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TITCHMARSH-WEYL THEORY FOR CANONICAL SYSTEMS

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Abstract. The main purpose of this paper is to develop Titchmarsh- Weyl theory of canonical systems. To this end, we first observe the fact that Schrödinger and Jacobi equations can be written into canonical systems. We then discuss the theory of Weyl m-function for canonical systems and establish the relation between the Weyl m-functions of Schrödinger equations and that of canonical systems which involve Schrödinger equations.

1. Introduction

The Titchmarsh-Weyl theory has been an essential tool in the spectral theory of Schrödinger operators, Jacobi operators and Sturn-Liouville differential operators. The origin of the theory goes back to 1910 when Weyl introduce this concept in his famous work in [16]. It was further studied by Titchmarsh [15] in 1962. The main object in the theory is the Titchmarsh-Weyl m-function which has close connection with the spectrum of the corresponding operators. Therefore, it is necessary to study the Titchmarsh-Weyl theory if we want to study direct and inverse spectral theory of such operators. The Titchmarsh-Weyl theory of Schrödinger and Jacobi equations has been studied very extensively. Only as a few reference, see [12, 13, 14]. There are several ways of defining these functions, but we give a basic definition here. For a one-dimensional Schrödinger expression $-\frac{d^2}{dx^2} + V(x)$ on a half-line $(0, \infty)$ with a bounded real-valued potential $V(x)$ that prevails limit point case, the Titchmarsh-Weyl m-function $m(z)$ may be defined as the unique coefficient, such that

$$f(x, z) = u(x, z) + m(z)v(x, z) \in L^2(0, \infty), \quad z \in \mathbb{C}^+,$$

where $u(x, z)$ and $v(x, z)$ are any two linearly independent solutions of

$$-y'' + V(x)y = zy$$

with some initial values $u(0, z) = v'(0, z) = 1, u'(0, z) = v(0, z) = 0$.

Likewise, for a Jacobi equation

$$a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n = zu_n$$

where, a_n, b_n are bounded sequence of real numbers, the Weyl m-function $m(z)$ is the unique coefficient such that

$$f_n(z) = u_n(z) + m(z)v_n(z) \in l^2(\mathbb{N})$$

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for $z \in \mathbb{C}^+$ where $u_n(z), v_n(z)$ are the basis of the solution space of Jacobi equation with the initial values $a_0 v_0(z) = u_1(z) = 0, v_1(z) = 1$ and $a_0 u_0 = -1$.

The main aim of this paper is to develop the Titchmarsh-Weyl theory for canonical systems and establish the relations between the Titchmersh-Weyl m-functions for the Schrödinger equations and that of canonical systems.

A canonical system is a family of differential equations of the form

$$Ju'(x) = zH(x)u(x), \quad z \in \mathbb{C} \tag{1.1}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $H(x)$ is a 2×2 positive semidefinite matrix whose entries are locally integrable. We also assume that there is no non-empty open interval I so that $H \equiv 0$ a.e. on I. The complex number $z \in \mathbb{C}$ involved in (1.1) is a spectral parameter. For fixed z, a vector valued function $u(.,z) : [0, N] \to \mathbb{C}^2$, $u(x,z) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$ is called a solution of (1.1)if u_1, u_2 are absolutely continuous and u satisfies (1.1). Consider the Hilbert space

$$L^2(H, R_+) = \Big\{ f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} : \int_0^\infty f(x)^* H(x) f(x) dx < \infty \Big\}$$

with the inner product $\langle f, g \rangle = \int_0^\infty f(x)^* H(x) g(x) dx$. Here $'*'$ denotes the complex conjugate transpose. Such canonical systems on the Hilbert space $L^2(H, R_+)$ have been studied by De snoo, Hassi, Remling and Winkler in [7, 8, 10, 17]. The canonical systems are closely connected with the theory of de Branges spaces and the inverse spectral theory of one dimensional Schrödinger operators, see [10]. We always get positive Borel measures, as the spectral measures, from Schrödinger operators. However, it is not always possible to get a potential that defines a Schrödinger operator, from a given positive Borel measure. This situation has been dealt in the inverse spectral theory of Schrödinger operators.

