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The general relativistic infinite plane

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Uniform fields are one of the simplest and most pedagogically useful examples in introductory courses on electrostatics or Newtonian gravity. In general relativity there have been several proposals as to what constitutes a uniform field. In this article we examine two metrics that can be considered the general relativistic version of the infinite plane with finite mass per unit area. The first metric is the 4D version of the 5D "brane" world models which are the starting point for many current research papers. The second case is the cosmological domain wall metric. We examine to what extent these different metrics match or deviate from our Newtonian intuition about the gravitational field of an infinite plane. These solutions provide the beginning student in general relativity both computational practice and conceptual insight into Einstein's field equations. In addition they do this by introducing the student to material that is at the forefront of current research.

I. INTRODUCTION

The homogeneous gravitational field (*i.e.* the $\vec{g}$ in $\vec{F} = m\vec{g}$) is one of the most commonly used examples in introductory mechanics courses. Students later find out that this is an approximation used when the observer is small compared to the radius of the spherical gravitating body on which they reside – if d is the size of the observer and R the radius of the spherical body then $d \ll R$. In Newtonian gravity a truly uniform field would be produced by an infinite plane with area mass density σ. Infinite planes of charge are also among the first examples students encounter in introductory electrostatics. Using the close connection between Newtonian gravity and electrostatics one can find the gravitational potential for such an infinite mass plane. The Newtonian gravitational potential, ϕ, for such a source is given by

$$\vec{g} = -\nabla\phi = -2\pi G\sigma[\Theta(z) - \Theta(-z)]\hat{z} \qquad \rightarrow \qquad \nabla^2\phi = 4\pi G\sigma\delta(z), \tag{1}$$

where $\Theta(u)$ is the step function ($\Theta(u) = 0$ for $u < 0$ and $\Theta(u) = 1$ for $u \geq 0$); $\delta(z)$ is the Dirac delta-function which results from differentiating the step function. The last equation is obtained by taking the divergence of the first equation. The solution, $\phi(z)$ to (1) is $\phi(z) = 2\pi G\sigma|z|$.

We want to construct the general relativistic version of the infinite plane. We find that there are two metrics which have some (but not all) of the characteristics of the Newtonian infinite plane. These two metrics are excellent examples to introduce beginning students to both calculational and conceptual aspects of general relativity. In addition one of the two examples has connections with current research in "brane" world models. This can help generate interest in beginning students of general relativity, by showing them they are not (in all cases) far removed from frontier research.

There have been many previous claims for constructing the general relativistic field for an infinite plane (a nice discussion and an extensive list of references on this subject can be found in [1]). However, none of these solutions has the correct matter configuration for an infinite plane, namely $\rho(z) \propto \delta(z)$ as in (1). In addition, one might expect that the general relativistic infinite plane would in some sense give rise to a "uniform" gravitational field. We will indeed find this to be the case since our first solution is a 4D version of the 2D "uniform" gravitational field introduced by Desloge [2]. There are, however, some important differences between the general relativistic infinite

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plane and the Newtonian infinite plane which arise from the fact that the field equations of general relativity imply the equations of motion. Thus one cannot put down an infinite plane and expect it to be static. We will find that in order to stabilize the general relativistic infinite plane one needs to introduce pressures/tensions and, in one case, a cosmological constant term.

In the end we find that the first solution is a 4D version of the 5D "brane world" models [3] [4] [5] which have provided a new solution to the hierarchy problem (i.e., the question or puzzle as to why the gravitational interaction is many orders of magnitude weaker than the three other known interactions) as well as other open questions in particle physics and cosmology. Reference [6] gives an excellent introduction to the topic of 5D and higher dimensional brane world models, their utility in solving the hierarchy problem, and other open questions in particle physics and cosmology that can be addressed by these models. The second solution is the domain wall solution first given in [7] and studied in some detail in [8] [9]. It is a simple solution which can again serve as an excellent calculational exercise for beginning students. Domain walls are not phenomenologically viable, but they are often grouped together with cosmic strings, which are being actively studied. As such the domain wall solution might provide a starting point for discussing exotic cosmological solutions [10]. In the concluding section we will give some discussion of the physical meaning of the brane world and domain wall metrics, and give some ideas for additional investigations one could assign as projects for students.

