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Elements of Study on Dynamic Materials

Marine Rousseau
Universite Pierre et Marie Curie, martine.rousseau@upmc.fr

Gerard A. Maugin
Universite Pierre et Marie Curie

Mihhail Berezovski
Tallinn University of Technology, berezovm@erau.edu

Martine Rousseau*, Gérard A. Maugin*, Mihhail Berezovski+

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***** Université Pierre et Marie Curie,
Institut Jean Le Rond d'Alembert, UMR CNRS 7190,
Case 162, Tour 55, 4 place Jussieu, 75252 Paris Cedex 05, France
E-mail of corresponding author: martine.rousseau@upmc.fr

+ Tallinn University of Technology
Center for Nonlinear Science, Akadeemia tee 21, 12618 Tallinn, Estonia

Abstract As a preliminary study to more complex situations of interest in small-scale technology, this paper envisages the elementary propagation properties of elastic waves in one-spatial dimension when some of the properties (mass density, elasticity) may vary suddenly in space or in time, the second case being of course more original. Combination of the two may be of even greater interest. Towards this goal a critical examination of what happens to solutions at the crossing of pure space-like and time-like material discontinuities is given together with simple solutions for smooth transitions and numerical simulations in the discontinuous case. The effects on amplitude, speed of propagation, frequency changes and the appearance of a Doppler like effect are demonstrated although the whole physical system remains linear.

Keywords Dynamic materials . Wave propagation . Elasticity . Inhomogeneity . Interfaces . Doppler effect

1. Introduction

We are interested in so-called *dynamic materials*. By these we understand materials whose characteristic properties (in the simplest case of elastic bodies, mass density and elasticity) may be made to vary in space or in time, or both, by an appropriate arrangement or control. Of course materially inhomogeneous materials are known in various forms, polycrystals, composites of the stratified type or so-called graded materials (with a more or less smooth gradient in their properties). We consciously ignore here inhomogeneous materials with stochastic properties. Materials inhomogeneous in time are not so frequent or are practically nonexistent in everyday conditions (room temperature, time scale in minutes or hours). We may conceive of some artificial means of causing these controlled changes in time, for instance, by the application of an external (non-mechanical) field, or through a phase transition. To avoid any misunderstanding, we specify that this should be realized in an infinitesimally short duration and over a sufficiently large material region if not over the whole specimen under consideration. Difficulties of realization cannot be overlooked. For instance, the following question may be raised: how can we change instantaneously everywhere the elasticity coefficients of an extended specimen?

This contribution of modest ambition presents the first steps towards building a general but difficult field of material dynamics as seen from the engineering viewpoint, noting, however, a strong influence of some previous works in physics and applied mathematics. In particular, the idea of dynamic materials was formulated by Blekhman and Lurie [1] who identified two kinds of dynamic materials: *activated dynamic materials* obtained by changing the material coefficients of the wave carrier medium in the absence of relative motion – the subject matter of the present work - and *kinetic dynamic materials* obtained by endowing the whole system or some of its regions with some prescribed relative motion. The book of Lurie [2] concerns the first type with applications mostly to electromagnetic materials. The second type in mechanics is best illustrated by the review of Vesnitskii and Metrikine [3] (and the many works of the Nizhny-Novgorod school referred to in that lengthy and detailed paper). That latter work emphasizes the interest to study so-called *transition radiation* in mechanical systems such as elastic waves generated by mechanical objects travelling in locally or periodically inhomogeneous elastic systems (e.g., train travelling on a railroad track supported by a more or less elastic ground) while recalling the relationship and differences with *transition radiation* in some electromagnetic systems previously studied in detail by Ginzburg *et al.* (e.g., [4]) although Vesnitski and Metrikine do not consider wave phenomena such as Cherenkov or bremsstrahlung radiation. The whole problem, however, presents analytical difficulties that cannot be overlooked. From the mathematical side, we note a few works on the existence of special solutions of wave systems that are inhomogeneous in space and time (e.g., Nadin [5] and other works by the same author). This obviously relates to some facets of our problem. Note also that the phenomenon of trapped modes of oscillations also partake of the same class of dynamical phenomena (see, e.g., Indeitsev and Osipova [6]).

Here, with a much lower ambition but envisioning small experimental elastic systems that could be realized in the laboratory, we first set forth in Section 2 the standard formulation of balance and conservation laws for elasticity with space and time inhomogneities. This helps one to classify the various dynamical behaviours in particular with the consideration of the conservation of energy, canonical momentum and action. Section 3 introduces the case of one-dimension in space on which the paper focuses. Here the origin of spatial inhomogeneities and explicit time dependence is specialized for the sake of simplicity, considering spatial dependence of the matter density only and time dependence of the elasticity properties only, favoring thus some kind of separation of space and time effects. Simple examples of smooth space-time variations of material properties are presented in Section 4 yielding exemplary analytical solutions that are of interest for further developments. The case of piecewise variations considered in Section 5 is of greater interest in that it allows us to critically examine what happens at junctions and accompanying jump conditions at pure space-like or pure time-like discontinuities that are typical of dynamic materials. If what happens at fixed or moving space-like discontinuities is well known, what happens at time-like layers across which properties change very rapidly (theoretically instantaneously) in time is much more original, yielding effects seldom envisaged in wave propagation in mechanical engineering, e.g., Doppler effect with a capture of energy and effects akin to transition radiation and the Cherenkov effect. Simple numerical simulations are reported in Section 6 first with regard to changes in amplitude and speed of propagation for various cases, and then relating to frequency variations and the associated Doppler effect. With this we pave the way for more extensive analytical and simulation works.

2. General problem

We are concerned with materials in bodies where the local balance of linear momentum at regular material points is written, in Piola-Kirchhoff format, in the form (no body force for the sake of simplicity)

$$\left.\frac{\partial}{\partial t}\mathbf{p}\right|_{\mathbf{X}} - div_R \mathbf{T} = 0 \qquad (1)$$

where we used the following notation. The placement function $\mathbf{x}=\bar{\mathbf{x}}(\mathbf{X},t)$ represents the direct deformation mapping between some reference configuration K_R and the actual configuration K_t at Newtonian time t. It is assumed to be sufficiently smooth. The material gradient ∇_R and the material divergence operator div_R are understood as acting from the left on, respectively, vectors and tensors or geometric objects considered as such for these operations. We have (T = transpose)

$$\mathbf{v}:=\left.\frac{\partial\bar{\mathbf{x}}}{\partial t}\right|_{\mathbf{X}},\mathbf{F}:=\left.\frac{\partial\bar{\mathbf{x}}}{\partial\mathbf{X}}\right|_{t}=\left(\nabla_R\bar{\mathbf{x}}\right)^T,J_F:=det\mathbf{F} \tag{2}$$

respectively the physical velocity, the direct deformation gradient, and the Jacobian determinant of **F**. The symbol $\mathbf{F}^{-1}$ stands for the inverse of **F**. the object **T** is a two-point tensor field called the **first Piola-Kirchhoff stress that** is not a tensor in the usual sense**.** In (1), **p** is the physical linear momentum - a vector in the actual configuration – and the object **T** is a two-point tensor field called the **first Piola-Kirchhoff stress that** is not a tensor in the usual sense**. These two quantities are defined by**

$$\mathbf{p}=\rho_0\mathbf{v}\ ,\ \mathbf{T}=J_F\mathbf{F}^{-1}\sigma \qquad , \tag{3}$$

where $\boldsymbol{\sigma}$ is the (here) supposedly symmetric Cauchy stress and ρ_0 is the matter density at material point **X** in K_R. The latter has to satisfy the continuity equation. For **T** we may have the decomposition

$$\mathbf{T}=\mathbf{T}^e+\mathbf{T}^{diss} \tag{4}$$

where T^e may be thought of as an «elastic» (i.e., energy derivable) quantity while $\mathbf{T}^{diss}$ is typically dissipative, i.e., we can write

$$\mathbf{T}^e=\frac{\partial W}{\partial\mathbf{F}}\ ,\ \Phi^{diss}=tr\left(\mathbf{T}^{diss}\dot{\mathbf{F}}\right)=tr\left(\mathbf{T}^{diss}\left(\nabla_R\mathbf{v}\right)^T\right) \tag{5}$$

where the former defines the constitutive law, *per se,* while the second represents the dissipation per unit material volume due to $\mathbf{T}^{diss}$. Leaving this last part outside our scope but noting that $\mathbf{T}^e$ may itself dissipate in certain material schemes (e.g., plasticity), we now need to specify the functional dependence of ρ_0 and W. Since here we contemplate no thermal effects, W simply stands for the specific potential energy (no mention of free or internal energy). We can distinguish between the following three main cases.

(i) In *spatially and temporally homogeneous materials*, we simply have

$$\rho_0=const.,\ W=\overline{W}(\mathbf{F}) \tag{6}$$

(ii) In materials that are *inhomogeneous from both inertial and elastic points of view*, we have

$$\rho_0=\bar{\rho}_0(\mathbf{X}),\ W=\overline{W}(\mathbf{F};\mathbf{X}) \tag{7}$$

In both cases (6) and (7) the mass conservation in the Piola-Kirchhoff format reads

$$\left.\frac{\partial \rho_0}{\partial t}\right|_{\mathbf{X}} = 0 \tag{8}$$

Equation (7)$_2$ means that $\overline{W}$ depends explicitly on the material point **X**. In particular, if proved useful we can define a material co-vector $\mathbf{f}^{inh}$ by

$$\mathbf{f}^{inh} := \left(\mathbf{v}^2/2\right)\nabla_R \bar{\rho}_0 - \left.\frac{\partial \overline{W}}{\partial \mathbf{X}}\right|_{expl} \tag{9}$$

where the last expression means the material gradient keeping the field **F** fixed.

