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Remling's Theorem on Canonical Systems

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ABSTRACT. In this paper, we extend the Remling's Theorem on canonical systems that the ω limit points of the Hamiltonian under the shift map are reflectionless on the support of the absolutely continuous part of the spectral measure of a canonical system.

Keywords: Canonical systems, absolutely continuous spectrum, reflectionless Hamiltonians.

1. INTRODUCTION

The main purpose of this paper is to extend the Remling's theorem on canonical systems. A canonical system is a system of first order differential equations of the following form

$$Ju'(x) = zH(x)u(x), \quad x \in \mathbb{R}. \tag{1.1}$$

Here $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the Hamiltonian $H(x)$ is a 2×2 positive semidefinite matrix whose entries are locally integrable and that there is no non-empty open interval I so that $H(x) \equiv 0$ *a.e.* on I. The complex number $z \in \mathbb{C}$ involved in (1.1) is a spectral parameter. For fixed $z \in \mathbb{C}$, a function $u(.,z) : [-N, N] \to \mathbb{C}^2$ is called a solution if u is absolutely continuous and satisfies (1.1). Consider the Hilbert space

$$L^2(H,\mathbb{R}) = \left\{ f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} : \|f\| < \infty \right\}$$

with an inner product $\langle f, g \rangle = \int_{\mathbb{R}} f(x)^* H(x) g(x) dx$. Such canonical systems (1.1) on $L^2(H,\mathbb{R})$ have been studied by Hassi, De snoo, Winkler, and Remling in the papers [7, 8, 11, 14] in various context. The Jacobi and Schrödinger equations can be written into canonical systems with appropriate choice of $H(x)$, see [1]. In addition, the canonical systems are closely connected with the theory of de Branges spaces and the inverse spectral theory of one dimensional Schrödinger equations, see [11].

We call a canonical system (1.1) *trace-normed* if $\operatorname{tr} H \equiv 1$ and a solution of (1.1) in $L^2(H,\mathbb{R})$ is called H-integrable. It has shown in [2, 3] that a *trace-normed* canonical system always implies a limit-point case at both end points $-\infty$ and ∞. This notion can also be found in [6]. This means that for $z \in \mathbb{C}^+$ (the complex upper half plane), there exist unique solutions $f_\pm(x,z)$ (up to multiplication by a constant) of (1.1) on $\mathbb{R}$ satisfying $f_+(x,z) \in L^2(H,\mathbb{R}_+)$ and $f_-(x,z) \in L^2(H,\mathbb{R}_-)$ where $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_- = (-\infty, 0)$.

In the spectral theory of Jacobi and Schrödinger operators, the Remling's theorem has revealed some new fundamental properties of absolutely continuous spectrum of Jacobi and Schrödinge operators, see [12]. The basic result says that the ω

limit points of the potential of a given Schrödinger equation, under the shift map are reflectionless on the support of the absolutely continuous part of the spectral measure of the Schrödinger operator. In this paper we extend the Remling's Theorem on canonical systems (1.1) when $\operatorname{tr} H \equiv 1$.

We consider the *trace-normend* canonical system (1.1) on $\mathbb{R}_+$ so that it prevails the limit-point case. The solution space of (1.1) is a two dimensional vector space. Let u_α, v_α be solutions of (1.1) with the initial values

$$u_\alpha(0,z) = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} \quad \text{and} \quad v_\alpha(0,z) = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}, \quad \alpha \in (0,\pi]. \tag{1.2}$$

As shown in [2], for fixed $z \in \mathbb{C}$, the Wronskian of any two solutions of (1.1), is independent of x, then

$$W(u_\alpha, v_\alpha) = v_\alpha(0,\bar{z})^* J u_\alpha(0,z) = u_{1\alpha}v_{2\alpha} - u_{2\alpha}v_{1\alpha} = 1.$$

For $z \in \mathbb{C}^+$, there exists a unique coefficient $m_\alpha(z) \in \mathbb{C}$ for which

$$f_\alpha = u_\alpha + m_\alpha(z)v_\alpha \in L^2(H, \mathbb{R}_+).$$

The coefficient $m_\alpha(z)$ is called a Weyl m function and is defined for $z \in \mathbb{C}^+$. For fixed $x \geq 0, u_\alpha(x,z)$ and $v_\alpha(x,z)$ are analytic functions of z. Therefore $m_\alpha(z)$ is an analytic in the upper half-plane, with Im $m_\alpha(z) > 0$, for detail see [2]. These are so called the Herglotz functions.

By the Herglotz representation theorem, $m_\alpha(z)$ they have unique integral representation of the form,

$$m_\alpha(z) = a + bz + \int_{\mathbb{R}} \Big(\frac{1}{t-z} - \frac{t}{t^2+1} \Big) d\nu_\alpha(t), \quad z \in \mathbb{C}^+$$

for some positive Borel measure ν_α on $\mathbb{R}$ with $\int \frac{1}{t^2+1} d\nu_\alpha < \infty$ and numbers $a \in \mathbb{R}$, $b \geq 0$. For $\alpha = 0$, we call the measure ν in above integral representation of $m(z)$ as the *spectral measure* of (1.1).

Recall that, the *essential support* of a Borel measure ρ on $\mathbb{R}$ is the complement of a largest open set $U \subset \mathbb{R}$ such that $\rho(U) = 0$. A Borel measure ρ on $\mathbb{R}$ is called *absolutely continuous* if $\rho(B) = 0$ for all Borel sets $B \subset \mathbb{R}$ of Lebesgue measure zero. By the Radon-Nikodym Theorem, ρ is absolutely continuous if and only if $d\rho = f(t)dt$ for some density $f \in L^1_{loc}(\mathbb{R})$, $f \geq 0$. If ρ is supported by a Lebesgue null set that is, there exists a Borel set $B \subset \mathbb{R}$ with $|B| = \rho(B^c) = 0$, then we say that ρ is *singular*.

