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The Dyad Theory of Classical and Quantum Physics

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Abstract

The 4-space equations of hydrodynamics, electrodynamics, and quantum mechanics are developed using the dyad calculus. Gravity is found to couple hydrodynamics, electrodynamics, and quantum mechanics through the 4-space metric and differential operator. Expanding the 4-space equations of electrodynamics a time varying metric is shown to dissipate electromagnetic energy.

I. Introduction

The theory of general relativity was developed by Einstein[1] using the component tensor calculus consistent with 4-space Riemann geometry. In this theory gravity is a geometric property of 4-space and the equations of physics are written as form invariant relations correct for all reference frame. While the component tensor calculus is the preferred method for the determination of the structure of 4-space geometry, where a coordinate frame is always assumed, this formalism does not produce the best representation of the equations describing a physical system. Two familiar alternatives to the component tensor calculus are the exterior calculus[2] and differential forms.[3] The principle advantage of these two methods over the component tensor calculus is the clear association between the physical constituents of the system and the 4-space objects of the mathematical formalism. A third method which preserve this association between the physical constituents of the system and the 4-space objects of the mathematical formalism was previously developed using dyads by Luehr and Rosenbaum.[4] Unique among the various alternatives the 4-space dyad calculus of Luehr and Rosenbaum is formally very similar to 3-space vector calculus and offers the most natural method for the development of form invariant equations from the familiar equations of physics written in an inertial frame.

The representation of 4-space physical relations in terms of dyads also produces an unambiguous demonstration of the connection between gravity, electrodynamics, and quantization. The phenomena of gravity is seen to be completely a property of 4-space geometry. Gravity is found to act in the equations of hydrodynamics, electrodynamics, and quantum mechanics through the 4-space metric and differential operator. This 4-space geometry is the geometry of Riemann and is indistinguishable from the 4-space geometry of general relativity. In term of the dyad calculus the relation between the metric and differential operator and a physical system is found in the consistency of the 4-space equations of the system with the Riemann geometry of 4-space and the differential operator and the metric of this 4-space. Unlike general relativity the expression of gravity through the metric and differential operator does not require the supposition of a gravitational field equation in the formalism of the dyad calculus. While it might be possible to write a gravitational field equation in terms of the dyad calculus the form invariant equations of hydrodynamics, electrodynamics, and quantum theory are independent of a separate equation for gravity.

Since the dyad calculus is less familiar than the tensor calculus this formalism will be presented here in a form suitable for the representation of 4-space physical relations. Having developed the dyad calculus the equations of hydrodynamics, electrodynamics, and quantum mechanics are written in terms of dyads. Since the 4-vectors in the dyads are independent of the local 4-space geometry it is sufficient to demonstrate that these equations are correct in an inertial frame to insure that the equations are also correct in all reference frames. Having developed the form invariant equations of electrodynamics and establishing the connection with gravity the equations of electrodynamics are expanded, assuming a metric that is time dependent, demonstrating that a time varying metric will dissipate electromagnetic energy.

II. 4-vectors and Dyad Calculus

Using the dyad calculus the equations describing a physical system are always form invariant in that the equations are identical in all reference frames. This form invariance insures that if the equations describing a system can be found under the conditions of any specific reference frame the equations will be the same for every other reference frame. In a Lorentz inertial frame (LIF) the physical equation can generally be represented as 3-space vector relations. By transforming these 3-space vector relation into the equivalent 4-space dyad relations the resulting equations are now correct in all systems and in particular where the local curvature of space might be very unlike a LIF.

The connection between the 3-space vector relations of the LIF and the form invariant 4-space relations of dyad calculus is achieved by retaining the base vectors, which leads to a natural connection between 3-space and 4-space mathematical objects and operators. In the dyad calculus geometric objects are independent of the reference frame $A^{\alpha}\mathbf{e}_{\alpha} = A^{\alpha'}\mathbf{e}_{\alpha'}$ and covariant and contravariant expressions are symmetric,

$$\mathbf{A} \equiv A^i \mathbf{e}_i = A_i \mathbf{e}^i; i = 1, 2, 3, \tag{1}$$

$$\underline{\mathbf{A}} \equiv A^\alpha \mathbf{e}_\alpha = A_\alpha \mathbf{e}^\alpha; \alpha = 0, 1, 2, 3. \tag{2}$$

The 3-space objects are represented in bold or with Latin indices and the 4-space objects are represented in bold and underlined or indicated by Greek indices. Repeated indices imply a summation over the range of the indices.