(iii) In materials that are simultaneously inhomogeneous in space and time, we would a priori write

$$\rho_0 = \tilde{\rho}_0(\mathbf{X}, t),\ W = \widetilde{W}(\mathbf{F}; \mathbf{X}, t) \tag{10}$$

although this is seldom done. The explicit dependence on time in the first expression opens up new horizons related to, e.g., the theory of *material growth* - more material of the same type is pushed into a material point **X** in the configuration K_R; cf. Epstein and Maugin [7]. The form (8) of the continuity equation is not longer valid. There exists a non-vanishing right-hand side. The system becomes thermodynamically open. The explicit dependence on time of the second expression in (10) leads to an evolution in time of, say, elasticity coefficients. This may represent the phenomenon of *ageing* [8] of which *creep* is an example. This, of course, does not conserve energy (see below). In view of the very structure of the strict conservation law that the balance law of physical linear momentum (1) achieves in the present paragraph, an interesting sub-case of (10) is

$$\rho_0 = \hat{\rho}_0(\mathbf{X}),\ W = \hat{W}(\mathbf{F}; t)\ . \tag{11}$$

This holds when there exists a purely inertial material inhomogeneity - i.e., inhomogeneous distribution of mass in K_R - and only a time evolution of the elasticity coefficients. This situation may be more easily realized experimentally than the general case (10). The time dependence in (11)$_2$ can only be through a relative time since the balance law has to comply with Galilean invariance (cf. Epstein and Maugin [9]). Also, the factors will not be affected by the differentiation in (1). Thus, when a single elasticity coefficient $\hat{E}(t)$ appears - case of linear isotropic bodies in small strains, and after application of the Helmholtz decomposition - equation (1) will yield for both the longitudinal and transverse components of the displacement an equation of the type ($\hat{E} = \lambda + 2\mu$ for $\mathbf{u}_L$ and $\hat{E} = \mu$ for $\mathbf{u}_T$)

$$\hat{\rho}_0(\mathbf{X})\frac{\partial^2 \mathbf{u}}{\partial t^2} - \hat{E}(t)\ \nabla_R.\left(\nabla_R \mathbf{u}\right) = \mathbf{0} \tag{12}$$

or

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \hat{c}^2(\mathbf{X}, t)\ \Delta\mathbf{u} = \mathbf{0},\quad \hat{c}^2(\mathbf{X}, t) := \frac{\hat{E}(t)}{\hat{\rho}_0(\mathbf{X})}\quad . \tag{13}$$

This introduces an interesting type of *linear* wave equation with a space-time dependent characteristic velocity. The apparent simplicity of linearity is superseded by the complexity brought in by space-time inhomogenity and pregnant of exotic wave-like effects (Cherenkov effect, Doppler-like effect, transition radiation; cf. Ginzburg and Tsytovich [4]). This is the essential subject matter of this contribution.

3. Conservation laws

Although written mathematically as a strict conservation law, equation (1) essentially represents three components of a field equation to be solved for the three components of the elastic displacement **u** with appropriate boundary and initial conditions, and perhaps additional jump relations at discontinuity surfaces. In addition there exist *conservation laws* that concern the *whole* physical system under study. When a variational formulation in the manner of Hamilton-Lagrange is used the field equations are none other than the Euler-Lagrange equations ((1) is an example of these) whereas it is the application of the celebrated Noether theorem (cf. Maugin [10]) that yields the conservation laws. In the presence of general dissipative processes, it is a direct manipulation of the field equations which allows one to deduce the expression of these conservation laws [10] with source terms appearing necessarily in these laws. The case (10) is amenable through a variational principle although not respecting energy conservation. We shall not repeat the detailed proof of this, but simply state the results. The three quantities whose conservation plays an important role in the present context are *energy*, *canonical momentum* because of their relation to invariance or lack of invariance under transformations of material coordinates and, because of its close relationship with wave-like processes and wave-mechanics formalism [11]-[12], *action*.
In agreement with (1) and (10), the Lagrangian density per unit reference volume to be considered reads

$$L(\mathbf{v},\mathbf{F};\mathbf{X},t)=K(\mathbf{v};\mathbf{X},t)-\hat{W}(\mathbf{F};\mathbf{X},t) \tag{14}$$

with kinetic energy

$$K(\mathbf{v};\mathbf{X},t)=\frac{1}{2}\hat{\rho}_0(\mathbf{X},t)\,\mathbf{v}^2 \tag{15}$$

Systems described by (14) are called *rheonomic* (here inhomogeneous) systems because of the explicit dependency on time (cf. Lanczos [13], following Boltzmann).

The Euler-Lagrange field equations at any regular material point deduced from the Lagrangian (14) are none other than (1). The *conservation laws of energy and canonical momentum* deduced from the application of Noether's theorem for time and material-space translations read

$$\left.\frac{\partial}{\partial t}H\right|_{\mathbf{X}}-\nabla_R.\mathbf{Q}=h \tag{16}$$

and

$$\left.\frac{\partial}{\partial t}\mathbf{P}\right|_{\mathbf{X}}-div_R\mathbf{b}=\mathbf{f}^{inh} \tag{17}$$

where $\mathbf{f}^{inh}$ has been defined in (9) and the energy (Hamiltonian) H, the material energy flux **Q**, the canonical (material) momentum **P**, the Eshelby (material) stress **b**, and the heat source h are given by

$$H=K+W \quad , \quad \mathbf{Q}=\mathbf{T}.\mathbf{v} \tag{18}$$

$$\mathbf{P} := -\rho_0 \mathbf{v}.\mathbf{F} \quad , \quad \mathbf{b} = -\left(L\mathbf{1}_R + \mathbf{T}.\mathbf{F}\right) \tag{19}$$

$$h := -\left.\frac{\partial L}{\partial t}\right|_{expl} = -\left(K/\rho_0\right)\left.\frac{\partial \hat{\rho}_0}{\partial t}\right|_X + \left.\frac{\partial \hat{W}}{\partial t}\right|_{expl} \tag{20}$$

Accordingly, energy and canonical momentum are not strictly conserved in rheonomic materially inhomogeneous systems. The loss of energy conservation is due to the explicit time dependence while the loss of conservation of material momentum is due to material inhomogeneities. In materials which exhibit dissipation of mechanical and thermal types, these two quantities are of course not conserved (as shown in Maugin [14]). Here the notation "*expl*" refers to the explicit derivative with respect to the variable, keeping the field fixed.

Conservation of action

In continuum mechanics the density of canonical action per unit reference volume will be defined by

$$A := \mathbf{P}.\mathbf{X} - Ht \tag{21}$$

If we remember that a *phase* for plane travelling waves is usually defined by

$$\varphi = \mathbf{K}.\mathbf{X} - \omega t \tag{22}$$

where $\mathbf{K}$ is a material wave vector and ω is a circular frequency, then the analogy between (21) and (22) is made crystal-clear. We recall that in elementary wave mechanics (Max Planck and Louis de Broglie), we have for a particle $H = \eta\omega$, $\mathbf{P} = \eta\mathbf{K}$, with η denoting the reduced Planck constant or action quantum, and thus $A = \eta\varphi$. From this we deduce [11]-[12] that the action (21) will play a prevailing role in wave studies in dynamic (bulky) materials. Indeed, on taking the scalar product of (17) by $\mathbf{X}$ in material space, and the product of (16) by t and subtracting the latter result from the former and performing a few manipulations, we will establish the following *non-strict* conservation law for the continuum action:

$$\left.\frac{\partial}{\partial t} A\right|_{\mathbf{X}} - \nabla_R . \left(\mathbf{b} . \mathbf{X} - \mathbf{Q}\, t\right) = \mathbf{f}^{inh}.\mathbf{X} - ht - \left(H + tr\mathbf{b}\right). \tag{23}$$

In the proof of this result we have accounted for the fact that $\mathbf{X}$ and t are independent of one another in the present space-time parametrization $\left(\mathbf{X}, t\right)$. Equation (23) becomes a *strict* conservation law (no source in the right-hand side) in the following circumstances. First, for a *scleronomic* (contrary of rheonomic) and homogeneous system, the first two terms in the right-hand side of this equation vanish. Second, as shown below (equation (29)) the last term within parentheses vanishes identically in the case of one space dimension – for which $tr\mathbf{b} = b = -H$. Otherwise (23) never is a strict conservation law. Furthermore, even *in quasi statics*, this equation with nonzero right-hand side plays a role in some mathematical proof. Indeed, the resulting identity for homogeneous materials then reads

$$\nabla_R . \left(\mathbf{b}.\mathbf{X}\right) - \left(W + tr\mathbf{b}\right) = 0 \ , \quad \mathbf{b} = W\mathbf{1}_R - \mathbf{T}.\mathbf{F} .$$

The integrated form of this over a material body is of interest (see Knops *et al* [15]).

Case of one space dimension
From here we shall consider the case of propagation in one space dimension with small strains only. Accordingly, we shall work in two-dimensional Euclidean space-time (x,t). This case which is the only manageable one is nonetheless misleading in many instances that we shall pinpoint to the reader.

With a standard notation (u = elastic displacement; derivatives indicated by a subscript x or t), the Lagrangian density (14) reduces to

$$L=\frac{1}{2}\hat{\rho}_0(x,t)(u_t)^2-\frac{1}{2}\hat{E}(x,t)(u_x)^2 \tag{24}$$

Equation (1) takes the form

$$\frac{\partial}{\partial t}\left(\hat{\rho}_0(x,t)\frac{\partial u}{\partial t}\right)-\frac{\partial}{\partial x}\left(\hat{E}(x,t)\frac{\partial u}{\partial x}\right)=0 \tag{25}$$

The canonical quantities **Q**, **P**, and **b** become scalars such that

$$H=\frac{1}{2}\hat{\rho}_0(x,t)(u_t)^2+\frac{1}{2}\hat{E}(x,t)(u_x)^2 \tag{26}$$

$$Q=-u_t\,\partial L/\partial u_x=\hat{E}\,u_t\,u_x \tag{27}$$

$$P=-u_x\,\partial L/\partial u_t=-\,\hat{\rho}_0\,u_t\,u_x \tag{28}$$

$$b=-(L-u_x\,\partial L/\partial u_x)=-H \tag{29}$$

The latter result is a peculiarity due to the one-dimensionality of the problem in space.
Equations (16) and (17) then read

$$\frac{\partial H}{\partial t}-\frac{\partial Q}{\partial x}=h:=-\left.\frac{\partial L}{\partial t}\right|_{expl} \tag{30}$$

and

$$\frac{\partial P}{\partial t}-\frac{\partial b}{\partial x}=f:=\left.\frac{\partial L}{\partial x}\right|_{expl}. \tag{31}$$

We have voluntarily reinstated the canonical definitions in eqs.(27) through (31). But the special forms of h and f are immediately read off from (9) and (20).