II. GENERAL RELATIVISTIC INFINITE PLANE

Starting with the Newtonian infinite plane we want to motivate the general relativistic infinite plane. In the weak field limit ($GM/rc^2 \ll 1$, where M is the mass of the gravitating object and r is some distance scale involved in the problem) the relationship between the Newtonian potential and the g_{00} component of the metric is $g_{00} = -(1+2\phi/c^2) \to -(1+2\phi)$ (from now on we set $c=1$). Thus, given that the Newtonian potential for the infinite plane is $\phi(z) = 2\pi G\sigma|z| \equiv g|z|$ one might be tempted to try the metric $g_{00} = -(1+2g|z|)$, $g_{ii} = +1$, where $i =, x, y, z$ and we have a gravitational acceleration $g = 2\pi G\sigma$. One can easily check that this does not work, since for this metric only the G_{xx} and G_{yy} components of the Einstein tensor are non-zero. Since $G_{\mu\nu} \propto T_{\mu\nu}$ this implies only the T_{xx} and T_{yy} components of the energy-momentum tensor are non-zero, whereas we wanted only $T_{00} \propto \rho$ to be non-zero. We will find that it is not possible to construct a static mass distribution without pressures/tensions or possibly a cosmological constant.

One may notice that the trial metric – $g_{00} = -(1+2g|z|)$, $g_{ii} = +1$ – can be seen as some limit of the Rindler metric [11] which is Minkowski spacetime as seen by an observer undergoing constant, linear acceleration,

$$ds^2 = -(1+gz)^2\, dt^2 + dx^2 + dy^2 + dz^2 \approx -(1+2gz)\, dt^2 + dx^2 + dy^2 + dz^2, \tag{2}$$

where the last approximation is $gz \ll 1$. If we ignore the lack of absolute value around z this is just the trial metric. However, it can be shown [2] [12] that the Rindler metric is not a solution to the Einstein field equations for any ponderable source of the gravitational field, i.e., it is a vacuum spacetime. This can be seen directly since the Rindler metric (2) can be obtained from Minkowski spacetime via the transformation $t' = [(1/a) + z]\sinh(at)$ and $z' = [(1/a) + z]\cosh(at)$.

Continuing our search of the general relativistic analog of the infinite plane we note that in [2] the g_{00} component for the uniform gravitation field metric was given by $g_{00} = e^{2gz}$ (in [2] the coordinate was x and $|g| = 1$). Also noting that $e^{2gz} \approx 1 + 2gz$ for $gz \ll 1$ we might try using $e^{2g|z|}$ for g_{00}. This same association between the approximate $(1+2gz)$ and exact (e^{2gz}) time component of the metric was given originally by Einstein [13]. What, if anything, do we do for g_{ii} where $i = x, y, z$? If we make the simplest guess that only g_{00} should be non-trivial we arrive at

$$ds^2 = -e^{2g|z|}dt^2 + dx^2 + dy^2 + dz^2. \tag{3}$$

This is a 4D version of the exotic Kaluza-Klein metric proposed by Visser [14] which is considered a precursor to the current higher dimensional brane world models. Again, if one calculates the Einstein tensor $G_{\mu\nu}$, the only non-zero terms are

$$G_{xx} = G_{yy} = \frac{\left(e^{g|z|}\right)''}{e^{g|z|}} = g\delta(z) + g^2, \tag{4}$$

where the primes indicate differentiation with respect to z. The $\delta(z)$ function appears because we are taking the second derivative of $|z|$. Adding a cosmological constant term to the field equations would give non-zero values of T_{00} and T_{zz}, but not of the form $\delta(z)$ needed for the mass distribution for an infinite plane.

