrow \infty \quad \text{and} \quad \theta \rightarrow \pi \quad \text{as} \quad (z - z_c)/\ell_c \rightarrow \infty
$$

(1.13)

The imaginary part of the integral for $z \gg z_c$ is then

$$
\mathcal{F}\{I\} \sim \varepsilon/c_iU_*'\theta = U''_c/U_c^3 H(z - z_c) \quad \text{as} \quad \xi \rightarrow \pm \infty
$$

(1.14)

where $H(z - z_c)$ is the Heaviside step function. The result given by (1.14) is remarkable since it is independent of $c_i$ which means even for a slowly-growing wave it leads to an out of phase contribution to the vertical velocity that ultimately leads to the same contribution to the wave transfer parameter for the wind-wave interaction. Using this expression, the ratio of the thickness of the shear stress layer surrounding the critical height as found by Miles (1957) is valid only when the waves grow sufficiently slowly such that

$$
c_i \ll U'_c z_c \sim U_*
$$

(1.15)

and hence the effects of the critical layer calculated by Miles (1957) are valid only in the limit $c_i/U_* \downarrow 0$.

BHC further argued that in the vicinity of the critical height the turbulent shear stress perturbation, $\Delta \tau$, can be modelled by an eddy viscosity model via

$$
\Delta \tau = \nu_e \frac{\partial \Delta u}{\partial z}
$$

(1.16)

where $\nu_e = \alpha U_* z$ is an eddy viscosity and $\alpha = 2\kappa$ is a parameter which reduces (1.16) to the mixing length model used in the inner region. Thus, there is a layer around the critical height whose thickness is $\ell_c$ in which the stresses cannot be neglected. By balancing the shear stress term in (1.6) with the gradient term, $\partial^2 \Psi/\partial z^2$, they then estimated

$$
\ell_c \sim z_c (\alpha/kz_c)^{1/2}
$$

Hence they found that the thickness of the shear stress layer surrounding the critical height is much smaller than the critical-layer thickness, i.e. $\ell_c \ll \ell_c$, provided

$$
(\alpha/kz_c)^{1/2} \ll c_i/U_*
$$

The main conclusions arrived by BHC, which has motivated the present investigation, is summarized below.

(a) From various studies and experimental data it is well known that $c_i$ varies with governing parameters according to

$$
c_i/U_* = s(U_*/c_i)\beta
$$

where $s \ll 1$ is the ratio of the air density, $\rho_a$, to that of water density, $\rho_w$, and $\beta$ is the energy-transfer parameter for the wind-wave interaction. Using this expression, the ratio of the thickness of an inertial critical layer, $\ell_c$ to the thickness of a stress-dominated critical layer, $\ell_r$, varies according to

$$
\ell_c/\ell_r \sim (kz_c/\alpha)^{1/2}s(U_*/c_i)\beta
$$

For growing waves in the ocean $s \sim 1/800$ and $\beta \approx O(25 - 30)$ then for intermediate waves when $kz_c \sim 1$ and $c_i/U_* \approx O(15 - 25)$, we see that $\ell_c/\ell_r \approx 1/(500\alpha^{1/2})$. We remark that for the mixing-length model $\alpha = 0.8$ and thus $\ell_c/\ell_r$ is likely to be small for a fully developed turbulent flow.
Hence, one is led to conclusion that in the intermediate regime the critical layer is dominated by effects of Reynolds stress and, contrary to Miles’ (1957) conclusion, there will not be any contribution from the critical layer to the wave growth. We further remark that for those ocean waves that are rapidly growing in time, such that \( c_i/U_* \sim 1 \), the inertial effects are dominant in the critical layer and thus the critical-layer mechanism (c.f. Miles 1957). This suggests that such ‘rapidly-growing’ waves might occur as a wave crest moves through a wave group. This is currently being investigated by the present authors.

(b) For slow waves, \( c_r/U_* \leq 15 \), the critical height \( z_c \) lies within the lower part of the inner region and thus the reverse-flow region, situated below \( z_c \) and the inner layer itself plays no significant dynamical role. In such circumstances, asymmetry in the flow is generated by the frictional effect of the shear stress through the inner region, resulting to lower wind speeds on the downwind side of the wave and consequently leads to a sheltering in the lee of the wave crest. This asymmetry results to an out-of-phase pressure perturbation which subsequently yields wave growth. We emphasize that in this case the air flow perturbations are similar to those over a stationary undulation, but in the range \( 1 \leq c_r/U_* \leq 15 \) the flow is similar to that over a rough surface except now the surface roughness is now effectively \( z_c \), which increases the value of \( \beta \).

It is to be noted that small corrections to the velocity of \( O(akc_rk_i) \) due to the orbital motions at the wave surface reduce \( \beta \).

For fast waves, on the other hand, the critical layer is far above the surface, \( k\zeta > 1 \), and again there is no significant dynamical role for the wave growth. In this scenario, the air above the wave flows largely against the wave which induces a ‘negative’ asymmetry from sheltering. Furthermore, orbital motions at the water surface generate additional air flow perturbations that contribute comparable ‘negative’ asymmetries. This ‘negative’ asymmetry causes an out-of-phase pressure perturbation which makes waves to decay.

Finally, between the two regions discussed above, there is also an intermediate region in which \( 15 \leq c_r/U_* \leq 30 \) and \( z_c \sim \ell \). In this region, numerical simulations show that as \( c_r/U_* \) increases from slow to the fast region, the reverse flow below the critical height becomes stronger and produces a ‘negative’ asymmetric displacement of streamlines upwind of the crest. However, above the critical height the asymmetric displacement is ‘positive’ downwind of the crest, similar to that for slow waves. Moreover, the critical-layer mechanism also displaces streamlines downwind of the crest. Therefore, as \( c_r/U_* \) increases across the intermediate region, the asymmetric component of the flow peaks to its maximum and then decreases to zero, with the wave growth following the same trends as that of the asymmetric component of the flow.

2. Shear stress model for unsteady wave growth

In this section we consider the perturbation Reynolds stresses in the flow of a turbulent wind over the surface wave

\[
z = a \cos[k(x - ct)] = h(x,t), \quad (ak \ll 1) \quad c = c_r + ic_i \quad (2.1)
\]

through an interpolation between inner, mixing-length approximation and an outer, rapid-distortion approximation. We will show that the wind-to-wave energy transfer predicted by this model is substantially larger than that predicted by either Miles’ (1957) quasi-laminar model (in which the perturbation Reynolds stresses are neglected) or Townsend’s viscoelastic model (Townsend 1972) and is in very good agreement with the model proposed by Belcher & Hunt (1993). Here we will point out, Townsend’s inner approximation differs from the conventional mixing-length approximation and yields a ratio of perturbation shear stress to perturbation shear that is negative for his choice of parameters.

