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A Dyad Theory of Hydrodynamics and Electrodynamics

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Abstract

The dyadic calculus is developed in a form suitable for the description of physical relations in curved space. The 4-space equations of hydrodynamics and electrodynamics are constructed using this dyadic calculus. As a demonstration of the relationship between gravity and electrodynamics a time varying metric is shown to generate electromagnetic radiation.

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1 Introduction

In the early part of the last century the theory of general relativity was developed by Einstein [1] using the component tensor calculus, consistent with 4-space Riemann geometry. Today, this component tensor calculus remains the most widely used method for the representation of physics in 4-space. However, the component tensor calculus is not completely suitable for the rigorous development of the mathematical formalism of general relativity and the associated 4-space Riemann geometry. To place general relativity and Riemann geometry on a rigorous mathematical foundation recourse is made to the theory of differential forms [2] [3] [4]. While differential forms can provide a rigorous foundation for the theory of general relativity this formalism is cumbersome or even useless in practical applications. The result is, that in practice, both the component tensor calculus and differential forms must be employed together in the theory of general relativity.

The goal here is to provide a bridge between the mathematically rigorous differential forms and more practically useful component tensor calculus. This will be achieved through the development of a middle ground in the formalism of the dyadic calculus. The dyadic calculus has been discussed by Goldstein [5] and used to represent the tensor as a second rank object in Euclidean 3-space. The formalism of the dyadic calculus was extended by Luehr and Rosenbaum [6] to provide a description of electrodynamics in Minkowski space using the traditional 3-vector electric and magnetic fields. However, this restriction to Minkowski space and 3-vector fields is only correct for inertial observers in flat space. In order to make the dyadic calculus a more useful tool the formalism must be further extended to describe 4-vector relations in curved 4-space.

One of the greatest advantages of the dyadic calculus over the component tensor calculus is the heuristic construction of the correct equations describing a physical system. Using the dyadic calculus the equations are always form invariant. That is the equations written in terms of 4-vectors and their direct products are identical in all reference frames. This form invariance insures that if the equations describing a system can be found under the conditions of any specific reference frame the equations will be the same for every reference frame. In an inertial frame and flat space the physical relations can generally be written as differential operations on 3-space objects. By transforming these 3-space relations into the equivalent 4-space relations the resulting equations are then correct in all reference frames and

in particular where the curvature of space might be very unlike flat space.

After developing the formal operations of the dyadic calculus, consistent with general relativity and curved 4-space, this formalism will be used to construct the hydrodynamic equations. These hydrodynamic equations, in curved space, are already well represented in the literature and written in the component tensor calculus [1] [2] [3] [4] [7]. These equations are found using the dyadic calculus as a demonstration of the heuristic construction of a physical theory in the formalism of the dyadic calculus. The electrodynamic equations are also found using this same heuristic construction. While the electrodynamic equations are also well represented in the literature [2] [7] the resulting equations, using the dyadic calculus, include an explicit representation of the electric and magnetic fields. This explicit dependence on the fields is less well known and is consistent with the electrodynamics of Ellis [8] [9].

The value of the dyadic calculus is in part aesthetic. Physical theories written in the dyadic calculus are highly symmetric. The dyadic calculus shares all the symmetries of the component tensor calculus and differential forms and is also symmetric with respect to covariant or contravariant coordinate transformations. Unlike the component tensor calculus, the dyadic calculus is completely symmetric with respect to flat space in the absence of gravity and curved space in the presence of gravity. This final symmetry leads to the expectation of a relationship between gravity and the electric and magnetic fields in the equations of electrodynamics. This relationship is demonstrated by considering the effect of a time varying metric on the electric and magnetic fields in the equations of electrodynamics. The time varying metric is shown to generate electromagnetic radiation. This gravity induced electromagnetic radiation has not been previously predicted.

2 Dyadic calculus

The dyadic calculus, as presented here, is an extension of the “intrinsic tensor techniques” of Luehr and Rosenbaum [6]. The “intrinsic tensor techniques” provide an intuitive representation of electrodynamics by expanding the differential operations of the vector calculus in Euclidean space to Minkowski space. While the differential operations of the dyadic calculus are consistent with the component tensor calculus, the dyadic calculus is more restrictive in requiring the representation of all physical relations in terms of 4-vectors.

It is this restriction to 4-vectors that retains the intuitive advantages of the “intrinsic tensor techniques” while also representing these relations in the curved space of general relativity.

