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# The Clock Paradox in a Static Homogeneous Gravitational Field

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**Abstract**

The *gedanken* experiment of the clock paradox is solved exactly using the general relativistic equations for a static homogeneous gravitational field. We demonstrate that the general and special relativistic clock paradox solutions are identical and in particular that they are identical for *finite* acceleration. Practical expressions are obtained for proper time and coordinate time by using the destination distance as the key observable parameter. This solution provides a formal demonstration of the identity between the special and general relativistic clock paradox with finite acceleration and where proper time is assumed to be the same in both formalisms. By solving the equations of motion for a freely falling clock in a static homogeneous field elapsed times are calculated for realistic journeys to the stars.

Key words: Clock paradox, special theory of relativity, general theory of relativity, hyperbolic motion, space exploration.

## 1 Introduction

Many mathematical solutions to the clock paradox are based on the formalism of special relativity. This solution parameterizes hyperbolic motion in terms of the proper time of an accelerating clock[1]. For decades the solution to the twin paradox, in terms of the formalism of general relativity, was also parameterized in terms of this proper time. Using this parameterization the general relativistic solution has been shown to be consistent with the special relativistic solution in the limit of infinite acceleration[2]. Recently the clock paradox has been solved in the formalism of general relativity for finite accelerations with the maximum relative velocity[3] as the parameter in the solution. Here we demonstrate that the general and special relativistic clock paradox solutions are identical with *finite* as well as infinite acceleration by using the destination distance as the key observable parameter.

Having reached its one hundredth anniversary the theory of relativity has become a cornerstone of modern physics. One of the earliest challenges to this theory was the clock paradox or twin paradox. While this paradox has long been resolved it remains an excellent vehicle for the study of relativity and in particular of the relationship between special and general relativity. The clock paradox is described by a *gedanken* experiment using identical twins with the first twin remaining at home and the second twin accelerating away and later returning. In order to describe a physically realistic trip the accelerating twin's journey will consist of four legs. During the first leg the twin travels in a space vehicle at a constant acceleration towards a distant star. The second leg begins where the traveling twin reverses thrust with constant deceleration. This deceleration causes the space ship to eventually come to rest relative to the distant star and the stay at home twin. The third leg consists of constant acceleration back toward home. Finally the fourth leg begins with constant deceleration and continues until the traveling twin is reunited with the stay at home twin.

The principle of equivalence allows this journey to be examined from a different point of view[4]. From the perspective of the accelerating twin the stay at home twin is moving in a static and homogeneous gravitational field (SHF). The equations of motion for a freely falling clock in this SHF, can be solved in terms of the coordinate distance of the accelerated twin for both leg 1 and 2 of the out going trip. With the destination distance in the SHF as a parameter, the two solutions can then be connected to obtain the elapsed time for a complete trip. Using coordinate distance as the parameter a coordinate transformation to a Lorentz inertial frame makes it possible to evaluate elapsed times and distances for realistic trips to the stars.

## 2 Static Homogeneous Field

Following Einstein's development of the equivalence principle[4][5] the acceleration experienced by the traveling twin is indistinguishable from station keeping (remaining motionless) in a gravitational field. From this perspective the traveling twin is station keeping while the stay at home twin falls freely in the field. Consider an orthogonal coordinate system with the gravitational field parallel to the $z$-axis. During the first and forth legs of the trip the field points in the negative direction. During the second and third legs the field is reversed and points in the positive direction.

The metric of a homogeneous field will be static and unchanged by coordinate transformations in any plane perpendicular to the acceleration. Under these conditions all derivatives in the field equations are zero except those parallel to the acceleration ($z$ axis). This allows the metric for a SHF to be written in the form,

$$-ds^2 = c^2 d\tau^2 = V(z)^2 c^2 dt^2 - dx^2 - dy^2 - U(z)^2 dz^2. \tag{1}$$

where $\tau$ is the proper time (time on a freely falling clock), $t$ and $x,\ y,\ z$ are the coordinate time and spatial coordinates of the station keeping twin.

