12-2004

Coordinate Conditions for a Uniformly Accelerated or Static Plane Symmetric Metric

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December 13, 2004

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Abstract

The coordinate conditions for three exact solutions for the metric components of a coordinate system with constant acceleration or of a static plane symmetric gravitational field are presented. First, the coordinate condition that the acceleration of light is constant is applied to the field equations to derive the metric of a coordinate system of constant acceleration. Second, the coordinate conditions required to produce the metrics of Rindler and Lass are applied to the field equations to calculate the components of these two metrics and the coordinate velocities and coordinate accelerations for light of these two metrics are compared to the coordinate system of constant acceleration.

I. Introduction

In developing the modern theory of gravitation Einstein[1,2] assumed that constant acceleration is indistinguishable from a uniform gravitational field. This assumption lead to the equivalence principle and ultimately to a geometric theory describing the kinematics of these indistinguishable systems. In this geometric theory the principle mathematical object for the description of particle trajectories is the metric, which defines the components of the 4-space invariant interval. There are at least three compelling reason for the continuing interest in the study of the relation between constant acceleration and a uniform field. First, the accelerated system is the most general form of motion for real physical systems. Second, the accelerated system is the historic basis of the modern theory of gravity. Finally, a good understanding of the accelerated system informs a broader understanding of general relativity as recognized by Misner et al[3] when they observed that "It will be helpful in many applications of gravitation theory".

Due to the central importance of the accelerated system for the understanding of general relativity, studies of the accelerated system are well represented in the physics literature[4,5,6,7].[8] However, there is no general agreement on the exact form of the metric components of an accelerated coordinate system. The reason for the appearance of different forms of the metric for an accelerated coordinate system is that the field equations do not uniquely determine the metric components. The problem of developing sufficient restrictions on the solution to the field equations in order to exactly determine the components of the metric is not peculiar to the static plane symmetric field. It is generally necessary to impose some additional restrictions on the components of the metric separate from the physical constraints. These restrictions are referred to as coordinate conditions and these together with the physical constrains uniquely determine the metric. Three different sets of coordinate conditions will be considered here for the field equations of a static and plane symmetric gravitational field. The first metric will be calculated by imposing the coordinate condition that the coordinate acceleration of light is a constant. This will be compared to the previously calculated metrics of Lass[4] and Rindler[5,6] and the corresponding coordinate conditions for these two metrics.

In order to determine the metric components of a static plane symmetric field or equally an accelerated coordinate system the field equations are solved, resulting in a relation between the metric components but not a unique solution. The solution to the field equations is also compared to the requirement that the metric of an accelerated coordinate system be conformal flat. The solution of the field equations for a static plane symmetric field[6] is found to provide a relation between the metric components that is indistinguishable from the conditions for a conformally flat metric.[7] This equivalence of the solution to the field equations and of a conformally flat metric is a formal demonstration of Einstein's equivalence principle. Finally, by applying appropriate coordinate conditions, the time and 3-space components of the metric are uniquely determined and the coordinate time is found to be the same for all coordinate conditions.

II. Coordinate Conditions and the Metric

The metric for any 4-space has 16 components which are in general independent. Of these 16 components 6 are uniquely determined by the field equations.[9] The complete determination of all 16 components requires the application of both physical restrictions on the metric and on the coordinates of the metric. The restrictions on the coordinates for the metric components are referred to as coordinate conditions. There are a great variety of possible coordinate conditions that can be applied to uniquely determine the metric components. Three will be considered here for the solution of the field equations for a static plane symmetric field. These coordinate conditions are that the coordinate acceleration of light is a constant, the application of the harmonic coordinate conditions, and the requirement that the coordinate 3-space interval is the same as the proper 3-space interval. The application of the later two coordinate conditions result in the Lass and Rindler metrics respectively.

