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On the Stability of a Microstructure Model

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Abstract

The asymptotic stability of solutions of the Mindlin-type microstructure model for solids is analyzed in the paper. It is shown that short waves are asymptotically stable even in the case of a weakly non-convex free energy dependence on microdeformation.

Keywords: wave propagation, microstructured solids, asymptotic stability, dispersion

1. Dispersive wave equation for microstructured solids

Equations of motion for a solid with a microstructure include both macroscopic and microscopic balances of linear momentum, which in the one-dimensional case without body forces can be represented as follows [1]:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x}, \tag{1}$$

$$I \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\partial \eta}{\partial x} + \tau, \tag{2}$$

where the macrostress σ, the microstress η, and the interactive force τ are defined as derivatives of the free energy function W

$$\sigma = \frac{\partial W}{\partial u_x}, \quad \eta = -\frac{\partial W}{\partial \varphi_x}, \quad \tau = -\frac{\partial W}{\partial \varphi}. \tag{3}$$

Here u is the displacement, ρ_0 is the matter density, I represents microinertia, φ is the microdeformation.

For the Mindlin-type microstructure model the free energy function can be chosen as quadratic [2]

$$W = \frac{\rho_0 c^2}{2} u_x^2 + A\varphi u_x + \frac{1}{2} B\varphi^2 + \frac{1}{2} C\varphi_x^2 + \frac{1}{2} D\psi^2. \tag{4}$$

Here c is the elastic wave speed, A, B, C, and D are material parameters, ψ is the auxiliary internal variable [3]. Due to definitions (3), equations of motion both for macroscale and for microstructure can be represented in the form, which includes only displacement and microdeformation

$$u_{tt} = c^2 u_{xx} + \frac{A}{\rho_0} \varphi_x, \tag{5}$$

$$I\varphi_{tt} = C\varphi_{xx} - Au_x - B\varphi, \tag{6}$$

where $I = 1/(R^2 D)$ and R is an appropriate constant.

The last system of equations yields the single dispersive wave equation [2]

$$u_{tt} = c^2 u_{xx} + \frac{C}{B} \left(u_{tt} - c^2 u_{xx}\right)_{xx} - \frac{I}{B} \left(u_{tt} - c^2 u_{xx}\right)_{tt} - \frac{A^2}{\rho_0 B} u_{xx}. \tag{7}$$

More particular cases of the dispersive wave equation (7) can be found in papers [4–9].

The comparison of results of direct computation of wave propagation in periodic laminates with predictions of the Mindlin-type microstructure model (5) - (6) [10] shows a good agreement for long waves, but it fails for waves which length is comparable with the size of layers. It was also shown [10, 11] that the discrepancy can be eliminated by a slight modification of the Mindlin-type model of a microstructure. The modification consists in the change of the sign in last two terms of right-hand side of Eq. (6) corresponding to the contribution of the interactive internal force. However, this may result in the lost of convexity in the free energy function (4). Convexity is a desired property of the free energy, because it provides stable solutions of corresponding mathematical problems. The stability for a non-convex free energy function requires a special consideration. This is why the stability of the Mindlin-type model of a microstructure (5) - (6) is analyzed in this paper. More precisely, the asymptotic stability of the model is considered in detail.

It should be noted that the similar problem with the choice of the sign of higher-order terms is appeared in the context of strain-gradient elasticity theories [12–15].

2. Asymptotic stability

The characteristic property of the Mindlin-type microstructure model is the quadratic form of the free energy function (4) which is assumed to be convex by default. Convexity requires that the matrix corresponding to this quadratic form

$$M = \begin{pmatrix} \rho_0 c^2 & A & 0 & 0 \\ A & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \tag{8}$$

must be positive definite. The requirement of the positive definiteness results in conditions

$$\rho_0 > 0, \quad \rho_0 c^2 B - A^2 > 0, \quad C > 0, \quad D > 0. \tag{9}$$

However, the free energy in a laminated composite used for the comparison with a microstructured medium [10, 11] may be not necessarily convex, while it is convex in each layer of the laminate. This may be a source of the discrepancy in results of numerical simulations of wave propagation in laminates performed by direct calculations and by the microstructure modeling [10].