The relation between $V(z)$ and $U(z)$ can be obtained from Einstein's field equations

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}_{\lambda}\right). \tag{2}$$

where $R_{\mu\nu} = 0$ in a source free region. The components of the Ricci tensor are,

$$R_{\mu\nu} = -\frac{\partial}{\partial x^{\alpha}}\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + \frac{\partial}{\partial x^{\nu}}\Gamma^{\alpha}_{\mu\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}, \tag{3}$$

and the Christoffel symbols in this equation are[6],

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\alpha}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} + \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}\right). \tag{4}$$

Using the requirement that only derivatives with respect to $z$ are non-zero leads to,

$$\Gamma^{0}_{03} = \Gamma^{0}_{30} = \frac{1}{V}\frac{\partial V}{\partial z}, \tag{5}$$

$$\Gamma^{3}_{00} = c^2 \frac{V}{U^2}\frac{\partial V}{\partial z}, \tag{6}$$

and,

$$\Gamma^3_{33} = \frac{1}{U}\frac{\partial U}{\partial z}. \tag{7}$$

All other Christoffel symbols are zero.

With these Christoffel symbols we obtain,

$$R_{00} = \frac{V}{U^3}\frac{\partial U}{\partial z}\frac{\partial V}{\partial z} - \frac{V}{U^2}\frac{\partial^2 V}{\partial z^2} \tag{8}$$

and

$$R_{33} = \frac{1}{V}\frac{\partial^2 V}{\partial z^2} - \frac{1}{UV}\frac{\partial U}{\partial z}\frac{\partial V}{\partial z}. \tag{9}$$

All other components of the Ricci tensor are equal to zero. Since the Ricci tensor is zero in source free space we obtain,

$$\frac{1}{U}\frac{\partial U}{\partial z} - \frac{1}{\frac{\partial V}{\partial z}}\frac{\partial^2 V}{\partial z^2} = 0 \tag{10}$$

for both Eqs. (8) and (9). The solution to Eq. (10) is[8],[9]

$$U = \frac{1}{\alpha}\frac{\partial V}{\partial z} \tag{11}$$

where $\alpha$ is a constant, which is easily obtained with dimensional analysis once $V(z)$ is chosen. Thus the static homogeneous field line element becomes

$$c^2 d\tau^2 = V^2 c^2 dt^2 - \frac{1}{\alpha^2}\left(\frac{\partial V}{\partial z}\right)^2 dz^2. \tag{12}$$

In order to calculate the equations of motion one chooses a $V(z)$ that is consistent with this metric. Two examples that exist in the literature are illustrated in the Lass[10] line element

$$c^2 d\tau^2 = e^{\frac{2g}{c^2}z} c^2 dt^2 - e^{\frac{2g}{c^2}z} dz^2. \tag{13}$$

and the Rindler[11] line element

$$c^2 d\tau^2 = \left(1 + \frac{g}{c^2}z\right)^2 c^2 dt^2 - dz^2. \tag{14}$$

These line elements describe the special case in which motion occurs parallel to the $z$-axis only. Due to the invariance of proper time any metric consistent with Eq. (12) can be used to calculate the difference in elapsed times for the clock paradox.

In what follows the Rindler metric Eq. (14) will be used to calculate these times. It is convenient to put this in dimensionless form by letting

$$\tau = \frac{c}{g}\tau', \quad t = \frac{c}{g}t', \quad z = \frac{c^2}{g}z', \tag{15}$$

where a prime designates the dimensionless quantity. If we let $g$ be the acceleration due to gravity at the surface of the earth and $c$ be the speed of light, then a time of 1 is $\sim 1$ year and a distance of 1 is $\sim 1$ light year. Dropping the primes the Rindler line element becomes

$$c^2 d\tau^2 = (1+z)^2 c^2 dt^2 - dz^2. \tag{16}$$

Consider the situation in which a test particle begins at rest at the origin and falls freely. The accelerating twin can be considered to be holding position at the origin. This twin's clock measures coordinate time. A clock that falls freely with the test particle can be identified with the stay at home twin. The equations of motion are obtained from the geodesic equations[6]