The equations of motion for a viscous incompressible fluid may be cast in Cartesian tensor
form as

\[
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial p_{ij}}{\partial x_j}
\]  

(2.2)

where \(x_i\) denotes the Cartesian coordinate, \(u_i\) a velocity component, \(p_{ij}\) component of a stress tensor and \(\rho\) the fluid density. Decomposing the variables according to

\[
\begin{align*}
    u_i &= U_i + u'_i + u''_i, \\
p_{ij} &= P_{ij} + p'_{ij} + p''_{ij}
\end{align*}
\]

(2.4)

where \(U_i + u'_i\) and \(P_{ij} + p'_{ij}\) represent a solution to (2.2), being functions of coordinates \(x_1\) and \(x_3\) and having mean values with respect to \(x_2\). Note that, \(U_i\) and \(P_{ij}\) represent the mean components, and \(u'_i\) and \(p'_{ij}\) the turbulent fluctuations. In (2.4) \(u''_i\) and \(p''_{ij}\) represent a small perturbation with respect to the solution of (2.2)–(2.3).

Substituting (2.4) in (2.2), neglecting second order terms in the perturbation flow, and noting the fact that the unperturbed flow satisfies (2.2), we obtain

\[
\frac{\partial u''_i}{\partial t} + (U_j + u'_j) \frac{\partial u''_i}{\partial x_j} + u''_j \frac{\partial (U_i + u'_i)}{\partial x_j} = \frac{1}{\rho} \frac{\partial p''_{ij}}{\partial x_j}
\]  

(2.5)

\[
\frac{\partial u''_i}{\partial x_i} = 0
\]  

(2.6)

Taking the mean values with respect to \(x_3\), the results can be expressed in the following form

\[
\begin{align*}
    \frac{\partial \overline{u''}_i}{\partial t} + U_j \frac{\partial \overline{u''}_i}{\partial x_j} + u''_j \frac{\partial U_i}{\partial x_j} &= \frac{1}{\rho} \frac{\partial (\overline{p''_{ij}} - r''_{ij})}{\partial x_j}, \\
    \frac{\partial u''_i}{\partial x_i} &= 0
\end{align*}
\]  

(2.7)

(2.8)

Invoking the equations of continuity for both \(u'_i\) and \(u''_i\) the perturbation Reynolds stress may be written as

\[
\overline{r''_{ij}} = \rho \left( \overline{u'_i u'_j} + \overline{u''_i u''_j} \right)
\]

(2.9)

\[
\overline{r''_{ij}} = \rho \left[ u'_i (u''_j - \overline{u''_j}) + u''_i (u'_j - \overline{u'_j}) \right]
\]  

(2.10)

with (2.10) follows from (2.9) by virtue of \(u''_0 = 0\).

If we now set \(x_1 = x\), \(x_3 = z\), \(U_1 = U(z)\), \(U_2 = U_3 = 0\), \(U'_1 = U, \overline{u'_3} = w, \overline{p''_{ij}} = -\delta_{ij} \varphi\), and \(\overline{r''_{ij}} = \rho \overline{u'_i u'_j}\), after taking the time average, we obtain the linearized, Reynolds-averaged equations governing \(u\) and \(w\), the \(x\) (horizontal) and \(z\) (vertical) components of the mean perturbation velocity, and the kinematic perturbation pressure \(p\) as

\[
\begin{align*}
    \frac{\partial u_x}{\partial t} + \frac{\partial u'_y}{\partial x} + \frac{\partial u'_z}{\partial z} &= 0, \\
    (U - c) u_x + U' w &= - \varphi_x + \sigma_x + \tau_x, \\
    (U - c) w_x &= \varphi_z + \sigma_z, \\
    \rho' &= \rho + \overline{w'^2} - \overline{(w^2)_0}, \\
    \sigma &= -(\overline{u'^2} - \overline{w'^2}) - \sigma_0, \\
    \tau &= -u'^{2}w' - \tau_0
\end{align*}
\]  

(2.11)

(2.12)

(2.13)

(2.14)
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\((w'^2)_0, \sigma_0\) and \(\tau_0\) are the unperturbed values of \(w'^2\), \(-u'^2 - w'^2\) and \(-u'w'\), and \(\sigma\) is Townsend’s \(\tau_n\).

In this paper we consider a turbulent shear flow blowing over the surface wave (2.1) whose mean velocity profile is given by

\[ U(z) = \left( \frac{\tau_0^{1/2}}{\kappa} \right) \log(z/z_0) \]  

(2.15)

where \(\tau_0\) is the kinematic shear stress in the basic flow, \(\kappa\) is Kármán’s constant, \(z\) is the elevation, \(c\) is the complex phase speed of the surface wave (2.1).

2.1. Energy-transport equation

As a first step toward a Reynolds-stress closure, Townsend points out the transport equation for the turbulent kinetic energy \(\frac{1}{2}q^2\) may be expressed in the form

\[ (U - c) \partial_z \left( \frac{1}{2} q^2 \right) = D + G - \varepsilon', \]  

(2.16)

where

\[ D = \rho \kappa \tau_0^{1/2} \partial_z \left[ z \partial_z \left( \frac{1}{2} q^2 \right) \right] \]  

(2.17)

represents diffusion†

\[ G = -u'^2 u_x - w'^2 w_z - u'w'(U' + u_z + w_x) - \tau U' \]  

(2.18)

\[ = \sigma_0 u_z + \tau_0 (u_z + w_x) + U' \tau \]  

(2.19)

represents generation (Lauder et al. 1975), (2.19) follows (2.18) through (2.11), (2.14) and linearization, and \(\varepsilon'\) represents dissipation (see below). Townsend neglects \(w_x\) in \(G\), although this appears to be inconsistent with his subsequent rapid-distortion approximation (see below).

Townsend’s approximation to the dissipation rate \(\varepsilon'\) involves \(h\), the surface displacement (2.1), but this may be eliminated through a transformation to wave-following coordinates, in which (Miles 1996, equation (3.4))

\[ \varepsilon' = \frac{1}{2} \tau_0 U' (e/e_0), \quad e \equiv q^2 - e_0, \]  

(2.20)

and \(e_0 \equiv q^2\) in the basic flow. Substituting (2.19) and (2.20) into (2.16), neglecting diffusion (see Miles 1996, §3), and multiplying the result by \(2a_1\), we obtain

\[ (\mathcal{D} + \lambda) a_1 e - 2\lambda (\tau - a_1 e) = 2\tau_0 [a_1 (u_z + w_x) + a_n u_z] \equiv \alpha_i \]  

(2.21)

where

\[ \mathcal{D} \equiv (U - c) \partial_z, \quad \lambda \equiv a_1 U', \quad a_1 \equiv \tau_0/e_0, \quad a_n = \sigma_0/e_0 \]  

(2.22)

The relaxation rate \(\lambda\) is a reciprocal measure of eddy life and dominates (is dominated by) \(\mathcal{D}\) in the inner (outer) domain.