Remark 3.1 For a scleronomic materially homogeneous system, both h and f vanish, ρ_0 and E are mere constants, and with $c_0^2=E/\rho_0$, we have that not only (2) reduces to the linear homogeneous wave equation

$$u_{tt}-c_0^2u_{xx}=0 \tag{32}$$

but, simultaneously, eqns.(30) and (31) take on the following remarkable form

$$\frac{\partial H}{\partial t}+c_0^2\frac{\partial P}{\partial x}=0\,,\quad \frac{\partial P}{\partial t}+\frac{\partial H}{\partial x}=0 \tag{33}$$

and by elimination, it is shown that both P and H satisfy the same equation

$$P_{tt}-c_0^2 P_{xx}=0\,,\ H_{tt}-c_0^2 H_{xx}=0 \tag{34}$$

This, interesting as it is, is however misleading for it is, like (29), an artifact of the one-dimensional formulation. In effect, the energy equation is normally a scalar one, while the balance of momentum is co-vectorial. The misleading symmetry induced by the one-dimensional nature between eqns. (33) was noticed by Hayes [16] (pp.23-24) when he wrote down these two equations as two quadratic-invariant equations deduced from (1) (without the present Eshelbian framework and its consequences $(33)_2$ in view). Hayes simply comments that the "freedom of generating new solutions by differentiation or integration must be kept in mind, as these generate new conservation laws". This is what happens in the theory of solitonic structures.

Remark 3.2 If the special case (11) of dependency holds good, then the source terms in (30) and (31) take on the special form

$$h=\frac{W}{\hat{E}}\hat{E}_t\qquad,\qquad f=\frac{K}{\hat{\rho}_0}\hat{\rho}_{0x} \tag{35}$$

while (13) is valid in the form

$$u_{tt}-\hat{c}^2(x,t)\,u_{xx}=0 \tag{36}$$

an equation that would lend itself to some type of space-time homogenization if periodicity is assumed in Euclidean space-time. However, in view of quantities that should be conserved across space-like and time-like discontinuities (see below) and the symmetry built in eq.(25), it might be preferable to rewrite the latter as two compatible first-order partial differential equations by introducing the auxiliary scalar field v so that

$$v_t=\sigma:=\hat{E}u_x\,,\quad v_x=p:=\hat{\rho}_0 u_t \tag{37}$$

equations that are valid in the general case (25). Lurie [2] has studied the homogenisation of system (36) and some asymptotic expansion of the u solution for long times compared to the period of the repeated motif (e.g., a checkerboard) in Euclidean 2D space-time.

Remark 3.3 Systems such as (25), (24) and (37) are symmetric in the interchange $x\to t$ and $E\to\rho$. This symmetry is an artefact of the one-dimensionality in space (and the quadratic form of the energy of linear elasticity that compares exactly to the quadratic nature of the kinetic energy). If we want to avoid this pitfall we need to consider two spatial dimensions at least in order to highlight the difference between scalar and vectorial-like quantities (see, e.g., in the problem of reflection and refraction).

Remark 3.4 Another way to look at the wave problem for the system (25) is as follows. We could envisage solution $u=\bar{u}(\varphi)$ where $\varphi=\bar{\varphi}(x,t)$ is a general *phase function* such that wave number and frequency are defined by [11]-[12]

$$k=\frac{\partial\bar{\varphi}}{\partial x}\ ,\ \omega=-\frac{\partial\bar{\varphi}}{\partial t} \tag{38}$$

Let a prime denote the derivative of $\bar{u}$ with respect to φ . Then one shows that (25) yields the equation

$$\left[\omega^2-c^2(x,t)\ k^2\right]\bar{u}''-\frac{1}{\rho}\left(\omega\frac{\partial\rho}{\partial t}\ +\ k\ \frac{\partial E}{\partial x}\right)\ \bar{u}'-\left(\frac{\partial\omega}{\partial t}+c^2\frac{\partial k}{\partial x}\right)\bar{u}'\ =0 \tag{39}$$

with $c^2(x,t)=E(x,t)/\rho(x,t)$. If the phase function itself satisfies the wave equation with wave velocity $c(x,t)$, then the last term in the left-hand side of (39) vanishes. In addition, if the special case (11) holds true, then the second term in (39) vanishes also identically and we are left with

$$\left(\omega^2-c^2(x,t)k^2\right)\bar{u}''=0 \tag{40}$$

from which we deduce that the phase velocity is such that

$$c_\varphi=\pm c(x,t)\ ,\ c(x,t)=\left(E(t)/\rho(x)\right)^{1/2} \tag{41}$$

Thus, contrary to the nonlinear case where the phase velocity would typically depend on the amplitude of the signal, here the phase velocity varies from point to point in space-time while being independent of this amplitude (the theory remains "linear").

4. Smooth space-time variation of material properties

Didenkulova *et al* [17] offer a nice discussion of the possibility to construct exact travelling wave solutions in the simpler case where c in (41) depends only on x (material inhomogeneity only). This case is already involved. One may first think of solutions of the type $u=u_1(x)u_2(t)$ of which a special case is

$$u(x,t)\ =A(x)\,\mathrm{e}^{-\mathrm{i}\psi(x)}\ \mathrm{e}^{\mathrm{i}\omega t}\ =A(x)\exp\mathrm{i}\left[\omega t-\ \psi(x)\right] \tag{42}$$

which is not *a priori* a plane travelling harmonic wave, but where the amplitude A and phase ψ may be assumed to vary slowly and approximate solutions can be obtained via asymptotic WKB solutions. However, it is proved [17] that all existing solutions of the type (42) are actually *travelling waves*, but obviously not necessarily monochromatic ones. For $c(x)\ \propto\ x^2$, an example of such solution is given by $A(x)\propto x$ and $\psi(x)\propto -1/x$. The situation is obviously more involved with (25). But again, if the space-time inhomogeneities are such that

$$\rho(x,t)=\rho_1(x)\,\rho_2(t),\ \ E(x,t)=E_1(x)E_2(t) \tag{43}$$

we are tempted to look for solutions of the product type $u=u_1(x)u_2(t)$. In which case a simple calculation allows one to show that $u_1(x)$ and $u_2(t)$ are given by the differential equations

$$\frac{\partial}{\partial x}\left(E_1(x)\frac{\partial u_1}{\partial x}\right)-\lambda^2\rho_1(x)u_1(x)=0 \tag{44}$$

$$\frac{\partial}{\partial t}\left(\rho_2(t)\frac{\partial u_2}{\partial t}\right)-\lambda^2 E_2(t)u_2(t)=0 \tag{45}$$

where λ^2 is the separation constant. For simple expressions of the data and initial and boundary conditions, these can be integrated in theory. For instance,

Example 1:
We consider the simpler case where

$$\hat{\rho}_0(x)=\rho_0 exp(Kx)\ ,\ \ \hat{E}(t)=E_0\, exp(\Omega t) \tag{46}$$

A simple algebra shows that u_2 and u_1 satisfy Bessel equations, hence the solutions

$$u_2(t)\ =U_2(T)=A J_0(T)+B\ Y_0(T) \tag{47}$$

and

$$u_1(x)=U_1(X)=C\ J_0(X)+D\ Y_0(X) \tag{48}$$

where A, B, C, and D are real quantities, we have set $c_0^2=E_0/\rho_0$ and $\omega_0=c_0 k_0$, and J_0 and Y_0 are Bessel functions of order zero with arguments

$$T=\frac{2\omega_0}{\Omega}\exp\left(\frac{1}{2}\Omega t\right),\ X=\frac{2k_0}{K}\exp\left(\frac{1}{2}Kx\right).$$

Remark 4.1 With c decreasing along x and increasing in time, we have a situation that favors the capture of energy and the convergence of space-time trajectories. For instance,

Example 2:
We consider the case where

$$\hat{\rho}_0(x)=\rho_0\left(1-\frac{x}{x_n}\right)^{-2},\hat{E}(t)=E_0\left(1-\frac{t}{t_n}\right)^{-2} \tag{49}$$

We set

$$c(x,t)=c_0\left|\frac{1-\frac{x}{x_n}}{1-\frac{t}{t_n}}\right|\ ,\ c_0=\left(\frac{E_0}{\rho_0}\right)^{\frac{1}{2}} \tag{50}$$

with $0 \le x \le l < x_n$, $0 \le t < t_n$. Space point l corresponds to the convergence of characteristic lines, hence a better concentration of the emitted acoustic energy. The wave solution is looked for by means of separate variables. Some lengthy calculations lead to the following solution

$$
\begin{aligned}
u_2(t) &= a_1\left|1-\frac{t}{t_n}\right|^{\frac{1}{2}\left(1-\sqrt{1-4\omega_0^2 t_n^2}\right)} + a_2\left|1-\frac{t}{t_n}\right|^{\frac{1}{2}\left(1+\sqrt{1-4\omega_0^2 t_n^2}\right)} \\
u_1(t) &= b_1\left|1-\frac{x}{x_n}\right|^{\frac{1}{2}\left(1-\sqrt{1-4k_0^2 x_n^2}\right)} + b_2\left|1-\frac{x}{x_n}\right|^{\frac{1}{2}\left(1+\sqrt{1-4k_0^2 x_n^2}\right)}
\end{aligned}
\tag{51}
$$

with real coefficients a_i and b_i. With

$$
c_0 = \frac{\omega_0}{k_0}\ , \quad x_n = c_0\, t_n \quad , \quad \omega_0 \le \frac{1}{2t_n} \tag{52}
$$

the exponentials in (51) are real, while they are complex for $\omega_0 > 1/2t_n$.

5. Piece-wise variations of material properties - What is conserved during propagation in dynamic materials

The question naturally occurs of what happens in general not only at regular points but also at the crossing of discontinuity lines of which special cases are purely space-like or purely time-like discontinuities or interfaces. An interface is called a discontinuity surface if it has no thickness. Here, it is said to be purely space-like when material properties vary only spatially across it. Similarly, it is said to be purely time-like if the material properties vary only timewise across it. In practice a mathematically zero-thickness space-like discontinuity Σ_x often is an interface or transition layer T_x across which properties vary smoothly, albeit rapidly, in space rather than abruptly. Similarly, as it is difficult to conceive a possible instantaneous change of properties solely in time across a purely time-like discontinuity Σ_t , such a change must practically occupy a short time duration, small but not nil, during which the change occurs smoothly. This transition time τ across a thin time-like layer T_t may be that required for the switching of a rapid phase transition. Brutal spatial or temporal changes without length scale or characteristic time duration should yield the consideration of generalized functions (distributions) of the Dirac and Heaviside types. We shall avoid this. But there remains the question of what are the conditions imposed on the field quantities (here, those derived from the particle motion) across Σ_x or Σ_t . The situation is quite different for an interface Σ_x and a time line Σ_t .

