8-1-2017

Traveling Wave Solutions for Wave Equations with Exponential Nonlinearities

S. C. Mancas
Embry-Riddle Aeronautical University, mancass@erau.edu

H. C. Rosu
IPICYT, Instituto Potosino de Investigacion Cientifica y Tecnologica

M. Perez-Maldonado
IPICYT, Instituto Potosino de Investigacion Cientifica y Tecnologica

Traveling wave solutions for wave equations with exponential nonlinearities

S C Mancas
Department of Mathematics,
Embry-Riddle Aeronautical University,
Daytona Beach, FL 32114-3900, USA
e-mail: mancass@erau.edu

H C Rosu and M. Pérez-Maldonado
[1] *IPICYT, Instituto Potosino de Investigacion Cientifica y Tecnologica,*
Camino a la presa San José 2055, Col. Lomas 4a Sección, 78216 San Luis Potosí, S.L.P., Mexico
e-mails: hcr@ipicyt.edu.mx, maximino.perez@ipicyt.edu.mx

We use a simple method which leads to the quadrature involved in obtaining the traveling wave solutions of wave equations with one and two exponential nonlinearities. When the constant term in the integrand is zero, implicit solutions in terms of hypergeometric functions are obtained while when that term is nonzero we give all the basic traveling wave solutions based on a detailed study of the corresponding elliptic equations of several well-known particular cases with important applications in physics.

PACS numbers: 02.30.Ik, 02.30.Gp, 02.30.Hq, 02.30.Jr

Keywords: Liouville equation; Tzitzéica, Dodd-Bullough; Dodd-Bullough-Mikhailov; sine-Gordon; sinh-Gordon; Weierstrass function

I. INTRODUCTION

In this paper, we will consider second-order differential equations with two exponential nonlinear terms

$$\psi_{uv} = \alpha e^{a\psi} + \beta e^{b\psi} \ , \tag{1}$$

where a and b are nonzero real constants, while α and β are real constants not simultaneously zero. This kind of equations can be turned into polynomial nonlinear equations

$$\frac{\partial^2}{\partial u \partial v} \log h = \alpha h^a + \beta h^b, \tag{2}$$

by using the change of variables $\psi = \log h$. Particular cases of this type of equations are the Tzitzéica equation [1–3] and its variants, such as the Dodd-Bullough equation [4] and the Dodd-Bullough-Mikhailov equation [5, 6], and also the sine-Gordon and the sinh-Gordon equations belong to this class, all of them with important applications in solid state physics, nonlinear optics, biological physics, and quantum field theory.

Along the two characteristics $z = u - \lambda v$, $t = u + \lambda v$, $\lambda \neq 0$, Eq. (1) becomes the nonlinear wave equation

$$\psi_{tt} - \psi_{zz} = \frac{1}{\lambda}\left(\alpha e^{a\psi} + \beta e^{b\psi}\right) \tag{3}$$

or in the logarithmic variable

$$h(h_{tt} - h_{zz}) - ({h_t}^2 - {h_z}^2) = \frac{h^2}{\lambda}\left(\alpha h^a + \beta h^b\right). \tag{4}$$

We are concerned with finding closed-form solutions of Eq. (4) using the traveling wave ansatz $h(z, t) = h(\xi)$ with $\xi = kz - \omega t$, and $k \neq \pm\omega$ which yields the following ordinary differential equation (ODE)

$$h h_{\xi\xi} - {h_\xi}^2 = \frac{h^2}{\lambda\gamma}\left(\alpha h^a + \beta h^b\right) \equiv f(h), \tag{5}$$

with $\gamma = \omega^2 - k^2 \neq 0$. This ODE will be analyzed in the next two sections of the paper, both in its full generality and simplified to the important cases mentioned above.

II. THE IMPLICIT SOLUTION

A simple method to solve ODEs of type (5) is to let $h_\xi = u(h(\xi))$, and use $h_{\xi\xi} = u\frac{du}{dh}$ [7], to obtain

$$hu\frac{du}{dh} - u^2 = f(h), \tag{6}$$

which can be turned into a Bernoulli equation using the substitution $u^2 = z$

$$\frac{dz}{dh} - \frac{2}{h}z = \frac{2}{h}f(h). \tag{7}$$

The solution for this equation is

$$z = h^2\left(c_0 + 2\int \frac{f(h)}{h^3}dh\right) \ , \tag{8}$$

and using back the transformations $u = \pm\sqrt{z} = \frac{dh}{d\xi}$ h is obtained by the quadrature

$$\int \frac{dh}{h\sqrt{c_1 + \frac{\alpha}{a}h^a + \frac{\beta}{b}h^b}} = \pm\sqrt{\frac{2}{\lambda\gamma}}\int d\xi \ . \tag{9}$$

In general, this quadrature can be performed only if $c_1 = 0$, when one obtains the following implicit solution involving the hypergeometric function

$$h^{-b}\sqrt{\frac{\alpha h^a}{a} + \frac{\beta h^b}{b}}\ {}_2F_1\left(1, 1 - \frac{a}{2(a-b)}; 1 - \frac{b}{2(a-b)}; -\frac{b\alpha}{a\beta}h^{a-b}\right) = \mp\frac{\beta}{\sqrt{2\lambda\gamma}}(\xi - \xi_0) \ , \tag{10}$$

which also implies $\beta \neq 0$. We will show in the next section how the quadrature in (9) is solved in the important particular cases mentioned in the introduction for $c_1 \neq 0$.

III. PARTICULAR CASES

A. Liouville equation

We start with the simplest case, which is the Liouville equation [8], one of the fundamental equations in nonlinear mathematical physics and differential geometry. It corresponds to $\alpha = 1,\ \beta = 0,\ a = 1,\ b = 0$ which is

$$\frac{\partial^2}{\partial u \partial v}\log h = h \ . \tag{11}$$

The quadrature in Eq. (9) takes the form

$$\int \frac{dh}{h\sqrt{c_1 + h}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0), \tag{12}$$

and the function h is obtained by solving the elliptic equation

$${h_\xi}^2 = a_3h^3 + a_2h^2 + a_1h + a_0 \tag{13}$$

with the coefficients given by the system

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 0 \\ a_2 &= \tfrac{2c_1}{\lambda\gamma} \\ a_3 &= \tfrac{2}{\lambda\gamma} \ . \end{aligned} \tag{14}$$