By Lebesgue's decomposition theorem, the spectral measure ν on $\mathbb{R}$ can uniquely decomposed into absolutely continuous and singular parts:

$$\nu = \nu_{ac} + \nu_s.$$

Let Σ_{ac} be the *essential support* of the absolutely continuous part ν_{ac}. Our aim is to show that that the ω limit points of the Hamiltonian under the shift map are reflectionless on Σ_{ac}.

2. Topologies on the space of Hamiltonians.

Let $\mathcal{V}$ denotes the set of all Hamiltonians $H(x)$ of canonical systems (1.1)

$$\mathcal{V} = \big\{ H(x) \text{ is a Hamiltonian in (1.1)}, \operatorname{tr} H(x) \equiv 1, H(x) \in L^1_{\text{loc}} \big\}$$

and $\mathcal{V}_\pm$ be the spaces of all Hamiltonians restricted on $\mathbb{R}_\pm$. A sequence of Hamiltonians H_n in $\mathcal{V}_+$ is said to converges H in *weak-$*$* if

$$\int_0^\infty \phi^* H_n \phi \to \int_0^\infty \phi^* H \phi$$

for all continuous functions $\phi = (\phi_1, \phi_2)^t$ with compact support on $[0, \infty)$. We would like to define a metric d on $\mathcal{V}_+$ so that the space $(\mathcal{V}_+, d)$ is a compact metric space. The process of defining such metric is adopted from [12].

Consider a countable dense (with respect to $\|.\|_\infty$) subset $\{\phi_n : \ n \in \mathbb{N}\} \subset C_c(\mathbb{R})$, the continuous functions of compact support. Let

$$\rho_n(H_1, H_2) = \Big| \int_0^\infty \phi_n^*(x)(H_1(x) - H_2(x))\phi_n(x)dx \Big|.$$

Then define a metric d on $\mathcal{V}$ as

$$d(H_1, H_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(H_1, H_2)}{1 + \rho_n(H_1, H_2)}.$$

Clearly d is a metric on $\mathcal{V}_+$ and $d(H_n, H) \to 0$ if and only if H_n converges to H in weak-$*$. Moreover, $(\mathcal{V}_+, d)$ is a compact metric space.

Let $m_\pm(x, z)$ denote the Weyl m functions on the half lines $\mathbb{R}_\pm$ for the Dirichlet boundary condition ($\alpha = 0$) at x, that is for $z \in \mathbb{C}^+, u_1(x, z) = v_2(x, z) = 0, v_1(x, z) = u_2(x, z) = 1$. These Weyl m functions are alternately defined as

$$m_\pm(x, z) = \pm \frac{f_{\pm_2}(x, z)}{f_{\pm_1}(x, z)}, \tag{2.1}$$

where $f_\pm(x, z) = u(x, z) \pm m_\pm(z)v(x, z)$ are the unique (upto a constant factors) $H-$ integrable solutions of (1.1) on $R_\pm$.

Lemma 2.1. *Suppose u_n be a solution of (1.1) with the Hamiltonian H_n having the same initial values $u_n(0)$ for all n . If H_n converges to H in weak-$*$, as $n \to \infty$, then the u_n has a subsequence which converges uniformly on any compact subsets of $[0; 1)$ to some solution u of (1.1).*

Proof. Suppose a sequence of Hamiltonians H_n converges in weak$-*$ to $H(x)$ in $\mathcal{V}_+$. Let u_n be the solution of canonical system with Hamiltonian $H_n(x)$. Let K be a compact subset of $\mathbb{C}^+$ contained in a ball $B(0, R)$ for some $R > 0$, $|z| < R$. Suppose a subinterval $[0, \eta]$ be such that $\eta = \frac{1}{8R}$. We claim that u_n has convergent subsequence on $[0, \eta]$. Define the operators $T_n : C[0, \eta] \longrightarrow C[0, \eta]$ by

$$T_n u(x) = -zJ \int_0^x H_n(t)u(t)dt.$$

Since

$$\begin{aligned}
\|T_n\| = &\sup_{\|u\|_\infty = 1} \| - zJ \int_0^x H_n(t)u(t)dt\| \\
\leq &|z|\|u\|_\infty \int_0^x |H_n(t)|dt \\
&\leq R4\eta = R4\frac{1}{8R} = \frac{1}{2},
\end{aligned}$$

$\|T_n\|$ are uniformly bounded. So the Neumann series $(1-T_n)^{-1} = \sum_{k=0}^{\infty} T_n^k$ is convergent. Here $u_n(x) = (1-T_n)^{-1}(u_0), \quad u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\|u_n\| \leq \|(1-T_n)^{-1}\|\|u_0\| = \|(1-T_n)^{-1}\| \leq \sum_{k=0}^{\infty} \|T_n\|^k = \sum_{k=0}^{\infty} (\frac{1}{2})^k = 2$. So $\{u_n(x) = n \in \mathbb{N}\}$ is uniformly bounded in n on $[0, \eta]$ and locally uniformly in z. Similar argument shows that u_n remains bounded on $[\eta, \eta + p]$ so that u_n are eventually bounded uniformly on $[0, N]$. Moreover, u_n are equicontinuous. Let $\epsilon > 0$ be given. Since u_n are solutions for the system 1.1 we have,

$$u_n(x) - u_n(x_0) = -zJ \int_{x_0}^{x} H_n(t) u_n(t) dt.$$

$$\begin{aligned} \|u_n(x) - u_n(x_0)\| &\leq |z| \|u_n\| \int_{x_0}^{x} |H_n(t)| dt \\ &= |z| \|u_n\| 4\eta \|x - x_0\| \\ &\leq R 2.4\eta \|x - x_0\|. \end{aligned}$$