The equivalence principle requires that the metric for an accelerating coordinate system is the same as that of a uniform and constant gravitational field. The metric of a uniform field will be static and unchanged by coordinate transformations in any plane perpendicular to the acceleration. Under these conditions all derivatives in the field equations are zero except parallel to the acceleration and in order to be consistent with a special relativistic momentarily co-moving frame (MCMF) we also require that $\frac{\partial g_{11}}{\partial z} = \frac{\partial g_{22}}{\partial z} = 0$. Under these restrictions the metric may be written, in plane symmetric or

Cartesian coordinates,

$$ds^2 = -c^2 d\tau^2 = g_{00} dt^2 + g_{11} dx^2 + g_{22} dy^2 + g_{33} dz^2, \tag{1}$$

or more conveniently in a system of units where $c = 1$,

$$ds^2 = -V^2 dt^2 + dx^2 + dy^2 + U^2 dz^2. \tag{2}$$

It is noteworthy that in this system of units the acceleration due to gravity at the surface of the earth is $g \simeq 1$.

This metric must as well match to first order $g_{00} \simeq -(1 + 2\mathbf{a} \cdot \mathbf{z})$, where $\mathbf{a}$ is the acceleration from the weak field approximation, $g_{00} \simeq -(1 + 2\phi)$. Here ϕ is the potential of the Newtonian solution for a static plane symmetric field.

III. The Field Equations and the Equivalence Principle

In order to calculate the metric of a plane symmetric field, the field equations must be solved,

$$R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}_{\lambda} \right), \tag{3}$$

where $R_{\mu\nu} = 0$ in a source free region. The RHS of this equation is determined by the mass-energy source terms and the LHS by the 4-space geometry. The components of the Ricci tensor are,

$$R_{\mu\nu} = -\frac{\partial}{\partial x^{\alpha}} \Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\mu\beta} \Gamma^{\beta}_{\nu\alpha} + \frac{\partial}{\partial x^{\nu}} \Gamma^{\alpha}_{\mu\alpha} - \Gamma^{\alpha}_{\mu\nu} \Gamma^{\beta}_{\alpha\beta}, \tag{4}$$

and the Christoffel symbols in this equation are,

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha} \left(\frac{\partial g_{\mu\alpha}}{\partial x^{\nu}} + \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right). \tag{5}$$

Using the requirement that only derivatives with respect to z are non-zero leads to,[6]

$$\Gamma^{0}_{03} = \Gamma^{0}_{30} = \frac{1}{V} \frac{\partial V}{\partial z},\ \Gamma^{3}_{00} = \frac{V}{U^2} \frac{\partial U}{\partial z},\ and,\ \Gamma^{3}_{33} = \frac{1}{U} \frac{\partial U}{\partial z}. \tag{6}$$

All other Christoffel symbols are zero.

In order to solve for the metric components we first calculate the relationship between the R_{00} component of the Ricci tensor and the derivatives of the metric, and substitute for the Christoffel symbols,

$$R_{00} = \frac{V}{U^3} \frac{\partial U}{\partial z} \frac{\partial V}{\partial z} - \frac{V}{U^2} \frac{\partial^2 V}{\partial z^2}. \tag{7}$$

A similar calculation for the relationship between the component of the Ricci tensor and the derivatives of the metric leads to,

$$R_{33} = \frac{1}{V} \frac{\partial^2 V}{\partial z^2} - \frac{1}{VU} \frac{\partial U}{\partial z} \frac{\partial V}{\partial z}. \tag{8}$$

Again all other components are found to be identically zero.

Since the Ricci tensor is zero in source free space we obtain from the expansion of the R_{00} equation,

$$\frac{\partial U}{\partial z} \frac{\partial V}{\partial z} - U \frac{\partial^2 V}{\partial z^2} = 0. \tag{9}$$

The R_{33} equation is identical to the R_{00} equation. The solution to these two equations is,[6]

$$U = \frac{1}{\alpha} \frac{\partial V}{\partial z} \tag{10}$$

where α is a constant. While this provides a relationship between the components of the metric, the differential equation is underdetermined, since both U and V are unknown.

The relationship between the metric components can also be established by requiring that the metric be conformally flat as developed previously by Tilbrook.[7] The restriction of conformal flatness on the metric assumes the existence of a diffeomorfism or coordinate transformation,

$$t' = t'(t, z), \quad x' = x, \quad y' = y, \quad and, \quad z' = z'(t, z), \tag{11}$$

where the prime coordinates are a Minkowski space,

$$ds^2 = dt'^2 + dx^2 + dy^2 + dz'^2, \tag{12}$$

and the unprimed coordinates are the accelerated coordinate system. The assumption of conformal flatness results in the same relationship for the metric components found from the solution to the field equations, which formally demonstrates the equivalence of a uniform plane symmetric gravitational field and rectilinear acceleration.