To get rid of the G_{xx}, G_{yy} terms one should take g_{xx}, g_{yy} as non-trivial. The simplest guess – that one should extend the "warp" factor $e^{2g|z|}$ to encompass $dx^2 + dy^2$ – turns out to be the correct one. The metric now becomes

$$ds^2 = e^{2g|z|}(-dt^2 + dx^2 + dy^2) + dz^2. \tag{5}$$

The components of the Einstein tensor, $G_{\mu\nu}$, for (5) can be calculated by hand. This provides a good exercise for a beginning general relativity student, which is non-trivial, yet simpler than the Schwarzschild metric. The non-zero components are

$$\begin{aligned} G_{xx} &= G_{yy} = -G_{00} = \left((e^{g|z|})'\right)^2 + 2e^{g|z|}(e^{g|z|})'' = 3g^2 e^{2g|z|} + 2g\delta(z) \\ G_{zz} &= 3\left(\frac{(e^{g|z|})'}{e^{g|z|}}\right)^2 = 3g^2. \end{aligned} \tag{6}$$

Now we finally see the presence of the $T_{00} \propto \delta(z)$ matter source implied by its presence in G_{00}. However, there are also pressures in the x, y directions since $G_{xx} = G_{yy} \neq 0 \rightarrow T_{xx} = T_{yy} \neq 0$, and the components G_{00}, G_{xx}, G_{yy} have an extra, non-δ function term – $\pm 3g^2 e^{2g|z|}$. Finally, G_{zz} is non-zero and equal to $3g^2$. These extra, non-δ function terms can be eliminated by the introduction of a cosmological constant term as follows

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - 8\pi G \lambda g_{\mu\nu} \qquad \text{where} \qquad \lambda = -\frac{3g^2}{8\pi G} \tag{7}$$

As can easily be checked, this is exactly the 4D version of the 5D brane models in [3] [4] [5]. The energy-momentum tensor for our system can be read off by comparing (6) and (7) with the result

$$T_{00} = -\frac{g}{4\pi G}\delta(z)\ , \qquad T_{xx} = T_{yy} = \frac{g}{4\pi G}\delta(z). \tag{8}$$

All other components are zero. Note that this is not exactly the same as the Newtonian case of the infinite plane where $T_{00} \propto \delta(z)$ and all other components of $T_{\mu\nu}$ zero. In the general relativistic case there are pressures in the x and y directions (the non-zero T_{xx}, T_{yy}) as well as a negative cosmological constant which is proportional to the square of the acceleration, g^2, and thus to the square of the area density, σ^2.

There is another plane solution to the Einstein field equations which has $T_{00} \propto \delta(z)$, and pressures in the perpendicular directions, but without a cosmological constant. It is the domain wall metric first given by Taub [7] and investigated further in [8]. The domain wall metric is

$$ds^2 = (1 - 2g|z|)^{-1/2}(-dt^2 + dz^2) + (1 - 2g|z|)(dx^2 + dy^2)\ , \tag{9}$$

where as before $g = 2\pi G\sigma$. Calculating the components of $G_{\mu\nu}$ for this metric again provides a simple calculational exercise for beginning students. The only non-zero components are

$$\begin{aligned} G_{00} &= -\frac{(1-2g|z|)''}{(1-2g|z|)} = +2g\delta(z) \\ G_{xx} &= G_{yy} = \frac{1}{4}(1-2g|z|)^{1/2}(1-2g|z|)'' = -\frac{1}{2}g\delta(z)\ . \end{aligned} \tag{10}$$

Using (10) in (7) but with $\lambda = 0$ the non-zero components of the energy-momentum tensor are

$$T_{00} = \frac{g}{4\pi G}\delta(z)\ , \qquad T_{xx} = T_{yy} = -\frac{g}{16\pi G}\delta(z). \tag{11}$$

Since the domain wall solution given in (10) (11) has no cosmological constant one might prefer this as the general relativistic version of the infinite plane. However, from (10) one can see that the domain wall solution has a singularity at $z = \pm\frac{1}{2g}$ whereas the brane solution (5) is everywhere non-singular. In the following section we examine in more detail the motion of a test particle in the two metrics from (5) and (9) to see which metric most closely corresponds to a Newtonian infinite mass plane in so far as giving a uniform acceleration.