In order to study a more general situation, let us assume that the conditions of convexity for the free energy in the microstructure model (9) are not fulfilled completely. Nevertheless, the microstructure model should produce asymptotically stable solutions. To verify the asymptotic stability, we return to the governing equations of the microstructure model (5) - (6) and consider exponential plane-wave solutions of the form $u = u_0 e^{\Gamma t + ikx}, \varphi = \varphi_0 e^{\Gamma t + ikx}$, where u_0 and φ_0 are constants (cf. [16]). The required asymptotic stability will be reached if $Re\,\Gamma$ is non-positive. This is equivalent to the non-positivity for $Im\,\Gamma^2$.

Introducing the exponential plane wave solutions into the system of equations (5), (6), we arrive at the system of linear equations for u_0 and φ_0

$$\begin{cases} \rho_0 u_0 \Gamma^2 + \rho_0 c^2 u_0 k^2 - A\varphi_0 ik = 0, \\ I\varphi_0 \Gamma^2 + C\varphi_0 k^2 + A u_0 ik + B\varphi_0 = 0. \end{cases} \tag{10}$$

The condition of the existence of a non-trivial solution is the vanishing of the determinant of this system of equations

$$(\rho_0 \Gamma^2 + \rho_0 c^2 k^2)(I\Gamma^2 + Ck^2 + B) - A^2 k^2 = 0. \tag{11}$$

Let $\Gamma^2 = v + iw$. Then Eq. (11) can be represented as the sum of real and imaginary parts

$$(\rho_0 v + \rho_0 c^2 k^2)(Iv + Ck^2 + B) + i\rho_0 w(Iv + Ck^2 + B)+ \\ + iIw(\rho_0 v + \rho_0 c^2 k^2) - \rho_0 I w^2 = A^2 k^2. \tag{12}$$

The imaginary part of the left-hand side of Eq. (12) should be zero yielding

$$(Iv + Ck^2 + B) + I(v + c^2 k^2) = 0. \tag{13}$$

This means that the real part of Γ^2 satisfies

$$2Iv = -Ck^2 - B - Ic^2 k^2. \tag{14}$$

Inserting the value of the real part of Γ^2 into the real part of Eq. (12)

$$(\rho_0 v + \rho_0 c^2 k^2)(Iv + Ck^2 + B) - \rho_0 I w^2 = A^2 k^2, \tag{15}$$

we obtain the condition for determining of the imaginary part of Γ^2

$$w^2 = -(Ic^2 k^2 + Ck^2 + B)^2/4I^2 - A^2 k^2/\rho_0 I. \tag{16}$$

As it was mentioned, the imaginary part of Γ^2 should be negative to provide the asymptotic stability of the microstructure model. This may be achieved by the choice of the negative sign in the square-root of the right hand side of Eq. (16)

$$w = -\sqrt{-(Ic^2 k^2 + Ck^2 + B)^2/4I^2 - A^2 k^2/\rho_0 I}. \tag{17}$$

Moreover, the imaginary part of Γ^2 is a real number. Therefore, the right hand side of Eq. (16) must be non-negative

$$-(Ic^2 k^2 + Ck^2 + B)^2/4I^2 - A^2 k^2/\rho_0 I \geq 0. \tag{18}$$

Rewriting Eq. (18) in the form

$$-A^2 k^2/\rho_0 I \geq (Ic^2 k^2 + Ck^2 + B)^2/4I^2, \tag{19}$$

we see that the latter condition is satisfied only if $I < 0$. Since signs of parameters I and D should coincide, we arrive at a non-convex free energy function. The conservation of hyperbolicity of the equation of motion for the microstructure (6) yields in the simultaneous negative sign for the material parameter C. It should be noted that the inequality (19) is violated for small wave numbers, where the pure convex microstructure model is valid.