Let $\delta = \frac{\epsilon}{8R\eta}$ then $\|u_n(x) - u_n(x_0)\| < \epsilon$, if $\|x - x_0\| < \delta$ for all n . By Arzella-Ascolli Theorem $\{u_n\}$ has convergent subsequence say $u_{n_j} \to u$. We show that u satisfies the canonical system corresponding to $H(x)$.

$$\begin{aligned} u_{n_j}(x) - u_{n_j}(0) &= -zJ \int_0^x H_{n_j}(t) u_{n_j}(t) dt \\ &= -zJ \int_0^x H_{n_j}\big((t) u_{n_j}(t) - u(t)\big) dt - zJ \int_0^x H_{n_j}(t) u(t) dt. \end{aligned}$$

Since $\| -zJ \int_0^x H_{n_j}\big((t)u_{n_j}(t) - u(t)\big) dt \| \leq |z| \|H_{n_j}\|_{L_1(0,x)} \|u_{n_j} - u\|$,
$\lim_{j\to\infty} -zJ \int_0^x H_{n_j}\big((t)u_{n_j}(t) - u(t)\big) dt = 0$. Hence, taking the limit as $j \to \infty$ we get, $u(x) - u(0) = \int_0^x H(t)u(t)dt$. Hence u is a solution of (1.1). $\square$

Let $\mathbb{H}$ denote the set of all Herglotz functions, that is $\mathbb{H} = \{F : \mathbb{C}^+ \to \mathbb{C}^+ :$ F is holomorphic $\}$ and $\overline{\mathbb{H}} \cup \mathbb{R} \cup \{\infty\}$. So $\{M_\pm = m_\pm^H(0, z)\} \subset \mathbb{H}$.

As a consequence of the Lemma 2.1, we have the following proposition. The detail proof of this proposition can be found in [9].

Proposition 2.2. *The maps $\mathcal{V}_\pm \longmapsto \overline{\mathbb{H}}$, $H_\pm \longmapsto M_\pm = m_\pm^H(0, z)$ are homeomorphism onto their images.*

3. Main theorem and its proof

Recall that $m_\pm(x, z)$ are Herglotz functions. So the boundary value of these m functions are defined by $m_\pm(x, t) \equiv \lim_{y\to 0} m_\pm(x, t + iy)$.

Definition 3.1. *Let $A \subset \mathbb{R}$ be a Borel set. We call a Hamiltonian $H \in \mathcal{V}$ reflectionless on A if*

$$m_+(x, t) = -\overline{m_-(x, t)} \tag{3.1}$$

for almost every $t \in A$ and for some $x \in \mathbb{R}$.

The set of reflectionless hamiltonian on A is denoted by $\mathcal{R}(A)$.

Proposition 3.2. *The definition (3.1) is independent of the choice of boundary condition and the choice of a point of a boundary condition.*

Proof. Suppose (3.1) is true for a boundary condition α at 0. Let $m_+^\alpha(z)$ be the unique coefficient such that $f(x,z) = u_\alpha(x,z) + m_+^\alpha(z) v_\alpha(x,z) \in L^2(H,\mathbb{R}_+)$. Suppose $T_\alpha(x,z) = \begin{pmatrix} u_{\alpha_1}(x,z) & v_{\alpha_1}(x,z) \\ u_{\alpha_2}(x,z) & v_{\alpha_2}(x,z) \end{pmatrix}$ with $T_\alpha(0,z) = \begin{pmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{pmatrix}$ and $T_\beta(x,z) = \begin{pmatrix} u_{\beta_1}(x,z) & v_{\beta_1}(x,z) \\ u_{\beta_2}(x,z) & v_{\beta_2}(x,z) \end{pmatrix}$ with $T_\beta(0,z) = \begin{pmatrix} \sin\beta & \cos\beta \\ -\cos\beta & \sin\beta \end{pmatrix}$. Then
$T_\alpha(x,z) = T_\beta(x,z) \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix}$, where $\gamma = \beta - \alpha$.

Here $m_+^\alpha(z) \in \mathbb{C}$ is a unique number such that

$$\begin{aligned}
&T_\alpha(x,z) \begin{pmatrix} 1 \\ m_+^\alpha(z) \end{pmatrix} \in L^2(H,\mathbb{R}_+). \\
\Rightarrow &T_\beta(x,z) \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} 1 \\ m_+^\alpha(z) \end{pmatrix} \in L^2(H,\mathbb{R}_+). \\
\Rightarrow &T_\beta(x,z) \begin{pmatrix} \cos\gamma + m_+^\alpha(z)\sin\gamma \\ -\sin\gamma + m_+^\alpha(z)\cos\gamma \end{pmatrix} \in L^2(H,\mathbb{R}_+). \\
\Rightarrow &(\cos\gamma + m_+^\alpha(z)\sin\gamma) T_\beta(x,z) \begin{pmatrix} 1 \\ \frac{-\sin\gamma + m_+^\alpha(z)\cos\gamma}{\cos\gamma + m_+^\alpha(z)\sin\gamma} \end{pmatrix} \in L^2(H,\mathbb{R}_+).
\end{aligned}$$