All the above metrics had $\delta-$ function sources (or, more correctly, distributions). In linear theories such as electromagnetism or Newtonian gravitation these types of sources do not present a major problem. But the nonlinear nature of general relativity suggests that one should proceed with caution when dealing with concentrated sources. The formalism for treating thin shell distribution such as those studied here can be found in [15]. However, the formalism for treating line and point distributions is more involved. An attempt to incorparate point distributions in general relativity can be found in [16]. A detailed study of the possible inclusion of point, line and string distributional sources in general relativity can be found in [17]. Starting from the notion of *regular* metrics, which satisfy the physically reasonable requirement that the curvature tensor - and, through Einstein's equations, the energy-momentum tensor as well - make sense as a distribution, Geroch and Traschen show that point particles and strings are not allowed as sources. Thin shells of matter or radiation, on the other hand, do admit a general formulation as distributional sources under this requirement of regularity.

III. PARTICLE TRAJECTORIES FOR INFINITE PLANE

In this section we study the geodesic motion of a test particle in each of the background metrics (5) and (9). For our purposes the best form of the geodesic equations is given on page 330 of reference [18] as

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}+\frac{\partial g_{\mu\nu}}{\partial x^\alpha}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau}-\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=0. \tag{12}$$

We first apply this to the "brane" metric of (5) restricting ourselves to $z>0$ so we drop the absolute value around z. The $\mu=0$ equation from (12) is

$$\frac{d^2t}{d\tau^2}+2g\frac{dz}{d\tau}\frac{dt}{d\tau}=0. \tag{13}$$

This has the solution [19] (we assume x and y are fixed; the consistency of this assumption may be verified directly from (12))

$$\frac{dt}{d\tau}=e^{-2gz} \tag{14}$$

From (12) the geodesic equation for the z-dimension is

$$\frac{d^2z}{d\tau^2}+\frac{1}{2}\frac{\partial(e^{2gz})}{\partial z}\left(\frac{dt}{d\tau}\right)^2-\frac{1}{2}\frac{\partial g_{ij}}{\partial z}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=0, \tag{15}$$

where i,j ranges over 1, 2. Substituting the result of (15), using $\frac{\partial g_{ij}}{\partial z}=2gg_{ij}=2ge^{2gz}\eta_{ij}$ (where $\eta_{ij}=+1$ if $i=j=1$ or $i=j=2$, and zero otherwise) we can write the above geodesic equation as

$$\frac{d^2z}{d\tau^2}=-ge^{-2gz}+ge^{2gz}\eta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\ , \tag{16}$$

The acceleration is not constant because of the e^{-2gz} factor in the first term. This runs counter to a naive extension of Newtonian intuition, as encapsulated in (1), that the acceleration should be constant. However, (16) is what is usually meant by *4-acceleration in special relativity*: it uses the proper time τ of the test particle undergoing geodesic motion. The observer who measures this acceleration is fixed at some $z\neq 0$. In the context of special relativity the acceleration defined in (16) is singled out by being covariant. In the context of general relativity this acceleration in (16) is not covariant and therefore does not hold the same priviledged role that it does in special relativity. Thus, in order to see the connection with the Newtonian case we follow [2] and look at the acceleration (referred to as "local" acceleration in [2]) measured with a system of clocks that are fixed with respect to the infinite plane at $z=0$ (due to general relativistic time dilation this observer is not the same as the previous observer fixed at $z\neq 0$). Time intervals dT measured with these clocks are related to proper time intervals via $d\tau=\sqrt{-g_{00}}dT=\sqrt{e^{2gz}}dT$ [2]. With this re-parameterization the geodesic equation (16) for the z-dimension becomes

$$e^{-gz}\frac{d}{dT}\left(e^{-gz}\frac{dz}{dT}\right)=-ge^{-2gz}+g\eta_{ij}\frac{dx^i}{dT}\frac{dx^j}{dT}. \tag{17}$$