2.2. Rapid-distortion approximation

To determine the outer departure of \(\tau/e\) and \(\sigma/e\) from their equilibrium values \(a_1\) and \(a_n\), Townsend posits the rapid-distortion approximations

\[ \mathcal{D} \begin{pmatrix} \tau_0 + \tau \\ e_0 + e \end{pmatrix} \approx \mathcal{D} \begin{pmatrix} \tau - a_1 e \\ e_0 \end{pmatrix} \sim A_1 u_z + A_2 w_x + A_3 u_x \]  

(2.23)

† Townsend chooses \(\rho = 0.3\) but states that ‘the value ... is not critical’.
and
\[ D \left\{ \sigma_0 + \sigma \right\} \approx D \left\{ \frac{\sigma - a_0 e}{e_0} \right\} \sim B_1 u_z + B_2 w_x + B_3 u_x \]
(2.24)
where \( A_{1,2,3} \) and \( B_{1,2,3} \) are ‘the incremental rates of change for suddenly imposed additional distortions’. He then ‘interpolates’ between (2.23)–(2.24) and the inner domain, in which (by hypothesis) \( \tau \to a_1 e \) and \( \sigma \to a_0 e \), by replacing \( D \) by \( D + \lambda \):
\[ (D + \lambda)(\tau - a_1 e) = e_0(A_1 u_z + A_2 w_x + A_3 u_x) \equiv \omega_0 \]
(2.25)
and
\[ (D + \lambda)(\sigma - a_0 e) = e_0(B_1 u_z + B_2 w_x + B_3 u_x) \equiv \omega_0 \]
(2.26)
However, the elimination of \( a_1 e \) between (2.21) and (2.25) in the inner domain (in which \( |\omega| \ll |\lambda| \)) yields
\[ \tau \to \lambda^{-1}(\omega_0 + 3\omega_0) \]
(2.27)
which differs from the mixing-length approximation obtained by invoking \( |\omega| \ll \lambda \) in (2.21):
\[ \tau \to a_1 e \to \lambda^{-1}/a_0 = 2v_0(u_z + w_x + \mathcal{H}u_x), \]
(2.28)
where
\[ \mathcal{H} = a_0/a_1, \quad v_0 = \tau_0/U' = \kappa^1 z_0. \]
(2.29)
Indeed, if (as typically assumed in the mixing-length approximation) \( |u_x, w_x| \ll |u_z| \), (2.27) implies
\[ \frac{\tau}{2v_0 u_z} \to 1 + \frac{3A_1}{2a_1}, \]
(2.30)
which reduces to the conventional mixing-length approximation for \( A_1 = 0 \) but is negative for Townsend’s values of \( a_1 \) and \( A_1 \), \( \frac{3}{8} \), and \(-0.03\).

3. Solution of boundary-value problem
Miles-Sajjadi theory of wave generation by turbulent wind (Miles 1996, Sajjadi 1998) reduces to the solution of the Orr-Sommerfeld-like equation
\[ (v_e \Phi'')'' = ik[(U - c)(\Phi'' - k^2 \Phi) - U'' \Phi] \equiv \mathcal{F}'' \]
(3.1)
subject to the boundary conditions
\[ \Phi = ac, \quad \Phi' = a(kc - U') \quad \text{on} \ \eta = 0 \]
(3.2)
\[ \Phi, \quad v_e \Phi \to 0 \quad \text{as} \ k \eta \to \infty \]
(3.3)
where \( \Phi = \Phi(\eta), U = U(\eta) \) and \( (\cdot)' \equiv d/d\eta \). In eqn (3.1) \( v_e \) is a complex eddy viscosity given by
\[ v_e = 2U_e^2(U' + ikV)^{-1}, \quad V = (U - c)/a_1 \]
(3.4)
where \( a_1 (\sim \kappa^2; \kappa = 0.4 \) is Kármán’s constant) is Townsend’s boundary-layer constant, and \( a, c \) and \( k \) are the amplitude, speed and wave number of the surface wave. The velocity profile has the logarithmic asymptote
\[ U \sim U_1 \ln(\eta/z_0), \quad (\eta \gg z_0) \]
(3.5)
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where

\[ U_1 = U_*/\kappa, \quad z_0 = \Omega U^2_1/g \]  

(3.6)

\( U_* \) is the wind friction velocity and \( \Omega \) is Charnock’s constant. We seek the impedance

\[ \alpha + i\beta = (\mathcal{P}_0 + i\mathcal{R})/(kaU^2_1) \]  

(3.7)

where

\[ \mathcal{P}_0 = kac^2 + (ik)^{-1}(\nu\Phi''\prime) \]  

and \( \mathcal{R}_0 = \nu\Phi'' \), on \( \eta = 0 \)  

(3.8)

are complex amplitudes of the wind-induced perturbation pressure and shear stress action on the wave.

3.1. Reduction to second-order differential equations

It is convenient to reduce the fourth-order differential equation (3.1) to the pair of second-order equations

\[ \mathcal{T}'' = (ik/\nu)(U - c)\mathcal{T} - ik[U'' + k^2(U - c)]\Phi \]  

(3.9)

\[ \Phi'' = \mathcal{T}/\nu \]  

(3.10)

for which the respective boundary conditions are

\[ \mathcal{T} = \mathcal{T}_0, \quad \mathcal{T}' = ik(\mathcal{P}_0 - kac^2) \]  

on \( \eta = 0 \)  

(3.11)

and (3.2). Note that, \( \mathcal{T}_0 \) and \( \mathcal{P}_0 \) are implicitly determined by the null conditions (3.3).

We render the formulation dimensionless by referring to \( \eta \) to \( k^{-1} \), \( c \), \( U \) and \( V \) to \( U_1 \), \( \Phi \) to \( aU_1 \), \( \mathcal{P}\ ) and \( \mathcal{T} \) to \( kaU^2_1 \), thereby reducing (3.4), (3.9)–(3.11) and (3.2) to

\[ \mathcal{T}'' = \frac{1}{2}iV(U' + iV)\mathcal{T} - i(U'' + U - c)\Phi \]  

(3.12)

\[ \Phi'' = (2k^2)^{-1}(U' + iV)\mathcal{T} \]  

(3.13)

\[ \mathcal{T} = \mathcal{T}_0, \quad \mathcal{T}' = ik(\mathcal{P}_0 - c^2), \quad \Phi = c, \quad \Phi' = c - U', \]  

on \( \eta = 0 \).  

(3.14)

3.2. Inner expansion for surface layer

A constant-stress interpolation between the logarithmic profile (3.5) and a viscous sublayer of vanishing thickness is given by (Rotta 1950)

\[ U/U_1 = \log(\zeta + \sqrt{\zeta^2 + 1}) - \frac{\zeta}{1 + \sqrt{\zeta^2 + 1}} \equiv \tilde{U}(\zeta), \quad \zeta = \frac{1}{2} e \frac{\eta}{z_0} \]  

(3.15)

in which

\[ \frac{dU}{d\zeta} = \frac{U_1}{1 + \sqrt{1 + \zeta^2}} \]

Note that \( U'' = \mathcal{O}(U_1/z_0^2) \) for \( \zeta = \mathcal{O}(1) \) and therefore, in contrast to the conventional Orr-Sommerfeld problem, is not negligible near the boundary, although it does vanish at \( \zeta = 0 \).