There is a close relation between operations of the dyad calculus and the vector calculus. This is achieved by the construction of second rank objects from 4-vectors using the definition of the direct product. This direct product forms a new object the dyad that does not exist in vector calculus,

$$\underline{\mathbf{AB}} \equiv \underline{\mathbf{A}} \otimes \underline{\mathbf{B}}. \tag{3}$$

This new object is the juxtaposition of two vectors and is sometimes referred to as a second rank object. The order of the vectors is important to the definition of the object and in the differential operations on the object.

The dyad calculus presented here closely follows Luehr and Rosenbaum.[4] The objects of the dyad calculus are all 4-vectors and the direct products of 4-vectors. This permits the formal preservation of the contributions from finite derivatives of the base vectors $\mathbf{e}_\alpha$,

$$\frac{\partial}{\partial x^\alpha} \mathbf{e}_\beta \equiv \mathbf{e}_\gamma \Gamma^\gamma_{\beta\alpha}, \tag{4}$$

where the coefficients $\Gamma^\gamma_{\beta\alpha}$ are the Christoffel symbols. This definition helps to provides an association of the dyad calculus with the component tensor calculus.

The definition of the inner product or dot product is very similar to the vector calculus operation $\underline{\mathbf{A}} \cdot \underline{\mathbf{B}} \equiv A^\alpha B^\beta g_{\alpha\beta} = A_\alpha B_\beta g^{\alpha\beta}$. The most significant difference is the negative signature of the metric in 4-space, which in a Lorentz inertial frame (LIF) can be expressed as $\mathbf{e}_0 \cdot \mathbf{e}_0 = \mathbf{e}^0 \cdot \mathbf{e}^0 = -\mathbf{e}_0 \cdot \mathbf{e}^0 = -1$. The 0 indices represent the time part. In this context the inner products of the remaining base vectors form the Kronecker delta, $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j$.

Differential operations are introduced by defining a 4-space differential operator $\underline{\boldsymbol{\nabla}} \equiv \frac{\partial}{\partial x^\delta}\mathbf{e}_\delta = \frac{\partial}{\partial x_\delta}\mathbf{e}^\delta$, where in a LIF for example $x^0 = -x_0 = -ct$ and $\underline{\boldsymbol{\nabla}} = -\frac{1}{c}\mathbf{e}_0\frac{\partial}{\partial t} + \boldsymbol{\nabla} = \frac{1}{c}\mathbf{e}^0\frac{\partial}{\partial t} + \boldsymbol{\nabla}$. The divergence of a 4-vector is defined as a natural extension of the vector calculus expansion $\underline{\boldsymbol{\nabla}} \cdot \underline{\mathbf{A}} \equiv \frac{\partial}{\partial x_\delta} g_{\delta\alpha} A^\alpha = \frac{\partial}{\partial x^\delta} g^{\delta\alpha} A_\alpha$. The divergence of the direct product of two vectors is defined in a similar fashion,

$$\underline{\boldsymbol{\nabla}} \cdot (\underline{\mathbf{A}\mathbf{B}}) \equiv (\underline{\mathbf{A}} \cdot \underline{\boldsymbol{\nabla}})\underline{\mathbf{B}} + \underline{\mathbf{B}}(\underline{\boldsymbol{\nabla}} \cdot \underline{\mathbf{A}}). \tag{5}$$

Unlike the dot product the cross product of vector calculus does not have such a close 4-space analogue. To develop a 4-space analogue to the cross product two further definitions, the wedge product and the dual, are required. The wedge product is the anti-symmetrization of the direct product of two vectors,

$$\underline{\mathbf{A}} \wedge \underline{\mathbf{B}} \equiv \underline{\mathbf{A}} \otimes \underline{\mathbf{B}} - \underline{\mathbf{B}} \otimes \underline{\mathbf{A}}. \tag{6}$$

The dual maps the 4-space object into the "dual" of the object,

$$dual(\underline{\mathbf{AB}}) \equiv \frac{1}{2}e^{\alpha\beta\gamma\delta}A_\alpha B_\beta \mathbf{e}_\gamma \mathbf{e}_\delta = \frac{1}{2}e_{\alpha\beta\gamma\delta}A^\alpha B^\beta \mathbf{e}^\gamma \mathbf{e}^\delta. \tag{7}$$