In the 3-space vector calculus geometric objects are represented as the product of direction unit vectors, in a convenient coordinate system, and the components as the magnitude of the object in that direction,

$$\mathbf{A} \equiv A^i \mathbf{e}_i = A_i \mathbf{e}^i; i = 1, 2, 3. \quad (1)$$

The 3-vectors will be written in bold or with Latin indices. These vectors are invariant in Euclidean space and a Galilean time transformation. However, the Galilean time transformation is only an approximation of the physically correct Lorentz time transformation. In order to construct geometric objects that are invariant under a Lorentz time transformation a fourth temporal direction must be included with the 3-space unit vectors,

$$\underline{\mathbf{A}} \equiv A^\alpha \mathbf{e}_\alpha = A_\alpha \mathbf{e}^\alpha; \alpha = 0, 1, 2, 3. \quad (2)$$

The 4-space objects will be represented in bold and underlined or indicated by Greek indices. These 4-vectors are invariant in Minkowski space, $A^\alpha \mathbf{e}_\alpha = A^{\alpha'} \mathbf{e}_{\alpha'}$, where the primed and the unprimed represent the components and basis vectors in two different Lorentz inertial frames (LIF). The LIF is assumed to not be rotating or accelerating and to be far enough away from any gravitational sources that the space can be considered flat. More importantly, for the present purpose, is that these 4-space objects are form invariant and represent the same physical phenomena in flat space of special relativity and in the curved space of general relativity.

In retaining the basis vectors there is a close relation between differential operations in the dyadic calculus and the vector calculus. This is achieved by the construction of second rank objects from 4-vectors using the definition of the direct product. This direct product produces a new object the dyad that does not exist in vector calculus,

$$\underline{\mathbf{AB}} \equiv \underline{\mathbf{A}} \otimes \underline{\mathbf{B}}. \quad (3)$$

This new object is the juxtaposition of two 4-vectors and is sometimes referred to as a second rank object. The order of the 4-vectors is important to the definition of the object and in the differential operations on the object. The objects of the dyadic calculus are all 4-vectors and the direct products

of 4-vectors and this preserves the contributions from derivatives of the base vectors \mathbf{e}_β [10],

$$\mathbf{e}_\gamma \Gamma_{\beta\alpha}^\gamma \equiv \frac{\partial}{\partial x^\alpha} \mathbf{e}_\beta, \quad (4)$$

where the $\Gamma_{\beta\alpha}^\gamma$ are the connection coefficients. The definition of the derivatives of the basis vectors provides a formal connection between the differential operations of the dyadic calculus and the covariant derivative of the component tensor calculus.

The scalar product in 4-space is very similar to the vector calculus operation $\underline{\mathbf{A}} \cdot \underline{\mathbf{B}} \equiv A^\alpha B^\beta g_{\alpha\beta} = A_\alpha B_\beta g^{\alpha\beta}$. The most significant difference is the negative signature of the metric in 4-space. This requires that a sign convention be adopted for either a negative space interval or negative time interval. Here the time interval is taken to be negative and in a LIF $g_{00} = -1$. The metric components are the scalar products of the basis vectors and the scalar product of the time basis, in a LIF, can be expressed as $g_{00} = \mathbf{e}_0 \cdot \mathbf{e}_0 = \mathbf{e}^0 \cdot \mathbf{e}^0 = -\mathbf{e}_0 \cdot \mathbf{e}^0$. In this context the scalar products of the remaining base vectors form the Kronecker delta, $\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j$. Differential operations are introduced by defining a 4-space differential operator $\underline{\nabla} \equiv \mathbf{e}^\delta \frac{\partial}{\partial x^\delta} = \mathbf{e}_\delta \frac{\partial}{\partial x_\delta}$, where in a LIF for example $x^0 = -x_0 = -ct$ and $\underline{\nabla} = -\frac{1}{c} \mathbf{e}_0 \frac{\partial}{\partial t} + \nabla = \frac{1}{c} \mathbf{e}^0 \frac{\partial}{\partial t} + \nabla$. The divergence of a 4-vector in curvilinear coordinates is defined as a natural extension of the vector calculus expansion,

$$\underline{\nabla} \cdot \underline{\mathbf{A}} \equiv \mathbf{e}_\delta \cdot \frac{\partial}{\partial x_\delta} A_\alpha \mathbf{e}^\alpha = \mathbf{e}^\delta \cdot \frac{\partial}{\partial x^\delta} A^\alpha \mathbf{e}_\alpha. \quad (5)$$

The divergence of the direct product of two vectors is defined in a similar fashion,

$$\underline{\nabla} \cdot (\underline{\mathbf{A}}\underline{\mathbf{B}}) \equiv (\underline{\mathbf{A}} \cdot \underline{\nabla})\underline{\mathbf{B}} + \underline{\mathbf{B}}(\underline{\nabla} \cdot \underline{\mathbf{A}}). \quad (6)$$