If we examine the question for piecewise constant variations in space and time with 1D space dimension and a space-time diagram, then discontinuities Σ_x are represented by straight lines parallel to the t-axis while discontinuities Σ_t are straight lines parallel to the x-axis (see Figure 1). Of course the 1D spatial situation here also may be misleading because what happens in space is essentially typically multidimensional with a *co-vectorial* connotation. In effect the dual of position is the canonical momentum according to the phase definition (21) and in terms of the wave vector a vector direction is involved with possible change of orientation across a two-dimensional interface surface Σ_x - e.g., in Snell-Descartes law - or across T_x). Quite differently, it is a *scalar* quantity, energy or frequency as shown by the duality present in (21), which is the essential evolving quantity at a time-like interface Σ_t or across T_t .

If numerical simulations are more easily carried with a wave front of a certain profile (see Section 6 below), the modelling of the perturbation by harmonic plane waves allows a certain physical understanding.

Case of a fixed material interface Σ_x or a transition layer T_x

This is represented by a point at x_0 in Figure 1(Part b). There is no matter transfer at that point although both density and elasticity may change abruptly across Σ_x . There is continuity of the normal surface traction. Both **P** and **K** are not kept unchanged at Σ_x . The only quantity that evolves is the gradient of displacement, as frequency is kept fixed and wave vector evolves (in 2D we obtain the Snell-Descartes laws). For a transition layer T_x where only the density ρ varies and does it slowly enough, application of the WKB method yields with a constant frequency ω a variable wave number $k(x)=\omega/c(x)$ with, in 2D, a curvature of rays - described by a relation of the form $\sin\theta(x)=(\sin\theta(0)/c(0))c(x)$ - and a *concentration of energy*. If both density and elasticity vary across T_x, then we obtain an inhomogeneous wave equation, a case not envisaged here.

Case of a time line Σ_t or a thin time-like layer T_t

In this case we consider only the possibility of an evolving elasticity (an evolving density would yield not only no mass conservation but also an inhomogeneous wave equation). This is represented by a point at t_0 in Figure1 (Part c). But here, analogous to the mass-flux condition and continuity of normal traction at Σ_x , we have continuity of the displacement gradient and of the displacement itself, respectively. The wave vector does not evolve but the velocity varies. For a "transition layer" T_t with a typical time scale τ much larger than the acoustic period, the equivalent of the WKB method in time yields $\omega(t)=k\,c(t)$ with k fixed and thus

$$\omega(t) = \omega_0 \frac{c(t)}{c(0)} \quad , \tag{53}$$

hence *a Doppler effect with capture of energy*. That is what distinguishes the two types of transitions. As a partial conclusion, spatial inhomogeneity allows the convergence or divergence of wave by conservation of the momentum, while a dynamic medium with time inhomogeneity allows capture of energy from the outside with a resulting change in frequency. The combination of the two in a true dynamic material may yield a concentration of energy although the system is fully linear.

Back to the 1D case where eqns.(25) and (37) hold, applying Remark 3.3, we will know what to do at Σ_t once we know the condition at Σ_x . In the latter case for a fixed Σ_x we have continuity of the mechanical traction, i.e., in terms of jumps,

$$[\hat{E}u_x]=0 \ at \ \Sigma_x \tag{54}$$

hence

$$[\hat{\rho}_0\, u_t] = 0 \; at \; \Sigma_t \tag{55}$$

In the special case (11), these two conditions reduce to

$$[u_x] = 0 \; at \; \Sigma_x \;\;,\;\; [u_t] = 0 \; at \; \Sigma_t \tag{56}$$

where square brackets denote the usual jump.

For illustrative purpose we consider the case of (54) and (55). For a **spatial interface** Σ_x , we must check at, say $x = x_0 = 0$, and **for all values** of *t*,

$$u_1 = u_2,\;\; u_{1x} = u_{2x} \tag{57}$$

with $u_1 = u_I + u_R$ and $u_2 = u_T$ (Figure 1, Part b) since in general there is one incoming wave (on side 1), one transmitted wave (on side 2) and a reflected wave (on side 1). Then we deduce immediately that, since $E_1 = E_2$ and $\rho_1 \neq \rho_2$, then $c_1 \neq c_2$, and

$$\omega_1 = \omega_2 = \omega \;\;,\;\; k_1 \equiv \frac{\omega_1}{c_1} \neq k_2 \equiv \frac{\omega_2}{c_2}\,. \tag{58}$$

The transmission and reflection coefficients are then given by

$$T_{12} = \frac{2\,\rho_1\, c_1}{\rho_1\, c_1 + \rho_2\, c_2} \;\;,\;\; R_{21} = \frac{\rho_1\, c_1 - \rho_2\, c_2}{\rho_1\, c_1 + \rho_2\, c_2} \tag{59}$$

Let $z_i = \rho_i\, c_i$ be the impedances. Then,

$$R_{21}^2 + \left(z_2/z_1\right) T_{12} = 1 \tag{60}$$

which stands for energy conservation. If we tune the impedances, reflection effects disappear and the whole of energy is transmitted.

For a **time-like interface** Σ_t situated, say, at $t = t_0 = 0$, we must check, **for all values of** *x*, (compare to (57); $\rho_1 = \rho_2$, $E_1 \neq E_2$) that

$$u_1 = u_2 \;\;,\;\; u_{1t} = u_{2t} \tag{61}$$

where region 1 is below Σ_t in space-time and region 2 is above. Here we should pay attention to the wording because we are no longer speaking of propagating waves since the transition across Σ_t occurs in the ideal case parodying the case (57)-(61) at zero propagation velocity (vertical lines crossing Σ_t). Nonetheless, a solution will exit at Σ_t (cf. Lurie [2], Ginzburg. and Tsytovich [4]) only if we have one "signal" A_1 coming from region 1 toward region 2, one signal A_2^+ continuing in region 2 and, what is more surprising, one signal A_2^- in region 2, oriented from region 2 to region 1 (as if it were coming from the future). That is, we have (cf. Figure 1, Part c): u_1 and $u_2 = u_+ + u_-$. A simple estimate yields (with $\rho_1 = \rho_2$, $E_1 \neq E_2$, $c_1 \neq c_2$)

$$T_{12}^+ = A_2^+ / A_1 = \frac{c_2 + c_1}{2c_2} \;\;,\;\; T_{12}^- = A_2^- / A_1 = \frac{c_2 - c_1}{2c_2} \tag{62}$$

Then at Σ_t , we have (since $c_1 \neq c_2$)

$$k_1 \equiv \frac{\omega_1}{c_1} = k = k_2 \equiv \frac{\omega_2}{c_2} \quad , \quad \omega_1 \neq \omega_2 \tag{63}$$

together with the local balance at Σ_t ($z_2/z_1 = c_2/c_1$)

$$1 + (z_2/z_1)(T_{12}^-)^2 = (z_2/z_1)(T_{12}^+)^2 \tag{64}$$

With perfect tuning of the impedance $[z] = 0 \, at \, \Sigma_t$, $T_{12}^- = 0$ and $T_{12}^+ = 1$, but this case is trivial.

Moving material interface

The problem becomes more involved when, although still one-dimensional in space (and therefore in fact still reduced to a point along the *x*-axis), the **spatial interface** moves in time in the material. The natural approach then is to consider the wave problem as seen by an observer moving with the material interface, i.e., to consider an instantaneous Galilean transformation such that the new space-time coordinates be

$$\xi = x - \hat{x}(t) \quad , \quad \tau = t \tag{65}$$

where the interface instantaneous position is given by $\hat{x}(t)$. With constant velocity V starting from $x = 0$ we have $\hat{x}(t) = Vt$. This velocity must be compared to the characteristic velocity of the material on its two faces (in principle at each instant of time) with the constraint that the considered mathematical system is *hyperbolic*. This matter is examined by Lurie [2] - Chapter 2 - who concludes that a necessary condition for the existence of a required solution at $\Sigma_{x,t}$ is that

$$(V^2 - c_1^2)(V^2 - c_2^2) > 0 \tag{66}$$

where c_1 and c_2 are the characteristic phase velocities on the two sides of the interface. Moving interfaces in 1D corresponding to phase changes are dealt with by Ericksen [18].We shall return to this in further work. For the time being, we shall enforce the conditions (54) through (56) in numerical examples.

6. Some numerical simulations

The goal of these numerical simulations is to check a possible concentration of acoustic energy in dynamic materials. The manifestation of this energy concentration is expected to be a signal localization and an amplification of signal amplitudes.

6.1. Amplitude and speed of propagation

Only the cases where matter density ρ depends only on x and elasticity coefficient E depends only on time are considered. This corresponds to the special case (11). Accordingly, the wave equation (36)

$$u_{tt} - \hat{c}^2(x, t) u_{xx} = 0 \tag{67}$$

is represented in the form of the hyperbolic system of the first-order equations

$$v_t - \hat{c}^2(x, t) \varepsilon_x = 0 \tag{68}$$

$$\varepsilon_t - v_x = 0 \tag{69}$$

where $v = u_t$, $\varepsilon = u_x$ and $\hat{c}^2(x,t) = E(t)/\rho(x)$.

The energy and canonical momentum balances take the form given in (30) and (31), that is, more explicitly

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho(x)u_t^2 + \frac{1}{2}E(t)u_x^2\right) - \frac{\partial}{\partial x}\left(E(t)u_x u_t\right) = \frac{\partial E(t)}{\partial t}\left(\frac{u_x^2}{2}\right) \tag{70}$$

and

$$\frac{\partial}{\partial t}\left(-\rho(x)u_x u_t\right) - \frac{\partial}{\partial x}b = \frac{\partial \rho(x)}{\partial x}\left(\frac{u_t^2}{2}\right) \tag{71}$$

where we recognize the total energy density of the elastic body at time t and the energy flux $Q = E(t)u_x u_t$, i.e., $Q = \sigma v$, where $\sigma = Eu_x$ is the elastic stress in 1D, and b is the reduced form of the Eshelby stress in 1D. The right-hand sides of (70) and (71) are, respectively, the explicit time derivative of the elastic energy (i.e., keeping the strain field u_x fixed) and the explicit space derivative of the kinetic energy (i.e., the gradient taken at fixed field u_t). This right-hand side is also positive for an increasing density with space, while with an increase in time of E the right-hand side of eq. (70) is also positive.

Here the (scalar) Eshelby stress is calculated as in eq.(29) with the peculiarity to be equal to minus the energy density. Keeping this in mind, we solve the system of equations (68) - (69) numerically by means of the conservative wave-propagation algorithm [19], [20]. This numerical scheme is stable up to the value of the Courant number equal to 1 and second-order accurate on smooth solutions.