The Levi-Civita tensor $e^{\alpha\beta\gamma\delta}$ is defined in terms of the permutations symbol $E^{\alpha\beta\gamma\delta}$ which is zero if any indices are repeated, one for even permutations of 0,1,2,3 and negative one for odd permutations, and $g \equiv \det(\mathbf{e}_\alpha \cdot \mathbf{e}_\beta)$ the determinant of the metric components,

$$e^{\alpha\beta\gamma\delta} \equiv -\frac{1}{\sqrt{-g}}E^{\alpha\beta\gamma\delta} = \sqrt{-g}E_{\alpha\beta\gamma\delta} = e_{\alpha\beta\gamma\delta}. \tag{8}$$

III. Hydrodynamics

The invariant equations of motion for a perfect fluid are found by first determining the correct equations for a LIF and applying the form invariance of the dyad relations to include all reference frames. The resulting equations written in terms of 4-space objects are therefore the same for all reference frames. Here it will be assumed that the fluid element in a LIF has 4-velocity $\underline{\mathbf{u}} = \gamma c \mathbf{e}_0 + \gamma u^c \mathbf{e}_c$ and 3-velocity $\mathbf{u}$, small compared to the speed of light $u \prec\prec c$. The 4-momentum of the perfect fluid is written as

$$\Omega = \gamma\left(\frac{P}{c} + \sigma c\right)\mathbf{e}_0 + \gamma\sigma u^a \mathbf{e}_a \tag{9}$$

where σ is the local density in the LIF and P is the pressure. Were the particles of the fluid all stationary relative to one another the fluid will have zero temperature and zero pressure. The 4-momentum for this condition will be defined as the zero temperature 4-momentum and written as

$$\underline{\mathbf{p}} = \gamma\sigma c\mathbf{e}_0 + \gamma\sigma u^b\mathbf{e}_b. \tag{10}$$

With the assumption of speeds small compared to light the derivative of $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ is $\frac{\partial\gamma}{\partial x^\alpha} = \frac{u}{c^2}\gamma^3\frac{\partial u}{\partial x^\alpha}$ and $u \prec\prec c \rightarrow \gamma \simeq 1$ which leads to $\frac{\partial\gamma}{\partial x^\alpha} = 0$. It is only necessary to show that the continuity and Euler equations for a perfect fluid in a LIF written as the direct products of 4-vectors are

$$\left(\underline{\boldsymbol{\nabla}} \cdot \underline{\mathbf{u}}\right)\Omega = \underline{\boldsymbol{\nabla}}\left(\underline{\mathbf{u}} \cdot \Omega\right) - \underline{\boldsymbol{\nabla}}\left(\underline{\mathbf{u}} \cdot \underline{\mathbf{p}}\right), \tag{11}$$

and the form invariance of the equations insures that this is correct in general. Expanding the space part,

$$\frac{\partial}{\partial t}\sigma u_d + \frac{\partial}{\partial x^a}u^a\sigma u_d = -\frac{\partial}{\partial x^d}P. \tag{12}$$

Expanding the time part and noting that $u^d \prec\prec c$,

$$\frac{\partial}{\partial t}\sigma + \frac{\partial}{\partial x^d}u^d\sigma = 0. \tag{13}$$

These are the same as the Euler and continuity equations for a perfect fluid in a LIF.

Due to the form invariance of the dyads the 4-space equations for hydrodynamics are the same for all reference frames and coordinate systems and in particular for any 4-space metric. For a given metric these equations can be expanded in component form,

$$\frac{\partial}{\partial x^{\delta}} u^{\delta}\Omega^{\eta} = \frac{\partial}{\partial x^{\delta}} g^{\delta\eta} u^{\alpha}\Omega_{\alpha} - \frac{\partial}{\partial x^{\delta}} g^{\delta\eta} u^{\alpha} p_{\alpha}. \tag{14}$$

The solution to the equations must be consistent for the 4-space metric and the mass-energy distribution of the system. Gravity as a purely geometric property of 4-space is then related to the mass-energy distribution of the system by associating a geometric object the Einstein tensor,

$$\frac{1}{8\pi} G^{\delta\eta} = u^{\delta}\Omega^{\eta} + g^{\delta\eta}\left(u^{\alpha} p_{\alpha} - u^{\alpha}\Omega_{\alpha}\right), \tag{15}$$

with the mass-energy distribution,

$$\frac{\partial}{\partial x^{\delta}} G^{\delta\eta} = 0. \tag{16}$$

This equation for the Einstein tensor and 4-space geometry provides a relation between the mass-energy distribution and 4-space geometry,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} R_{\alpha\beta}, \tag{17}$$

where,

$$G^{\delta\eta} = g^{\delta\mu} g^{\eta\nu} G_{\mu\nu}. \tag{18}$$

The components of the Ricci tensor in this equation are

$$R_{\mu\nu} = -\frac{\partial}{\partial x^\alpha}\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha} + \frac{\partial}{\partial x^\nu}\Gamma^\alpha_{\mu\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} \tag{19}$$

and the Christoffel symbols in this equation are

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\alpha}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right). \tag{20}$$

By developing a solution to the 4-space equations the geometry is determined by the mass-energy distribution and gravity is completely a consequence of this geometry.