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu_{\mu\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau} = 0, \quad x^0 = t, \quad x^3 = z. \tag{17}$$

Using the dimensionless Rindler Christoffel symbols and Eq. (17) the equation of motion for coordinate time is

$$\frac{\partial t}{\partial \tau} = \frac{K}{(1+z)^2} \tag{18}$$

Where $K$ is a constant of motion obtained from the initial conditions. Also from Eq. (17) the equation of motion for $z$ is

$$\frac{\partial^2 z}{\partial \tau^2} + (1+z)\left(\frac{\partial t}{\partial \tau}\right)^2 = 0. \tag{19}$$

This can be put into a form that is easily solved by substituting Eq. (18) into Eq. (19). With this one obtains

$$\frac{\partial^2 z}{\partial \tau^2} + \frac{K^2}{(1+z)^3} = 0. \tag{20}$$

Next we will solve Eq. (20) for the first leg of the *gedanken* experiment. During the first leg the falling twin begins at the origin and at rest making $K = 1$. With this the solution to Eq. (20) is

$$z = -1 + \sqrt{1 - \tau^2}. \tag{21}$$

This result can then be substituted into Eq. (18) to obtain

$$t = \tanh^{-1}(\tau). \tag{22}$$

It is noteworthy that this solution to the field equations has a vanishing Riemann tensor as shown by Desloge[7]. This vanishing of the Riemann tensor would be expected from the symmetry of the Rindler line element[6], which is indistinguishable from a 2-dimensional problem, and following Weinberg[6] cannot be associated with a "true gravitational field". From physical considerations this is perhaps not surprising since there can be no physically real source for a homogeneous gravitational field.

## 3 Round Trip in a Static Homogeneous Field

In order to calculate the full round-trip time the equations of motion for the second leg must be obtained. The constant of motion for this leg can be determined by observing that the falling clock comes to rest when it reaches a maximum distance in the SHF. Figure 1 is a plot of legs one and two for the outbound trip. Notice that the falling clock comes to rest at a maximum distance $D$. With this $K = (1 + D)$ in Eq. (20).

The equations of motion for leg two can be derived by the same method as the first leg with the exception that the acceleration has the opposite sign and the initial conditions must match the position, velocity, proper and coordinate time at the end of the first leg. The equations obtained in this way are

$$z = 1 - \sqrt{(1+D)^2 - \left(\tau - \sqrt{D(D+4)}\right)^2} \tag{23}$$

and

$$t = \tanh^{-1}\left(\frac{\tau - \sqrt{D(D+4)}}{1+D}\right) + 2\tanh^{-1}\left(\frac{\sqrt{D(D+4)}}{2+D}\right). \tag{24}$$

Where $D$ is the maximum distance attained by the falling twin. In order to match the end of the first leg to the beginning of the second leg the reversal of the direction of acceleration must occur at

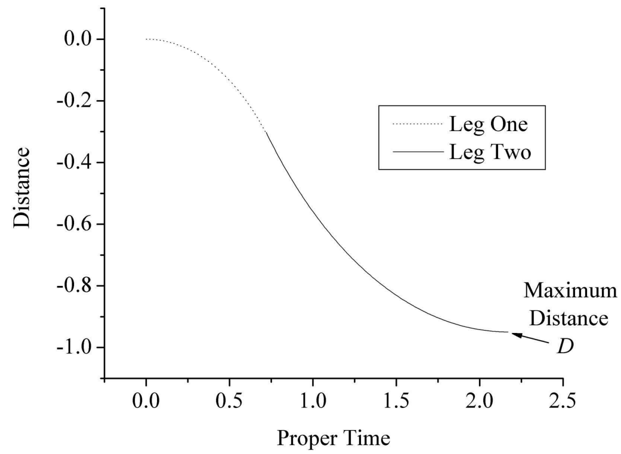


Figure 1: This is a plot of the falling twin's distance from the origin versus proper time for legs one and two of the outbound trip. Notice that this twin comes to rest at a maximum distance