In order to place in evidence the effects of the space and time inhomogeneities on the amplitude and speed of propagation in the one-dimensional setting, we examine the propagation of a bell-shaped pulse excited at the left boundary of the computational domain by non-zero strain for a limited time interval $0 < t < 180\Delta t$, such that

$$\varepsilon(0,t) = \frac{1}{2}\left(1+\cos\left[\omega_0(t-90\Delta t)\right]\right), \quad \omega_0 = \frac{\pi}{90\Delta t}, \quad \varepsilon(x,0) = 0 \ . \tag{72}$$

The initial stress pulse amplitude is equal to 0.9 and the width of the pulse is 180 space steps. The space step is chosen so as to have a Courant number equal to one. The velocity of propagation is dictated by the medium. The input is such that only a wave propagating to the right is initiated.

Case 1: ***Increasing density with traveled space with fixed elasticity coefficient***

We start calculations with the fixed value of the dimensionless stiffness E = 0.9. The dimensionless density increases in space from the initial value ρ = 0.9 for 10% every 100 space steps. Results of computations of the stress pulse shape after 1600 time steps are shown in Fig. 2. The shape of the actual pulse is compared with the reference pulse shown by a dashed curve. The reference pulse shape was calculated in the pure homogeneous medium with values ρ = 0.9 and E = 0.9 that provides the dissipationless propagation of the initial pulse. Here we have only spatial inhomogeneity and this does not lead to a gain or loss of energy. As to the energy which must be conserved, we have checked the energy contained in the reference signal and the one obtained by adding the energy in the travelling signal plus the energy contained in the small ripples (that are due to the space discretization of density). The calculated ratio of actual and reference energies is equal to 0.9655. This is due to loss of the second-order accuracy in the numerical scheme at discontinuities each 100 space steps. For the bell-shaped signal, one can observe a retardation and a clear localisation of the signal accompanied by its amplification because of reflections at boundaries between computational cells with different densities. In the considered case with increasing density, each increase in density leads to the corresponding increase in impedance. Consequently, the transmission coefficient is always more than 1 at boundaries between computational cells with different densities.

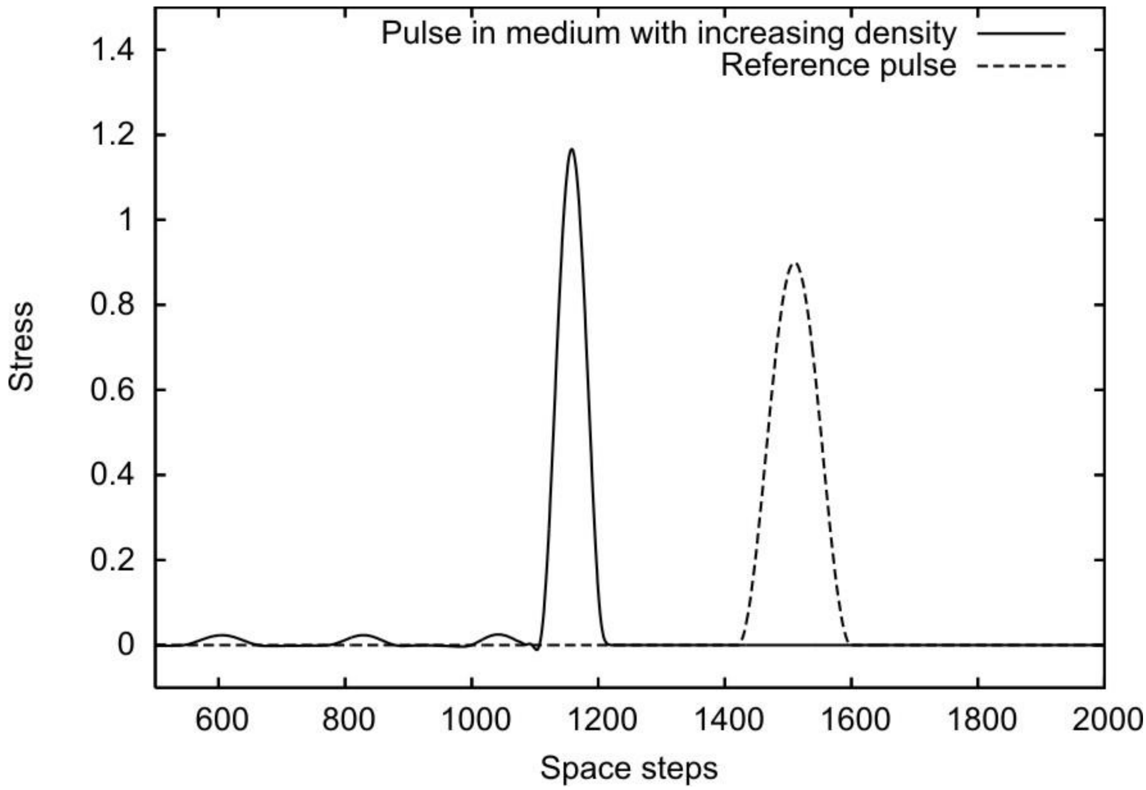

Figure 2: Pulse propagation in medium with increasing density.

Case 2: ***Increasing elasticity coefficient in time while keeping fixed density***

Next, we calculate the propagation of the same pulse in another case. Here the density has the constant values ρ = 0.9, but the non-dimensional stiffness increases every 100 time steps by 1% starting from the initial value E= 0.9. After 1600 time steps its value is equal to 1.055.The increase in the stiffness leads to the corresponding increase in the characteristic velocity. Accordingly, the Courant number is equal to 0.93 to provide the stability of computations. The continuity of velocity is provided each time of the stiffness increase. Results of numerical calculations of the pulse shape after 1600 time steps are shown in Fig. 3. Here we see no ripples since the discretization of the varying quantity (stiffness) is not in space but in time. The actual signal travels faster than the reference signal calculated with the same Courant number. The energy is increased since we have to input energy to make the elasticity coefficient grow. The calculated ratio of actual and reference energies is equal to 1.0846. This additional energy manifests itself in an amplification of the pulse amplitude together with its acceleration. The width of the pulse does not change.

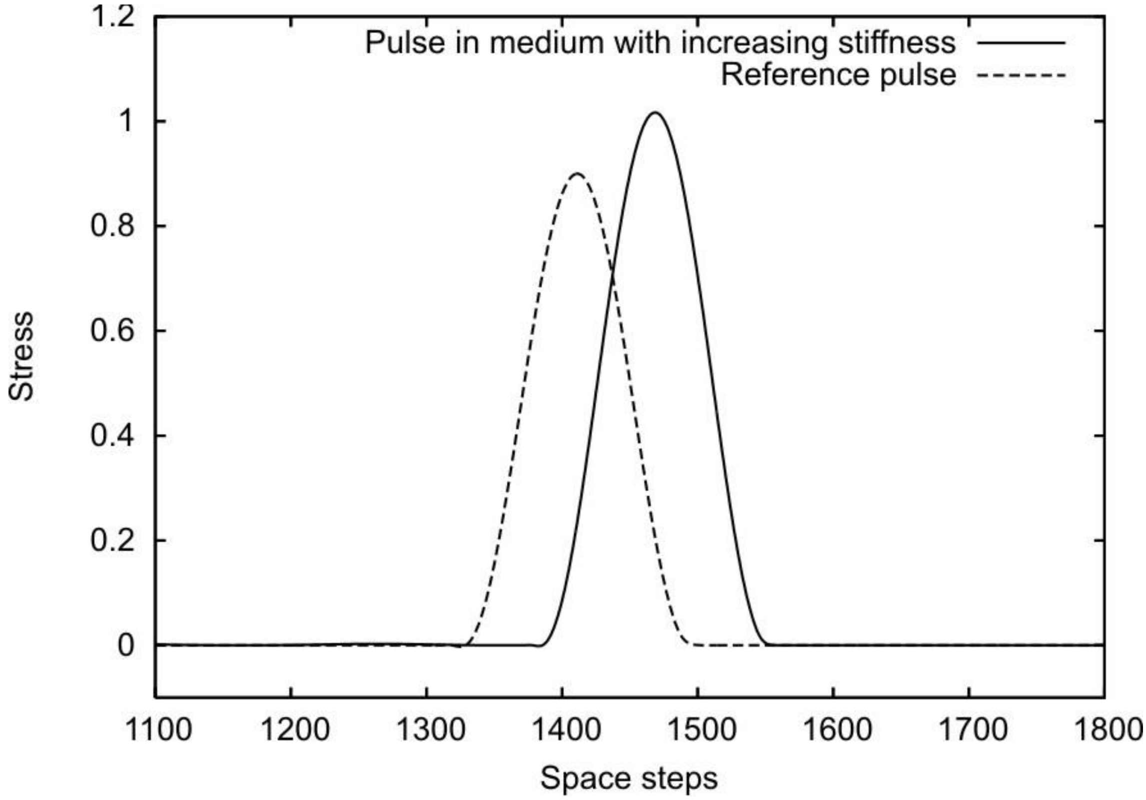

Figure 3: Pulse propagation in medium with increasing stiffness.

Case 3: ***Increasing elasticity coefficient in time and density in space***

(a) *Increasing in stiffness is slower than increasing in density*
Now we combine cases 1 and 2 for different growth rates. First we simply apply the growth rates both for density and stiffness used in the cases 1 and 2. The result of calculations with the Courant number equal to 0.93 presented in Fig. 4 shows the superposition of the foregoing two pictures.