Having written the equations of hydrodynamic in the form of the dyad calculus gravity is found to be a purely geometric phenomena and acts in the equations of hydrodynamics through the metric and the differential operator. As a property of 4-space geometry the action of gravity through the metric and the differential operator must be the same for all physical relations. In particular the relation of gravity to electrodynamics and quantization will be the same action through the metric and differential operator and will provide a connection between the equations of hydrodynamic, electrodynamics, and quantization.

IV. Electrodynamics

The dyad calculus as it is developed here makes it possible to write a form invariant expression of the Maxwell equations in terms of the direct products of 4-vector electric $\underline{\mathbf{E}}$ and magnetic $\underline{\mathbf{B}}$ fields with the 4-velocity of the observer $\underline{\mathbf{u}}$. The Maxwell equation as developed here were written in a similar form as direct products between the 4-velocity and 4-space fields by Ellis.[5] While Ellis used the component methods the 4-space objects are the same.

The inhomogeneous equations of electrodynamics are written in terms of the wedge product of the 4-velocity $\underline{\mathbf{u}}$ and the 4-vector $\underline{\mathbf{E}}$ field and the dual of this wedge product with the 4-vector $\underline{\mathbf{B}}$ field.

$$\underline{\boldsymbol{\nabla}} \cdot (\underline{\mathbf{u}} \wedge \underline{\mathbf{E}}) + \underline{\boldsymbol{\nabla}} \cdot dual(\underline{\mathbf{u}} \wedge \underline{\mathbf{B}}) = -\frac{4\pi}{c}\underline{\mathbf{J}}. \tag{21}$$

The homogeneous equations are written as the wedge product with the 4-vector $\underline{\mathbf{B}}$ field and the dual of the wedge product with the 4-vector $\underline{\mathbf{E}}$ field.

$$\underline{\boldsymbol{\nabla}} \cdot (\underline{\mathbf{v}} \wedge \underline{\mathbf{B}}) = \underline{\boldsymbol{\nabla}} \cdot dual(\underline{\mathbf{v}} \wedge \underline{\mathbf{E}}). \tag{22}$$

In this form the physical content of the equations is independent of the observer and the reference frame and depends only on the source terms $\underline{\mathbf{J}}$.

Since these equations are form invariant it will suffice to show that the equations are correct for a LIF to demonstrate that the equations are correct for all reference frames. For a LIF the 3-velocity is zero, $\mathbf{u}$ =0 and $\underline{\mathbf{u}} = c\mathbf{e}_0 = -c\mathbf{e}^0$. Here c is the speed of light in a suitable system of units. With this definition the source terms are $\underline{\mathbf{J}} \equiv \rho c\mathbf{e}_0 + \mathbf{J} = -\rho c\mathbf{e}^0 + \mathbf{J}$. The fields are similarly expressed as a sum of a time and space part. Substitution of these expressions into the general form of the Maxwell equations and collecting space and time terms reduces to the expected form of the equations for an observer at rest in a LIF. The connection between 4-space and 3-space is facilitated by recognizing the relation between the Levi-Civita tensors, $e^{0123} = -e^{123}$ assuming constant base vectors. In a LIF this relation leads to an expression for the negative of the curl in terms of the dual,

$$\underline{\boldsymbol{\nabla}} \cdot dual(\mathbf{e}_0 \wedge \mathbf{A}) = -\boldsymbol{\nabla} \times \mathbf{A}. \tag{23}$$

Expanding the inhomogeneous equations in a LIF,

$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - (\nabla \cdot \mathbf{E})\,\mathbf{e}_0 - \nabla \times \mathbf{B} = -\frac{4\pi}{c}\left(\mathbf{J} + \rho c \mathbf{e}_0\right). \tag{24}$$

Expanding the homogeneous equations in a LIF,

$$(\nabla \cdot \mathbf{B})\,\mathbf{e}_0 - \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}. \tag{25}$$

These equations are correct for a LIF and the form invariance of the dyads insures that these equations are correct for all reference frames.