Unlike the scalar product, the vector product of vector calculus does not have an obvious 4-space analogue. To define a 4-space analogue to the vector product two further operations, the wedge product and the dual, are required. The wedge product is the anti-symmetrization of the direct product of two vectors,

$$\underline{\mathbf{A}} \wedge \underline{\mathbf{B}} \equiv \underline{\mathbf{A}} \otimes \underline{\mathbf{B}} - \underline{\mathbf{B}} \otimes \underline{\mathbf{A}}. \quad (7)$$

The dual maps the 4-space object into the ‘‘dual’’ of the object,

$$dual(\underline{\mathbf{A}}\underline{\mathbf{B}}) \equiv \frac{1}{2} e^{\alpha\beta\gamma\delta} A_\alpha B_\beta \mathbf{e}_\gamma \mathbf{e}_\delta = \frac{1}{2} e_{\alpha\beta\gamma\delta} A^\alpha B^\beta \mathbf{e}^\gamma \mathbf{e}^\delta. \quad (8)$$

The Levi-Civita tensor,

$$e^{\alpha\beta\gamma\delta} \equiv -\frac{1}{\sqrt{-g}}E^{\alpha\beta\gamma\delta} = \sqrt{-g}E_{\alpha\beta\gamma\delta} = e_{\alpha\beta\gamma\delta}, \quad (9)$$

is defined in terms of the permutations symbol $E^{\alpha\beta\gamma\delta}$ which is zero if any indices are repeated, one for even permutations of 0, 1, 2, 3 and negative one for odd permutations. The coefficient $g \equiv \det(\mathbf{e}_\alpha \cdot \mathbf{e}_\beta)$ is the determinant of the metric components.

3 Hydrodynamics

Satisfying the requirement of form invariance, requiring that the equations are the same in any reference frame, is where the formalism of the component tensor calculus and the dyadic calculus most differ. Form invariance is satisfied in the component tensor calculus by changing from the partial derivative in the flat space of inertial reference frames to the covariant derivative in more general curved spaces. In the dyadic calculus, form invariance is insured by calculating the derivatives of the basis vectors in the partial derivatives. As a demonstration of the application of form invariance, using the dyadic calculus, the equations of motion for a perfect fluid will be developed heuristically by determining these equations in flat space. The form invariance of the 4-space objects insures that these objects are exactly the same in curved space. Using the dyadic calculus the correct equations in flat space and special relativity are the same equations in curved space and general relativity.

Assume that the volume element of a perfect fluid, in a LIF, has nonzero temperature or pressure P , 3-velocity \mathbf{u} , and 4-velocity $\underline{\mathbf{u}} = \gamma\mathbf{e}_0 + \gamma u^c\mathbf{e}_c$ in a system of units with the speed of light $c = 1$. Define the finite temperature mass current density of this fluid as

$$\underline{\Omega} = \gamma\tilde{\sigma}\mathbf{e}_0 + \gamma\tilde{\sigma}u^a\mathbf{e}_a, \quad (10)$$

where $\tilde{\sigma} = (P + \sigma)$ is the local mass-energy density, σ is the local rest mass density, and P is the pressure. If the particles of the fluid could be made stationary, relative to one another, each particle would have exactly the same motion in a chosen frame. While these particles would have a different motion in some other frame the particles relative motion, in that frame, will still be

the zero. This permits the definition of a second physical object, the zero temperature mass current density, which in the LIF is

$$\underline{\mathbf{p}} = \gamma\sigma\mathbf{e}_0 + \gamma\sigma u^b\mathbf{e}_b. \quad (11)$$

It is now only necessary to show that, in the low energy limit, the continuity and Euler equations for a perfect fluid in a LIF and written as 4-vector relations are

$$\underline{\nabla}(\underline{\mathbf{u}} \cdot \underline{\Omega}) - \underline{\nabla}(\underline{\mathbf{u}} \cdot \underline{\mathbf{p}}) = \underline{\nabla} \cdot (\underline{\mathbf{u}}\underline{\Omega}). \quad (12)$$

Form invariance insures that this expression is correct in any reference frame and in curved space.