Summarizing, we conclude that asymptotically stable solutions are provided for sufficiently large wave numbers by a microstructure model with a (non-convex) free energy function

$$W = \frac{\rho_0 c^2}{2} u_x^2 + A\varphi u_x + \frac{1}{2} B\varphi^2 - \frac{1}{2} C\varphi_x^2 - \frac{1}{2} D\psi^2, \tag{20}$$

with $C > 0$ and $D > 0$.

Keeping the definitions of stresses, we have in this case

$$\sigma = \frac{\partial W}{\partial u_x} = \rho_0 c^2 u_x + A\varphi, \quad \eta = -\frac{\partial W}{\partial \varphi_x} = C\varphi_x. \tag{21}$$

The expression for the interactive internal force τ is not changed

$$\tau = -\frac{\partial W}{\partial \varphi} = -Au_x - B\varphi. \tag{22}$$

The derivative of the free energy with respect to the auxiliary internal variable gives

$$\xi = -\frac{\partial W}{\partial \psi} = D\psi. \tag{23}$$

The evolution equations for the microdeformation φ and for the auxiliary internal variable ψ can be rewritten as [17]

$$\varphi_t = RD\psi, \quad \psi_t = -R(\tau - \eta_x). \tag{24}$$

This leads to the hyperbolic equation for the microdeformation [17]

$$\varphi_{tt} = -R^2 D(\tau - \eta_x), \tag{25}$$

and allows us to represent the equations of motion both for macro- and microstructure in the form

$$u_{tt} = c^2 u_{xx} + \frac{A}{\rho_0}\varphi_x, \tag{26}$$

$$I\varphi_{tt} = C\varphi_{xx} + Au_x + B\varphi, \tag{27}$$

where $I = 1/(R^2 D)$.

It easy to see that the sign in last two terms of right-hand side of Eq. (27) corresponding to the contribution of the interactive internal force is changed in the comparison with Eq. (6), as well as it follows from results of numerical simulations [10].

3. Dispersion relations

To illustrate the difference in the original and modified microstructure models, we analyze the corresponding dispersion relations. We will consider the dispersive wave equations for both original and modified microstructure models in parallel

$$u_{tt} = c^2 u_{xx} + \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} - \frac{I}{B}\left(u_{tt} - c^2 u_{xx}\right)_{tt} - \frac{A^2}{\rho_0 B} u_{xx}. \tag{28}$$

$$u_{tt} = c^2 u_{xx} - \frac{C}{B}\left(u_{tt} - c^2 u_{xx}\right)_{xx} + \frac{I}{B}\left(u_{tt} - c^2 u_{xx}\right)_{tt} - \frac{A^2}{\rho_0 B} u_{xx}. \tag{29}$$

To simplify the matter, we introduce new coefficients following [2]

$$c_1^2 = \frac{C}{I}, \quad c_A^2 = \frac{A^2}{\rho B}, \quad p^2 = \frac{I}{B}. \tag{30}$$

The constants c_1 and c_A are velocities while p has the dimension of time. Then Eqs. (28) and (29) are rewritten in the form

$$u_{tt} = (c^2 - c_A^2) u_{xx} + p^2 c_1^2 \left(u_{tt} - c^2 u_{xx}\right)_{xx} - p^2 \left(u_{tt} - c^2 u_{xx}\right)_{tt}, \tag{31}$$

$$u_{tt} = (c^2 - c_A^2) u_{xx} - p^2 c_1^2 \left(u_{tt} - c^2 u_{xx}\right)_{xx} + p^2 \left(u_{tt} - c^2 u_{xx}\right)_{tt}. \tag{32}$$