For convenience, denote $r = \frac{1}{\lambda\gamma}$, and $p = \frac{2c_1}{\lambda\gamma}$; then Eq. (13) becomes the reduced elliptic equation

$$h_\xi^2 = 2rh^3 + ph^2 \ , \tag{15}$$

with soliton solution if $p > 0$, or periodic solution if $p < 0$

$$\begin{array}{ll} h(\xi) = -\frac{p}{2r}\mathrm{sech}^2\left[\frac{1}{2}\sqrt{p}(\xi-\xi_0)\right] \ , & p > 0 \\ h(\xi) = -\frac{p}{2r}\sec^2\left[\frac{1}{2}\sqrt{-p}(\xi-\xi_0)\right] \ , & p < 0 \ , \end{array} \tag{16}$$

which by using system (14) give the solutions

$$\begin{array}{ll} h(\xi) = -c_1\mathrm{sech}^2\left(\sqrt{\frac{c_1}{2\lambda\gamma}}(\xi-\xi_0)\right) \ , & \frac{c_1}{2\lambda\gamma} > 0 \\ h(\xi) = -c_1\sec^2\left(\sqrt{\frac{-c_1}{2\lambda\gamma}}(\xi-\xi_0)\right) \ , & \frac{c_1}{2\lambda\gamma} < 0. \end{array} \tag{17}$$

Since we are in the case $\beta = 0$, we cannot apply (10) when $c_1 = 0$, but the integration of (12) corresponds to the most degenerate case of the Weierstrass elliptic equation (25) when both germs g_2 and g_3 are zero, which leads to the rational solution

$$h(\xi) = \frac{2\lambda\gamma}{(\xi-\xi_0)^2} \ . \tag{18}$$

Plots of all these three types of Liouville solutions are given in Fig. (1).

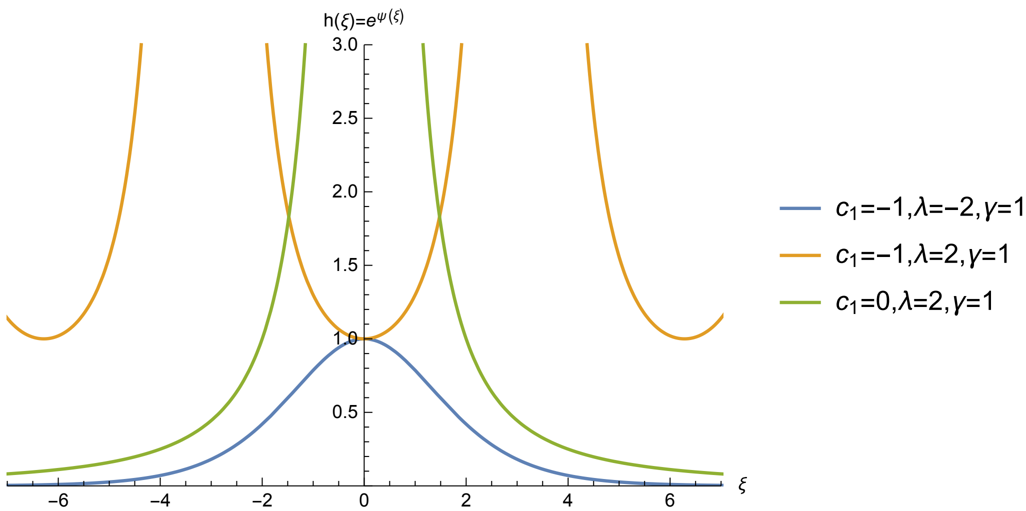

FIG. 1: The soliton and periodic solutions (17) and the rational solution (18) of the Liouville equation.

B. The Tzitzéica equation

Tzitzéica's equation,

$$\frac{\partial^2}{\partial u \partial v}\log h = h - \frac{1}{h^2} \ , \tag{19}$$

emerged in 1907-1910 in the area of geometry, but only after eighty years it has been found to have applications in physics. For example, Euler's equations for an ideal gas with a special equation of motion can be reduced to the Tzitzéica equation, and a 2+1-dimensional system in magneto-hydrodynamics has been shown to be in one-to-one correspondence with it [9, 10]. Very recently, dark optical solitons and traveling waves of Tzitzéica type have been also discussed in the literature [11, 12].

For Tzitzéica's equation, we identify the constants in (1) as $\alpha = 1$, $\beta = -1$, $a = 1$, $b = -2$ which gives the quadrature

$$\int \frac{dh}{\sqrt{h^3 + c_1 h^2 + \frac{1}{2}}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi-\xi_0). \tag{20}$$

In implicit form h satisfies Eq. (10) which simplifies to

$$h\,{}_2F_1\left(\frac{1}{2},\frac{1}{3};\frac{4}{3};-2h^3\right)=\pm\frac{\xi-\xi_0}{\sqrt{\lambda\gamma}}. \tag{21}$$

The solution h is obtained explicitly by solving the elliptic equation (13) with coefficients given by the system

$$\begin{aligned} a_0 &= \tfrac{1}{\lambda\gamma}\\ a_1 &= 0\\ a_2 &= \tfrac{2c_1}{\lambda\gamma}\\ a_3 &= \tfrac{2}{\lambda\gamma}\ . \end{aligned} \tag{22}$$

which becomes

$$h_\xi{}^2 = 2rh^3+ph^2+r\ . \tag{23}$$

Using the scale shift transformation

$$h(\xi)=\frac{1}{r}\left(2\wp(\xi;g_2,g_3)-\frac{p}{6}\right), \tag{24}$$

Eq. (23) becomes the Weierstrass equation

$$\wp_\xi{}^2 = 4\wp^3-g_2\wp-g_3\ . \tag{25}$$

The germs of the Weierstrass function are given by

$$\begin{aligned} g_2 &= \tfrac{a_2^2-3a_1a_3}{12}=\tfrac{p^2}{12}=2(e_1^2+e_2^2+e_3^2)\\ g_3 &= \tfrac{9a_1a_2a_3-27a_0a_3^2-2a_2^3}{432}=-\tfrac{1}{4}\left(r^3+\tfrac{p^3}{54}\right)=4e_1e_2e_3 \end{aligned} \tag{26}$$

and together with the modular discriminant

$$\Delta = g_2{}^3-27g_3{}^2=-\frac{r^3}{16}(p^3+27r^3)=16(e_1-e_2)^2(e_1-e_3)^2(e_2-e_3)^2 \tag{27}$$

are used to classify the solutions of Eq. (23), where the constants e_i are the roots of the cubic polynomial

$$s_3(t)=4t^3-g_2t-g_3=4(t-e_1)(t-e_2)(t-e_3)=0. \tag{28}$$

Case (1). If $\Delta\equiv 0\Rightarrow p=-3r\Rightarrow c_1=-\frac{3}{2}$. This degenerate case implies that $s_3(t)$ has repeated root of multiplicity two. Then the Weierstrass solutions can be simplified since $\wp$ degenerates into *hyperbolic* or *trigonometric* functions. Because of the degeneracy Eq. (20) can be factored as

$$\int\frac{dh}{\sqrt{(h-1)^2\left(h+\frac{1}{2}\right)}}=\pm\sqrt{\frac{2}{\lambda\gamma}}(\xi-\xi_0). \tag{29}$$