There is a one to one correspondence between positive Borel measures and canonical systems with $\operatorname{tr} H(x) \equiv 1$, see [17]. As we show that the Jacobi equations and Schrödinger equations can be written into canonical systems, we believe that canonical systems can be useful tools for inverse spectral theory of one dimensional Schrödinger operators. Thus, it is a natural context to consider the spectral theory of such systems. Morever, in order to discuss the spectral theory of canonical systems we need the corresponding spectral measure. The canonical systems can not be considered as an eigenvalue equation of an operator as $H(x)$ in the equation (1.1) is not invertible in general. Therefore, as in the case of Jacobi and Schrödinger operators, we can not use the spectral theorem to obtain spectral measures for canonical systmes. However, an alternate way to get the spectral measure is through Titchmarsh-Weyl m-functions. These m-functions are holomorphic functions mapping upper half-plane to itself; these are so called the Herglotz functions. In 1911, Gustav Herglotz, proved that every Herglotz function has integral representation with positive Borel measure, see [6]. For different version of the theorem, see [2]. The Borel measures in the integral representation of m-functions of the canonical systems are called the spectral measures for canonical systems which are not discussed in this paper though.

Moreover, the canonical systems contains the Jacobi and Schrödinger equations. Therefore it is also natural to think about extending the theories form Jacobi and Schrödinger operators to canonical systems.

First, we observe that the Jacobi and Schrödinger equations can be written into canonical systems.

2. Relation between Schrödinger equations, Jacobi equations and canonical systems

In this section, we show that the canonical systems contains the Jacobi and Schrödinger equations. More precisely, we show that the Jacobi and Schrödinger equations can be written as canonical systems. Let

$$-y'' + V(x)y = zy \tag{2.1}$$

be a Schrödinger equation. Suppose $u(x,z)$ and $v(x,z)$ are the linearly independent solutions of (2.1), with initial values $u(0,z) = v'(0,z) = 1, u'(0,z) = v(0,z) = 0$. Then $u_0 = u(x,0)$ and $v_0 = v_0(x,0)$ are solutions of $-y'' + V(x)y = 0$. Let

$$H(x) = \begin{pmatrix} u_0^2 & u_0v_0 \\ u_0v_0 & v_0^2 \end{pmatrix}$$

then the Schrödinger equation (2.1) is equivalent with the canonical system,

$$Jy'(x) = zH(x)y(x). \tag{2.2}$$

Let

$$T(x) = \begin{pmatrix} u(x,0) & v(x,0) \\ u'(x,0) & v'(x,0) \end{pmatrix}.$$

Then, if y solves equation (2.1) then $U(x,z) = T^{-1}(x)\begin{pmatrix} y(x,z) \\ y'(x,z) \end{pmatrix}$ solves the canonical system (2.2).

Alternate approach. Let

$$-y'' + V(x)y = z^2y \tag{2.3}$$

be a Schrödinger equation such that $-\frac{d^2}{dx^2} + V(x) \geq 0$ and $y(x,z)$ be its solution. Then $y_0 = y(x,0)$ be a solution of $-y'' + V(x)y = 0$. Let $W(x) = \frac{y_0'}{y_0}$ then $W^2(x) + W'(x) = V(x)$ so that equation (2.3) becomes

$$-y'' + (W^2 \pm W')y = z^2y.$$

Proposition 2.1. *The Schrödinger equation*

$$-y'' + (W^2 + W')y = z^2y \tag{2.4}$$

is equivalent with the canonical system

$$Ju'(x) = zH(x)u(x), \quad H(x) = \begin{pmatrix} e^{2\int_0^x W(t)dt} & 0 \\ 0 & e^{-2\int_0^x W(t)dt} \end{pmatrix}. \tag{2.5}$$

Proof. We claim that (2.4) is equivalent to the Dirac system

$$Ju' = \begin{pmatrix} z & W \\ W & z \end{pmatrix} u. \tag{2.6}$$

If y is a solution of (2.4), then $u = \begin{pmatrix} y \\ -\frac{1}{z}(-y' + Wy) \end{pmatrix}$ is a solution of (2.6). Also if $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a solution of (2.6) then u_1 is a solution of (2.4). Next we show that the Dirac system (2.6) is equivalent with the canonical system (2.5). For if u is a solution of (2.6) then $T_0 u$, where $T_0 = \begin{pmatrix} e^{-\int_0^x W(t)dt} & 0 \\ 0 & e^{\int_0^x W(t)dt} \end{pmatrix}$ is a solution of (2.5). Conversely if u is a solution of the canonical system (2.5) then $T_0^{-1}u$ is a solution of the Dirac system (2.6). $\square$

Proposition 2.2. *The Schrödinger equation*

$$-y'' + (W^2 - W')y = z^2 y \tag{2.7}$$

is equivalent with the canonical system

$$Ju'(x) = zH(x)u(x) \tag{2.8}$$

where

$$H(x) = \begin{pmatrix} e^{-2\int_0^x W(t)dt} & 0 \\ 0 & e^{2\int_0^x W(t)dt} \end{pmatrix}.$$

Proof. The Schrödinger equation (2.7) is equivalent with the Dirac system

$$Ju' = \begin{pmatrix} z & -W \\ -W & z \end{pmatrix} u. \tag{2.9}$$

In other words, if y is a solution of the Schrödinger equation (2.7) then $u = \begin{pmatrix} zy \\ y' + Wy \end{pmatrix}$ is a solution of the Dirac system (2.9). Conversely, if $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a solution of the Dirac system (2.9) then u_1 is a solution to the Schrödinger equation (2.7).