The inner and outer length scales are \( z_0 \) (or, more conveniently, \( \tilde{z}_0 \equiv 2z_0/c \) and \( k^{-1} \), and the introduction of the inner variable \( \zeta \) and the small parameter

\[ \varepsilon \equiv k\tilde{z}_0 = (2\Omega/e)c^{-2} \ll 1 \]  

(3.16)

in (3.12), (3.13) and (3.14) leads to the inner expansions

\[ \Phi = c + (\varepsilon c - \frac{1}{2})\zeta + (\varepsilon/2k^2) \left[ \mathcal{R}_0 \int_0^\zeta U \, d\zeta + i\mathcal{T}(\zeta) \right] + \mathcal{O}(\varepsilon^2), \]  

(3.17)
where
\[ \mathcal{G}(\zeta) = \int_0^\zeta d\zeta_1 \int_0^{\zeta_1} U'(\zeta_2) d\zeta_2 \int_0^{\zeta_1} \left[ \frac{1}{2}(c - U + \zeta U') - cU' \right] d\zeta_3, \] (3.18)

and
\[ \mathcal{T} = \mathcal{T}_0 + i \left[ \frac{1}{2}(c + U)\zeta - cU - \int_0^\zeta U d\zeta \right] + \mathcal{O}(\varepsilon) \] (3.19)

Letting \( \zeta \to \infty \) in (3.17) and (3.18) and invoking \( \zeta = \eta / \varepsilon \), we obtain
\[ \Phi \sim c + \left( c - \frac{1}{2} \varepsilon^{-1} \right) \eta + \frac{1}{2} \kappa^{-1} \left[ \mathcal{T}_0 \eta(U - 1) + \frac{1}{4} i \varepsilon^{-1} \eta^2 (c - U + \frac{7}{2}) \right]. \] (3.20)

Hence, the problem is reduced in determining \( \mathcal{T} \) and \( \mathcal{T}_0 \). The form given by (3.19) is not very convenient for this task. A better approach is to determine \( \mathcal{T} \) in the inner via shear-stress-layer approximation.

In this region, the complex amplitude of the wind-induced perturbation shear stress may be expressed as (Sajjadi 1998)
\[ T = \nu e \left[ U \left( \frac{\Phi}{U} \right)' - 2U'(\frac{\Phi}{U})' \right], \quad U \equiv U - c \] (3.21)

Rearranging this equation, we obtain
\[ \frac{U T}{\nu e} = U \Phi'' - U'' \Phi. \] (3.22)

Under the assumption that \( k \eta \ll 1 \) in this region, we may neglect \( k^2 \mathcal{U} \Phi \) in (3.1) and upon combining the result with (3.22) we arrive at the shear-stress-layer approximation
\[ \mathcal{T}'' - (i k \mathcal{U} / \nu e) \mathcal{T} = 0; \quad (k \eta \ll 1) \] (3.23)

whose asymptotic solution for \( k \eta \downarrow 0 \), which may be expressed in terms of of modified Bessel function of the first kind (Sajjadi 2010), is
\[ \mathcal{T}(\eta) = -2 \left\{ 1 + 4 \eta K_0'(\eta) \right\} + \mathcal{O}(\varepsilon), \] (cf. Appendix A of BH)† whence for growing waves we obtain
\[ \tau = -2 e^{k c \text{ct}} \mathcal{T}(\eta) e^{i k(x - c \text{ct})}. \]

The real part of the complex amplitude of shear stress as a function of non-dimensional height is shown in figure 3. This figure clear that the shear stress becomes negative after decending from a maxima and then rising again. The latter part is not depicted in this figure.

However, for the purpose of calculating the energy-transfer parameter, from wind to wave \( \beta \), we multiply (3.23) by \( \mathcal{T} \) and integrate by parts over \( 0 < \eta < \infty \) to obtain
\[ \langle (\mathcal{T} \mathcal{T}') \rangle_0 = -\int_0^\infty \left[ \mathcal{T}^2 + (i k \mathcal{U} / \nu e) \mathcal{U}^2 \right] d\eta \] (3.24)

The integral (3.24) is stationary with respect to first-order variations in \( \mathcal{T} \) about the true solution (3.23).

Substituting the trial function
\[ \mathcal{T} = \mathcal{T}_0 e^{-k \eta / \delta} \] (3.25)

where \( \delta \) is a (complex) free parameter to be determined, and \( \nu e \) given by (3.4), with \( a_1 = \kappa^2 \)

† The asymptotic solution given here reduces to that given by BH in the limit as \( c_i \downarrow 0 \).
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Figure 3. The real part of the complex amplitude of $\mathcal{F}$ as a function of non-dimensional height $k\eta$.

into (3.24) and invoking the condition $\partial (\mathcal{F}_0'/\mathcal{F}_0)/\partial \delta = 0$ we obtain

$$\frac{\mathcal{F}_0'}{\mathcal{F}_0} = -\frac{ik}{4\kappa^2} \left[ (1 + \delta_s) \left( L_0^2 + \frac{1}{6} \pi^2 \right) + 2\delta_s^{-1} - \hat{c}^2 \right]$$

(3.26)

$$(L_0^2 + 2L_\delta + \frac{1}{6} \pi^2) \delta_s^2 + 2L_\delta \delta_s - 2 = 0, \quad L_\delta \equiv L_0 + \ln \delta, \quad \delta_s \equiv i\kappa^{-2} \delta$$

(3.27a–c)

where $L_0 = \Lambda^{-1} = -\gamma - \ln 2k$, $k = \Omega \hat{c}^{-2} \hat{c}$, $\hat{c} = c/U_1$, $\gamma = 0.5772$ is the Euler’s number and $\Omega$ is the Charnock’s constant.

Solving (3.27a) as a quadratic in $\delta, L_\delta$ and letting $\delta_s \to 0$ we get

$$\frac{i\delta L_\delta}{\kappa^2} = \sqrt{3} - 1 + O(\delta)$$

(3.28)

4. Outer approximation above inertial critical-layer

In contrast to the inner region, the solution in the outer region is very straightforward. In the outer domain $k\eta \gg 1$, $U$ may be approximated by (3.5), and $\Phi$ admits the Green-Liouville approximation (Olver 1974, chap. 6)

$$\Phi \sim \mathcal{F}(\mathcal{Z}) \exp \left[ \int_0^\eta \mathcal{W}(\mathcal{Z}) d\eta \right], \quad \mathcal{Z}(\eta) \sim (1/\kappa^2) \ln(\eta/\eta_c)$$

(4.1)

Substituting (4.1) into the dimensionless counterpart of (3.1), we obtain

$$\mathcal{W}^4 + \frac{1}{2} \mathcal{Z}^2 (\mathcal{W}^2 - 1) = 0$$

(4.2)

and

$$\frac{\mathcal{W}(\mathcal{W}^2 + \mathcal{Z}^2)}{\mathcal{F}} \frac{d\mathcal{F}}{d\mathcal{Z}} + \left( 6\mathcal{W}^2 + \frac{1}{2} \mathcal{Z}^2 \right) \frac{d\mathcal{W}}{d\mathcal{Z}} + \frac{(i\kappa^2 \mathcal{W}^2 - 2) \mathcal{W}^3}{\mathcal{Z}} = 0$$

