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Difference of two weighted composition operators on Bergman spaces

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- $H(\mathbb{D})$: All Analytic functions on \mathbb{D}

- **Weighted Bergman space**

$$A_{\alpha}^p = \left\{ f \in H(\mathbb{D}) : \int_{\mathbb{D}} |f(z)|^p dA_{\alpha}(z) < \infty \right\}, \text{ where}$$

$$dA_{\alpha}(z) = (\alpha + 1) (1 - |z|^2)^{\alpha} dA(z), \alpha > -1$$

- $A_0^2 = A^2$ (Bergman Space)

Definitions

- φ : **Analytic** from $\mathbb{D} \rightarrow \mathbb{D}$

Definition

The composition operator with symbol φ :

$$C_\varphi : H(\mathbb{D}) \rightarrow H(\mathbb{D}), \quad C_\varphi(f) = f \circ \varphi$$

- u : **Measurable** from $\mathbb{D} \rightarrow \mathbb{C}$

Definition

The weighted composition operator with weight u and symbol φ :

$$uC_\varphi : H(\mathbb{D}) \rightarrow \text{All measurable functions on } \mathbb{D}, \quad uC_\varphi(f) = u(f \circ \varphi)$$

Question

When is $uC_\varphi - vC_\psi$ compact? (Assume u and v are **analytic**)

It is known that:

Theorem (Z. Čučković and R. Zhao, 2007)

- $1 < p \leq q < \infty$

Then uC_φ is **compact** from A_α^p into A_β^q if and only if

$$\lim_{|z| \rightarrow 1^-} \int_{\mathbb{D}} \left(\frac{1 - |z|^2}{|1 - \bar{z}\varphi(w)|^2} \right)^{\frac{(2+\alpha)q}{p}} |u(w)|^q dA_\beta(w) = 0.$$

Question

When is $C_\varphi - C_\psi$ compact?

Theorem (J. Moorhouse, 2005)

$C_\varphi - C_\psi$ is **compact** on A_α^2 if and only if **both**

$$\lim_{|z| \rightarrow 1^-} |\sigma(z)| \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0, \quad \lim_{|z| \rightarrow 1^-} |\sigma(z)| \frac{1 - |z|^2}{1 - |\psi(z)|^2} = 0.$$

Here $\sigma(z) = \frac{\varphi(z) - \psi(z)}{1 - \overline{\varphi(z)}\psi(z)}$, $z \in \mathbb{D}$

Note: $|\sigma|$ is often referred to as the **Cancellation Factor**.

Connection between the Difference operator and Weighted Composition operators

The next theorem links the two types of operators.

Theorem (E. Saukko, 2011)

- $1 < p \leq q < \infty$

Then $C_\varphi - C_\psi$ is compact from A_α^p into A_β^q if and only if σC_φ and σC_ψ are **both** compact from A_α^p into $L^q(A_\beta)$.

Another Version of Saukko's theorem

Theorem (Another Version)

$C_\varphi - C_\psi$ is compact from A_α^p into A_β^q if and only if each of the following holds:

(a)

$$\lim_{|z| \rightarrow 1^-} \int_{\mathbb{D}} \left(\frac{1 - |z|^2}{|1 - \bar{z}\varphi(w)|^2} \right)^{\frac{(2+\alpha)q}{p}} |\sigma(w)|^q dA_\beta(w) = 0,$$

(b)

$$\lim_{|z| \rightarrow 1^-} \int_{\mathbb{D}} \left(\frac{1 - |z|^2}{|1 - \bar{z}\psi(w)|^2} \right)^{\frac{(2+\alpha)q}{p}} |\sigma(w)|^q dA_\beta(w) = 0$$

Answering the Original Question

Question

When is $uC_\varphi - vC_\psi$ compact? (Assume u and v are **analytic**)

Definition

For $\gamma \in \mathbb{R}$, $M(\gamma)$ is defined as follows:

$$M(\gamma) = \{f : \|f(z)(1 - |z|^2)^\gamma\|_{L^\infty} < \infty\}$$

Compactness of $uC_\varphi - vC_\psi$

- $0 < p \leq q < \infty$
- $\frac{2+\alpha}{p} \leq \frac{2+\beta}{q}$
- $u, v \in \mathbf{M}\left(\frac{2+\beta}{q} - \frac{2+\alpha}{p}\right)$