The Dirac system (2.9) is equivalent with the canonical system (2.8). If u is a solution of the Dirac system (2.9) then $y = T_0 u$,

$$T_0 = \begin{pmatrix} e^{\int_0^x W(t)dt} & 0 \\ 0 & e^{-\int_0^x W(t)dt} \end{pmatrix}$$

is a solution of the canonical system (2.8). Conversely if u is a solution of the canonical system (2.8) then $T_0^{-1}u$ is a solution of the Dirac system (2.9). $\square$

Let a Jacobi equation be

$$a(n)u(n+1) + a(n-1)u(n) + b(n)u(n) = zu(n). \tag{2.10}$$

This equation can be written as

$$\begin{aligned} \begin{pmatrix} u(n) \\ u(n+1) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\frac{a(n-1)}{a(n)} & \frac{z-b(n)}{a(n)} \end{pmatrix} \begin{pmatrix} u(n-1) \\ u(n) \end{pmatrix} \\ &= [B(n) + zA(n)] \begin{pmatrix} u(n-1) \\ u(n) \end{pmatrix}. \end{aligned}$$

Where

$$B(n) = \begin{pmatrix} 0 & 1 \\ -\frac{a(n-1)}{a(n)} & \frac{-b(n)}{a(n)} \end{pmatrix}, \quad A(n) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{a(n)} \end{pmatrix}.$$

Suppose $p(n,z)$ and $q(n,z)$ be the solutions of (2.10) such that $p(0,z)=1, p(1,z)=1$ and $q(0,z)=0, q(1,z)=1$. So that $p_0(n)=p(n,0)$ and $q_0(n)=q(n,0)$ be the solutions of equation (2.10) when $z=0$. Then

$$\begin{pmatrix} p_0(n) \\ p_0(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{a(n-1)}{a(n)} & \frac{-b(n)}{a(n)} \end{pmatrix} \begin{pmatrix} p_0(n-1) \\ p_0(n) \end{pmatrix}.$$

(similar expression for $q_0(n)$). Let

$$T(n) = \begin{pmatrix} p_0(n-1) & q_0(n-1) \\ p_0(n) & q_0(n) \end{pmatrix},$$

$T(1)=1$. Then we have the relation $T(n+1)=B(n)T(n)$. Now define $U(n,z)=T^{-1}(n+1)Y(n,z)$,

$$Y(n,z) = \begin{pmatrix} p(n-1,z) & q(n-1,z) \\ p(n,z) & q(n,z) \end{pmatrix}.$$

Then $U(n,z)$ solves an equation of the form

$$J\big(U(n+1,z) - U(n,z) = zH(n)U(n,z)\big) \tag{2.11}$$

where $H(n) = JT^{-1}(n+1)A(n)T(n)$. Suppose for each $n \in \mathbb{Z}$, on $(n,n+1)$, H has the form

$$H(x) = h(x)P_\phi, \quad P_\phi = \begin{pmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}$$

for some $\phi \in [0,\pi)$ and some $h \in L_1(n,n+1), h \geq 0$. (We may choose $h(x) \equiv 1$ on $(n,n+1)$ for each $n \in \mathbb{Z}$) Then the canonical system (1.1) reads

$$u'(x) = -zh(x)JP_\phi u(x).$$

Since the matrices on the right-hand side commute with one another for different values of x, the solution is given by

$$u(x) = \exp\Big(-z\int_a^x h(t)dtJP_\phi\Big)u(a).$$

However, $P_\phi JP_\phi = 0$, we see that the exponential terminates and we obtain

$$u(x) = \Big(1 - z\int_a^x h(t)dtJP_\phi\Big)u(a). \tag{2.12}$$

Clearly equation (2.12) is equivalent with the equation (2.11).