$$z = -\frac{D}{2+D}. \tag{25}$$

By letting $z = -D$ in Eq. (23) one sees that the total outbound trip time is

$$\tau_{outbound} = \sqrt{D\,(D+4)} \tag{26}$$

which corresponds to a coordinate time of

$$t_{outbound} = 2\tanh^{-1}\left(\frac{\sqrt{D\,(D+4)}}{2+D}\right) = 2\cosh^{-1}\left(1+\frac{D}{2}\right). \tag{27}$$

where the last two expressions in Eq. (27) are mathematically equal due to a hyperbolic functional identity. The return trip time is equal to the outbound trip time because there is a one-to-one correspondence between velocity and position for the outbound and inbound trips, thus the total round-trip time is twice these values.

# 4 Special Relativity and Hyperbolic Motion

The *gedanken* experiment of the clock paradox can also be solved within the formalism of special relativity[1][8]. Following Misner et al[1] assign to the accelerating twin a 4-space velocity $u^\mu$ and 4-space acceleration $a^\mu = \frac{d}{d\tau'} u^\mu$ where $u^\mu u_\mu = -1$ and $a^\mu a_\mu = g^2$ or in terms of our dimensionless parameters $a^\mu a_\mu = 1$. The invariance of the square of the 4-space acceleration and the orthogonality of the 4-space velocity and 4-space acceleration $a^\mu u_\mu = 0$ leads to a set of differential equations with solution

$$Z = \cosh\left(\tau'\right) - 1, \tag{28}$$

and

$$T = \sinh\left(\tau'\right). \tag{29}$$

Capital letters are used for the coordinates to distinguish these from the general relativity solution. The proper time in these equations is the time of a clock moving with the accelerated twin[1]. The proper time for the accelerated twin is the same as coordinate time (station keeping clock) in the SHF Eq. (27), by the equivalence principle. Substituting Eq. (27) into Eqs. (28) and (29) the equations for the outbound trip distance and time for the stay at home twin can be written as

$$Z_{outbound} = 2\cosh\left(\cosh^{-1}\left(1 + \frac{D}{2}\right)\right) - 1 = D, \tag{30}$$

and

$$T_{outbound} = 2\sinh\left(\cosh^{-1}\left(1 + \frac{D}{2}\right)\right) = \sqrt{D\left(D+4\right)}. \tag{31}$$

Comparing Eqs. (26) and (27) to Eqs. (30) and (31) demonstrates that the formalisms of special and general relativity have the same solutions for elapsed times and distances.

# 5 Elapsed Time and Distance in a Lorentz Inertial Frame

The clock paradox *gedanken* experiment has been solved here using the destination trip distance in the SHF as the parameter. This permits a simple coordinate transformation from the SHF to a Lorentz inertial frame (LIF) at the end of leg 2 where the twins are comoving with the destination star. This transformation can be written as

$$T'_{outbound} = (1 + D)\sinh\left(t_{outbound}\right), \tag{32}$$

and

$$L'_{outbound} = (1 + D) \cosh (t_{outbound}) - 1. \tag{33}$$

By equating the square line element in the SHF and LIF

$$-ds^2 = dT'^2 - dx^2 - dy^2 - dZ'^2 = (1 + z)\, dt^2 - dx^2 - dy^2 - dz^2. \tag{34}$$

Eqs. (32) and (33) can be shown to be the correct transformation from the SHF to the LIF by direct substitution into Eq. (34 ).