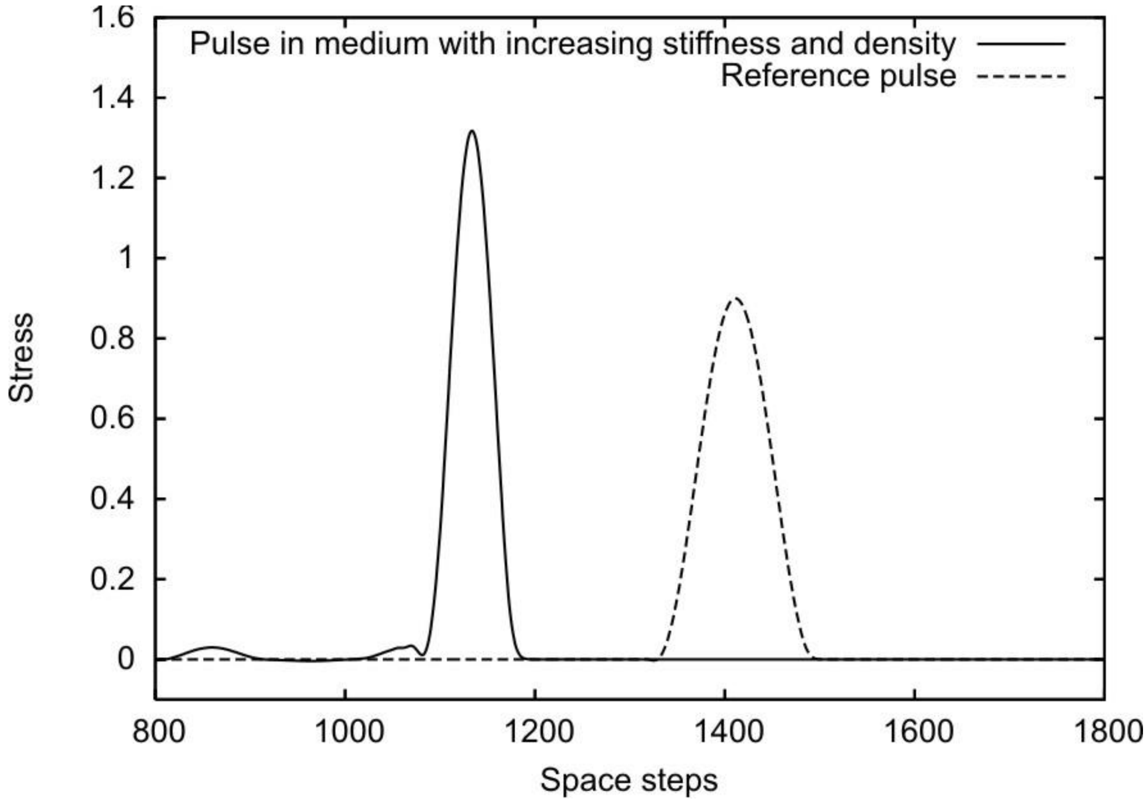

Figure 4: Pulse propagation in medium with increasing stiffness and density (increasing in stiffness is slower than increasing in density).

The energy is still increased in this case reflecting in the increased amplification and decreased retardation of the pulse. The calculated ratio of actual and reference energies is equal to 1.0812.

(b) *Increasing in stiffness is the same as increasing in density*

If we decrease the growth rate for the density from 10 % to 1% for every 100 space steps, we obtain a faster pulse comparing to the previous case (Fig. 5). The calculated ratio of actual and reference energies is again equal to 1.0846, which reflects in the amplification of the amplitude. In this case of identical same growth rates for both density and stiffness, the retardation and acceleration of the pulse practically compensate each other.

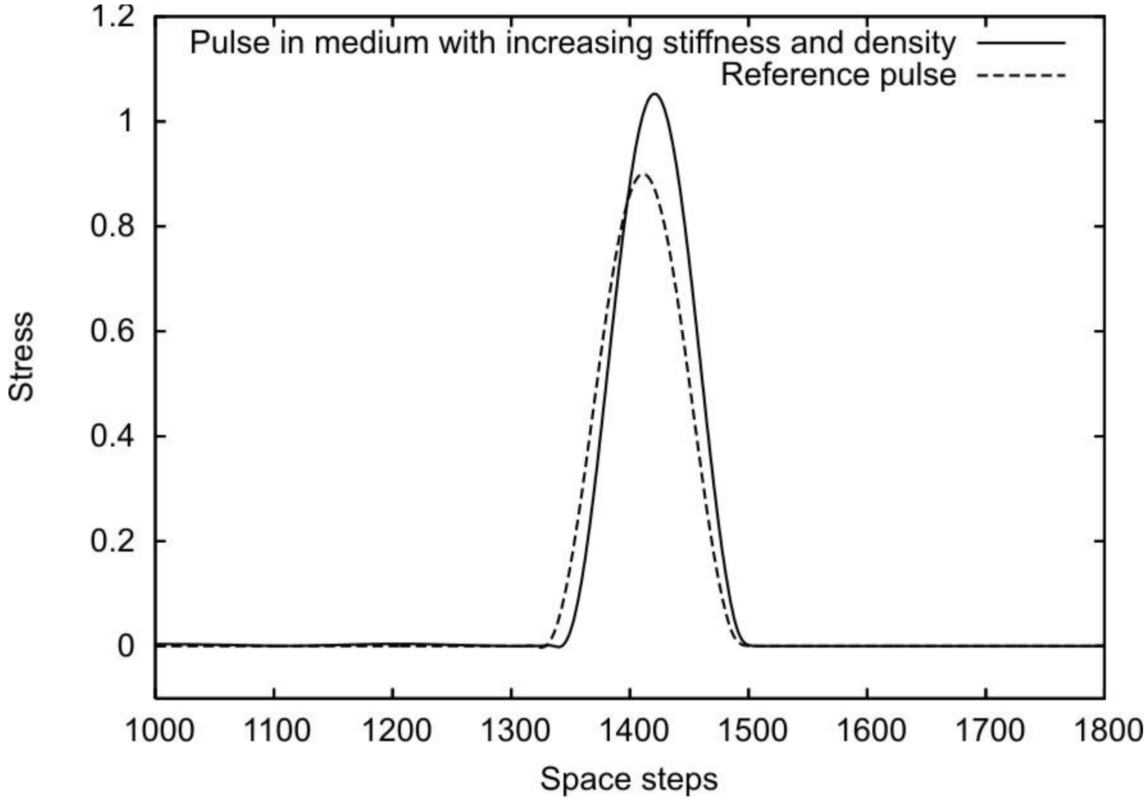

Figure 5: Pulse propagation in medium with increasing stiffness and density (increasing in stiffness is the same as increasing in density).

(c) *Increasing in stiffness is faster than increasing in density*

At last, we examine a slightly higher growth rate for the stiffness than for the density. Namely, we apply 2% increase in the stiffness every 100 time steps, keeping 1% increase for the density every 100 space steps. The Courant number here is equal to 0.877 due to the stiffness increase. The reference pulse is calculated with the same Courant number. The result is shown in Fig. 6. It looks like the one in Case 2, because of the leading role of the stiffness variation.

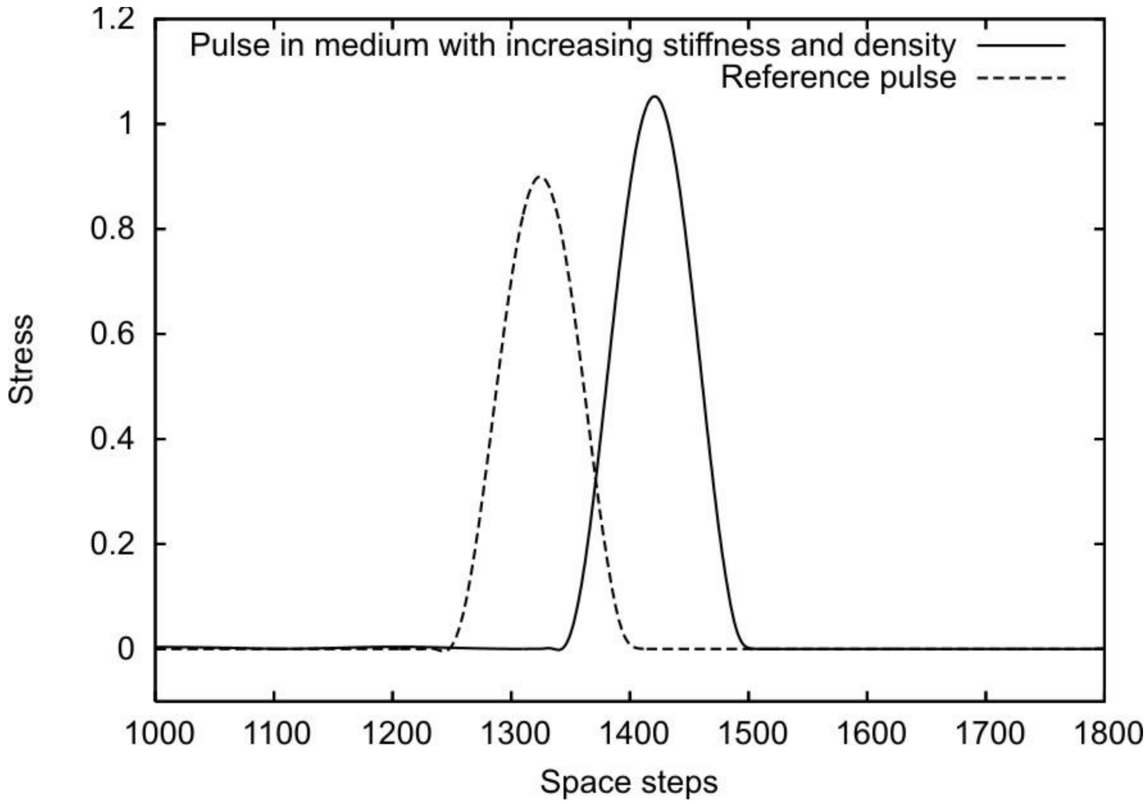

Figure 6: Pulse propagation in medium with increasing stiffness and density (increasing in stiffness is faster than increasing in density).

Thus, numerical calculations confirm theoretical predictions for the concentration of acoustic energy in dynamic materials. Increasing of the density in space leads to retardation and localisation (see Figure 4 the main part of the signal) of the pulse whereas the increase in stiffness results in its acceleration and an increase in the energy content of the signal while the other changes, if any, are not markedly visible. Variation of the rates of these factors can be used to obtain a desired shape and localization of the signal.

6.2. Frequency variations and Doppler effect

In order to demonstrate numerically the pure effect of time variation of the elasticity, we observe at a fixed spatial point the signal that passes by after alteration by successive increases of elasticity. At the left boundary one applies a sinusoidal load such as $\varepsilon(0,\bar{t}) = 0.9 \ \sin\left(\pi\bar{t}/64\right)$. The dimensionless density is also taken as $\bar{\rho} = 0.9$, so that the initial characteristic velocity is $c_0 = 1$. Symbols with an over bar are dimensionless. With the prescribed initial signal the initial circular frequency clearly is $\bar{\omega}_0 = \pi/64$. The corresponding frequency is $\bar{f}_0 = \bar{\omega}_0/2\pi = 1/128$. This provides the scaling of period T_0 and frequency f_0 such that $T_0 = 128\,\Delta t$ and $f_0 = 1/128\,\Delta t$. The recording point is chosen as $x = 1024\ \Delta x$ for real time

$t=4096\,\Delta t$ or nondimensional space $\bar{x}=x/\Delta x=1024$ for nondimensional time $\bar{t}=4096$. Δt and Δx are the time and space increments in the numerical scheme. With Courant number $N_C=0.8197$ to guarantee the finite-difference convergence, $\Delta t=1/3072$ or $3072\,\Delta t=1$ and $\Delta x=1/3072\times 0.8197$.