3. Weyl theory of canonical systems

For any $z \in \mathbb{C}$, the solution space of the canonical system (1.1) is a two dimensional vector space. Suppose f and g are solutions of (1.1), the Wronskian is defined as

$$W_x(f,g) = f_1(x,z)g_2(x,z) - f_2(x,z)g_1(x,z) = g(x,\bar{z})^*Jf(x,z)$$

Lemma 3.1. *The Wronskian $W_x(f,g)$ is constant for all x.*

Proof. If f and g are solutions of equation (1.1), then $Jf'(x,z) = zH(x)f(x,z)$ and $Jg'(x,z) = zH(x)g(x,z)$. Here $Jf'(x,\bar{z}) = \bar{z}H(x)f(x,\bar{z})$ and $-g'(x,z)^*J = \bar{z}g(x,z)^*H(x)$. From these two equations we have the following two equations

$$-g'(x,z)^*Jf(x,\bar{z}) = \bar{z}g(x,z)^*H(x)f(x,\bar{z}),$$

$$g(x,z)^* J f'(x,\bar{z}) = \bar{z} g(x,z)^* H(x) f(x,\bar{z}).$$

On subtraction we obtain, $\frac{d}{dx}(g(x,z)^* J f(x,\bar{z})) = 0$. It follows that $g(x,z)^* J f(x,\bar{z}))$ is constant and so is the Wronskian $W_x(f,g)$. $\square$

Let us write $\tau y = zy$ if and only if $Jy' = zH(x)y$. Suppose f and g are solutions of (1.1) then we have the following identity.

Lemma 3.2 (Green's Identity)**.**

$$\int_0^N (f^* H(x)\tau g - (\tau f)^* H(x) g) dx = W_0(\bar{f}, g) - W_N(\bar{f}, g)$$

Proof. Note that

$$\begin{aligned}
\int_0^N (f^* H(x)\tau g - (\tau f)^* H(x) g) dx &= \int_0^N (f^* H(x) z g - (zf)^* H(x) g) dx \\
&= \int_0^N (f^* z H(x) g - (zH(x) f)^* g) dx \\
&= \int_0^N (f^* J g' + f'^* J g) dx \\
&= \int_0^N (f^* J g' + f'^* J g) dx \\
&= \int_0^N \frac{d}{dx}(f^* J g) \\
&= W_0(\bar{f}, g) - W_N(\bar{f}, g)\,.
\end{aligned}$$

This completes the proof. $\square$

For any $z \in \mathbb{C}^+$, we want to define a coefficient $m(z)$ such that $f(x,z) = u(x,z) + m(z)v(x,z) \in L^2(H, R_+)$ for any linearly independent solutions $u(x,z), v(x,z)$ of (1.1). This leads us defining Weyl m functions $m_N(z)$ on compact interval $[0,N]$. Let u_α, v_α be the solution of (1.1) with the initial values

$$u_\alpha(0,z) = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix}, \quad v_\alpha(0,z) = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}, \quad , \alpha \in (0,\pi]. \tag{3.1}$$

For $z \in \mathbb{C}^+$, want to define $m_\alpha(z) \in \mathbb{C}$ as the unique coefficient for which

$$f_\alpha = u_\alpha + m_\alpha(z) v_\alpha \in L^2(H, \mathbb{R}_+).$$

Consider a compact interval $[0,N]$ and let $z \in \mathbb{C}^+$, define the unique coefficient $m_N^\beta(z)$ as follows, $f(x,z) = u(x,z) + m_N^\beta(z) v(x,z)$ satisfying

$$f_1(N,z)\sin\beta + f_2(N,z)\cos\beta = 0.$$

Clearly this is well defined because $u(x,z)$ does not satisfies the boundary condition at N. Otherwise $z \in C^+$ will be an eigenvalue for some self-adjoint relation of the system (1.1) as explained in [1]. From the boundary condition

$$f_1(N,z)\sin\beta + f_2(N,z)\cos\beta = 0$$

at N we obtain

$$m_N^\beta(z) = -\frac{u_1(N,z)\sin\beta + u_2(N,z)\cos\beta}{v_1(N,z)\sin\beta + v_2(N,z)\cos\beta}.$$

Since z, N, β varies, $m_N^\beta(z)$ becomes a function of these arguments, and since u_1, u_2, v_1, v_2 are entire function of z, it follows that $m_N^\beta(z)$ is meromorphic function of z. Rewrite the above equation in the form

$$m_N^\beta(z) = -\frac{u_1 t + u_2}{v_1 t + v_2}, \quad t = \tan\beta, \quad t \in \mathbb{R} \cup \{\infty\}.$$

This is a fractional linear transformation. As a function of $t \in \mathbb{R}$ it maps real line to a circle.