(4.3)
from which it follows that
\[ \mathcal{W}^2 = \frac{1}{4} \left[ -\mathcal{Z}^2 \pm (\mathcal{Z}^4 + 8\mathcal{Z}^2) \right], \]  
(4.4)
\[ \frac{d\mathcal{W}}{d\mathcal{Z}} = \frac{-\mathcal{Z}(\mathcal{W}^2 - 1)}{\mathcal{W}(4\mathcal{W}^2 + 2\mathcal{Z})} = \frac{\pm 2\mathcal{W}^3}{\mathcal{Z}^2 (4\mathcal{W}^2 + 2\mathcal{Z})^{1/2}} \]  
(4.5)
and
\[ \frac{d\ln \mathcal{F}}{d\mathcal{Z}} = -\frac{\mathcal{W}^2(4\mathcal{W}^2 - 2\mathcal{Z})}{\mathcal{F}(4\mathcal{W}^2 + 2\mathcal{Z})^2} - \frac{i\kappa^2 \mathcal{W}^3}{\mathcal{F}(4\mathcal{W}^2 + 2\mathcal{Z})}, \]  
(4.6)
We restrict further consideration to the asymptotic limit \( V \to \infty \), for which the admissible roots of (4.3) may be approximated by
\[ \mathcal{W} \sim -1, \quad -i\mathcal{Z} / \sqrt{2} \]  
(4.7)
(the roots +1 and +i\( \mathcal{Z} / \sqrt{2} \) are ruled out by the null condition at \( \mathcal{Z} = \infty \)). The corresponding approximations to (4.6) are
\[ \frac{d\ln \mathcal{F}}{d\mathcal{Z}} \sim \frac{1 + i\kappa^2}{\mathcal{Z}^2}, \quad -\frac{3}{2} \mathcal{Z}^{-1} - 2^{-3/2} \kappa^2, \]  
(4.8)
the integration of which leads, through (4.1), to
\[ \Phi \sim C_1 e^{-\eta} + C_2 \mathcal{Z}^{-3/2} \exp \left[ -\frac{i\eta (\mathcal{Z}^2 - \kappa^{-2})}{\sqrt{2}} - \frac{\kappa^2 \mathcal{Z}^2}{2\sqrt{2}} \right], \]  
for \( \mathcal{Z} \to \infty \) (4.9)
where \( C_1 \) and \( C_2 \) are constants. [The term \(-\frac{1}{2} \mathcal{Z}^{-2} \), derived from the real part of (4.8), has been neglected in the exponent in (4.9) since it is dominated by the error implicit in the approximation (4.7).] The corresponding approximation to \( \mathcal{F} \), obtained through substitution of (4.9) into (3.13) is
\[ \mathcal{F} \sim 2i\kappa^2 \left[ C_1 \mathcal{Z}^{-1} e^{-\eta} - \frac{1}{2} C_2 \mathcal{Z}^{-1/2} \exp \left\{ -\frac{i\eta (\mathcal{Z}^2 - \kappa^{-2})}{\sqrt{2}} \right\} \right]. \]  
(4.10)
Note that, using the standard exponential substitution of the Liouville-Green method for the asymptotic solution of (3.1), we obtain the following expansion of the phase function
\[ \Phi \sim \tilde{\eta} \right\} \]  
(4.11)
where \( \tilde{\eta}_1 \) and \( \gamma_2 \) are real, \( \tilde{\eta} \approx k\eta \) and \( \text{Ei} \) is the exponential integral. Formally there is an infinite number of terms with coefficients \( \theta_i \) that precede the determination of the \( \gamma_i \). However, as only the first of these, namely \( \theta_1 \), enters into the formula for \( \gamma_1 \), the first two into \( \gamma_2 \) (although the \( \theta \) contributions happen to cancel), the first four into \( \gamma_3 \), and so on. The result is depicted in figure 4 where we see that an initial exponential decay follows by an algebraic tail. It should be noted that this is agreement with numerical simulation of Ierley and Miles (2001), and the expression (4.11) is originally found by them. However, the results of figure 4 is drawn from the analytical expression given by the equation (4.11).

Since the governing equation is fourth order, we find \( \theta \) by solving for the roots of the fourth degree polynomial
\[ \theta_1^2 \left[ (2\kappa^4 + A_1) \theta_1^2 + 1 \right] = 0 \]  
Two roots are zero and the other two constitute an imaginary pair. The double zero leads to a particularly simple result at next order: \( \theta_2^2 - 1 = 0 \), thus one pair of solutions is approximately
The variation of the log of modulus square for perturbation stream function $\Phi$ against the non-dimensional distance $k\eta$ showing the initial exponential decay follows by an algebraic tail. The lower figure is a close of the upper figure in the range $6 \leq k\eta \leq 20$.

$\exp(\pm \eta)$. The pure imaginary root pair exhibits weak algebraic growth (or decay), as reflected in $\gamma_1$. A suitable boundary condition for (3.1) is to match the decaying exponential and the decaying algebraic solutions. For this purpose, we employ the following expressions

$$\vartheta_1 = -(A_1 + 2\kappa^4)^{-1/2}$$
$$\vartheta_2 = (1 + \kappa^4)(A_1 + 2\kappa^4)^{-1/2} - \frac{A_3 + B_1}{2(2\kappa^4 + A_1)} \quad \frac{b(4\kappa^4 + A_1)}{2(2\kappa^4 + A_1)}$$
$$\gamma_1 = -\frac{1}{2}\kappa^2(A_1 + 2\kappa^4)^{-1/2}; \quad \gamma_2 = -\frac{3}{2}$$

after selecting $\tilde{a}_1 = \kappa^2$. 

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*Asymptotic analysis for generation of unsteady waves by turbulence*
5. Effect of the inertial critical-layer

5.1. Comparison with BHC

In the case of slow waves, where \( c \ll 1 \), the perturbation shear stress in (3.1) can be neglected in the outer region and thus \( \Phi \) will satisfy the Rayleigh equation

\[
(U - c)(\Phi'' - k^2\Phi) - U''\Phi = 0
\]

where now we shall assume \( c = c_s \) and \( c_i = 0 \). The corresponding expressions for unsteady waves will be given in the next subsection.

As was shown by Sajjadi (1988), the leading order solution to (5.1) is

\[
\Phi = (U - c)e^{-k\eta} \left[ A + \Phi_U U'_c e^{k\eta} \int_0^\infty \left\{ \frac{1}{(U - c)^2} - 1 \right\} d\eta \right]
\]

(5.2)

The integral is (5.2) is regular at the critical height and hence, by indenting the path of integration in (5.2) under the singularity \( \eta = \eta_c \), we obtain

\[
\Phi = (U - c)e^{-k\eta} \left[ A + \Phi_U U'_c e^{k\eta} \left( \int_0^\infty \left\{ \frac{1}{(U - c)^2} - 1 \right\} d\eta - I \right) \right]
\]

(5.3)

where \( A \) is constant which can be determined from the boundary conditions and

\[
I = \lim_{\varphi \to 0} \int_{\eta - \varphi}^{\eta + \varphi} \left\{ \frac{1}{(U - c)^2} - 1 \right\} d\eta
\]

(5.4)

Expanding \( U(\eta) \) in a Taylor expansion in the vicinity of the critical point, setting \( \eta = \eta_c \varphi e^{i\theta} \), where \( \varphi \equiv c_i/U_s \ll 1 \), and

\[
\tan \theta = -c_i/U'_c (\eta - \eta_c)
\]

(5.5)

then (5.4) becomes

\[
I = \frac{1}{U'_c^2} \left\{ \lim_{\varphi \to 0} \int_{\eta_c - \varphi}^{\eta_c + \varphi} \frac{d\eta}{(\eta - \eta_c)^2} \right\} + i\pi \frac{U''_c}{U'_c}
\]

\[
= i\pi U''_c
\]

(5.6)

which agrees with the result obtained by Belcher et al. (1999).