Accordingly, $f_0=1/128\,\Delta t=3072/128=24$, as indicated by the central value of the reference spectrum in Figure 9. The arrival time of the first signal at the observation point is $\bar{t}_i=\bar{x}/0.8197=1024/0.8197$ or nondimensional time $\bar{t}_i=1249.23$ (see Figure 7). But this signal has already been altered 12 times with an elasticity increased by 1% every 100 time steps, and after the total time duration of observation, i.e., $4096-1249\approx 2840$, it has been altered 40 times and the observed period is of the order of 22-23. Of course, an increase in the measured stress is observed (Figure 8). An *FFT* analysis of the observed signal effected by a standard MATLAB procedure shows that we obtain a shift in normalized frequency spectrum in accord with the Doppler effect formula for an observer in motion (but here it is the velocity of the signal that is observed). This differs from the more well known Doppler effect where the source is moving. Indeed, we have the standard formula

$$\frac{f}{f_0}=\frac{c(t)}{c_0}\approx 1+\frac{\Delta c(t)}{c_0}.$$

For the initial signal we had $f_0=24$. After one increase in stiffness we have $f_{n=1}=24(1+0.04)$. When the first signal reaches the observation point this corresponds to $f_{n=12}=24\times 1.06=25.64$, and after 40 increases, we have $f_{n=40}=24\times 1.20=28.80$. Of course this variation is in fact continuous from 24 to 28.80, but the three relevant values are evidenced in Figure 9 (they have no equivalent to the left of $f_0=24$). Here we must account for the fact that even the initial reference signal cannot be represented by a single spectral line. We conclude that our simple-minded approach corroborates the predicted effect.

7. More general schemes and conclusion

Lurie, in a series of papers [21], [22] and a book [2], has advocated the consideration of a checkerboard of space-time variations of material parameters. But he also provided an interesting result obtained by homogenisation for long time (and space) propagation where both material coefficients are fast periodic functions of the characteristic right-running variable. In our notation (Equations 30-31), the substance of his result (Equations 2.92 on p.44 in Lurie [2]) is that after homogenisation, we obtain the balance of energy and canonical momentum in the source-free form (superimposed tilde corresponds to the zeroth-order asymptotic homogenized solution)

$$\frac{\partial \tilde{H}}{\partial t}-\frac{\partial \tilde{Q}}{\partial x}=0,\quad \frac{\partial \tilde{P}}{\partial t}-\frac{\partial \tilde{b}}{\partial x}=0 \tag{73}$$

which is tantamount to saying that the looked for effects disappear altogether by successive increases and decreases that compensate each other since the inequalities requested in Sections 4 and 5 above must be satisfied (cf. Figure 10). In general, however, in order to realize practically the phenomenon, it remains to find a way to cause a sufficiently rapid and sizeable change in time of the elasticity properties by action of an external field causing, e.g., a fast phase transition. As a preparation for this, in the light of simple analytical solutions and a critical examination of what occurs at space-like and time-like discontinuities, we have here established the general tendencies of the acceleration and slow down of propagating pulses as well as their possible increase in amplitude due to the propagation through a series of material interfaces and a succession of periodic increases in energy input in a one-dimensional (in space) model characteristic of dynamic materials where some minimum separation of space and time effects can be expected.