The solution to the field equations for a plane symmetric field or equally the requirement that the accelerated coordinate system is conformally flat does not uniquely determine the components of the metric for an accelerated coordinate system. To uniquely determine the metric components some ancillary conditions to the physical restrictions on the metric are required. These added restrictions on the metric components are provided by requiring that the coordinate system satisfy some additional constraints. These added constraints are the coordinate conditions for the metric. The components of the metric of the accelerated coordinate system are restricted by the relation between the time and space components $U = \frac{1}{\alpha}\frac{\partial V}{\partial z}$ and are not explicitly time dependent. Any two accelerated coordinate systems that satisfy the relation between the metric components will have a time independent coordinate transformation and the space part of the metric will only be position dependent, $V(z) = V(Z)$. The coordinate time for any accelerated coordinate system is then independent of the coordinate conditions. This is shown by first writing the metric for some accelerated coordinate system,

$$ds^2 = -V(T)^2\, dT^2 + U(Z)^2\, dZ^2. \tag{13}$$

For some second accelerated coordinate system the spacial part of the metric components can be equated $V(z) = V(Z)$, as well as the spacial part of the metric,

$$U(Z)\, dZ = \frac{1}{\alpha}\frac{\partial V(z)}{\partial z}\frac{dz}{dZ}\frac{dZ}{dz}dz, \tag{14}$$

and the metric expressed in terms of the space part of this coordinate system as

$$ds^2 = -V(z)^2\, dT^2 + U(z)^2\, dz^2. \tag{15}$$

It follows from the invariance of the 4-space interval that the coordinate time for all accelerated coordinate systems are the same, $dT^2 = dt^2$, provided that the acceleration and the velocity of the reference frames are the same.

IV. Constant Acceleration of Light

The Einstein field equations do not uniquely determine the components of the metric since both U and V are unknown in the differential equations. An exact solution for the metric components can be obtained by requiring that the metric produce the null line element of light. Assume for example the null interval of light moving in the direction z,

$$0 = -V^2 dt^2 + U^2 dz^2. \tag{16}$$

Now recalling that $U = \frac{1}{\alpha}\frac{\partial V}{\partial z}$ leads to an equation for the metric components,

$$0 = -V^2 dt^2 + \frac{1}{\alpha^2}\left(\frac{\partial V}{\partial z}\right)^2 dz^2. \tag{17}$$

This equation can be rearranged and solved for the velocity,

$$\frac{dz}{dt} = \alpha \left(\frac{\partial \ln (V)}{\partial z}\right)^{-1}. \tag{18}$$

Here we will look for solutions in which the acceleration is a constant. This is equivalent to looking for a coordinate system in which the acceleration with respect to coordinate time t is constant. Differentiating the velocity with respect to time t and setting $\frac{d^2 z}{dt^2} = -a$ we obtain a differential equation, which has the solution,

$$V = e^{-\nu} e^{\sqrt{\mu + 2\frac{\alpha^2}{a} z}}. \tag{19}$$

The constants α, μ, and ν are determined by requiring that the metric components become that of Minkowski coordinates in the limit of zero acceleration and that the metric agree with the weak field limit to first order and we have the solution for the metric,

$$ds^2 = -e^{-2} e^{2\sqrt{1+2az}} dt^2 + dx^2 + dy^2 + \frac{e^{-2} e^{2\sqrt{1+2az}}}{1+2az} dz^2. \tag{20}$$

The velocity of light in the accelerated coordinate system can be calculated by setting the 4-space interval to zero , $\frac{dz}{dt} = \sqrt{1+2az}$, and the coordinate velocity of light is found to be position dependent in the accelerated coordinate system.