As also pointed out by Belcher et al. (1999), for a logarithmic mean velocity profile \( U(\eta) \) yields \( \tan \theta = \varphi \eta_c/(\eta - \eta_c) \). Hence \( \theta \) varies between 0 and \( \pi \) as \( (\eta - \eta_c)/\ell_c \) tends to \( \pm \infty \), respectively. Note that, the transition between these limiting values occurs across the layer of thickness \( \ell_c = \varphi \eta_c \). Note also, the significance of the term \( iU''_c/U'_c^3 \) in the solution for \( I \) is that it leads to an out of phase contribution to the wave induced vertical velocity that gives rise to the same contribution to the wave growth from Miles (1957) critical-layer mechanism.

The result of the present analysis confirms the earlier finding of Belcher et al. (1999) in that Miles (1957) solution is only valid when the waves grow significantly slowly such that

\[
c_i \ll U'_c \eta_c \sim U_s
\]

(5.7)

As in Belcher et al. (1999), our analysis also shows that when inertial effects controls the behaviour around the critical layer, there is a smooth behaviour around the critical layer of thickness

\[
\ell_c \sim c_i/U'_c \sim \eta_i c_i/U_s
\]

(5.8)

Hence this proves the effects of critical layer, as calculated by Miles (1957), are only valid in the limit \( c_i/U_s \downarrow 0 \).
5.2. Steady monochromatic waves

For comparison with the waves that are unsteady, we calculate the energy-transfer parameter due to critical layer, $\beta_c$, for steady monochromatic waves. Thus, we let $\Phi = -\mathcal{M}$. Thus, (3.1) becomes

$$[\nu_c(\mathcal{M}' + 2U^* \mathcal{M}' + U'' \mathcal{M})]' = ik[(\mathcal{M}'^2 + k^2 \mathcal{M}^2)]'$$

(5.9)

In quasi-laminar limit the left-hand side of (5.9) is negligible and thus (5.9) reduces to

$$(\mathcal{M}'^2 + k^2 \mathcal{M}^2) = 0$$

(5.10)

Multiplying (5.10) by $\mathcal{M}$, integrating by parts over $0 < \eta < \infty$, and invoking the inner limits $\mathcal{M} \to a$ and $\mathcal{M}'^2 \mathcal{M}' \to P_0$ and a null condition at $\eta = \infty$, we obtain

$$a P_0 = -\int_0^\infty \mathcal{M}'^2 (\mathcal{M}'^2 + k^2 \mathcal{M}^2) d\eta$$

(5.11)

Using the simplest admissible trial function for the variational integral (5.11), i.e.

$$\mathcal{M} = a e^{-k \eta / \zeta}$$

(5.12)

where $\zeta$ is a free parameter. Substituting (5.12) into (5.11) together with the approximation $\nu \approx U_1 \ln(\eta / \eta_c)$ we get

$$P_0 - P/aU_1^2 = -k(\zeta^{-2} + 1) \int_0^\infty e^{-2k \eta / \zeta} \ln^2(\eta / \eta_c) d\eta$$

$$= -\xi_c(\zeta^{-2} + 1) \int_0^\infty e^{-\zeta t} \ln^2 t dt$$

$$= -\xi + \xi^{-1} \left\{ \frac{\pi^2}{6} + \ln^2 \left( \frac{2\gamma \xi_c}{\zeta} \right) \right\}$$

(5.13)

where $\xi_c \equiv k \eta_c$. It then follows from the variational condition $\partial P_0 / \partial \zeta = 0$ that

$$\xi^2 = \frac{L_\zeta^2 - 2L_\zeta + \pi^2 / 6}{L_\zeta^2 + 2L_\zeta + \pi^2 / 6}$$

(5.14)

where $L_\zeta \equiv 2\gamma \xi_c / \zeta$, and $\zeta = O(1)$.

The corresponding, quasi-laminar approximation to the energy-transfer parameter may be calculated from (5.3), which implies $\Phi_c = P_0 / U_1 \approx P_0 / U_0$, and (5.6), which yields

$$\beta_c = \pi \xi_c [\Phi_c / U_1 a]^2 = \pi \xi_c^3 |\mathcal{M}'|^2 = \frac{1}{4} \pi (\xi + \xi^{-1})^2 (L_\zeta + \frac{1}{6} \pi^2)^2$$

(5.15)

$$= \pi \xi_c^3 L_\zeta^4 [1 - (4 - \frac{1}{4} \pi^2) \Lambda^2 + O(\Lambda^4)]$$

(5.16)

where $\Lambda = L_\zeta^{-1} = -\gamma - \ln(2\kappa), L_0 = L_\delta - \ln \delta$, and $\delta L_\delta / k^2 = (\sqrt{3} - 1) + O(\delta)$.

To obtain the corresponding expression for the component of the energy-transfer parameter, $\beta_T$, due to turbulence, we multiply (3.1) by $-\mathcal{M}$, integrating over $0 < z < \infty$, invoking the conditions

$$\mathcal{M} = a, \quad \mathcal{M}' = ka, \quad \mathcal{T}' = ik[\mathcal{M}_0 - kac]^2$$

on $z = 0$ and the null condition for $z \to 0$, we obtain

$$\int_0^\infty \mathcal{M} \mathcal{T}'' dz = ka[\mathcal{T}_0 - i \mathcal{M}_0] + i(kac_c)^2 + \int_0^\infty \mathcal{M}'' \mathcal{T} dz$$

$$= i(kac_c)^2 + ik \int_0^\infty \mathcal{T} (\mathcal{M}'^2 + k^2 \mathcal{M}^2) dz,$$

with $c = c_r + ic_i$. Then, in the limit as $s \equiv \rho_a / \rho_w$, where $\rho_a$ and $\rho_w$ are densities of the air and
Thus, the expression (5.11) now reads by setting

\[ \alpha + i \beta = \left( c^2 - c_w^2 \right)/sU_1^2 = \left( \mathcal{R}_0 + i \mathcal{R}_0 \right)/kaU_1^2 \Rightarrow \left( \mathcal{R}_0 + i \mathcal{R}_0 \right), \]

(5.17)

where \( c \) is the complex wave speed, \( c_w = \sqrt{g/k} \) is the speed of water waves in the absence of the airflow above it and the suffix zero denotes evaluation at \( z = 0 \),

\[ \alpha_T + i \beta_T = (kaU_1)^{-2} \int_0^\infty \left\{ i V e \left[ \mathcal{M}'^2 + 2U \mathcal{M}' + \mathcal{M}'' + U'' \mathcal{M}'' \right] - k \mathcal{M}' \left( \mathcal{M}'^2 + k^2 \mathcal{M}'' \right) \right\} dz. \]

(5.18)

The above integral can be evaluated asymptotically whose imaginary part yields

\[ \beta_T = 5\kappa^2 L_0^{-1}. \]

(5.19)

Therefore, in summary, the energy-transfer parameter from wind to surface waves for steady monochromatic waves may be given by the following formulae.

\[ \beta = \beta_T + \beta_c, \quad \beta_T = \frac{5\kappa^2}{\Lambda}, \quad \beta_c = \frac{5}{2} \pi \Lambda L_0^4 \left[ 1 - \left( \frac{4 - \pi^2}{3} \right) e^3 \right] \]

\[ \Lambda = L_0^{-1}, \quad L_0 = -\gamma - \log \Lambda, \quad \mathcal{W} = k_0 e^{c \gamma / U_\lambda} (U_\lambda/c_r)^2, \quad U_\lambda = 2U_\lambda \]

where \( \kappa = 0.4 \) is von Kármán's constant and \( \gamma = 0.5772 \) is Euler's constant.