Depending on the sign of g_3 we have the sub-cases:

Case (1a). $r>0$ with $g_2>0\Rightarrow g_3=-\frac{r^3}{8}<0\Rightarrow\lambda\gamma>0$, so Eq. (29) has the soliton solution

$$h(\xi)=\frac{1}{2}\left[-1+3\tanh^2\left(\frac{1}{2}\sqrt{\frac{3}{\lambda\gamma}}(\xi-\xi_0)\right)\right]. \tag{30}$$

By letting $e_1=e_2=\hat{e}>0$ then $e_3=-2\hat{e}<0$, hence

$$\begin{aligned} g_2 &= 12\hat{e}^2>0\\ g_3 &= -8\hat{e}^3<0 \end{aligned} \tag{31}$$

the Weierstrass $\wp$ solution to Eq. (25) reduces to

$$\wp(\xi;12\hat{e}^2,-8\hat{e}^3)=\hat{e}+3\hat{e}\ \mathrm{csch}^2(\sqrt{3\hat{e}}\xi)\ . \tag{32}$$

For $\hat{e}=\frac{1}{4\lambda\gamma}>0$, the Weierstrass solution replaced by (32) gives

$$h(\xi)=1+\frac{3}{2}\mathrm{csch}^2\left(\frac{1}{2}\sqrt{\frac{3}{\lambda\gamma}}(\xi-\xi_0)\right). \tag{33}$$

Case (1b). $r < 0$ with $g_2 > 0 \Rightarrow g_3 = -\frac{r^3}{8} > 0 \Rightarrow \lambda\gamma < 0$, so Eq. (29) has the periodic solution

$$h(\xi) = -\frac{1}{2}\left[1 + 3\tan^2\left(\frac{1}{2}\sqrt{\frac{3}{-\lambda\gamma}}(\xi - \xi_0)\right)\right]. \tag{34}$$

By letting $e_2 = e_3 = -\tilde{e} < 0$ with $\tilde{e} > 0$, then $e_1 = 2\tilde{e} > 0$, hence

$$\begin{array}{l} g_2 = 12\tilde{e}^2 > 0 \\ g_3 = 8\tilde{e}^3 > 0 \end{array} \tag{35}$$

the Weierstrass $\wp$ solution reduces to

$$\wp(\xi; 12\tilde{e}^2, 8\tilde{e}^3) = -\tilde{e} + 3\tilde{e}\ \csc^2(\sqrt{3\tilde{e}}\xi). \tag{36}$$

For $\tilde{e} = -\frac{1}{4\lambda\gamma} > 0$, the Weierstrass solution replaced by (36) gives

$$h(\xi) = 1 - \frac{3}{2}\csc^2\left(\frac{1}{2}\sqrt{\frac{3}{-\lambda\gamma}}(\xi - \xi_0)\right)\ . \tag{37}$$

All the Tzitzéica solutions corresponding to these cases are displayed in Fig. (2).

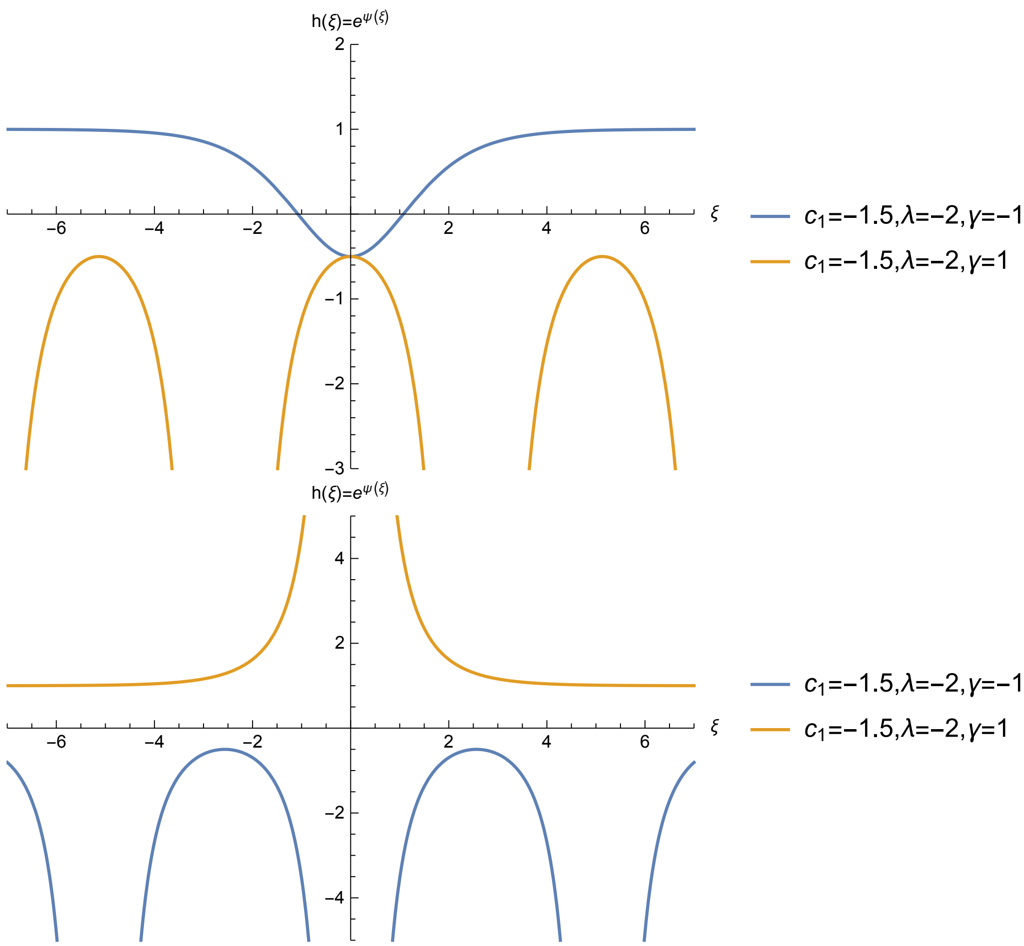

FIG. 2: The Tzitzéica soliton (30) and periodic solution (34) (top). The Tzitzéica solution (33) and periodic solution (37) (bottom).