Let $C_N(z) = \{m_N^\beta(z) : 0 \le \beta < \pi\}$. So for fixed $z \in C^+$, $C_N(z)$ is a circle. Hence for any complex number $m \in \mathbb{C}$,

$$m \in C_N(z) \Leftrightarrow \operatorname{Im} \frac{u_2 + m v_2}{u_1 + m v_1} = 0 \tag{3.2}$$

Lemma 3.3. *The equation of the circle $C_N(z)$ is given by $|m - c|^2 = r^2$ where*

$$c = \frac{W_N(u, \bar{v})}{W_N(\bar{v}, v)}, \quad r = \frac{1}{|W_N(\bar{v}, v)|}\,. \tag{3.3}$$

Proof. Suppose $m \in C_N(z)$. By (3.2) we obtain,

$$\begin{aligned}
&\operatorname{Im} \frac{u_2 + m v_2}{u_1 + m v_1} = 0 \\
&\Rightarrow \frac{u_2 + m v_2}{u_1 + m v_1} - \frac{\bar{u}_2 + \bar{m}\bar{v}_2}{\bar{u}_1 + \bar{m}\bar{v}_1} = 0 \\
&\Rightarrow (u_2 + m v_2)(\bar{u}_1 + \bar{m}\bar{v}_1) - (\bar{u}_2 + \bar{m}\bar{v}_2)(u_1 + m v_1) = 0 \\
&\Rightarrow m\bar{m} W_N(\bar{v}, v) - m W_N(v, \bar{u}) - \bar{m} W_N(u, \bar{v}) + u_2\bar{u}_1 - \bar{u}_2 u_1 = 0 \\
&\Rightarrow m\bar{m} - m\frac{W_N(v, \bar{u})}{W_N(\bar{v}, v)} - \bar{m}\frac{W_N(u, \bar{v})}{W_N(\bar{v}, v)} + \frac{W_N(\bar{u}, u)}{W_N(\bar{v}, v)} = 0 \\
&\Rightarrow m\bar{m} - m\frac{W_N(v, \bar{u})}{W_N(\bar{v}, v)} - \bar{m}\frac{W_N(u, \bar{v})}{W_N(\bar{v}, v)} + \frac{W_N(u, \bar{v})}{W_N(\bar{v}, v)}\frac{W_N(\bar{u}, v)}{W_N(v, \bar{v})} \\
&\quad - \frac{W_N(u, \bar{v})}{W_N(\bar{v}, v)}\frac{W_N(\bar{u}, v)}{W_N(v, \bar{v})} + \frac{W_N(\bar{u}, u)}{W_N(\bar{v}, v)} = 0 \\
&\Rightarrow m\bar{m} - m\frac{W_N(v, \bar{u})}{W_N(\bar{v}, v)} - \bar{m}\frac{W_N(u, \bar{v})}{W_N(\bar{v}, v)} + \frac{W_N(u, \bar{v})}{W_N(\bar{v}, v)}\frac{W_N(\bar{u}, v)}{W_N(v, \bar{v})} \\
&\quad - \frac{1}{W_N(\bar{v}, v)W_N(v, \bar{v})} = 0 \\
&\Rightarrow m\bar{m} - m\bar{c} - \bar{m}c + c\bar{c} = r^2, \quad c = \frac{W_N(u, \bar{v})}{W_N(\bar{v}, v)}, \quad r = \frac{1}{|W_N(\bar{v}, v)|} \\
&\Rightarrow \quad |m - c|^2 = r^2\,.
\end{aligned}$$

This completes the proof. □

Now suppose $f(x, z) = u(x, z) + m_N^\beta(z) v(x, z)$, then $m = m_N^\beta$ is an interior of C_N if and only if

$$|m - c|^2 < r^2 \Leftrightarrow \frac{W_N(\bar{f}, f)}{W_N(\bar{v}, v)} < 0 \tag{3.4}$$

Using the Green's identity we have,

$$W_N(\bar{f}, f) = 2i \operatorname{Im} m(z) - 2i \operatorname{Im} z \int_0^N f^*(x) H(x) f(x) dx, \tag{3.5}$$

$$W_N(\bar{v},v) = -2i\operatorname{Im} z \int_0^N v^*(x)H(x)v(x)dx,$$
$$\frac{W_N(\bar{f},f)}{W_N(\bar{v},v)} = \frac{-\operatorname{Im} m(z) + \operatorname{Im} z \int_0^N f^*(x)H(x)f(x)dx}{\operatorname{Im} z \int_0^N v^*(x)H(x)v(x)dx}.$$

Hence from (3.4) we see that $\frac{W_N(\bar{f},f)}{W_N(\bar{v},v)} < 0$ if and only if

$$\int_0^N f^*(x)H(x)f(x)dx < \frac{\operatorname{Im} m(z)}{\operatorname{Im} z}.$$

Thus it follows that m is an interior of C_N if and only if

$$\int_0^N f^*(x)H(x)f(x)dx < \frac{\operatorname{Im} m(z)}{\operatorname{Im} z}, \tag{3.6}$$

and $m \in C_N(z)$ if and only if

$$\int_0^N f^*(x)H(x)f(x)dx = \frac{\operatorname{Im} m(z)}{\operatorname{Im} z}. \tag{3.7}$$