This simplifies to

$$\frac{d^2z}{dT^2} = -g + g\left(v_z^2 + e^{2gz}(v_x^2+v_y^2)\right) \tag{18}$$

where $v_x \equiv \frac{dx}{dT}$, $v_y \equiv \frac{dy}{dT}$ and $v_z \equiv \frac{dz}{dT}$. Note that the acceleration in (18) is not covariant, even in the special relativistic sense. The first term in (18) gives the constant acceleration toward the plane at $z=0$ which one would expect from the Newtonian case. However, unlike the Newtonian case, both the 4-acceleration (16) as well as the acceleration (18) measured with clocks fixed relative to the plane have a dependence on the velocity, and do not therefore reproduce the Newtonian result in general.

In (18) one can see that the velocities in the plane (i.e. v_x, v_y) scale exponentially as one moves off of the plane $z=0$. The same is true of momenta and some other physical quantities. It is exactly this property of the metric (5) that makes it useful for addressing the hierarchy problem. In 5D it can be arranged so that the effective 4D Newton's constant, G_4, is related to the underlying 5D Newton's constant, G_5, via an exponential scaling, $G_4 \propto e^{-kz}G_5$, where k is some constant and z is now the 5^{th} dimension. In this picture G_5 can be on the order of other couplings (electroweak, strong) and G_4 is small because of the exponential suppression due to the metric. To actually arrange this one needs two planes – one with $+\sigma$ and one with $-\sigma$. The details of this can be found in [6].

We now study the motion of a test particle on the domain wall background of (9). The $\mu=0$ component of (12) is

$$\frac{d^2t}{d\tau^2} + \frac{g}{\sqrt{1-2gz}}\frac{dz}{d\tau}\frac{dt}{d\tau} = 0. \tag{19}$$

This has the solution

$$\frac{dt}{d\tau} = (1-2gz)^{1/2}\,. \tag{20}$$

Next, from (12), and using (20) the geodesic equation for the z-dimension is

$$\frac{d^2z}{d\tau^2} = -\frac{1}{2}g - \frac{1}{2}g\,(1-2gz)^{-1}\left(\frac{dz}{d\tau}\right)^2 - g\,(1-2gz)^{1/2}\,\eta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}, \tag{21}$$

The first term on the right hand side of (21) shows that the domain wall metric gives a uniform 4-acceleration of $g/2$ toward the plane. The remaining terms on the right hand side show (as in the case of the "brane" metric) that contrary to Newtonian intuition the 4-acceleration has velocity dependent terms. Switching to time intervals measured with clocks fixed with respect to the plane at $z=0$ via the transformation $d\tau = \sqrt{-g_{00}}dT = (1-2gz)^{-1/4}dT$ we find

$$\frac{d^2z}{dT^2} = -\frac{g}{2}\,(1-2gz)^{-1/2} - g\,(1-2gz)^{1/2}\left(v_x^2+v_y^2\right)\ , \tag{22}$$

where $v_x \equiv \frac{dx}{dT}$ and $v_y \equiv \frac{dy}{dT}$. Therefore the "local" acceleration is not constant, but diverges as one approaches $z = \frac{1}{2g}$.