Theorem (Acharyya and Wu, 2017)

$uC_\varphi - vC_\psi : A_\alpha^p \rightarrow A_\beta^q$ is **compact** if and only if each of the following holds:

(a)

$$\lim_{|z| \rightarrow 1^-} |\sigma(z)| \left(|u(z)| \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + |v(z)| \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0,$$

(b)

$$\lim_{|z| \rightarrow 1^-} (1 - |\sigma(z)|^2)^{\frac{2+\alpha}{p}} |u(z) - v(z)| \left(\frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0$$

Compactness of $uC_\varphi - vC_\psi$

Theorem (Acharya and Wu, 2017)

$uC_\varphi - vC_\psi : A_\alpha^p \rightarrow A_\beta^q$ is **compact** if and only if each of the following holds:

(a)

$$\lim_{|z| \rightarrow 1_-} |\sigma(z)| \left(|u(z)| \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + |v(z)| \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0,$$

(b)

$$\lim_{|z| \rightarrow 1_-} (1 - |\sigma(z)|^2)^{\frac{2+\alpha}{p}} |u(z) - v(z)| \left(\frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0$$

Proof: " \implies " Suppose $uC_\varphi - vC_\psi : A_\alpha^p \rightarrow A_\beta^q$ is compact.

Let $\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$. Note that $k_a, \varphi_a k_a \rightarrow 0$ weakly. Thus

$$\lim_{|a| \rightarrow 1_-} \|uC_\varphi(k_a) - vC_\psi(k_a)\|_{q,\beta} = 0, \quad \lim_{|a| \rightarrow 1_-} \|uC_\varphi(\varphi_a k_a) - vC_\psi(\varphi_a k_a)\|_{q,\beta} = 0$$

Compactness of $uC_\varphi - vC_\psi$

Apply the lemma:

Lemma

Suppose $0 < p < \infty$ and $0 < r < 1$. There is a constant $C > 0$ such that for any $z \in \mathbb{D}$ and $f \in A_\alpha^p$

$$|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{2+\alpha}} \int_{\Delta(z,r)} |f(w)|^p dA_\alpha(w).$$

Also, use the elementary facts that $C_\varphi(\varphi_{\varphi(z)})(z) = 0$ and $|C_\psi(\varphi_{\varphi(z)})(z)| = |\sigma(z)|$, and a chain of inequalities to obtain

$$\lim_{|z| \rightarrow 1^-} \frac{|\sigma(z)| |u(z)| (1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\varphi(z)|^2)^{\frac{2+\alpha}{p}}} = 0,$$

$$\lim_{|z| \rightarrow 1^-} \frac{|u(z) - v(z)| (1 - |\sigma(z)|^2)^{\frac{2+\alpha}{p}} (1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\psi(z)|^2)^{\frac{2+\alpha}{p}}} = 0.$$

Similarly

$$\lim_{|z| \rightarrow 1^-} \frac{|\sigma(z)||v(z)|(1-|z|^2)^{\frac{2+\beta}{q}}}{(1-|\psi(z)|^2)^{\frac{2+\alpha}{p}}} = 0,$$

$$\lim_{|z| \rightarrow 1^-} \frac{|u(z) - v(z)|(1-|\sigma(z)|^2)^{\frac{2+\alpha}{p}}}{(1-|\varphi(z)|^2)^{\frac{2+\alpha}{p}}}(1-|z|^2)^{\frac{2+\beta}{q}} = 0.$$

\Leftarrow (**has root in Moorhouse and Saukko's work:**)

It is sufficient to show that for any sequence $\{f_n\}$ in A_α^p with $\|f_n\|_{p,\alpha} \leq 1$ and $f_n(z) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on any compact set of \mathbb{D} , we have

$$\|(uC_\varphi - vC_\psi)(f_n)\|_{q,\beta} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Partition the disk into E and E' , with $E = \{z \in \mathbb{D} : |\sigma(z)| < \frac{2-\sqrt{3}}{2}\}$