For $z \in \mathbb{C}^+$, the radius of the circle $C_N(z)$ is

$$r_N(z) = \frac{1}{|W_N(\bar{v},v)|} = \frac{1}{2\operatorname{Im} z \int_0^N v^*(x)H(x)v(x)dx}. \tag{3.8}$$

Now let $0 < N_1 < N_2 < \infty$. Then if m is inside or on C_{N_2}

$$\int_0^{N_1} f^*(x,z)H(x)f(x,z)dx < \int_0^{N_2} f(x,z)^*H(x)f(x,z)dx \le \frac{\operatorname{Im} m}{\operatorname{Im} z}$$

and therefore m is inside C_{N_1}. Let us denote the interior of $C_N(z)$ by $\operatorname{Int}C_N(z)$ and suppose $D_N(z) = C_N(z) \cup \operatorname{Int}C_N(z)$. Then

$$m \in D_N(z) \Leftrightarrow \int_0^N f^*(x)H(x)f(x)dx \le \frac{\operatorname{Im} m(z)}{\operatorname{Im} z}.$$

These are called the Weyl Disks. These Weyl Disks are nested. That is $D_{N+\epsilon}(z) \subset D_N(z)$ for any $\epsilon > 0$. From (3.8) we see that $r_N(z) > 0$, and $r_N(z)$ decreases as $N \to \infty$. So $\lim_{N\to\infty} r_N(z)$ exists and

$$\lim_{N\to\infty} r_N(z) = 0 \Leftrightarrow v \notin L^2(H,\mathbb{R}_+).$$

Thus for a given $z \in \mathbb{C}^+$ as $N \to \infty$ the circles $C_N(z)$ converges either to a circle $C_\infty(z)$ or to a point $m_\infty(z)$. If $C_N(z)$ converges to a circle, then its radius $r_\infty = \lim r_N$ is positive and (3.8) implies that $v \in L^2(H,\mathbb{R}_+)$. If $\tilde{m}_\infty$ is any point on $C_\infty(z)$ then $\tilde{m}_\infty$ is inside any $C_N(z)$ for $N > 0$. Hence

$$\int_0^N (u + \tilde{m}_\infty v)^* H(u + \tilde{m}_\infty v) < \frac{\operatorname{Im} \tilde{m}_\infty}{\operatorname{Im} z}$$

and letting $N \to \infty$ one sees that $f(x,z) = u + \tilde{m}_\infty v \in L^2(H,\mathbb{R}_+)$. The same argument holds if $\tilde{m}_\infty$ reduces to a point m_∞. Therefore, if $\operatorname{Im} z \ne 0$, there always exists a solution of (1.1) of class $\in L^2(H,\mathbb{R}_+)$. In the case $C_N(z) \to C_\infty(z)$ all solutions are in $L^2(H,\mathbb{R}_+)$ for $\operatorname{Im} z \ne 0$ and this identifies the limit-circle case with the existence of the circle $C_\infty(z)$. Correspondingly, the limit-point case is identified with the existence of the point $m_\infty(z)$. In this case $C_N(z) \to m_\infty$ there results $\lim r_N = 0$ and (3.8) implies that v is not of class $L^2(H,\mathbb{R}_+)$. Therefore in this situation there is only one linearly independent solution of class $L^2(H,\mathbb{R}_+)$. In the limit circle case $m \in C_N$ if and only (3.7) holds. Since $f(x,z) = u(x,z) + mv(x,z)$, it follows that m is on C_∞ if and only if

$$\int_0^\infty f(x,z)^* H f(x,z) dx = \frac{\operatorname{Im}(m(z))}{\operatorname{Im} z}. \tag{3.9}$$

From (3.5), it follows that m is on the limit circle if and only if $\lim_{N\to\infty} W_N(\bar{f}, f) = 0$. From the above discussion we proved the following theorem. This theorem is well known in the Weyl theory of Schrödinger operators, Jacobi operators and Sturm-Liouville differential operators.

Theorem 3.4. (1) *The limit-point case* ($r_\infty = 0$) *implies* (1.1) *having precisely one* $L^2(H,R_+)$ *solution.*
(2) *The limit-circle case* ($r_\infty > 0$) *implies that all solutions of* (1.1) *are in* $L^2(H,\mathbb{R}_+)$.

The identity (3.7) shows that $m_N^\beta(z)$ are holomorphic functions mapping upper-half plane to itself. The poles and zeros of these functions lie on the real line and are simple. In the limit-point case, the limit $m_\infty(z)$ is a holomorphic function mapping upper-half plane to itself. In limit-circle case, each circle $C_N(z)$ is traced by points $m = m_N^\beta(z)$ as β ranges over $0 \le \beta < \pi$ for fixed N and z. Let z_0 be fixed, $\operatorname{Im} z_0 > 0$. A point $\tilde{m}_\infty(z_0)$ on the circle $C_\infty(z_0)$ is the limit point of a sequence $m_{N_j}^{\beta_j}(z)$ with $N_j \to \infty$ as $j \to \infty$.