Case (2). If $\Delta \neq 0 \Rightarrow p \neq -3r \Rightarrow c_1 \neq -\frac{3}{2}$ we include two particular solutions which will fix the integration constant c_1 as follows: the equianharmonic ($g_2 = 0$) and lemniscatic case ($g_3 = 0$), respectively.

i) For the equianharmonic case $g_2 = 0 \Rightarrow p = 0 \Rightarrow g_3 = -\frac{r^3}{4}$. Because $\Delta = -\frac{27}{16}r^6 < 0$, then $s_3(t)$ has a pair of conjugate complex roots, and since $c_1 = 0$ the solution to Eq. (20) reduces to

$$h(\xi) = 2\lambda\gamma\wp\left(\xi - \xi_0; 0, -\frac{1}{4\lambda^3\gamma^3}\right). \tag{38}$$

ii) For the lemniscatic case $g_3 = 0 \Rightarrow p = -3\sqrt[3]{2}r \Rightarrow g_2 = \frac{3\sqrt[3]{4}}{4}r^2$. Because $\Delta = \frac{27}{16}r^6 > 0$, then $s_3(t)$ has three distinct real roots given by $e_3 = -\frac{\sqrt{g_2}}{2}$, $e_2 = 0$, and $e_1 = \frac{\sqrt{g_2}}{2}$. Although the Weierstrass unbounded function has poles aligned on the real axis of the $\xi - \xi_0$ complex plane, we can choose ξ_0 in such a way to shift these poles a half of period above the real axis, so that the elliptic function simplifies using the formula [13]

$$\wp(\xi; g_2, 0) = e_3 + (e_2 - e_3)\mathrm{sn}^2\left(\sqrt{e_1 - e_3}(\xi - \xi_0'); m\right) \tag{39}$$

with elliptic modulus $m = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}$. Using the values of the roots together with $\xi_0' = 0$ we obtain

$$\wp(\xi; g_2, 0) = -\frac{\sqrt{g_2}}{2}\mathrm{cn}^2\left(\sqrt[4]{g_2}\xi; \frac{\sqrt{2}}{2}\right). \tag{40}$$

Because $c_1 = -\frac{3}{\sqrt[3]{4}}$ the solutions for the lemniscatic case are reduced using the transformation (24) to periodic cnoidal waves, and they become

$$h(\xi) = \frac{1}{\sqrt[3]{4}}\left[1 - \sqrt{3}\ \mathrm{cn}^2\left(\frac{\sqrt[4]{3}}{\sqrt[3]{2}\sqrt{\lambda\gamma}}\xi; \frac{\sqrt{2}}{2}\right)\right]. \tag{41}$$

iii) For the most general case, $g_2 \neq 0$, $g_3 \neq 0$ the general solution to Eq. (20) is

$$h(\xi) = \lambda\gamma\left[2\wp\left(\xi - \xi_0; \frac{{c_1}^2}{3\lambda^2\gamma^2}, -\frac{4{c_1}^3 + 27}{108\lambda^3\gamma^3}\right) - \frac{c_1}{3\lambda\gamma}\right]. \tag{42}$$

The equianharmonic, lemniscatic, and Weierstrass solutions of Tzitzéica's equation are displayed in Fig. (3).

C. The Dodd-Bullough equation

This variant of Tzitzéica's equation has been introduced in the first dedicated study of the polynomial conserved quantities of the sine-Gordon equation [4]. Its form in the h variable is

$$\frac{\partial^2}{\partial u \partial v}\log h = -h + \frac{1}{h^2}, \tag{43}$$

thus, we identify the constants to be $\alpha = -1,\ \beta = 1,\ a = 1,\ b = -2$ which gives the quadrature

$$\int \frac{dh}{\sqrt{-h^3 + c_1 h^2 - \frac{1}{2}}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0). \tag{44}$$

In implicit form, the h functions satisfies Eq. (10) which simplifies to

$$h\,{}_2F_1\left(\frac{1}{2}, \frac{1}{3}; \frac{4}{3}; -2h^3\right) = \mp\frac{\xi - \xi_0}{\sqrt{-\lambda\gamma}}. \tag{45}$$

Using Tzitzéica solutions provided that $r \to -r \Rightarrow \lambda\gamma \to -\lambda\gamma$ and $c_1 \to -c_1$, we have:

Case (1). If $\Delta \equiv 0 \Rightarrow c_1 = \frac{3}{2}$, then

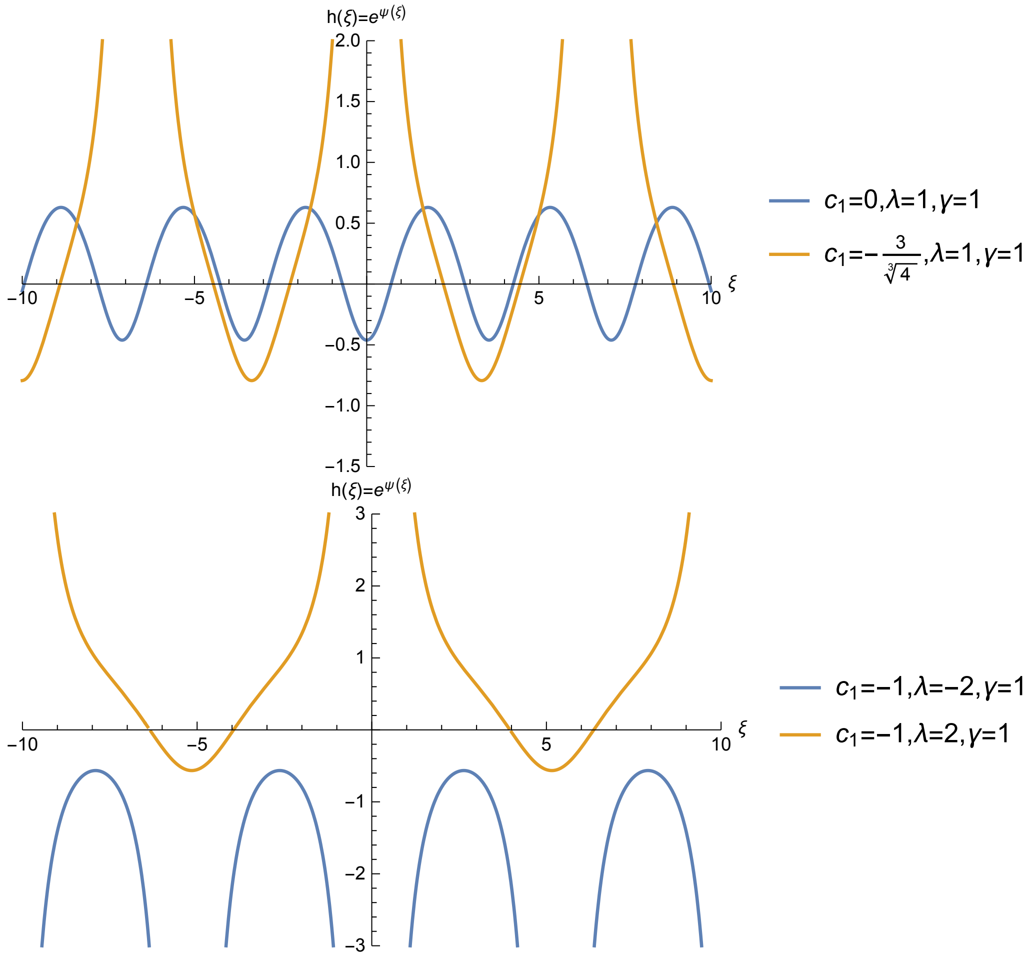

FIG. 3: The equianharmonic and lemniscatic solutions (38), (41) of the Tzitzéica equation (top). The Weierstrass solution (42) of the Tzitzéica equation (bottom).

Case (1a). $\lambda\gamma < 0$ so Eq. (44) has soliton solution

$$h(\xi) = \frac{1}{2}\left[-1 + 3\tanh^2\left(\frac{1}{2}\sqrt{\frac{3}{-\lambda\gamma}}(\xi - \xi_0)\right)\right] , \tag{46}$$

while the second linearly independent solution corresponding to (33) is

$$h(\xi) = 1 + \frac{3}{2}\text{csch}^2\left(\frac{1}{2}\sqrt{\frac{3}{-\lambda\gamma}}(\xi - \xi_0)\right) . \tag{47}$$

Case (1b). $\lambda\gamma > 0$ so Eq. (44) has periodic solution

$$h(\xi) = -\frac{1}{2}\left[1 + 3\tan^2\left(\frac{1}{2}\sqrt{\frac{3}{\lambda\gamma}}(\xi - \xi_0)\right)\right] \tag{48}$$

and the second linearly independent solution corresponding to (37) is

$$h(\xi) = 1 - \frac{3}{2}\csc^2\left(\frac{1}{2}\sqrt{\frac{3}{\lambda\gamma}}(\xi - \xi_0)\right) . \tag{49}$$

Case (2). If $\Delta \neq 0 \Rightarrow c_1 \neq \frac{3}{2}$ then the Weierstrass solution of Eq. (44) is

$$h(\xi) = -\lambda\gamma\left[2\wp\left(\xi - \xi_0; \frac{{c_1}^2}{3\lambda^2\gamma^2}, -\frac{4{c_1}^3 - 27}{108\lambda^3\gamma^3}\right) - \frac{c_1}{3\lambda\gamma}\right] . \tag{50}$$

i) For the equianharmonic case, $c_1 = 0$, we have:

$$h(\xi) = -2\lambda\gamma\wp\left(\xi - \xi_0; 0, \frac{1}{4\lambda^3\gamma^3}\right) . \tag{51}$$

ii) For the lemniscatic case, $c_1 = \frac{3}{\sqrt[3]{4}}$, we have:

$$h(\xi) = \frac{1}{\sqrt[3]{4}}\left[1 - \sqrt{3}\ \mathrm{cn}^2\left(\frac{\sqrt[4]{3}}{\sqrt[3]{2}\sqrt{-\lambda\gamma}}\xi; \frac{\sqrt{2}}{2}\right)\right] . \tag{52}$$

Respecting the rules of changing the signs of the parameters, the Dodd-Bullough solutions are identical to the Tzitzéica solutions and consequently we will not plot them here.

D. The Tzitzéica-Dodd-Bullough equation

This variant equation reads

$$\frac{\partial^2}{\partial u \partial v}\log h = h + \frac{1}{h^2}, \tag{53}$$

thus, we identify the constants to be $\alpha = 1,\ \beta = 1,\ a = 1,\ b = -2$ which gives the quadrature

$$\int \frac{dh}{\sqrt{h^3 + c_1 h^2 - \frac{1}{2}}} = \mp\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0) . \tag{54}$$

We can use the Dodd-Bullough solutions, Eqs. (46)-(50,) with $h \to -h$ and $\xi \to -\xi$.

E. The Dodd-Bullough-Mikhailov equation

The last Tzitzéica variant equation reads

$$\frac{\partial^2}{\partial u \partial v}\log h = -h - \frac{1}{h^2}, \tag{55}$$

thus the constants are identified as $\alpha = -1,\ \beta = -1,\ a = 1,\ b = -2$ which gives the quadrature

$$\int \frac{dh}{\sqrt{-h^3 + c_1 h^2 + \frac{1}{2}}} = \mp\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0) . \tag{56}$$

From the polynomial in the integrand, one can see that the solutions of this variant equation are obtained from Tzitzéica solutions by $h \to -h$ and $\xi \to -\xi$.

F. The sine-Gordon equation

According to [14], Tzitzéica's equation is the "nearest relative" of the well-known sine-Gordon equation which can be written as

$$\frac{\partial^2}{\partial u \partial v}\log h = \frac{1}{2i}\left(h^i - h^{-i}\right) = \sin(\log h) . \tag{57}$$

Thus, we identify the constants to be $\alpha = \frac{1}{2i},\ \beta = -\frac{1}{2i},\ a = i,\ b = -i$ which gives the quadrature

$$\int \frac{dh}{h\sqrt{c_1 - \cos(\log h)}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0), \tag{58}$$

that is equivalent to

$$\int \frac{d\psi}{\sqrt{c_1 - \cos(\psi)}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0). \tag{59}$$

In implicit form, h satisfies Eq. (10) which simplifies to

$$h^i \sqrt{\cos(\log h)}\ {}_2F_1\left(1, \frac{3}{4}; \frac{5}{4}; -h^{2i}\right) = \mp\frac{\xi - \xi_0}{2\sqrt{2\lambda\gamma}}. \tag{60}$$

For the special case of $c_1 = 1$, Eq. (59) simplifies to

$$\int \frac{d\psi}{\sin\frac{\psi}{2}} = \pm\frac{2}{\sqrt{\lambda\gamma}}(\xi - \xi_0). \tag{61}$$

for $\lambda\gamma > 0$, and using the identity $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$ we obtain the kink-antikink solutions

$$\psi(\xi) = 4\arctan\left(e^{\pm\frac{\xi-\xi_0}{\sqrt{\lambda\gamma}}}\right). \tag{62}$$

When $c_1 = -1$, Eq. (59) simplifies to

$$\int \frac{d\psi}{\cos\frac{\psi}{2}} = \pm\frac{2}{\sqrt{-\lambda\gamma}}(\xi - \xi_0) \tag{63}$$

for $\lambda\gamma < 0$, and using the identity $\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \frac{1-\cos\theta}{\sin\theta}$ the shifted kink-antikink solution is obtained

$$\psi(\xi) = -\pi + 4\arctan\left(e^{\pm\frac{\xi-\xi_0}{\sqrt{-\lambda\gamma}}}\right)\ . \tag{64}$$

Plots of the $c_1 = \pm 1$ arctan solutions (62) and (64) are displayed in Fig. (4).