As an illustration of the similarity between the dyadic calculus and the more familiar vector calculus the equation for a perfect fluid will be expanded in some detail. Using the definition of the divergence of the direct product, the right hand side can be expanded as

$$\underline{\nabla} \cdot (\underline{\mathbf{u}}\underline{\Omega}) = (\underline{\mathbf{u}} \cdot \underline{\nabla})\underline{\Omega} + \underline{\Omega}(\underline{\nabla} \cdot \underline{\mathbf{u}}). \quad (13)$$

Assuming a LIF the equation can be expanded and assuming the low energy limit terms proportional to $\frac{1}{c^2}$ can be dropped,

$$\begin{aligned} -\frac{\partial}{\partial x^a}P\mathbf{e}_a &= \left(\frac{\partial}{\partial t}\sigma + u^d\frac{\partial}{\partial x^d}\sigma + \sigma\frac{\partial}{\partial x^d}u^d \right)\mathbf{e}_0 \\ &+ \left(\frac{\partial}{\partial t}\sigma u^a + \sigma u^a\frac{\partial}{\partial x^d}u^d + u^d\frac{\partial}{\partial x^d}\sigma u^a \right)\mathbf{e}_a. \end{aligned} \quad (14)$$

Note that for low energies $\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} \simeq 1$ and the derivatives of γ are $\frac{\partial\gamma}{\partial x^\alpha} = \frac{u}{c^2}\gamma^3\frac{\partial u}{\partial x^\alpha} \simeq 0$. Equating the terms for the space part, this expression can also be written as a 3-space dyadic equation,

$$\frac{\partial}{\partial t}\sigma\mathbf{u} + \nabla \cdot (\sigma\mathbf{u}\mathbf{u}) = -\nabla P. \quad (15)$$

Collecting terms for the time part,

$$\frac{\partial}{\partial t}\sigma + \nabla \cdot (\sigma\mathbf{u}) = 0. \quad (16)$$

These are the Euler and continuity equations for a perfect fluid in a LIF and the low energy limit.

4 Electrodynamics

The dyadic calculus, as it is presented here, makes it possible to write a form invariant expression of the Maxwell equations in terms of the direct products between a 4-velocity and the 4-vector electric $\underline{\mathbf{E}}$ and magnetic $\underline{\mathbf{B}}$ fields. The 4-velocity $\underline{\mathbf{u}}$ is the velocity of the volume element where the electric and magnetic fields are defined. The Maxwell equations were previously written in a similar form, as direct products between the 4-velocity and 4-space fields, by Ellis [8]. While Ellis writes the equations of electrodynamics using the component tensor calculus the 4-space objects are the same.

The inhomogeneous equations of electrodynamics are written in terms of the wedge product of the 4-velocity $\underline{\mathbf{u}}$ and the 4-vector $\underline{\mathbf{E}}$ field and the dual of this wedge product with the 4-vector $\underline{\mathbf{B}}$ field,

$$\underline{\nabla} \cdot (\underline{\mathbf{u}} \wedge \underline{\mathbf{E}}) + \underline{\nabla} \cdot dual(\underline{\mathbf{u}} \wedge \underline{\mathbf{B}}) = -4\pi \underline{\mathbf{J}}. \quad (17)$$

The homogeneous equations are written as the wedge product with the 4-vector $\underline{\mathbf{B}}$ field and the dual of the wedge product with the 4-vector $\underline{\mathbf{E}}$ field,

$$\underline{\nabla} \cdot (\underline{\mathbf{u}} \wedge \underline{\mathbf{B}}) = \underline{\nabla} \cdot dual(\underline{\mathbf{u}} \wedge \underline{\mathbf{E}}). \quad (18)$$

In this form the physical content of the equations are independent of the observer and the reference frame and depend only on the source terms $\underline{\mathbf{J}}$.

Since these equations are form invariant it will suffice to show that the equations are correct for a LIF and flat space to demonstrate that the equations are correct for curved space as well. In a LIF take the 3-velocity as zero, $\mathbf{u} = 0$ and $\underline{\mathbf{u}} = \mathbf{e}_0 = -\mathbf{e}^0$. A system of units is assumed where the speed of light $c = 1$. With this definition the source terms are $\underline{\mathbf{J}} = \rho \mathbf{e}_0 + \mathbf{J} = -\rho \mathbf{e}^0 + \mathbf{J}$. The fields are similarly expressed as a sum of a time and space part. Substituting these expressions into the general form of the Maxwell equations and collecting space and time terms reduces to the expected form of the equations for an observer at rest in a LIF. The connection between 4-space and 3-space objects and operators is facilitated by recognizing the relation between the Levi-Civita tensors in a LIF and assuming constant base vectors, $e^{0123} = -e^{123}$. This relation leads to an expression for the negative of the curl in terms of the dual,

$$\underline{\nabla} \cdot dual(\mathbf{e}_0 \wedge \mathbf{A}) = -\nabla \times \mathbf{A}. \quad (19)$$