We assume the solution of each equation above in the form

$$u(x,t) = \hat{u}\, exp\left[i(kx - \omega t)\right], \tag{33}$$

with wave number k and frequency ω. Introducing the latter into Eqs. (31) and (32), the following dispersion relations are obtained

$$\omega^2 = \left(c^2 - c_A^2\right)\ k^2 + p^2\ \left(\omega^2 - c^2\ k^2\right)\ \left(\omega^2 - c_1^2 k^2\right), \tag{34}$$

$$\omega^2 = \left(c^2 - c_A^2\right)\ k^2 - p^2\ \left(\omega^2 - c^2\ k^2\right)\ \left(\omega^2 - c_1^2 k^2\right). \tag{35}$$

Introducing dimensionless quantities

$$\xi = p c_0 k, \quad \eta = p\omega, \tag{36}$$

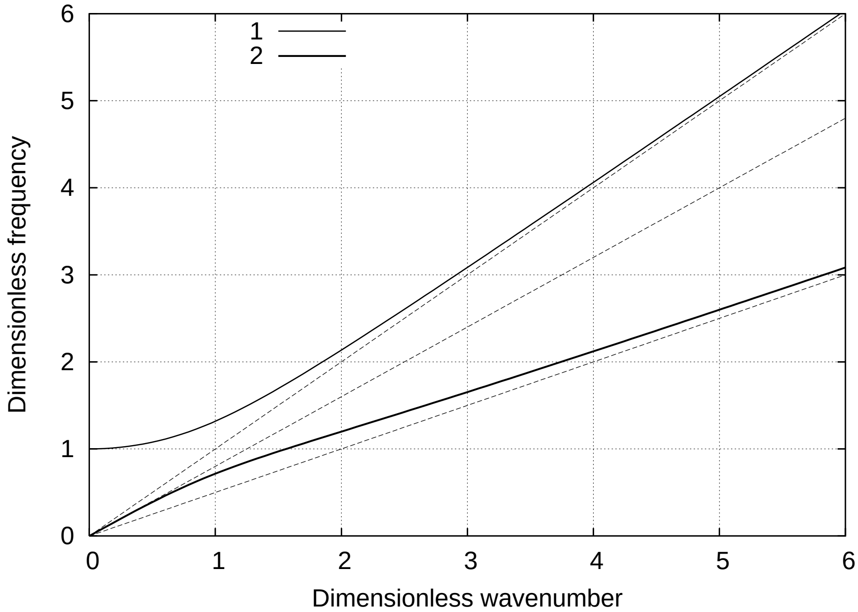

Figure 1: Dispersion curves for the original microstructure model (28) for $\gamma_A = 0.6, \gamma_1 = 0.5$: 1 – optical branch, 2 – acoustical branch; dashed lines – asymptotes to dispersion curves.

and dimensionless parameters

$$\gamma_1 = c_1/c, \quad \gamma_A = c_A/c, \tag{37}$$

we can reduce Eqs. (34) and (35) to

$$\eta^2 = \left(1 - \gamma_A^2\right)\xi^2 + \left(\eta^2 - \xi^2\right)\left(\eta^2 - \gamma_1^2\xi^2\right), \tag{38}$$

$$\eta^2 = \left(1 - \gamma_A^2\right)\xi^2 - \left(\eta^2 - \xi^2\right)\left(\eta^2 - \gamma_1^2\xi^2\right). \tag{39}$$

In order to visualize the dispersion curves, the dispersion relations are rearranged as follows

$$\left(\eta^2 - \xi^2\right)^2 + \left(\eta^2 - \xi^2\right)\left(\xi^2 - \gamma_1^2\xi^2 - 1\right) - \gamma_A^2\xi^2 = 0, \tag{40}$$

$$\left(\eta^2 - \xi^2\right)^2 + \left(\eta^2 - \xi^2\right)\left(\xi^2 - \gamma_1^2\xi^2 + 1\right) + \gamma_A^2\xi^2 = 0, \tag{41}$$

respectively.