When $c_1 \neq \pm 1$, then ψ satisfies the elliptic equation

$$\psi_\xi{}^2 = \frac{2}{\lambda\gamma}(c_1 - \cos\psi), \tag{65}$$

with solutions given by

$$\psi(\xi) = 2\ \mathrm{am}\left(\pm\sqrt{\frac{c_1 - 1}{2\lambda\gamma}}(\xi - \xi_0); \frac{2}{1 - c_1}\right) \tag{66}$$

where the Jacobi amplitude $\psi = \mathrm{am}(\xi; m)$ is the inverse of the elliptic integral of the first kind $F(\psi; m) = \xi$ given by Eq. (59) [15, 16]. For plots of the sine-Gordon Jacobi amplitude solutions, see Fig. (5).

G. The sinh-Gordon equation

The hyperbolic version of the sine-Gordon equation, the sinh-Gordon equation, is also extensively used in integrable quantum field theory [17, 18], kink dynamics [19], and hydrodynamics [20]. Here, we will use the parametrization in [17] and write the corresponding equation in the variable h in the form

$$\frac{\partial^2}{\partial u \partial v}\log h = \frac{1}{2}\left(h^2 - h^{-2}\right) = \sinh(2\log h)\ . \tag{67}$$

Thus, we identify the constants to be $\alpha = \frac{1}{2},\ \beta = -\frac{1}{2},\ a = 2,\ b = -2$ which gives the quadrature

$$\int \frac{dh}{h\sqrt{c_1 + \frac{1}{2}\cosh(2\log h)}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0), \tag{68}$$

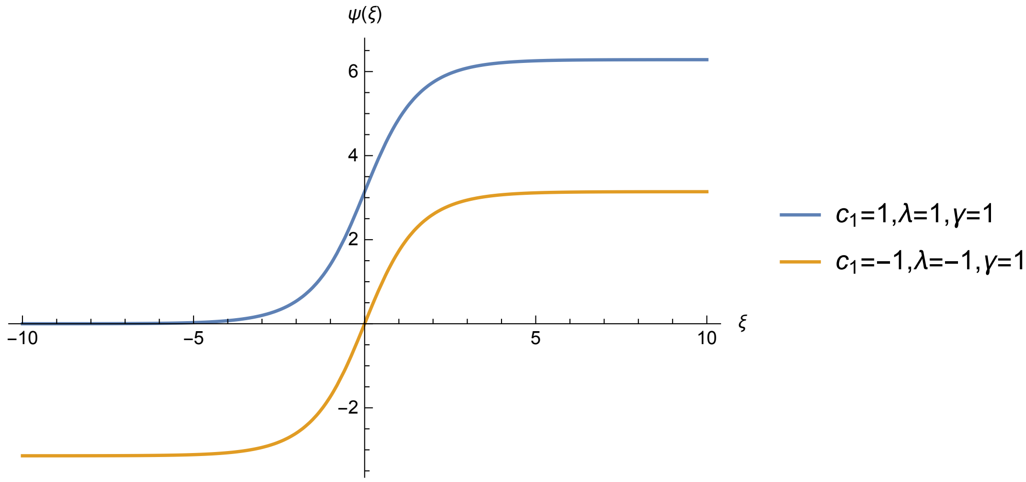

FIG. 4: The arctan kink solutions (62) and (64) of the sine-Gordon equation.

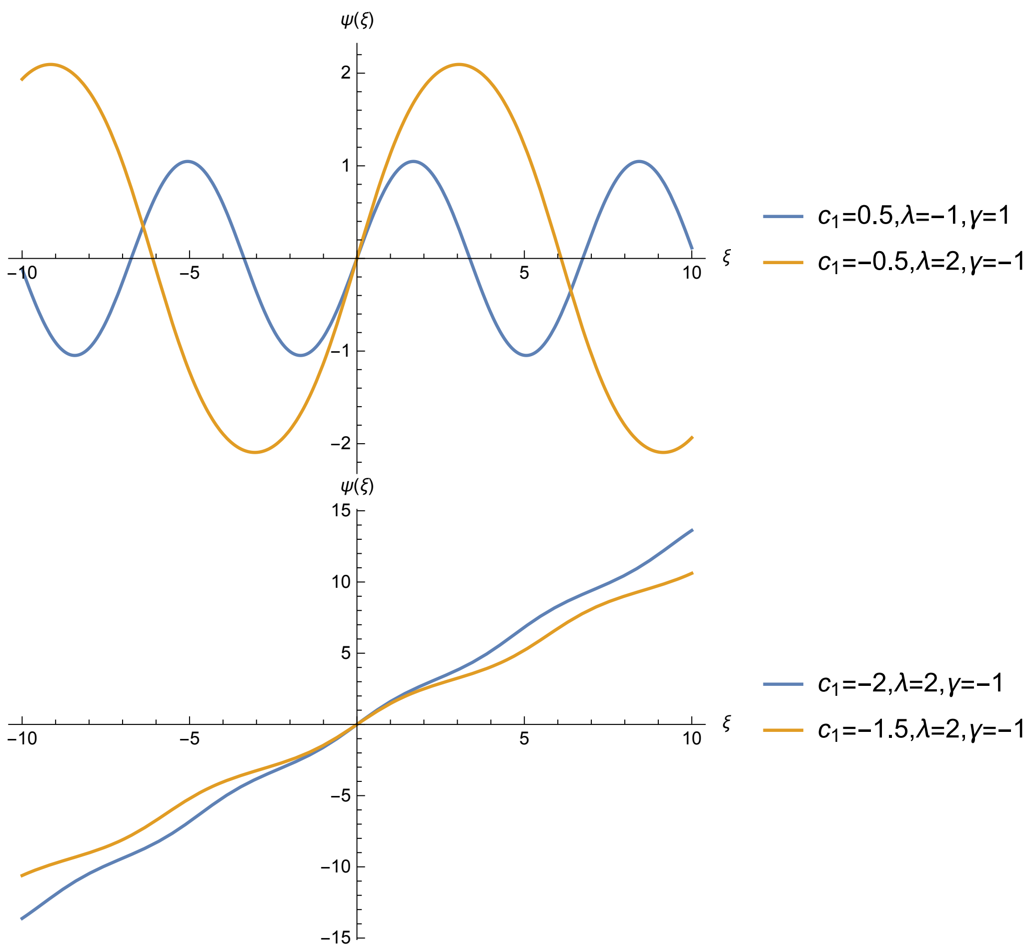

FIG. 5: The Jacobi amplitude solution (66) of the sine-Gordon equation for $|m| > 1$ (top) and $|m| < 1$ (bottom).