Compactness of $uC_\varphi - vC_\psi$

We can write

$$(uC_\varphi - vC_\psi)(f_n) = (uC_\varphi - vC_\psi)(f_n)\chi_{E'} + (u - v)C_\psi(f_n)\chi_E + u(C_\varphi - C_\psi)(f_n)\chi_E.$$

Therefore we need to establish the following three statements.

$$\lim_{n \rightarrow \infty} \|(uC_\varphi - vC_\psi)(f_n)\chi_{E'}\|_{q,\beta} = 0,$$

$$\lim_{n \rightarrow \infty} \|(u - v)C_\psi(f_n)\chi_E\|_{q,\beta} = 0,$$

$$\lim_{n \rightarrow \infty} \|u(C_\varphi - C_\psi)(f_n)\chi_E\|_{q,\beta} = 0.$$

The first two statements are true, due to the following lemma.

Compactness of $uC_\varphi - vC_\psi$

Lemma

Suppose $s, t > 0$, ω is a nonnegative locally bounded measurable function on \mathbb{D} , φ is a holomorphic self map of \mathbb{D} , and

$$\lim_{|z| \rightarrow 1^-} \omega(z) \frac{(1 - |z|^2)^s}{(1 - |\varphi(z)|^2)^t} = 0.$$

- (a) If $\beta > s - 1$, then the measure $\varphi_*(\omega, A_\beta)$ is a compact $(2 + \beta + t - s)$ -Carleson measure.
- (b) If $\beta > -1$ and $\omega \in M(\gamma)$ with $\gamma < 1 + \beta$, then the measure $\varphi_*(\omega, A_\beta)$ is a compact $(2 + \beta - \gamma + \epsilon(\gamma + t - s))$ -Carleson measure for any $\epsilon \in (0, \min\{\frac{1 + \beta - \gamma}{s - \gamma}, 1\})$ if $\gamma < s$, or $\epsilon \in (0, 1)$ if $\gamma \geq s$.

Compactness of $uC_\varphi - vC_\psi$

To prove the third statement

$$\lim_{n \rightarrow \infty} \|u(C_\varphi - C_\psi)(f_n)\chi_E\|_{q,\beta} = 0,$$

we apply Fubini, the previous lemma, and the following lemma:

Lemma

Let $0 < p \leq q < \infty$. There exists a constant $C > 0$, such that for all $a \in \mathbb{D}$, $z \in \Delta(a, \frac{2-\sqrt{3}}{2})$, and $f \in A_\alpha^p$ with $\|f\|_{p,\alpha} \leq 1$

$$|f(z) - f(a)|^q \leq C \frac{|\varphi_a(z)|^q}{(1 - |a|^2)^{(2+\alpha)q/p}} \int_{\Delta(a, \frac{1}{2})} |f(w)|^p dA_\alpha.$$

The theorem of Moorhouse

Theorem (Acharya and Wu, 2017)

$uC_\varphi - vC_\psi : A_\alpha^p \rightarrow A_\beta^q$ is **compact** if and only if each of the following holds:

(a)

$$\lim_{|z| \rightarrow 1^-} |\sigma(z)| \left(|u(z)| \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + |v(z)| \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0,$$

(b)

$$\lim_{|z| \rightarrow 1^-} (1 - |\sigma(z)|^2)^{\frac{2+\alpha}{p}} |u(z) - v(z)| \left(\frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\varphi(z)|^2)^{\frac{2+\alpha}{p}}} + \frac{(1 - |z|^2)^{\frac{2+\beta}{q}}}{(1 - |\psi(z)|^2)^{\frac{2+\alpha}{p}}} \right) = 0$$

Corollary (J. Moorhouse, 2005)

$C_\varphi - C_\psi$ is **compact** on A_α^2 if and only if **both**

$$\lim_{|z| \rightarrow 1^-} |\sigma(z)| \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0, \quad \lim_{|z| \rightarrow 1^-} |\sigma(z)| \frac{1 - |z|^2}{1 - |\psi(z)|^2} = 0.$$

Hilbert-Schmidt operator (definition)