It has been shown in [1] that the canonical system with $\operatorname{tr} H \equiv 1$ implies the limit-point case. This means that, for $z \in \mathbb{C}^+$, there is a unique $L^2(H,\mathbb{R}_+)$ solution of canonical systems (1.1). In addition, it has been shown that if the solutions of canonical systems (1.1) are in $L^2(H,\mathbb{R}_+)$ for fixed $z_0 \in \mathbb{C}$ then it has all solutions in $L^2(H,\mathbb{R}_+)$ for all $z \in \mathbb{C}$. It also follows that if $H(x)$ in (1.1) has $\operatorname{tr} H \ge 1$ then it prevails the limit point case.

We would like to remark that the canonical systems (1.1) can be changed into equivalent canonical systems with the Hamiltonian H having trace norm 1. More precisely by a change of variable

$$t(x) = \int_0^x \operatorname{tr} H(s) ds, \tag{3.10}$$

a canonical system (1.1) can be reduced to a system with $\operatorname{tr} H \equiv 1$ which imply limit-point case. For if, $\widetilde{H}(t) = \frac{1}{\operatorname{tr} H(x)} H(x(t))$ so that $\operatorname{tr} \widetilde{H}(t) \equiv 1$. Further, let $u(x,z)$ be a solution of

$$Ju' = zHu$$

and put $\widetilde{u}(t,z) = u(x(t),z)$. Then $\widetilde{u}(t,z)$ solves

$$J\widetilde{u}' = z\widetilde{H}\widetilde{u}.$$

Their corresponding Weyl m functions on $[0,N]$ are related as follows,

$$\begin{aligned}
\tilde{m}_N^\beta(z) &= -\frac{\tilde{u}_1(N,z)\sin\beta + \tilde{u}_2(N,z)\cos\beta}{\tilde{v}_1(N,z)\sin\beta + \tilde{v}_2(N,z)\cos\beta} \\
&= -\frac{u_1(x(N),z)\sin\beta + u_2(x(N),z)\cos\beta}{v_1(x(N),z)\sin\beta + v_2(x(N),z)\cos\beta} \\
&= m_{x(N)}^\beta(z)
\end{aligned}$$

This shows that we obtain same Weyl circles even after changing the variable. The m function $\tilde{m}_N^\beta(z)$ of new system is obtained by changing the point of boundary condition from N to $x(N)$ of original system. Let $x(t)$ be the inverse function and define the new Hamiltonian $\widetilde{H}(t) = \frac{1}{\operatorname{tr} H(x)} H(x(t))$ so that $\operatorname{tr} \widetilde{H}(t) \equiv 1$. Let $u(x,z)$ be the solution of the original system

$$Ju' = zHu$$

and put $\widetilde{u}(t,z) = u(x(t),z)$. Then $\widetilde{u}(t,z)$ solves the new equation

$$J\widetilde{u}' = z\widetilde{H}\widetilde{u}.$$

Their corresponding Weyl m- functions on a compact interval $[0,N]$ are the same up to the change of the point of boundary condition, i.e. $\tilde{m}_N^\beta(z) = m_{x(N)}^\beta(z)$.

3.1. **Relation between Weyl m-functions.** We next observe the relation between the Weyl m-functions for Schrödinger equations and the canonical systems.

Theorem 3.5. *For $z \in \mathbb{C}^+$, let $m_s(z), m_c(z)$ denote the Weyl m-functions corresponding to the Schrödinger equation* (2.1) *and the canonical system* (2.2) *respectively. Then $m_s(z) = m_c(z)$.*

Proof. Let

$$T_s(x,z) = \begin{pmatrix} u(x,z) & v(x,z) \\ u'(x,z) & v'(x,z) \end{pmatrix}, \quad T_c(x,z) = \begin{pmatrix} u_1(x,z) & v_1(x,z) \\ u_2(x,z) & v_2(x,z) \end{pmatrix}$$

be the transfer matrices corresponding to the Schrödinger equation (2.1) and the canonical system (2.2) respectively. Let $T_0(x) = T_s(x,0)$ then in (2.2), $H(x) = T_0^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_0$. Here $m_s(z)$ is such that $(1,0)T_s(x,z)\begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} \in L^2(R_+)$ and $m_c(z)$ is such that $T_c(x,z)\begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} \in L^2(H,R_+)$. Note that $T_s(x,z) = T_0(x)T_c(x,z)$. It follows that

$$\begin{aligned}
&\int_0^\infty (1,\bar{m}_s)T_s^*(x,z)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_s(x,z)\begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty \\
&\Rightarrow \int_0^\infty (1,\bar{m}_s)T_c^*(x,z)T_0^*(x)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_0(x)T_c(x,z)\begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty \\
&\Rightarrow \int_0^\infty (1,\bar{m}_s)T_c^*(x,z)HT_c(x,z)\begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty.
\end{aligned}$$