Since $m_+^\beta(z)$ be the unique coefficient such that $T_\beta(x,z) \begin{pmatrix} 1 \\ m_+^\beta(z) \end{pmatrix} \in L^2(H,\mathbb{R}_+)$ we must have,

$$m_+^\beta(z) = \frac{-\sin\gamma + m_+^\alpha(z)\cos\gamma}{\cos\gamma + m_+^\alpha(z)\sin\gamma} = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} m_+^\alpha(z).$$

On the other hand, exactly in the same way,

$$\begin{aligned}
&T_\alpha(x,z) \begin{pmatrix} 1 \\ -m_-^\alpha(z) \end{pmatrix} \in L^2(H,\mathbb{R}_-). \\
\Rightarrow &T_\beta(x,z) \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} 1 \\ -m_-^\alpha(z) \end{pmatrix} \in L^2(H,\mathbb{R}_-), \text{ where } \gamma = \beta - \alpha. \\
\Rightarrow &T_\beta(x,z) \begin{pmatrix} \cos\gamma - m_-^\alpha(z)\sin\gamma \\ -\sin\gamma - m_-^\alpha(z)\cos\gamma \end{pmatrix} \in L^2(H,(-\infty,0]) \\
\Rightarrow &(\cos\gamma - m_-^\alpha(z)\sin\gamma) T_\beta(x,z) \begin{pmatrix} 1 \\ \frac{-\sin\gamma - m_-^\alpha(z)\cos\gamma}{\cos\gamma - m_-^\alpha(z)\sin\gamma} \end{pmatrix} \in L^2(H,\mathbb{R}_-) \\
\Rightarrow &-m_-^\beta(z) = \frac{-\sin\gamma - m_-^\alpha(z)\cos\gamma}{\cos\gamma - m_-^\alpha(z)\sin\gamma} = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} m_-^\alpha(z).
\end{aligned}$$

Let $P_+(0,z) = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$ and $P_-(0,z) = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix}$, so that
$m_-^\beta(z) = P_-(0,z) m_-^\alpha(z)$, $m_+^\beta(z) = P_+(0,z) m_+^\alpha(z)$ and

$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} P_+(0,z) = P_-(0,z)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. By simple calculation we can see that

$$m_+^\beta(t) = -\overline{m_-^\beta(t)}.$$

Similarly, equation 3.1 is independent of the choice of the point. Suppose $T_0(x,z) = \begin{pmatrix}u_1(x,z) & v_1(x,z)\\ u_2(x,z) & v_2(x,z)\end{pmatrix}$ be solutions with the boundary conditions at 0. Then $T_0(x,z) = T_a(x,z)\begin{pmatrix}u_1(a,z) & v_1(a,z)\\ u_2(a,z) & v_2(a,z)\end{pmatrix}$. Suppose $m_\pm(0,z) \in \mathbb{C}$ be the unique coefficients such that $f_\pm(x,z) = u(x,z) \pm m_\pm(0,z)v(x,z) \in L^2(H,\mathbb{R}_\pm)$.
In another way, $T_0(x,z)\begin{pmatrix}1\\ \pm m_\pm(0,z)\end{pmatrix} \in L^2(H,\mathbb{R}_\pm)$.
$\Rightarrow T_a(x,z)\begin{pmatrix}u_1(a,z) & v_1(a,z)\\ u_2(a,z) & v_2(a,z)\end{pmatrix}\begin{pmatrix}1\\ \pm m_\pm(0,z)\end{pmatrix} \in L^2(H,\mathbb{R}_\pm)$.
$\Rightarrow m_\pm(a,z) = \frac{u_2(a,z)\pm m_\pm(0,z)v_2(a,z)}{u_1(a,z)\pm m_\pm(0,z)v_1(a,z)} = \begin{pmatrix}v_2(a,z) & \pm u_2(a,z)\\ \pm v_1(a,z) & u_1(a,z)\end{pmatrix} m_\pm(0,z)$.
Let $T_\pm(z) = \begin{pmatrix}v_2(a,z) & \pm u_2(a,z)\\ \pm v_1(a,z) & u_1(a,z)\end{pmatrix}$, then $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} T_+(z) = T_-(z)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.
By calculation we see that

$$m_+(a,t) = -\overline{m_-(a,t)}$$

□

For a Hamiltonian $H \in \mathcal{V}$, a shift map S_x on $\mathcal{V}$ is defined by $S_x(H) = H(x+t)$.

Definition 3.3. *The ω limit set of a Hamiltonian $H \in \mathcal{V}$ under the shift map is defined as,*

$$\omega(H) = \{W \in \mathcal{V} : \textit{ there exist } x_n \to \infty \textit{ so that } d(S_{x_n}H, W) \to 0\}.$$

Clearly the ω limit set $\omega(H)$ is a nonempty compact subset of $\mathcal{V}$. Now we state our main theorem.

Theorem 3.4. *Let $H \in \mathcal{V}_+$ be a (half line) Hamiltonian, and let Σ_{ac} be the essential support of the absolutely continuously part of the spectral measure. Then*

$$\omega(H) \subset \mathcal{R}(\Sigma_{ac}).$$

This theorem is an extension of the Remling's theorem for Jacobi and Schrödinger equations, see [12, 13], and the proof follows in a similar fashion. However, we present the proof in details so that it is easily readable for general readers. First we extend the Breimesser-Pearson Theorem from a Schrödinger equations to canonical systems.

3.1. **Breimesser-Pearson Theorem on canonical systems.** For $z = x+iy \in \mathbb{C}^+$, $\omega_z(S) = \frac{1}{\pi}\int_S \frac{y}{(t-x)^2+y^2}dt$, denotes the harmonic measure in the upper-half plane. For any $G \in \mathbb{H}$ and $t \in \mathbb{R}$ we define $\omega_{G(t)}(S)$ as the limit

$$\omega_{G(t)}(S) = \lim_{y\to 0+} \omega_{G(t+iy)}(S).$$

For complete description about the Herglotz functions and harmonic measures, see [12].