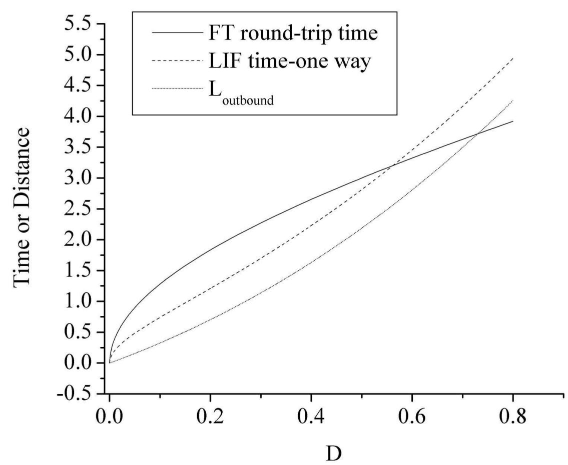


Figure 2: FT round-trip time is a plot of 2 times Eq. (26) and provides the round trip proper time for the freely falling twin. LIF time one-way is a plot of the outbound trip coordinate time in a Lorentz inertial frame Eq. (32). $L_{outbound}$ is a plot of the distance of the destination in the LIF Eq. (33).

Figure 2 provides a plot of elapsed times and distances in both the SHF and LIF for distances up to $D = 0.8$. In this Figure the outbound coordinate time in the LIF has been plotted from Eq. (32). The round-trip proper time is plotted in the same Figure as 2 times Eq. (26). Examination of this plot indicates that the outbound coordinate time in the LIF is greater than the round-trip proper time in the SHF for a one way trip distance greater than $D = 0.56$ and $L_{ouutbound} = 2.6$. The magnitude of the minimum distance for this to occur can also be found analytically by setting 2 times Eq. (26) equal to Eq. (32)

$$2\sqrt{D\,(D+4)} = (1+D)\sinh\left(2\cosh^{-1}\left(1+\frac{D}{2}\right)\right)$$

and solving for $D = \frac{1}{2}\sqrt{17} - \frac{3}{2}$.

Figure 3 provides a plot of elapsed times in the SHF and for distances in the LIF up to $D = 16$. In order to fit the one-way distance in the plot $L_{outbound}$ has been divided by 100. Examination of this plot indicates that for distance greater than $L_{outbound} = 2,500$ the time differences between the stay at home and the accelerating twin become appreciable.

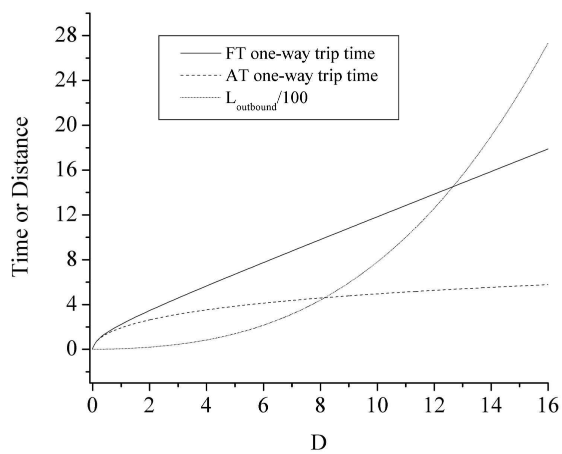


Figure 3: In order to show the LIF one-way distance the plot shows $L_{outbound}$ divided by 100. FT one-way trip time is the one-way (SHF) falling clock time. AT one-way trip time is the one-way trip (SHF) coordinate time Eq. (27).

The plot of Figure 3 extends to a distance that is $\sim 1/10$ the distance from the earth to the center of the Milky Way galaxy (assuming an acceleration of $g = 1$). At this distance the time difference between the two twins is about 20 years. However, it would seem that the first generation to achieve this sustained acceleration would have access to $\sim 10^7$ star systems in this time interval.

## 6 Conclusions

The *gedanken* experiment of the clock paradox has been solved exactly, by parameterizing the solution in terms of the maximum trip distance. The solution was arrived at independently using the formalisms of special and general relativity and these solutions are shown to be identical. We have also shown that for a one-way trip of sufficient distance the outbound trip coordinate time in the LIF is greater than the round-trip proper time in the SHF. Transforming the maximum trip distance and one-way elapsed time in the SHF to a LIF the elapsed times and distances for realistic journeys to the stars were calculated.

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