To more quickly arrive at the acceleration for a given metric one may directly use the definition for the initial, local gravitational acceleration given in [2]

$$g(z) \equiv -\frac{1}{\sqrt{-g_{00}}}\left(\frac{d\sqrt{-g_{00}}}{dz}\right). \tag{23}$$

Applying (23) to the "brane" world (5) and domain wall metrics (9) yields

$$g_{Brane}(z) = -g[\Theta(z) - \Theta(-z)]\ , \qquad g_{Domain-Wall}(z) = -\frac{g}{2(1-2g|z|)}[\Theta(z)-\Theta(-z)]\ . \tag{24}$$

Thus using (23) as our definition we find that it is the brane world metric which gives a uniform acceleration, while the acceleration of the domain wall diverges at $z = \frac{1}{2g}$. This frame dependence of the acceleration, which is not an issue in the case of the Newtonian infinite plane, is of course unavoidable and expected in the general relativistic case.

IV. DISCUSSION OF THE SOURCE TERMS AND ENERGY CONDITIONS

From the previous discussion one finds that the two metrics put forward as general relativisitic versions of the Newtonian infinite plane have energy-momentum tensors ((8) and (11)) which include not only mass-energy density terms (i.e. T_{00}) but have pressure or tension terms (i.e. T_{ii}). In contrast, the matter source of the Newtonian plane is only from a mass-energy density term.

One can give a physical motivation for the appearance of pressures or tensions (i.e. negative pressures) in the static, general relativistic solutions: a matter source with only a plane of mass-energy – $T_{00} \propto \delta(z)$ – is not stable, but will collapse under its gravitational self-attraction. To have a static configuration one must stabilize the mass-energy density by pressures/tensions (and in the case of the "brane" world metric by a cosmological constant). Further, these pressures/tensions play a significant role in the total gravitational field. In [9] it was shown that in order for an observer to remain stationary next to a plane with planar mass-energy density σ and planar tensions τ, they would have to accelerate *away* from the plane if $(\sigma - 2\tau) > 0$ and *toward* the plane if $(\sigma - 2\tau) < 0$. In either case the magnitude of the acceleration would be proportional to $|\sigma - 2\tau|$. In other words, tensions are gravitationally repulsive, while pressures (negative tensions) are gravitationally attractive; positive mass-energy densities are gravitationally attractive, while negative mass-energy densities are gravitationally repulsive.

We can use this conclusion to further understand the results of the previous section for the accelerations of particles that are near the plane. If one is near the plane $z = 0$, the acceleration for a test particle in the "brane" world metric is $\simeq g$ and toward the plane (see (16) (18)), while a test particle in the domain wall metric sees an acceleration of $\simeq g/2$ and also toward the plane (see (21) (22)). For the "brane" world sources in (8) we have $\sigma = -\frac{g}{4\pi G}$ and $\tau = -\frac{g}{4\pi G}$ (since tensions act as negative pressures). Thus for the "brane" world sources $(\sigma - 2\tau) = \frac{g}{4\pi G} > 0$. The domain wall sources have $\sigma = \frac{g}{4\pi G}$ and $\tau = \frac{g}{16\pi G}$ which yields $(\sigma - 2\tau) = \frac{g}{8\pi G} > 0$. Thus both metrics give an attractive acceleration toward $z = 0$, but when one is near the plane the "brane" world acceleration is twice that of th domain wall acceleration. The reason for the acceleration toward the plane is different in the two cases: (i) for the "brane" world source the attraction due to the pressures dominates the repulsion from the negative mass-energy density; (ii) For the domain wall source the attraction from the positive mass-energy density dominates the repulsion coming from the tension.

The unusual source terms for these plane solutions (especially the negative mass-energy density of the "brane" world solution) provide a nice segue for the introduction of energy conditions. Three principal energy conditions are the weak energy condition

$$T_{\mu\nu} V^\mu V^\nu \geq 0 \qquad \rightarrow \qquad \rho \geq 0 \;\; \text{and} \;\; \rho + p_i \geq 0 \; , \tag{25}$$

the strong energy condition

$$\left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right) V^\mu V^\nu \geq 0 \qquad \rightarrow \qquad \rho + p_i \geq 0 \quad \text{and} \quad \rho + \sum_i p_i \geq 0 \; , \tag{26}$$

and the dominant energy condition

$$T_{\mu\nu} V^\mu V^\nu \geq 0 \qquad \text{and} \;\; T_{\mu\nu} V^\nu \quad \text{not} \;\; \text{timelike} \qquad \rightarrow \qquad \rho \geq 0 \quad \text{and} \quad -\rho \geq p_i \geq \rho \; . \tag{27}$$