Theorem 3.5. *Consider the canonical system (1.1) on R_+. Let Σ_{ac} denotes the essential support of absolutely continuous part of Spectral measure then for any $A \subset \Sigma_{ac}, |A| < \infty$ and $S \subset \mathbb{R}$, we have*

$$\lim_{N\to\infty} \Big(\int_A \omega_{m_-(N,t)}(-S)dt - \int_A \omega_{m_+(N,t)}(S)dt \Big) = 0.$$

Moreover, the convergence is uniform in S.

Here $m_+(N,z)$ and $m_-(N,z)$ are m functions for canonical systems defined on $[0,N]$ and $[N,\infty)$ respectively. Therefore, $m_-(N,z)$ in this theorem is different from how we defined in equation (2.1). This theorem is an extension of Breimesser-Pearson Theorem from one dimensional Schrödinger equations to canonical systems. For Schrödinger equations, see [4] and it has been further extended for Jacobi equations, see [12].

Lemma 3.6. [12] *Let $A \subset \mathbb{R}$ be a Borel Set with $|A| < \infty$. Then*

$$\lim_{y\to 0+} \sup_{F\in\mathcal{H}; S\subset\mathbb{R}} \Big| \int_A \omega_{F(t+iy)}(S)dt - \int_A \omega_{F(t)}(S)dt \Big| = 0$$

.

Definition 3.7. *If $F_n, F \in \mathbb{H}$, we say that $F_n \to F$ in value distribution if*

$$\lim_{n\to\infty} \int_A \omega_{F_n(t)}(S)dt = \int_A \omega_{F(t)}(S)dt \tag{3.2}$$

for all Borel set $A, S \subset \mathbb{R}, |A| < \infty$.

Notice that if the limit in the value distribution exists, it is unique.

Theorem 3.8. [12] *Suppose $F_n, F \in \mathcal{H}$, and let a_n, a, and ν_n, ν be the associated numbers and measures, respectively, from the integral representation of Herglotz function. Then the following are equivalent:*

1. *$F_n(z) \to F(z)$ uniformly on compact subsets of $\mathbb{C}^+$;*
2. *$a_n \to a$ and $\nu_n \to \nu$ weak * on $\mathcal{M}(\mathbb{R}_\infty)$, that is,*

$$\lim_{n\to\infty} \int_{\mathbb{R}_\infty} f(t)d\nu_n(t) = \int_{\mathbb{R}_\infty} f(t)d\nu(t)$$

   for all $f \in C(\mathbb{R}_\infty)$;

3. *$F_n \to F$ in value distribution.*

The hyperbolic distance of two points $w, z \in \mathbb{C}^+$ is defined as

$$\gamma(w,z) = \frac{|w-z|}{\sqrt{\mathrm{Im}w}\sqrt{\mathrm{Im}z}}.$$

Hyperbolic distance and harmonic measure are intimately related as follows,

$$|\omega_w(S) - \omega_z(S)| \le \gamma(w,z)$$

for any $z, w \in \mathbb{C}^+$ and any Borel set $S \subset \mathbb{R}$. Moreover, if $F(z) = \alpha(z) + i\omega_z(S)$, $\alpha(z)$ is a harmonic conjugate of $\omega_z(S)$ we have

$$|\omega_w(S) - \omega_z(S)| \le \frac{|\omega_w(S) - \omega_z(S)|}{\sqrt{\omega_w(S)}\sqrt{\omega_z(S)}} \le \gamma(F(w), F(z)) \le \gamma(w,z). \tag{3.3}$$

Lemma 3.9. *Let $u(.,z), v(.,z)$ be the solution of the canonical system 1.1, subject to the condition $u(0,z) = \begin{pmatrix}1\\0\end{pmatrix}, v(0,z) = \begin{pmatrix}0\\1\end{pmatrix}$. Let w be any constant such that $Imw \geq 0$, for any $N > 0$, and all $z \in \mathbb{C}^+$, we have the estimate,*

$$\gamma\Big(-\frac{v_2(N,z)}{v_1(N,z)}, -\frac{u_2(N,z)+\bar{w}v_2(N,z)}{u_1(N,z)+\bar{w}v_1(N,z)}\Big) \leq \frac{1}{\sqrt{I(I+1)}},$$

where $I = I(N,z)$ is the integral defined by $I(N,z) = (\ Imz)\int_0^N Im(u^*Hv)dx$.

Proof. Denote the wronskian $W_N(f,g) = f_1(N)g_2(N) - f_2(N)g_1(N))$. Using the Greens's Identity we have,

$$\int_0^N v^*Hvdx = \frac{1}{2i\ \mathrm{Im}z}W_N(v,\bar{v}), \tag{3.4}$$

$$\int_0^N \mathrm{Im}(u^*Hv)dx = -\frac{1}{2\ \mathrm{Im}z}\Big(1-\ \mathrm{Re}W_N(\bar{u},v)\Big) = \frac{1}{2\ \mathrm{Im}z}\Big(1-\ \mathrm{Re}W_N(u,\bar{v})\Big), \tag{3.5}$$

$$|W(u,\bar{v})|^2 = 1 - W(u,\bar{u})W(v,\bar{v}). \tag{3.6}$$

Now at $x = N$, we have,

$$\gamma^2\Big(-\frac{v_2}{v_1}, -\frac{u_2+\bar{w}v_2}{u_1+\bar{w}v_1}\Big) = -\frac{4}{W(v,\bar{v})W(u+\bar{w}v,\bar{u}+w\bar{v})}.$$