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Additional figures below

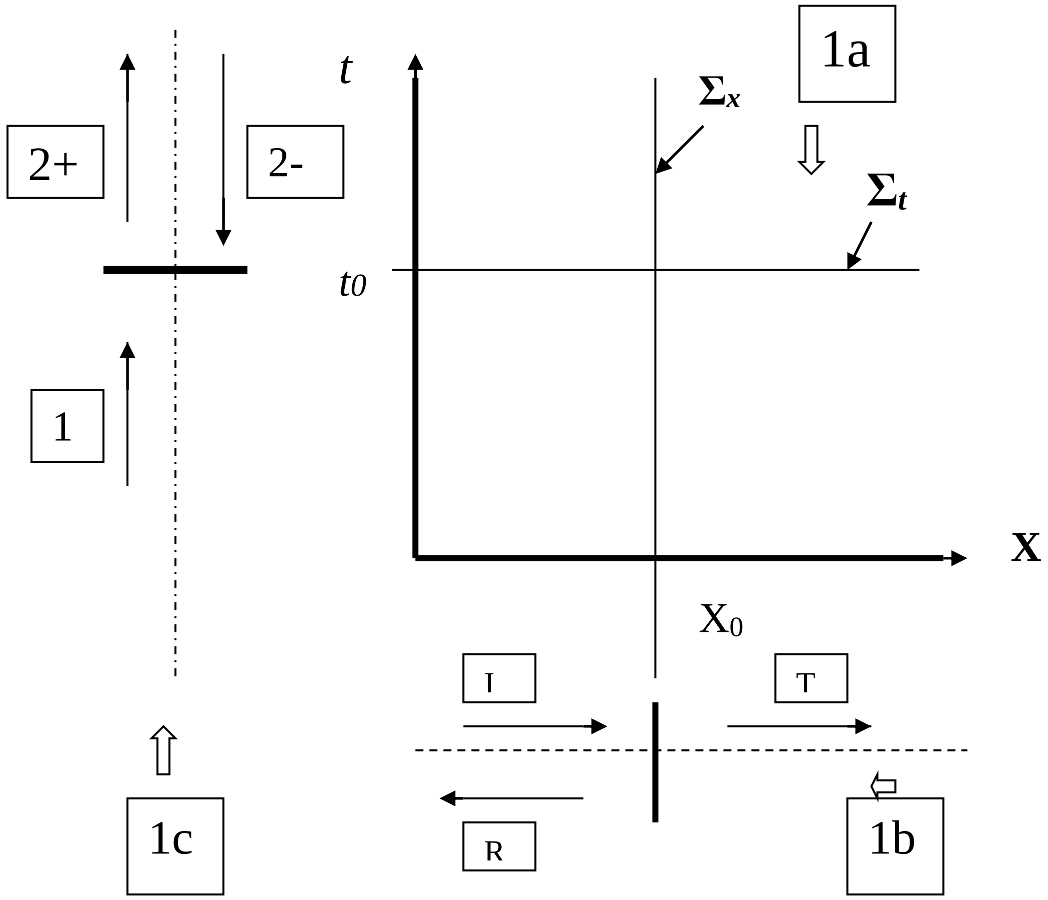

Figure 1: Space-like and time-like discontinuities in dynamic materials

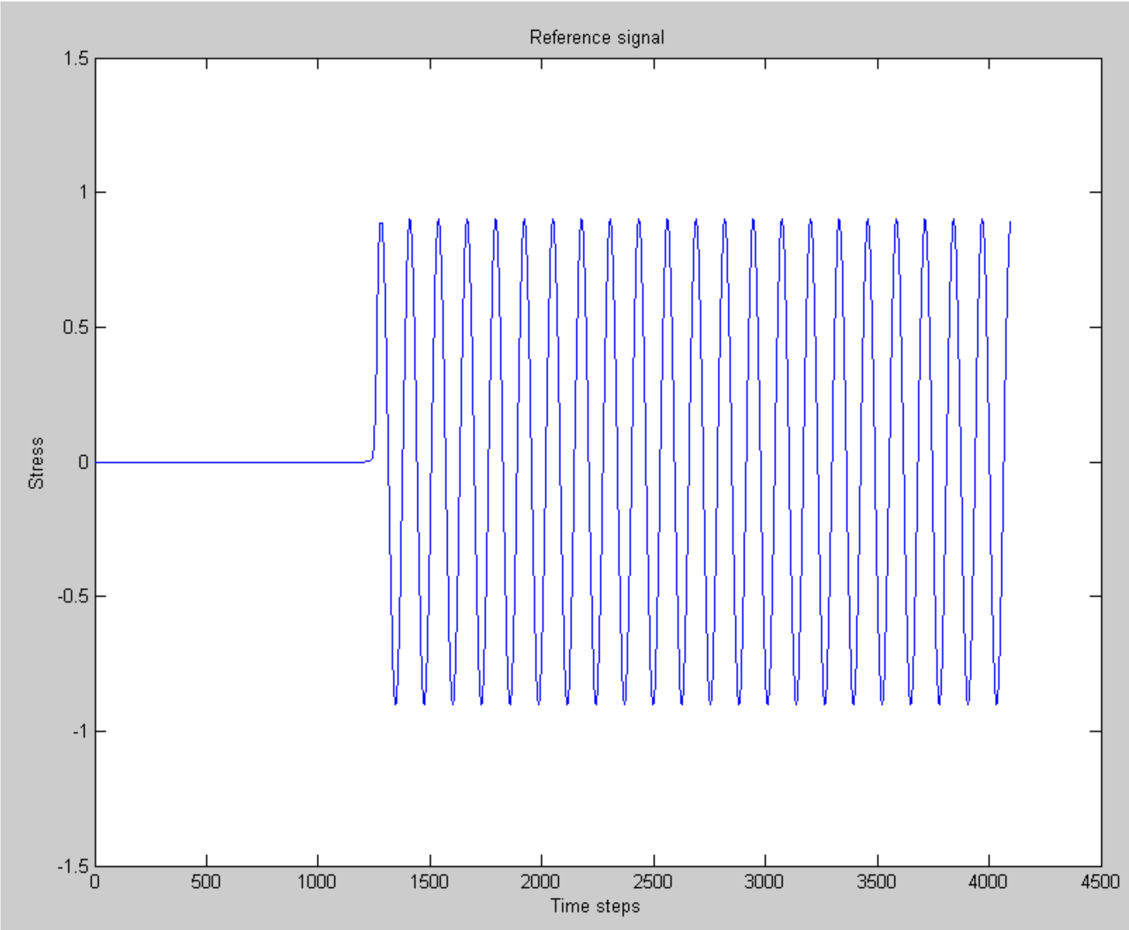

Figure 7: Reference signal

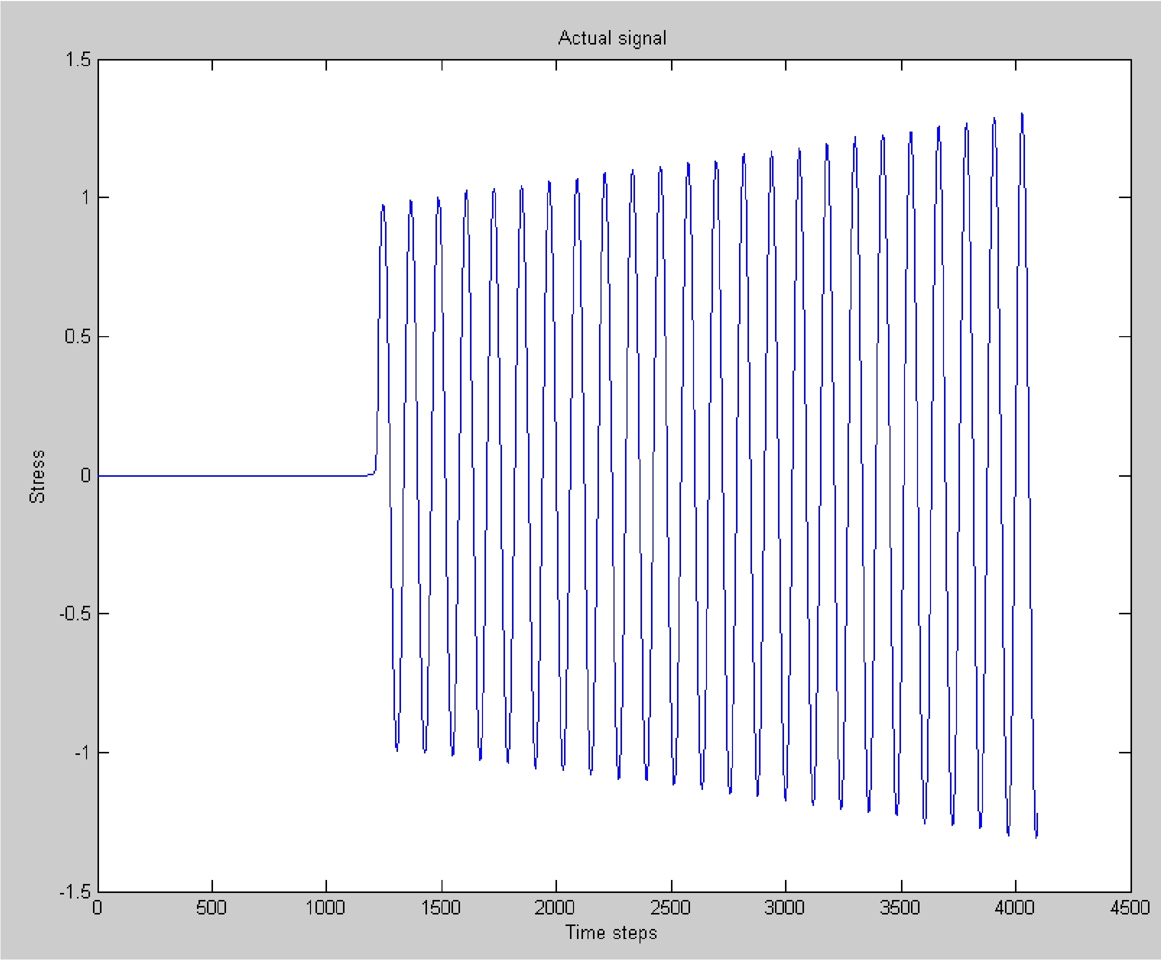

Figure. 8: Sinusoidal signal in material with increasing stiffness

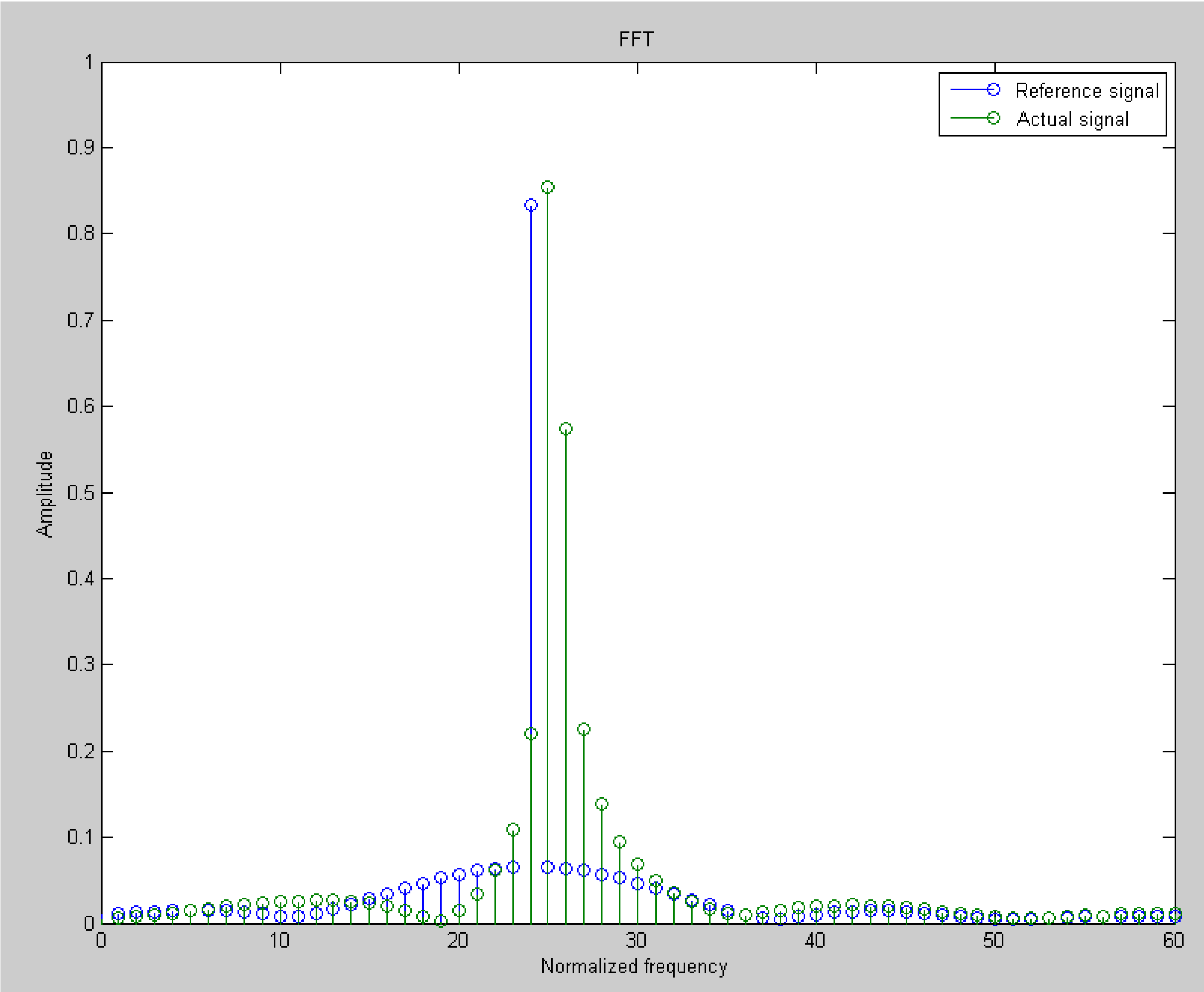

Figure 9: Frequency shift for dynamic material

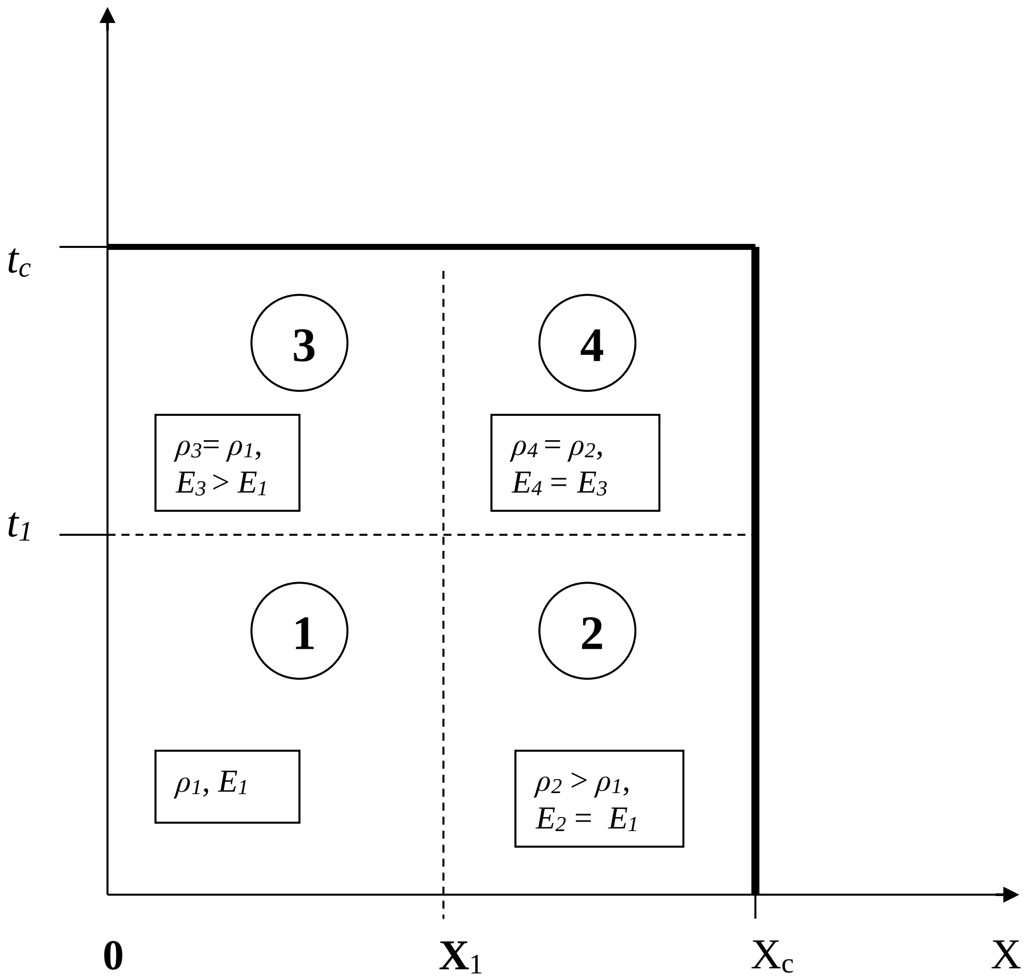

Figure 10: A typical unit cell in a space-time checkerboard dynamic material