In the above V^μ is a timelike vector and in the strong energy condition $T = -T_{00} + \sum_i T_{ii}$. The first statement of each condition is given in terms of $T_{\mu\nu}$ while the second statement is given in terms of densities and pressures – $T_{00} \rightarrow \rho$ and $T_{ii} \rightarrow p_i$. The dominant energy condition implies the weak condition but not the strong condition.

The energy-momentum tensor of the domain wall (11) satisfies all three energy conditions. The "brane" world sources (8) violate the dominant and weak conditions yet satisfies the strong condition. Thus one might again prefer the domain wall metric over the "brane" world metric, since it does not violate any of the energy conditions. Nevertheless there are experimentally confirmed cases where all the above energy conditions are violated. The prototypical example is the Casimir effect [20]. Further discussion about energy conditions and their actual or potential violation can be found in [21] [22].

V. SUMMARY AND CONCLUSIONS

In this article we have presented two metrics – (5) and (9) – which can to some extent be considered as general relativistic versions of the Newtonian infinite plane. Each has a planar mass-energy density – $T_{00} \propto \delta(z)$ – and each has (in some frame) an (initially) uniform acceleration toward the plane. For the "brane" world metric this is the frame fixed to the plane (see (18)) while for the domain wall solution it is the proper frame (see (21)). There are many points in which these general relativistic planes differ from the Newtonian plane: (i) both require pressures/tensions in order to have a static solution; (ii) the "brane" world solution in addition requires a cosmological constant; (iii) the acceleration (in any frame) has velocity dependent terms (iv) the acceleration is frame dependent.

Although neither solution is new, both provide good examples for beginning general relativity students in terms of calculations and to illustrate conceptual as well as physical points regarding general relativity. The metrics (5) and (9) are simple enough to provide a doable exercise of calculating the Einstein tensor, $G_{\mu\nu}$. Both metrics provide a relatively straightforward application of the geodesic equations (12). There are additional projects or exercises that one could assign to students that would help to give a deeper understanding of the physical meaning of the solutions: (i) In this paper we have computed accelerations toward the plane $z = 0$ only. One can have students probe the physical meaning of both metrics further by having them calculate the solutions to the geodesic equations (12) in the x or y directions for each metric under the initial conditions $\frac{dx}{d\tau} = 0$ and $\frac{dy}{d\tau} = 0$. They could then be asked to explain the apparent discrepancy between these solutions and the picture emerging from a consideration of (the Newtonian approximation to) tidal forces as given on page 41 of [18],

$$f^i = -mR^i{}_{0k0}\, x^k. \tag{28}$$

Since $R^x{}_{0x0}$ and $R^y{}_{0y0}$ are non-vanishing this clearly gives $f^x = f^y \neq 0$, implying the existence of forces that should make two test particles with initial separations $\Delta x_0 \neq 0$ and/or $\Delta y_0 \neq 0$ approach each other as they fall toward the plane. Such forces are, of course, completely absent in the case of the Newtonian plane. (ii) It is apparent that the domain wall metric (9) has a horizon at $z = \frac{1}{2g}$, but it is less obvious that the brane world metric (5) also possesses a horizon at $z = \pm\infty$ where the metric becomes infinite. Even though the distance from $z = \pm\infty$ to $z = 0$ is infinite the proper time required to cover this interval is finite. The easiest way to see this is to alter the metric (5) by letting $g \rightarrow -g$ so that particles are repelled from the $z = 0$ plane and accelerate toward $z = \infty$. Solving the geodesic equation (15) for the initial conditions $v_y = v_x = 0$ and $z = 0$ at $t = 0$ yields [6]

$$z(T) = \frac{1}{2g} \ln\left(1 + g^2 T^2\right) \tag{29}$$

The proper time for this particle can be determined via

$$d\tau^2 = -e^{-2g|z(T)|} dT^2 + \left(\frac{dz(T)}{dT}\right)^2 dT^2 \tag{30}$$