Therefore,

$$\gamma^2\Big(-\frac{v_2}{v_1}, -\frac{u_2+\bar{w}v_2}{u_1+\bar{w}v_1}\Big) \leq -\frac{4}{W(v,\bar{v})W(u+\bar{w}v,\bar{u}+w\bar{v})}.$$

Let w be real. The denominator on the right side is of the form $A + Bw + Cw^2$, where $A \geq 0, C \geq 0$ and B is real. The denominator has minimum value $A - \frac{B^2}{4C}$. Hence,

$$\begin{aligned}\gamma^2 &\leq \frac{4}{-W(v,\bar{v})W(u,\bar{u}) - \frac{\big(W(v,\bar{v})(W(u,\bar{v})-W(\bar{u},v))\big)^2}{4(-W(v,\bar{v})^2)}} \\ &\leq \frac{-4}{W(v,\bar{v})(W(u,\bar{v})+\ \mathrm{Im}(W(u,\bar{v}))\big)^2}.\end{aligned}$$

Using equation (3.1) we get,

$$\begin{aligned}\gamma^2 &\leq -\frac{4}{1-|W(u,\bar{v})|^2 + (\ \mathrm{Im}(W(u,\bar{v}))\big)^2)} \\ &= \frac{-4}{1-(\mathrm{Re}W(u,\bar{v}))^2}.\end{aligned}$$

Here,

$$\begin{aligned}1-(\mathrm{Re}W(u,\bar{v}))^2 &= (1-(\mathrm{Re}W(u,\bar{v})))(1+(\mathrm{Re}W(u,\bar{v}))) \\ &= \Big(-2\ \mathrm{Im}z\int_0^N \mathrm{Im}(u^*Hv)dx\Big)\Big(1+2\ \mathrm{Im}z\int_0^N \mathrm{Im}(u^*Hv)dx\Big).\end{aligned}$$

Therefore,

$$\gamma^2 \leq \frac{1}{I(1+I)} \text{ where } I = \text{ Im}z \int_0^N \text{ Im}(u^* Hv)dx.$$

If w is not real, $w = \text{ Re}w + iY, Y > 0$ then $u - iYv$ is also a solution and we have,

$$|W(u-iYv,\bar{v})|^2 = 1 - W(u-iYv,\bar{u}+iY\bar{v})W(v,\bar{v}).$$

Also from above equation,

$$\begin{aligned}\gamma^2 &\leq \frac{-4}{W(v,\bar{v})W(u,\bar{u}) + (\text{ Im}(W(u,\bar{v}))^2) + Y^2W(v,\bar{v})^2 + 2iY\text{Re}W(u,\bar{v})W(v,\bar{v})} \\ &\leq \frac{-4}{W(u-iYv,\bar{u}+iY\bar{v})W(v,\bar{v}) + \Big(\text{ Im}W(u-iYv,\bar{v})\Big)^2}.\end{aligned}$$

Since the equation (3.1) is valid for $u - iYv$ we get,

$$\begin{aligned}\gamma^2 &\leq \frac{-4}{1 - \big(\text{ Re}W(u-iYv,\bar{v})\big)^2} \\ &= \frac{-4}{\big(1+\text{ Re}W(u-iYv,\bar{v})\big)\big(1-\text{ Re}W(u-iYv,\bar{v})\big)} \\ &= \frac{-4}{\Big(1+\text{ Re}\big(W(u,\bar{v})-iYW(v,\bar{v})\big)\Big)\Big(1-\text{ Re}\big(W(u,\bar{v})-iYW(v,\bar{v})\big)\Big)} \\ &= \frac{-4}{\big(1-\text{ Re}W(u,\bar{v})-Y\text{ Im}W(v,\bar{v})\big)\big(1+\text{ Re}W(u,\bar{v})+Y\text{ Im}W(v,\bar{v})\big)} \\ &= \frac{-4}{\Big(-2\text{ Im}z\int_0^N \text{ Im}(u^*Hv)dx - \frac{Y}{i}W(v,\bar{v})\Big)\Big(2\text{ Im}z\int_0^N \text{ Im}(u^*Hv)dx + 2 + \frac{Y}{i}W(v,\bar{v})\Big)} \\ &= \frac{1}{I'(I'+1)},\end{aligned}$$

where $I' = \text{ Im}z \int_0^N \text{ Im}(u^*Hv)dx + 2 + \frac{Y}{2i}W(v,\bar{v})$. Notice that $I' \geq I$ since $W(v,\bar{v}) = 2i\text{ Im}z\int_0^N v^*Hvdx \geq 0$. Hence the lemma is proved for general case. $\square$

Corollary 3.10. *Let K be a compact subset of $\mathbb{C}^+$. Then with the notation above, we have*

$$\lim_{N\to\infty}\gamma\Big(-\frac{v_2(N,z)}{v_1(N,z)}, -\frac{u_2(N,z)+\bar{w}v_2(N,z)}{u_1(N,z)+\bar{w}v_1(N,z)}\Big) = 0,$$

uniformly in $z \in K, w \in \bar{\mathbb{C}^+}$.