5.3. Unsteady waves

The generalization of the results just obtained above follows a similar development, but with the exception that \( c_i \neq 0 \). Here we shall present results for Stokes waves being a sum of two harmonics \( (n = 1, 2) \). Hence, we consider the surface wave expressed as

\[ z = \text{Re}\{ae^{i(x-cr)} + \frac{1}{2} k a^2 e^{2i(x-cr)}\}, \quad c = c_r + ic_i \]

Note that, results for monochromatic waves follows immediately from what will be developed by setting \( n = 1 \) and ignoring the second harmonic.

Therefore, we begin by using the expression (5.11), but now we take

\[ \mathcal{M} = U - c_r - ic_i \]

(5.20)

Thus, the expression (5.11) now reads

\[ a \mathcal{R}_{n0} = -\int_0^\infty [(U - c_r)^2 - 2ic_i(U - c_r) - c_i^2](\mathcal{M}_n^2 + k_n^2 \mathcal{M}_n) d\eta \]

(5.21)

where \( k_1 \equiv k \) and \( k_2 \equiv 2k \).

Once again using a similar admissible trial function as that given by (5.12), namely

\[ \mathcal{M}_n = ae^{kn \eta / \varphi_n} \quad (n = 1, 2) \]

the variational integral (5.21) becomes

\[ a \mathcal{R}_{n0} = a^2 k_n^2 (\varphi_n^{-2} + 1) \left\{ \int_0^\infty U_1^2 \ln^2(\eta / \varphi_n) e^{-2k_n \eta / \varphi_n} d\eta \right. \]

\[ + 2ic_i \int_0^\infty U_1 \ln(\eta / \varphi_n) e^{-2k_n \eta / \varphi_n} d\eta + c_i^2 \int_0^\infty e^{-2k_n \eta / \varphi_n} d\eta \right\} \]

\[ \text{† The detail evaluations may be obtained found in the appendix of the paper by Sajjadi (2007).} \]
Evaluating the integrals, we obtain

$$e^{\mathcal{H}_{n0}} = \frac{1}{2} \left( \frac{\pi^2}{6} + \log^2 \left( \frac{2k_n \eta_c}{\zeta_n} \right) \right)$$

where

$$c_i = c_i/U_1.$$

As before, applying the variational condition \( \partial \mathcal{H}_{n0}/\partial \zeta_n = 0 \) yields

$$L_{n\zeta}^2 - 2(1 + i\tilde{c})L_{n\zeta} + \left( \tilde{c}_i^2 + 2i\tilde{c}_i + \frac{\pi^2}{6} \right)$$

whence

$$\zeta_n^2 = \frac{L_{n\zeta}^2 - 2c_iL_{n\zeta} + \left( \tilde{c}_i^2 + 2i\tilde{c}_i + \frac{\pi^2}{6} \right)}{L_{n\zeta}^2 + 2\mathcal{E}_i L_{n\zeta} + \left( \tilde{c}_i^2 - 2i\tilde{c}_i + \frac{\pi^2}{6} \right)}$$

where \( \mathcal{E}_i = 1 + \tilde{c}_i \), the superscript * denotes the complex conjugate, and \( L_{n\zeta} = 2\gamma \eta_c \zeta_n \) where \( L_{n\zeta} = -\gamma - \log(2k_n \eta_c) \).

The expression (5.23) may be approximated to \( O(\lambda_n^2) \), to give

$$\zeta_n = \frac{1 - \mathcal{E}_i \lambda_n}{1 + \mathcal{E}_i \lambda_n} = 0(\lambda_n^2).$$

where \( \lambda_n = L_{n0}^{-1} \), and therefore we obtain the following expression for the energy-transfer parameter to the two \((n = 1, 2)\) harmonics of the wave

$$\beta_{nc} = \pi \left( \frac{\zeta_n L_{n\zeta}}{2\gamma} \right)^3 \left[ 1 - \left( 4 - \frac{\pi^2}{3} + 10\tilde{c}_i^2 \right) \lambda_n^2 + O(\lambda_n^4) \right]$$

Note that, for a steady wave \((c_i = 0)\) and a monochromatic wave \((n = 1\) and ignoring the second harmonic of the Stoke wave\) the expression (5.24) reduces to (5.16). We remark that for a steady Stokes wave we only need to assume \( c_i = 0 \).

In a similar manner, for an unsteady wave, we adopt the complex amplitude of surface shear stress is given by (3.24) but with \( \mathcal{U} \) given by (5.20) and the following modification for the expression for the eddy viscosity, given by (3.4), namely

$$\nu_{ne} = 2U_1^2 \left[ \eta^{-1} + (i k_n / \kappa^2) \ln(\eta / \eta_c) \right] + k_n c_i / \kappa^2 \right]^{-1}$$

Thus, upon substituting (5.25), and using an equivalent trial function to that given by (3.25), namely

$$\mathcal{F}_n = \mathcal{F}_{n0} e^{-k_n \eta / \delta_n}$$

the ‘unsteady’ version of the integral (3.24) becomes

$$\left( \mathcal{F}_n, \mathcal{F}_n' \right)_0 = -k_n \int_0^\infty \left[ k_n / \delta_n^3 + U_1^2 \left( i \mathcal{U}^2 - 2k_n \mathcal{U}^2 \right) \right] \mathcal{F}_{n0} e^{-2k_n \eta / \delta_n} d\eta$$

Invoking the condition \( \partial(\mathcal{F}_n / \mathcal{F}_n) / \partial \delta_n = 0 \), after substituting \( \mathcal{U} \) and (5.26), and evaluating
the integral \( \mathcal{F}_n / \mathcal{F}_0 \) we obtain
\[
(\mathcal{F}_n / \mathcal{F}_0) = \frac{1}{2i\delta_n} - \frac{(c_r + ic_i)^2}{4U_*^2} + \frac{\delta_n}{4\kappa^2} \left( \frac{1}{\delta_n} - \frac{ik}{4\kappa^2} \right) \left( \frac{\pi^2}{6} + L_{\delta_n}^2 \right)
\]
\[
= \frac{1}{4\kappa^2} \left[ \frac{2}{\delta_n} - \frac{(c_r + ic_i)^2}{U_*^2} + (1 + \delta_n) \left( \frac{\pi^2}{6} + L_{\delta_n}^2 \right) \right] \quad (5.28)
\]
where \( \delta_n = i\delta_n / \kappa^2 \).

and \( L_{\delta_n} = L_{n0} + \ln \delta_n \).