which is equivalent to

$$\int \frac{d\psi}{\sqrt{c_1 + \frac{1}{2}\cosh(2\psi)}} = \pm\sqrt{\frac{2}{\lambda\gamma}}(\xi - \xi_0). \tag{69}$$

In implicit form, h satisfies Eq. (10) which simplifies to

$$h\,{}_2F_1\left(\frac{1}{2},\frac{1}{4},\frac{5}{4};-h^4\right)=\pm\frac{\xi-\xi_0}{\sqrt{2\lambda\gamma}}. \tag{70}$$

For the special case of $c_1=-\frac{1}{2}$, Eq. (69) simplifies to

$$\int\frac{d\psi}{\sinh\psi}=\pm\sqrt{\frac{2}{\lambda\gamma}}(\xi-\xi_0). \tag{71}$$

with the kink-antikink solutions

$$\psi(\xi)=2\ \mathrm{arctanh}\left(e^{\pm\sqrt{\frac{2}{\lambda\gamma}}(\xi-\xi_0)}\right). \tag{72}$$

When $c_1=\frac{1}{2}$, Eq. (69) simplifies to

$$\int\frac{d\psi}{\cosh\psi}=\mathrm{gd}(\psi)=\pm\sqrt{\frac{2}{\lambda\gamma}}(\xi-\xi_0), \tag{73}$$

where the Gudermannian function $\mathrm{gd}(\psi)=2\ \arctan\left(\tanh\frac{\psi}{2}\right)$ gives also the kink-antikink solutions

$$\psi(\xi)=2\ \mathrm{arctanh}\left[\tan\left(\pm\frac{1}{\sqrt{2\lambda\gamma}}(\xi-\xi_0)\right)\right]. \tag{74}$$

The arctanh solutions are plotted in Fig. (6).

When $c_1\neq\pm\frac{1}{2}$, then ψ satisfies the elliptic equation

$$\psi_\xi{}^2=\frac{2}{\lambda\gamma}\left(c_1+\frac{1}{2}\cosh(2\psi)\right), \tag{75}$$

with solutions given by

$$\psi(\xi)=i\ \mathrm{am}\left(\pm\sqrt{\frac{2c_1+1}{-\lambda\gamma}}(\xi-\xi_0);\frac{2}{2c_1+1}\right)\ . \tag{76}$$

Plots of the Jacobi amplitude solution are given in Fig. (7).

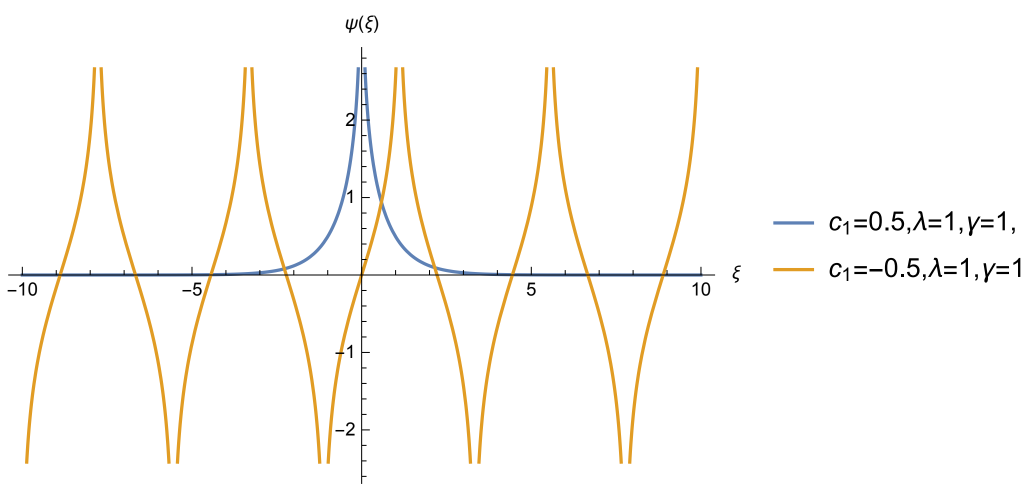

FIG. 6: The arctanh solutions (72) and (74) of the sinh-Gordon equation.