Proof. From above lemma we have

$$\gamma\Big(-\frac{v_2(N,z)}{v_1(N,z)}, -\frac{u_2(N,z)+\bar{w}v_2(N,z)}{u_1(N,z)+\bar{w}v_1(N,z)}\Big) \leq \frac{1}{\sqrt{I(I+1)}},$$

where $I = I(N,z)$ is the integral defined by $I(N,z) = (\text{ Im}z)\int_0^N Im(u^*Hv)dx$. Want to show that $I \to \infty$ as $N \to \infty$. We have,

$$\int_0^N v^*Hvdx = \frac{1}{2i\text{ Im}z}W_N(v,\bar{v})$$

$$\int_0^N \text{Im}(u^*Hv)dx = -\frac{1}{2i\ \text{Im}z}\big(1 - \ \text{Re}W_N(u,\bar{v})\big).$$

Now lets look at the ratio

$$\begin{aligned}\frac{2\ \text{Im}z\int_0^N \ \text{Im}(u^*Hv)dx + 1}{2i\ \text{Im}z\int_0^N v^*Hvdx} &= \frac{W_N(u,\bar{v}) + W_N(\bar{u},v)}{2iW_N(v,\bar{v})} \\ &= \frac{W_N(u,\bar{v})}{2iW_N(v,\bar{v})} - \frac{W_N(\bar{u},v)}{2iW_N(\bar{v},v)} \\ &= \ \text{Im}C\end{aligned}$$

where C is the center of the Weyl circle. More details about these Weyl circles can be found in [1]. Since the center of the Weyl circle is continuously depend on z it is uniformly bounded on a compact subset of $\mathbb{C}^+$. So

$$\int_0^N \ \text{Im}(u^*Hv)dx + 1 = \ \text{Im}C\int_0^N v^*Hvdx \to \infty \text{ as } N \to \infty.$$

This implies that $I \to \infty$ as $n \to \infty$. $\square$

We are now ready to prove Theorem 3.5.

Proof of Theorem 3.5 : Let $A \subset \Sigma_{ac}$, $|A| < \infty$ and let $\epsilon > 0$ be given. We first define a partition $A = A_0 \cup A_1 \cup A_2, ... \cup A_N$ of disjoint subsets such that $|A_0| < \epsilon, A_j$ is bounded for $j \geq 1$. We also require that $m_+(t) \equiv \lim_{y\to 0+} m_+(t+iy)$ exists and $m_+(t) \in \mathbb{C}^+$ on $\bigcup_{j=1}^N A_j$. To find A_j's with these properties, first of all put all $t \in A$ for which $m_+(t)$ does not exist or does not lie in $\mathbb{C}^+$ into A_0. Then pick (sufficiently large) compact subset $K \subset \mathbb{C}^+, K' \subset \mathbb{R}$ so that $A_0 = \{t \in A : m_+(t) \notin K \text{ or } t \notin K'\}$ satisfies $|A_0| < \epsilon$. Subdivide K into finitely many subsets of hyperbolic diameter less than ϵ, then take the inverse images under m_+ of these subsets, and finally intersect with K' to obtain the A_j for $j \geq 1$. It is then true that $m_+(N,t)$ exists and lies in $\mathbb{C}^+$ for arbitrary $N \in \mathbb{R}$ if $t \in \bigcup_{j=1}^N A_j$. Moreover, we need $m_j \in \mathbb{C}^+$ such that

$$\gamma(m_+(t), m_j) < \epsilon,$$

such m_j can be defined as $m_j = m_+(t_j)$ for any fixed $t_j \in A_j$. By Lemma 3.6, there is a number $y > 0$ such that , for arbitrary Herglotz function F, for any Borel subset S of $\mathbb{R}$ and for all $j = 1, 2, ..., n$ we have the estimate

$$\Big|\int_{A_j} \omega_{F(t+iy)}(S)dt - \int_{A_j} \omega_{F(t)}(S)dt\Big| \leq \epsilon|A_j|. \tag{3.7}$$

We can define y for each value of j ; so y is a function of j. However, by taking the minimum value of $y(j)$ as j runs from 1 to n we my assume y is independent of j. Let $M_j(N,z) = \frac{u_2(N,z)+\bar{m}_j v_2(N,z)}{u_1(N,z)+\bar{m}_j v_1(N,z)}$ for any $z \in \mathbb{C}^+$. We shall complete the proof of the theorem by showing that, for $j \geq 1$,

(i): $\int_{A_j} w_{m_+(N,t)}(S)dt$ is close to the integral $\int_{A_j} \omega_{\overline{M_j}(N,t)}(S)dt$

where $M_j(N,t) = \frac{u_2(N,t)+\bar{m}_j v_2(N,t)}{u_1(N,t)+\bar{m}_j v_1(N,t)}$ and that

(ii): $\int_A \omega_{m_-(N,t)}(-S)dt$ is close to the same integral for all N sufficiently large.

Proof of (i): We have

$$m_+(N,t) = \frac{u_2(N,t) + m_+(t)v_2(N,t)}{u_1(N,t) + m_+(t)v_1(N,t).}$$

Hence, for fixed N and t, the mapping from $m_+(t)$ to $m_+(N,t)$ is a Mobius transformation with real coefficients and discriminant$u_1v_2 - v_1u_2 = 1.$ and γ is invariant under Mobius transformations. Now from 3.1 we see that

$$\gamma\Big(m_+(N,t), \frac{u_2(N,t) + m_jv_2(N,t)}{u_1(N,t) + m_jv_1(N,t)}\Big) \leq \epsilon \text{ for } j \geq 1 \text{ and } t \in A_j.$$

By equation 3.3 we see that,

$$\Big|\omega_{m_+(N,t)}(S) - \omega_{M_j(N,t)}(S)\Big| \leq \epsilon,$$

and integration with respect to t over A_j gives the estimate

$$\Big|\int_{A_j} \omega_{m_+(N,t)}(S)dt - \int_{A_j} \omega_{\overline{M_j}(N,t)}(S)dt\Big| \leq \epsilon|A_j|. \tag{3.8}$$