V. Quantum Mechanics

The action of gravity through the metric and differential operator provides a natural connection between gravity and quantization. The differential operator in the equations of quantization is the same as that of hydrodynamics and electrodynamics. Gravity is then connected to quantization in the same fashion as with hydrodynamics and electrodynamics. The equation for quantization have been previously developed in a form completely consistent with the dyad calculus.[6] The continuity equation for quantum states is written as

$$\underline{\boldsymbol{\nabla}} \cdot \underline{\mathbf{j}} = 0 \tag{26}$$

The 4-current vector for bosons as in the Klein-Gordon equation can be expanded as

$$\underline{\mathbf{j}} = \mathbf{j} + \rho \mathbf{e}^0 \tag{27}$$

Where the 3-current in a system of units with $\hbar = c = 1$ is

$$\mathbf{j} = \frac{1}{2im} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right) \tag{28}$$

and the density

$$\rho = \frac{i}{2m} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right). \tag{29}$$

The 4-current vector for fermions as in the Dirac equation can be expanded in terms of the Dirac matrices $\boldsymbol{\alpha}$ as

$$\underline{\mathbf{j}} = \psi^\dagger \boldsymbol{\alpha} \psi + \psi^\dagger \psi \mathbf{e}^0, \tag{30}$$

where ψ is now a spinor.

VI. Gravity and Electromagnetic Radiation

Having established the relation between gravity and electrodynamics it is possible to examine how these phenomena are related. In particular it is possible to determine the affect of gravity on electricity and magnetism in the action of gravity through the metric and differential operator in the equations of electrodynamics. As a practical example of this connection the affect of a time varying metric will be considered. Assuming weak but time dependent gravity in some region of space, the metric is written as $g_{00} = -(1 + h(t))$, where $h \prec\prec 1$, $g_{ii} \simeq 1$ and $g_{ij} = 0$ if $i \neq j$. Also assuming $\underline{\mathbf{u}} = c\mathbf{e}_0 = -c\mathbf{e}^0$, and $\underline{\mathbf{J}} = 0$ the inhomogeneous equations for electrodynamics can be expanded as

$$-c\mathbf{E} \cdot (\nabla \times \mathbf{B}) + \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = -E^2 \frac{\partial h}{\partial t} \tag{31}$$

and the homogeneous equations are similarly expanded as

$$c\mathbf{B} \cdot (\nabla \times \mathbf{E}) + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = -B^2 \frac{\partial h}{\partial t}. \tag{32}$$

Adding the equations and dividing by $\frac{1}{4\pi}$,

$$\frac{c}{4\pi} \nabla \cdot (\mathbf{E} \times \mathbf{B}) = -\frac{1}{4\pi} \left(B^2 + E^2\right) \frac{\partial h}{\partial t} - \frac{1}{8\pi} \frac{\partial}{\partial t} \left(B^2 + E^2\right) \tag{33}$$

Defining the internal energy as $U = \frac{1}{8\pi}\left(E^2 + B^2\right)$,

$$\frac{c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{B})\simeq-\frac{\partial}{\partial t}\left(1+2h\right)U. \tag{34}$$

Demonstrating that a time varying metric will dissipate electromagnetic radiation.

VII. Conclusion

Using the dyad calculus the association of gravity with hydrodynamics, electrodynamics, and quantization has been established through the metric and 4-space differential operator. Gravity is found to be completely geometric and formally represented in the dyad calculus by 4-space Riemann geometry. Having established the connection between gravity and electrodynamics a time varying metric is shown to dissipate electromagnetic energy.

VIII. Acknowledgment

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References

[1]Albert Einstein, *The Meaning of Relativity*, Princeton University Press, 1945.

[2]Charles W. Misner, Kips S. Thorne, John Archibald Wheeler, *Gravitation*, New York: W. H. Freeman, 1973, pp. 141.

[3]Bernard F. Schutz, *Geometrical methods of mathematical physics*, Cambridge University Press, 1980.

[4]C. P. Luehr and M. Rosenbaum, *Intrinsic Vector and Tensor Techniques in Minkowski Space with Application to Special Relativity*, Vol. 9, Journal of Mathematical Physics, February 1968, pp. 284-298.

[5]G. F. R. Ellis, *Relativistic Cosmology*, Cargese Lectures in Physics, edited by Evry Schatzman, 1973, pp. 1-60.

[6]Hans A. Bethe and Roman Jackiw, *Intermediate Quantum Mechanics*, Addison-Wesley Longman, 1997, pp. 341-366.