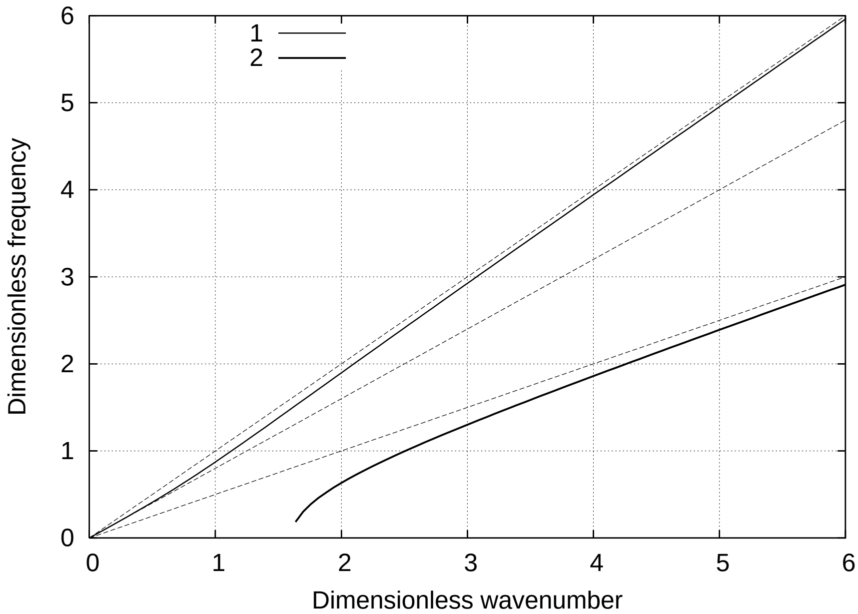

Figure 2: Dispersion curves for the modified microstructure model (29) for $\gamma_A = 0.6, \gamma_1 = 0.5$: 1 – optical branch, 2 – acoustical branch; dashed lines – asymptotes to dispersion curves.

Solving quadratic equations (40) and (41), we will have

$$\eta^2 = \xi^2 - \frac{\xi^2 - \gamma_1^2\xi^2 - 1}{2} \pm \sqrt{\frac{(\xi^2 - \gamma_1^2\xi^2 - 1)^2}{4} + \gamma_A^2\xi^2}, \tag{42}$$

$$\eta^2 = \xi^2 - \frac{\xi^2 - \gamma_1^2\xi^2 + 1}{2} \pm \sqrt{\frac{(\xi^2 - \gamma_1^2\xi^2 + 1)^2}{4} - \gamma_A^2\xi^2}. \tag{43}$$

The dispersion curves corresponding to relations (42) and (43) are presented in Figs. 1 and 2.

Figure 1 is completely similar to that presented in [2] for other values of dimensionless parameters. Figure 2 demonstrates a different behavior of dispersion curves, especially for the acoustical branch: it exists only for sufficiently large wave numbers and is placed below the asymptote.

4. Conclusions

It is clear that the convex Mindlin-type microstructure model (4) - (6) (with positive values of parameters A, B, C and D) is stable for all wave numbers. This model is working well for waves which length is longer than the size of inhomogeneity. However, the solution of Eqs. (5) - (6) is stable but incorrect for a certain range of wave numbers. In particular, such solution cannot reproduce correctly the results of direct numerical simulations of wave propagation in periodic laminates [10] if the wavelength is comparable with the size of layers. The slightly modified microstructure model (20), (26)- (27), which provides much more good agreement with direct numerical simulations [11], is not strictly convex but asymptotically stable for sufficiently large wave numbers.

The convexity of the free energy is a natural requirement providing stability conditions in homogeneous solids. Inhomogeneous solids may have, in principle, more weak requirements for the free energy convexity [18]. As it is shown, the asymptotic stability of wave propagation in microstructured solids can be provided for sufficiently large wave numbers by a microstructure model with not necessary strictly convex free energy function.

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