Using (29) in (30) and integrating gives a proper time of $\tau = \frac{\pi}{2g}$ for the particle to move from $z = 0$ to $z = \infty$. Thus for either an attracting brane $(+g)$ or a repelling brane $(-g)$ the particle will cover the infinite distance in finite proper time. Thus even though the horizons are at infinity they are nevertheless important in considerations of this metric. This also allows one to introduce a similiar feature for the Schwarzschild solution where it takes an infinite time as seen by an outside stationary observer for a particle to fall through the horizon at $r = 2M$, but the proper time, as seen by an obsever who falls with the particle, is finite. (iii) Instead of studying the motion of point particles in these background metrics as was done in section III, one can instead investigate the wave equations of various particles (spin $0, \frac{1}{2}, 1, 2$). For example, one could look at the wave equation of a spin-0 particle for the brane world metric. This is done by modifiying the Klein-Gordon equation for the spin-0 field, Φ, as

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi) + m_0^2 \Phi = 0 \ , \tag{31}$$

where g is the determinant of the metric, and m_0 is the mass of the field Φ. The equation for a spin-2 tensor field is the same as that for the spin-0 field. The 4D wave equations for spin-$\frac{1}{2}$ and spin-1 fields are similiar. A guide or hint for obtaining these wave equations in the brane world background is that they can be found by reducing from the explicit 5D wave equations – for spin $0, 2$, see the second article in [5], for spin $\frac{1}{2}$ see [23], and for spin 1 see [24]. To simplify the problem, one should first try massless fields (i.e. set $m_0 = 0$ in wave equations like (31)). One should also use separation of variables of the fields with the x and y directions being plane waves, e.g., in wave equations such as (31), try $\Phi(t, x, y, z) \propto e^{i(Et - p_x x - p_y y)} \psi(z)$ where E is the energy of the particle and p_x, p_y are the x, y momenta. In this way one obtains a differential equation for $\psi(z)$ and one can investigate if a particular spin field is trapped or confined near the plane at $z = 0$ i.e. does the function $\psi(z)$ fall off as $z \rightarrow \infty$. In the 5D case this gave rise to some unusual and still puzzling results: spin 0 and 2 fields are trapped when one has a repelling brane $(-g)$ [5]; spin $\frac{1}{2}$ fields are trapped by an attracting brane $(+g)$ [23]; spin 1 fields are trapped by neither brane [24]. It would be interesting to see if/how these results from 5D carry over into 4D and if it is possible to reconcile the trapping or non-trapping behavior of the wave fields with the respective particle results from the geodesic equations when the metric given in

(5) is attractive or repulsive (obtained from the former by making the replacement $g \to -g$ in (5)). (iv) In section III we studied the geodesic equations for a massive particle. One could redo the analysis for a massless particle.

With the 4D brane solution (5) in hand one can also begin to discuss (either via class lecture or through assigned exercises) how the 5D versions reproduces effective 4D gravity at low energies and how these solutions are used to address the heirarchy problem. Although the domain wall solutions are not phenomenologically viable (if they existed one would have seen evidence for them in, among other things, the fluctuations of the cosmic microwave background), they nevertheless would provide an introduction to other exotic cosmological solutions, such as cosmic strings or monopoles, which are still possibilities and continue to be experimentally sought.

Conceptually, these examples may be used to show that the acceleration will in general contain velocity terms, which is distinct from the Newtonian case. In addition, one sees how different types of energy-momentum gravitate – positive mass-density and pressures lead to gravitational attraction, while negative mass-density and tensions lead to repulsion. Finally, one can introduce various energy conditions by examining the energy-momentum tensor for each solution.

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