This holds for all $j = 1, 2, ...n$.
Proof of (ii): For $j \geq 1$, define the subset A_j^y of $\mathbb{C}^+$, consisting of all $z \in \mathbb{C}^+$ of the form $z = t + iy$, for $t \in A_j$. Thus A_j^y is the translation of A_j by distance y above the real z-axis. Since A_j is bounded, A_j^y is contained in a compact subset of $\mathbb{C}^+$. Hence by Corollary 3.1 there a positive number N_0 such that for $j \geq 1, N \geq N_0$ and $z \in A_j^y$ we have the estimate

$$\gamma\Big(-\frac{v_2(N,z)}{v_1(N,z)}, -\frac{u_2(N,z) + \bar{m}_jv_2(N,z)}{u_1(N,z) + \bar{m}_jv_1(N,z)}\Big) \leq \epsilon. \tag{3.9}$$

As in the case of y we may choose N_0 to be independent of j. Let $m_-(N,z) = -\frac{v_2(N,z)}{v_1(N,z)}$. Following the similar argument to that in the proof of (i), for any $z = t+iy$ we have the estimate

$$\Big|\int_{A_j} \omega_{m_-(N,z)}(-S)dt - \int_{A_j} \omega_{-M_j(N,z)}(-S)dt\Big| \leq \epsilon|A_j|,$$

valid for $j \geq 1$ and $N \geq N_0$. Now by Lemma 3.6, equation 3.1 we have,

$$\Big|\int_{A_j} \omega_{m_-(N,t)}(-S)dt - \int_{A_j} \omega_{-M_j(N,t)}(-S)dt\Big| \leq 3\epsilon|A_j|.$$

Now using the identity $\omega_{-w}(S) = \omega_{\bar{w}}(S)$

$$\Big|\int_{A_j} \omega_{m_-(N,t)}(-S)dt - \int_{A_j} \omega_{\overline{M_j}(N,t)}(S)dt\Big| \leq 3\epsilon|A_j|, \tag{3.10}$$

which holds for all $j \geq 1$ and $N \geq N_0$ and completes the proof of (ii). Combining the inequalities 3.1 and 3.1 now yields, for $j \geq 1$ and $N \geq N_0$,

$$\Big|\int_{A_j} \omega_{m_-(N,t)}(-S)dt - \int_{A_j} \omega_{m_+(N,t)}(S)dt\Big| \leq 4\epsilon|A_j|. \tag{3.11}$$

Noting that A_0 was chosen such that $|A_0| \leq \epsilon|A|$ we now have for all $N \geq N_0$,

$$\begin{aligned}
&\Big|\int_A \omega_{m_-(N,t)}(-S)dt - \int_A \omega_{m_+(N,t)}(S)dt\Big| \\
&\leq \sum_{j=0}^{n}\Big|\int_{A_j} \omega_{m_-(N,t)}(-S)dt - \int_{A_j} \omega_{m_+(N,t)}(S)dt\Big| \\
&\leq 2|A_0| + 4\epsilon\sum_{j=0}^{n}|A_j| \leq \epsilon|A_j| \leq 6\epsilon|A|.
\end{aligned}$$

Since ϵ was arbitrary, the theorem follows.

Proof of Theorem 3.4:

Let $W \in \omega(H)$. Then there exists a sequence $x_n \to \infty$ such that $d(S_{x_n}H, W) \to 0$. Then by Proposition 2.2 we have that

$$m_\pm(x_n, z) \to M_\pm(z) \ \ (n \to \infty),$$

uniformly on compact subset of $\mathbb{C}^+$. Here $M_\pm(z) = m_\pm^W(0, z)$ are the m functions of the whole line Hamiltonian W. By Theorem 3.8 we see that

$$m_\pm(x_n, z) \to M_\pm(z) \ \ (n \to \infty),$$

in value distribution. That is

$$\lim_{n\to\infty}\int_A \omega_{m_\pm(x_n,t)}(S)dt = \int_A \omega_{M_\pm(t)}(S)dt$$

for all Borel sets $A, S \subset \mathbb{R}, |A| < \infty$. Also by Theorem 3.5 we have

$$\int_A \omega_{M_-(t)}(-S)dt = \int_A \omega_{M_+(t)}(S)dt.$$

By Lebesgue differentiation theorem,

$$\omega_{M_-(t)}(-S) = \omega_{M_+(t)}(S) \tag{3.12}$$

for $t \in \Sigma_{ac}$ and all intervals S with rational end points. We can also assume that $M_\pm(t) = \lim_{y\to 0+} M(t + iy)$ exists for these t. Moreover, if $M_-(t) \in \mathbb{R}$, then, by choosing small intervals about this value for $-S$, we see that $M_+(t) = -M_-(t)$. If $M_-(t) \in \mathbb{C}$, then

$$\begin{aligned}
\omega_{M_-(t)}(-S) &= \int_{(-S)} \frac{v}{(t-u)^2 + v^2}dt \\
&= -\int_{(S)} \frac{v}{(t+u)^2 + v^2}dt \\
&= \omega_{-\overline{M_-(t)}}(S).
\end{aligned}$$

By 3.12 we get,

$$M_+(t) = -\overline{M_-(t)}. \tag{3.13}$$

In the case when $M_-(t) \in \mathbb{R}$ we already have $M_+(t) = -M_-(t)$. So 3.13 holds for almost every $t \in \Sigma_{ac}$, that is $W \in \mathcal{R}(\Sigma_{ac})$. This completes the proof.

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