- X : Separable Hilbert space
- $\{e_j\}$: Orthonormal basis

Definition

T is **Hilbert-Schmidt** if

$$\|T\|_{HS(X)} = \left\{ \sum_{j=0}^{\infty} \|Te_j\|^2 \right\}^{\frac{1}{2}} < \infty$$

Notational Simplicity: $\|T\|_{HS(X)} = \|T\|_{HS}$

Theorem (B.R. Choe, T. Hosokawa and H. Koo, 2010)

Let $\alpha \geq -1$. Consider $C_\varphi - C_\psi$ acting on A_α^2 . Then

$$\|C_\varphi - C_\psi\|_{HS}^2 \asymp \int_{\mathbb{D}} \frac{|\sigma^2(\mathbf{z})| dA_\alpha(z)}{(1 - |\varphi(z)|^2)^{2+\alpha}} + \int_{\mathbb{D}} \frac{|\sigma^2(\mathbf{z})| dA_\alpha(z)}{(1 - |\psi(z)|^2)^{2+\alpha}}$$

Here $\alpha = -1$ corresponds to H^2 , and the comparability constants depend only on α .

Theorem (Acharyya and Wu, 2017)

- $\mathbb{E} : \mathbb{D}$ or \mathbb{T}
- $u, v : \mathbf{Measurable}$
- $uC_\varphi - vC_\psi$ acting from $A_\alpha^2 \rightarrow L^2(\mu)$

Then

$$\|uC_\varphi - vC_\psi\|_{HS}^2 \asymp \int_{\mathbb{E}} |\sigma|^2 \left(\frac{|u|^2}{(1 - |\varphi|^2)^{2+\alpha}} + \frac{|v|^2}{(1 - |\psi|^2)^{2+\alpha}} \right) d\mu \\ + \int_{\mathbb{E}} (1 - |\sigma|^2)^{2+\alpha} |u - v|^2 \left(\frac{1}{(1 - |\varphi|^2)^{2+\alpha}} + \frac{1}{(1 - |\psi|^2)^{2+\alpha}} \right) d\mu$$

A Key Lemma:

Lemma (Acharyya and Wu, 2017)

- For $z, w \in \mathbb{D}$, define $\rho = \frac{z-w}{1-\bar{z}w}$
- $\alpha > -2$

Then

$$\begin{aligned} |A|^2 K_z^{(\alpha)}(z) + |B|^2 K_w^{(\alpha)}(w) + 2\Re\left(A\bar{B}K_w^{(\alpha)}(z)\right) \asymp \\ |\rho|^2 \left(|A|^2 K_z^{(\alpha)}(z) + |B|^2 K_w^{(\alpha)}(w)\right) \\ + (1 - |\rho|^2)^{2+\alpha} |A + B|^2 \left(K_z^{(\alpha)}(z) + K_w^{(\alpha)}(w)\right). \end{aligned}$$

Corollary

Theorem (Acharyya and Wu, 2017)

$$\|uC_\varphi - vC_\psi\|_{HS}^2 \asymp \int_{\mathbb{E}} |\sigma|^2 \left(\frac{|u|^2}{(1 - |\varphi|^2)^{2+\alpha}} + \frac{|v|^2}{(1 - |\psi|^2)^{2+\alpha}} \right) d\mu \\ + \int_{\mathbb{E}} (1 - |\sigma|^2)^{2+\alpha} |u - v|^2 \left(\frac{1}{(1 - |\varphi|^2)^{2+\alpha}} + \frac{1}{(1 - |\psi|^2)^{2+\alpha}} \right) d\mu$$

Corollary

Consider the following operators

$$\sigma uC_\varphi, \sigma vC_\psi, (1 - |\sigma|^2)^{1+\frac{\alpha}{2}}(u - v)C_\varphi \text{ and } (1 - |\sigma|^2)^{1+\frac{\alpha}{2}}(u - v)C_\psi$$

from A_α^2 or H^2 to $L^2(\mu)$. Then $uC_\varphi - vC_\psi$ is Hilbert-Schmidt if and only if all of the four operators are Hilbert-Schmidt.

Thank You!