Expanding the inhomogeneous equations in a LIF,

$$\frac{\partial \mathbf{E}}{\partial t} - (\nabla \cdot \mathbf{E}) \mathbf{e}_0 - \nabla \times \mathbf{B} = -4\pi (\mathbf{J} + \rho \mathbf{e}_0). \quad (20)$$

Expanding the homogeneous equations in a LIF,

$$(\nabla \cdot \mathbf{B}) \mathbf{e}_0 - \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}. \quad (21)$$

These equations are correct for flat space and the form invariance of the dyads insures that these equations are correct for all reference frames and curved spaces.

5 Gravity and electromagnetic radiation

Having established the form invariant equations for electrodynamics, it is evident that the magnitude of the 4-space electric and magnetic fields must be related to the local curvature of space and the gravitational sources associated with that curvature. As a practical example of this connection, between gravity and electromagnetism, the effect of a time varying metric on the electric and magnetic fields will be considered. Assuming time dependent gravity, in some region of space, the metric is written as $g_{00} = -(1 + h(t))$, and, $g_{ii} \simeq 1$, $g_{ij} = 0$ if $i \neq j$. Assume that the field is weak, $h \ll 1$, and that the spacial derivatives of the field are small compared to the time derivative, $\frac{\partial}{\partial x^i} h \ll \frac{\partial}{\partial t} h$. This metric is similar to the metric of the usual weak field limit, except that the field here is time dependent and independent of position. Also assume that there are no electromagnetic sources, $\mathbf{J} = 0$. The choice of reference frames is made where the volume element is stationary, $\mathbf{u} = \mathbf{e}_0 = -\mathbf{e}^0$.

The time dependent gravity and the equations of electrodynamics are operationally related by the time dependence of the temporal basis vector. The effect of the time varying metric on the equations of electrodynamics can be calculated explicitly by equating the time derivatives of the basis vectors with the time derivative of the metric component. The only non zero contribution is from the temporal unit basis in the first term on the left hand side of the ‘‘inhomogeneous equations’’, $(-\mathbf{e}_0 \cdot \frac{\partial}{\partial t} \mathbf{e}_0 \mathbf{E}) = \frac{\partial}{\partial t} \mathbf{E} - \mathbf{E} (\mathbf{e}_0 \cdot \frac{\partial}{\partial t} \mathbf{e}_0)$. This last expression, in parentheses, on the right hand side can be rewritten in terms of the time variation in gravity, $(\frac{\partial}{\partial t} e_0) \cdot e_0 = \frac{1}{2} \frac{\partial}{\partial t} g_{00} = -\frac{1}{2} \frac{\partial}{\partial t} h$. The ‘‘inhomogeneous equations’’ can then be expanded, retaining the time

dependence in the metric and taking the scalar product with the electric field as,

$$-\mathbf{E} \cdot (\nabla \times \mathbf{B}) + \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{2} E^2 \frac{\partial h}{\partial t}. \quad (22)$$

The homogeneous equations are similarly expanded,

$$\mathbf{B} \cdot (\nabla \times \mathbf{E}) + \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} = -\frac{1}{2} B^2 \frac{\partial h}{\partial t}. \quad (23)$$

Adding the equations and dividing by $\frac{1}{4\pi}$,

$$\frac{1}{4\pi} \nabla \cdot (\mathbf{E} \times \mathbf{B}) = -\frac{1}{8\pi} (B^2 + E^2) \frac{\partial h}{\partial t} - \frac{1}{8\pi} \frac{\partial}{\partial t} (B^2 + E^2), \quad (24)$$

where the identity $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$ has been used to simplify the left hand side. Substituting the local energy density $U = \frac{1}{8\pi} (E^2 + B^2)$ and the Poynting vector $\mathbf{S} = \frac{1}{4\pi} \mathbf{E} \times \mathbf{B}$,

$$\nabla \cdot \mathbf{S} = \frac{\partial}{\partial t} (U g_{00}). \quad (25)$$

This demonstrates that time variations in gravity will generate electromagnetic radiation.

6 Conclusion

The dyadic calculus, in the present form, offers a middle ground between the computational utility of the component tensor calculus and the mathematical rigor of differential forms. As a demonstration of the heuristic construction of a physical theory, in curved space, the equations of hydrodynamics and electrodynamics were developed using the dyadic calculus. Recognizing the connection between gravity and the electric and magnetic fields, in the equations of electrodynamics, gravity was shown to be a potential source of electromagnetic radiation.

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