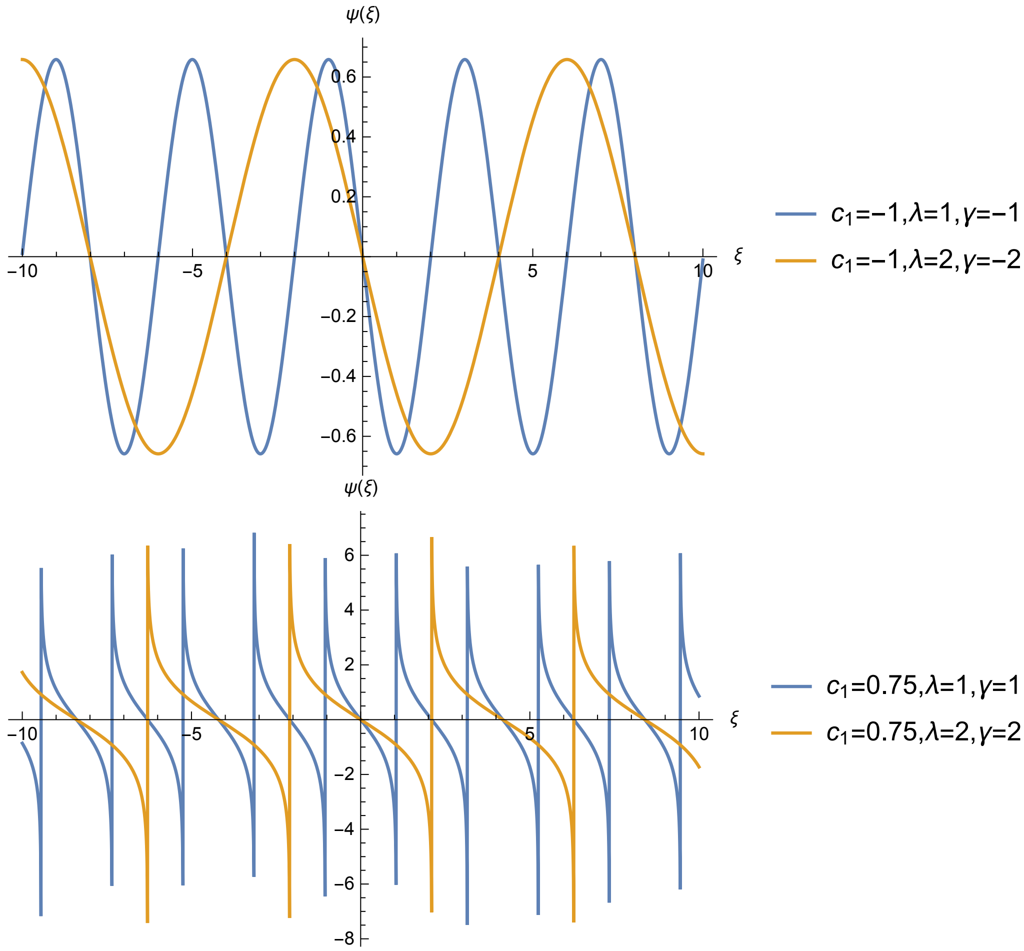

FIG. 7: The Jacobi amplitude solution (76) of the sinh-Gordon equation for $|m| > 1$ (top) and $|m| < 1$ (bottom).

IV. CONCLUSION

In summary, we have used a very simple method to obtain the quadrature of the general form of wave equations with two exponential nonlinearities whose particular cases correspond to celebrated equations in mathematical physics, such as Liouville, Tzitzéica and its variants, sine-Gordon, and sinh-Gordon equations. All the basic soliton, periodic and Weierstrass solutions of these equations are obtained consistently in the traveling variable by a thorough analysis of the elliptic equation. Particular implicit solutions in terms of a generic hypergeometric function are also obtained through a direct integration. Some of these solutions, such as the Weierstrass solutions of the Tzitzéica class of equations and the amplitude Jacobi solutions of the sine/sinh-Gordon equations cannot be obtained by other methods in the literature, for example the well-known tanh method, although they can also be obtained via the integral bifurcation method [21]. Of course, for more complicated multiple-soliton (multi-phase soliton) solutions, one should use Darboux and Bäcklund transformations [14, 22, 23].

Acknowledgements

SCM would like to acknowledge partial support from the Dean of Research & Graduate Studies at Embry-Riddle Aeronautical University while on a short visit to IPICyT, San Luis Potosí, Mexico. MPM thanks CONACyT for a doctoral fellowship.

[1] Tzitzéica G, "Sur une nouvelle classe de surfaces" C.R. Acad. Sci. Paris **144** 1257 (1907).
[2] Tzitzéica G, "Sur une nouvelle classe de surfaces" Rend. Circ. Mat. Palermo **25** 180 (1908); "ibid" **28** 210 (1909).
[3] Tzitzéica G, "Sur une nouvelle classe de surfaces" C.R. Acad. Sci. Paris **150** 955 and 1227 (1910).
[4] Dodd R K and Bullough R K, "Polynomial conserved quantities for the sine-Gordon equations" Proc. R. Soc. A **352** 481 (1977).
[5] Mikhailov A V, "Integrability of a two-dimensional generalization of the Toda chain" JETP Lett. **30** 414 (1979).
[6] Wazwaz A M, "The tanh method: solitons and periodic solutions for the Dodd-Bullough-Mikhailov and the Tzitzeica-Dodd-Bullough equations" Chaos, Solitons & Fractals **25** 55 (2005).
[7] Mancas S C and Rosu H C, "Integrable dissipative nonlinear second order differential equations via factorizations and Abel equations" Phys. Lett. A **377** 1434 (2013).
[8] Liouville J, "Sur l'équation aux différences partielles $d^2 \ln \lambda/dudv \pm \lambda/2a^2 = 0$" J. Math. Pures Appl. **18** 71 (1853).
[9] Gaffet B, "SU(3) symmetry of the equations of unidimensional gas flow, with arbitrary entropy distribution" J. Math. Phys. **25** 245 (1984).
[10] Gaffet B, "A class of 1-d gas flows soluble by the inverse scattering transform" Physica D **26** 123 (1987).
[11] Manafian J and Lakestanhi M, "Dispersive dark optical soliton with Tzitzeica type nonlinear evolution equations arising in nonlinear optics" Opt. Quant. Electron. **48** 116 (2016).
[12] Hosseini K, Bekir A, and Kaplan M, "New exact traveling wave solutions of the Tzitzeica-type evolution equations arising in non-linear optics" J. Mod. Opt. **64** 1688 (2017).
[13] Whittaker E T and Watson G N, A Course of Modern Analysis Chap XX (Cambridge: Cambridge University Press, 1958).
[14] Boldin A Yu, Safin S S, and Sharipov R A, "On an old article of Tzitzeica and the inverse scattering method" J. Math. Phys. **34** 5801 (1993).
[15] Robertson J S, "Gudermann and the simple pendulum" The College Math. J. **28** 271 (1997).
[16] Mondaini L, "The rise of solitons in sine-Gordon field theory: From Jacobi amplitude to Gudermannian function" J. Appl. Math. Phys. **2** 1202 (2014)
[17] Bajnok Z, Rim C and Zamolodchikov Al, "Sinh-Gordon boundary TBA and boundary Liouville reflection amplitude" Nucl Phys. B **796** 622 (2008).
[18] Perring J K and Skyrme T H, "A model unified field equation" Nucl Phys. **31** 550 (1962).
[19] Fu Z, Liu S, and Liu S, *Exact solutions to double and triple sinh-Gordon equations* Z. Naturforsch. **59a** 933 (2004).
[20] Andreev V K, Kaptsov O V, Pukhnachov V V and Rodionov A A, Applications of Group-Theoretical Methods in Hydrodynamics (Dordrecht: Kluwer, 1998).
[21] Rui W, "Exact traveling wave solutions for a nonlinear evolution equation of generalized Tzitzéica-Dodd-Bullough-Mikhailov type" J. Appl. Math. **2013** 395628 (2013).
[22] Brezhnev Yu M, "Darboux transformation and some multi-phase solutions of the Dodd-Bullough-Tzitzeica equation: $U_{xt} = e^U - e^{-2U}$" Phys. Lett. A **211** 94 (1996).
[23] Conte R, Musette M and Grundland A M, "Bäcklund transformation of partial differential equations from the Painlevé-Gambier classification. II Tzitzéica equation" J. Math. Phys. **40** 2092 (1999).