Solving the equation \( (5.29) \) for \( \delta_n \), we find that
\[
\delta_n = (\sqrt{3} - 1)(L_{n0} + \ln \delta_n)^{-1} \quad (5.30)
\]

Note that, \( \delta_n \) may be complex and \( |\delta_n| \ll 1 \) (but strictly not equal to zero), we may use the expansion for \( \ln \delta_n \), given by
\[
\ln \delta_n = 2 \sum_{m=1}^{\infty} \frac{1}{2m-1} \left( \frac{\delta_n - 1}{\delta_n + 1} \right)^{2m-1} \div 2 \frac{\delta_n - 1}{\delta_n + 1} + O \left[ \left( \frac{\delta_n - 1}{\delta_n + 1} \right)^3 \right], \quad 0 < |\delta_n| \leq 2
\]

we may cast \( (5.30) \) as
\[
\delta_n = (\sqrt{3} - 1) \left[ L_{n0} + 2 \left( \frac{\delta_n - 1}{\delta_n + 1} \right) \right]^{-1} \quad (5.31)
\]

Hence, the asymptotic evaluation of integrals in \( (5.27) \) yields to the following expression (for further details see the appendix of Sajjadi (2007))
\[
\beta_n = 2\kappa^2 \left[ 1 + (\sqrt{3} + 1) \left( 1 - \delta_n \right) L_{|\delta_n|} - (\sqrt{3} - 1) \left( L_{|\delta_n|} + 4\ln 2 \right) \right.
\]
\[
- 4 \left( \delta_{\hat{n}} - \delta_2 \right)] - 4\delta_n L_{n0}^2 \quad (5.32)
\]

where \( \delta_n = \mathcal{F} \{ \delta_n \} \),
\[
\delta_p = \delta \left( \frac{2\alpha_1}{ipL_p} \right) \approx 1 - \frac{ipL_p}{2\alpha_1}, \quad L_p = -\gamma - \ln (2kp) = L_0 + \ln p
\]

and
\[
\delta'(\Theta_n) = \Theta_n e^{\Theta_n} \int_0^{\tau} r^{-1} e^{-r} dt = \Theta_n e^{\Theta_n} E_1(\Theta_n)
\]

From these expressions we obtain (see Sajjadi (2007)) we obtain
\[
\beta_n = \frac{1}{2} \left( \Lambda_n + \Lambda_n^{-1} \right) \left( L_{\Theta_n}^2 + \frac{\pi^2}{6} \right), \quad \beta_{st} = \frac{4\kappa^2}{\Lambda_n}
\]

which are in reasonable agreement with (5.19) and (5.24).

6. Results and conclusions

In figure 5, we show comparison of the energy-transfer rate, \( \beta \), between the present result for a monochromatic unsteady (growing) wave, both analytically and numerically, and those calculated by Miles (1957) and Janssen (1991) for the steady wave counterpart. Miles and Janssen both assume that the drag \( C_D \), and thence \( \beta \), is dominated by the limiting inviscid wave growth
mechanisms for unsteady waves (where $c_i$ is a viscous unsteady (growing) wave, where $c_s$ is the stress closure model for s-stress closure model for $c_i/U_1$ and $c_i/U_1$ is the Charnock’s (1955) constant. Thick solid line, parameterization of Miles (1957) formula, for $c_i = 0$, $c_r = 0$, given by Janssen (1991): $\beta = 1.2 \kappa^{-2} \eta_c \log^4 \eta_c$, where $\eta_c = \min \left\{ 1, k_00^{e \left( \frac{U_1}{c_i} \right)^{e \left( \frac{U_1}{c_i} \right)}} \right\}$. Thin solid line, present formulation: $(\beta_T + \beta_r)$ for $c_i \neq 0$, $c_r \neq 0$. Numerical simulation using Launder, Reece & Rodi (1975) Reynolds-stress closure scheme for the energy transfer parameter, $\beta$, shown in figure 5. This shows consistency between these models and the unimportance of very small $c_i$ for which viscous processes are significant.

Figure 6 shows comparison of $\beta_i$ as a function of wave age $c_i/U_1$, calculated according to (Conte & Miles 1959) for numerical solution of inviscid Orr-Sommerfeld equation, against the numerical solution of equation (1.6) for $c_i/U_1 = 0.01$, $k_00 = 10^{-4}$ and $c_i < 2$. We remark that increasing $c_i/U_1$ from 0.01 to 0.1 (not shown here) makes no significant difference in the magnitude of $\beta_i$. We conclude therefore for a finite value of $c_r$ the right-hand side of equation (1.6) is dominant and therefore the magnitude of $\beta_i$, calculated from the solution of (1.6), is practically zero over a wide range of the wave age, in particular for a ‘young’ wave, where $c_i/U_1 < 2$. We thus conclude that the critical-layer mechanism plays an insignificant role for $c_i/U_1 < 9$, and very little effect for $9 \leq c_i/U_1 \leq 10.5$.

We remark that these parameterizations have been incorporated and tested in a spectral wave models, WaveWatch and WindWave, which shows superior results when compared with field data (see Fitzpatrick et al. 2002 and Sajjadi et al. 2002).

In conclusion we adopted an asymptotic multi-deck solution for turbulent shear flows over unsteady surface waves, in the limits of low turbulent stresses and small wave amplitude. The structure of the flow is defined, using an eddy-viscosity turbulence model, in terms of asymptotically-matched thin-layers, namely the surface layer and a critical layer. Solutions for both inner and outer regions are constructed through an interpolation between an inner, mixing-length and an outer, rapid-distortion approximations. The results particularly demonstrate the physical importance of the singular flow features and physical implications of the elevated critical layer in the
Figure 6. Component of energy transfer parameter, $\beta_c$, by different models of critical layer mechanisms for unsteady waves (where $c_i \ll U_\ast$) as a function of the wave age $c_r/U_1$. •, numerical solution of inviscid Orr-Sommerfeld equation by Conte & Miles (1959) for $c_i = 0$ and $\nu_e = 0$ using the singular critical layer approach; ○ numerical solution of equation (1) for $c_i \neq 0$ and $\nu_e \neq 0$. Taken from Sajjadi et al. (2014).

limit of the unsteadiness tending to zero. These agree with the variational mathematical solution of Miles (1957) for small but finite growth rate. However, the results obtained here, are not consistent physically or mathematically with his analysis in the limit of growth rate tending to zero. In the present study it is shown that in the limit of zero growth rate the effect of the elevated critical layer is eliminated by finite turbulent diffusivity, so that the perturbed flow and the drag force on determined by the asymmetric and sheltering flow in the surface shear layer and its matched interaction with the upper region, as physically demonstrated by Sajjadi, Hunt and Drul-lion (2014). The results for an unsteady monochromatic waves is also extended to those growing Stokes waves. Thus, estimation can be made as to what precentage of total energy transfer from wind goes to each harmonic of a Stokes wave.

References

Asymptotic analysis for generation of unsteady waves by turbulence