V. Lass Metric

Another unique solution can be obtained by choosing a coordinate system that satisfies the harmonic coordinate conditions (HCC). The restrictions on the metric components required to satisfy the HCC can be written in the form,[9]

$$\frac{\partial \sqrt{g} g^{\lambda\alpha}}{\partial x^{\alpha}} = 0. \tag{21}$$

where $g = -Det\,(g_{\mu\nu})$. The metric components are functions of z only and the metric is also diagonal which insures that the only nonzero terms are $\lambda = 3$ and also noting that for a diagonal metric $g^{33} = -g_{33}$,

$$\frac{\partial}{\partial \xi}\left(\frac{\sqrt{g}}{g_{33}}\right) = 0. \tag{22}$$

This expression can be integrated once, with C a constant, solving the differential equation,

$$V\,(\xi) = e^{C\xi}. \tag{23}$$

Matching the weak field limit $C = a$ and noting that locally $g_{33}\,(\xi = 0) = 1$ requiring $\alpha = a$ uniquely determines the metric,

$$ds^2 = -e^{2a\xi}dt^2 + dx^2 + dy^2 + e^{2a\xi}d\xi^2. \tag{24}$$

Note that for light $\frac{d\xi}{dt} = 1$ and $\frac{d^2\xi}{dt^2} = 0$. This is the same property as a photon in a special relativistic coordinate system moving with a constant velocity. The assumption of the HCC leads to a zero coordinate acceleration for light which is consistent with a momentarily co-moving frame (MCMF). The coordinate transformation from the MCMF to the accelerated coordinate system is only a function of the spatial coordinates,

$$\xi = \frac{-1 + \sqrt{1 + 2az}}{a}. \tag{25}$$

VI. Rindler Metric

The final coordinate condition that will be considered is the requirement that the coordinate 3-space interval is equal to the proper 3-space interval.[357] This coordinate condition results in a unique solution to the field equation and the Rindler metric. Recall that $U = \frac{1}{\alpha}\frac{\partial V}{\partial Z}$ and to insure that the coordinate condition is satisfied set $U = 1$. Consistent with the weak field limit the metric is

$$ds^2 = -\left(1 + aZ\right)^2 dt^2 + dx^2 + dy^2 + dZ^2. \tag{26}$$

The coordinate transformation from the Rindler coordinates to MCMF of the Lass coordinates is

$$\xi = \frac{\ln\left(1 + aZ\right)}{a}. \tag{27}$$

In the Rindler metric the coordinate velocity of light is $\frac{dZ}{dt} = 1 + aZ$ and the coordinate acceleration of light is $\frac{d^2Z}{dt^2} = a + a^2 Z$. Both the coordinate velocity and coordinate acceleration are position dependent in the Rindler coordinate system.

VII. Conclusion

The field equations for the static plane symmetric field were solved and provide a restriction on the components of the metric that is identical to the restriction of a conformally flat metric. In order to uniquely determine the components of the metric the appropriate coordinate conditions restricting the metric to the accelerated coordinates, the Lass coordinates, and the Rindler coordinates were applied to the field equations in order to uniquely determine the components of the metric. While the three coordinate conditions resulted in different 3-space intervals the coordinate time was found to be the same for all three metrics and in general the same for all accelerated coordinate systems with the same velocity and acceleration.

References

[1]Albert Einstein, *The Meaning of Relativity*, Princeton University Press, 1945.

[2]Albert Einstein, H. A. Lorentz, H. Weyl, H. Minkowski, *The Principle of Relativity*, Dover Publications, 1952.

[3]Misner, C. W., Thorne, K. S., and Wheeler, J. A, *Gravitation*, Freeman, San Francisco, pp. 163-174, 1973.

[4]Harry Lass, *Accelerating Frames of Reference and the Clock Paradox*, American Journal of Physics, Vol. 31, pp. 274-276, 1963.

[5]W. Rindler, *Kruskal Space and Uniformly Accelerated Frame*, American Journal of Physics, Vol. 34, pp. 1174-1178, 1966.

[6]F. Rohrlich, *The Principle of Equivalence*, Annals of Physics, Vol. 22, pp. 169-191, 1963.

[7]David Tilbrook, *General Coordinatisations of the Flat Space-time of Constant Proper-acceleration*, Australian Journal of Physics, Vol. 50, pp. 851-868, 1997.

[8]Richard C. Tolman, *Relativity, Thermodynamics, & Cosmology*, Oxford at the Claredon Press, pp. 193-197, 1934.

[9]Steven Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, 1972.