Since the Weyl m- function $m_c(z)$ is uniquely defined we must have $m_s(z) = m_c(z)$. □

Theorem 3.6. *For $z \in \mathbb{C}^+$, let $m_s(z^2), m_c(z)$ denote the Weyl m-functions corresponding to the Schrödinger equation* (2.7) *and the canonical system* (2.8) *respectively. Then $m_s(z^2) = zm_c(z)$.*

Proof. Note that, since $H(x) = \begin{pmatrix} e^{2\int_0^x W(t)dt} & 0 \\ 0 & e^{-2\int_0^x W(t)dt} \end{pmatrix}$, it follows that $f \in L^2(H,\mathbb{R}_+)$ if and only if

$$\int_0^\infty |f_1|^2 e^{2\int_0^x W(t)dt} dx < \infty, \quad \int_0^\infty |f_2|^2 e^{-2\int_0^x W(t)dt} dx < \infty.$$

Let $T_s(x,z^2), T_d(x,z)$ and $T_c(x,z)$ denote the transfer matrices of the Schrödinger equation (2.4), the Dirac system (2.6) and the canonical system (2.5) respectively. Then

$$T_s(x,z^2) = \begin{pmatrix} u(x,z^2) & v(x,z^2) \\ u'(x,z^2) & v'(x,z^2) \end{pmatrix},$$

$$T_d(x,z) = \begin{pmatrix} u(x,z^2) & zv(x,z^2) \\ \frac{u'(x,z^2)-W(x)u(x,z^2)}{z} & v'(x,z)-W(x)v(x,z), \end{pmatrix},$$

$$T_c(x,z) = T_0 T_d(x,z).$$

It follows that

$$T_d(x,z) = \begin{pmatrix} z & 0 \\ -W & 1 \end{pmatrix} T_s(x,z^2) \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix}.$$

So $T_d(x,z) = T_0^{-1} T_c(x,z)$ and

$$T_s(x,z^2) = \frac{1}{z} \begin{pmatrix} 1 & 0 \\ W & z \end{pmatrix} T_d(x,z) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we have

$$\begin{aligned}
&\int_0^\infty (1, \bar{m}_c(z)) T_c^*(x,z) H(x) T_c(x,z) \begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} dx < \infty \\
&\Rightarrow \int_0^\infty (1, \bar{m}_c(z)) T_c^*(x,z) \Big[T_0^{-1}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_0(x)^{-1} \\
&\quad + T_0(x)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T_0(x)^{-1} \Big] T_c(x,z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty. \\
&\Rightarrow \int_0^\infty (1, \bar{m}_c(z)) T_d^*(x,z) T_0(x) T_0(x)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
&\quad \times T_0(x)^{-1} T_0(x) T_d(x,z) \begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} dx < \infty \\
&\Rightarrow \int_0^\infty (1, \bar{m}_c) \begin{pmatrix} /z & 0 \\ 0 & 1 \end{pmatrix} T_s^*(x,z^2) \begin{pmatrix} \bar{z} & W \\ 0 & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ -W & 1 \end{pmatrix} T_s(x,z^2) \begin{pmatrix} 1/z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} dx < \infty \\
&\Rightarrow \int_0^\infty \begin{pmatrix} 1/z & \bar{m}_c \end{pmatrix} T_s^*(x,z^2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T_s(x,z^2) \begin{pmatrix} 1/z \\ m_c(z) \end{pmatrix} dx < \infty.
\end{aligned}$$

Since the Weyl m- function $m_c(z)$ is uniquely defined we must have $m_s(z^2) = zm_c(z)$. □

Suppose

$$H_+ = \begin{pmatrix} e^{2\int_0^x W(t)dt} & 0 \\ 0 & e^{-2\int_0^x W(t)dt} \end{pmatrix}, \quad H_- = \begin{pmatrix} e^{-2\int_0^x W(t)dt} & 0 \\ 0 & e^{2\int_0^x W(t)dt} \end{pmatrix}$$

in the canonical system (2.5) and (2.8) respectively. The following lemma shows the relation between their Weyl m- functions.

Theorem 3.7. *If m_{c_+} and m_{c_-} are the Weyl m- function corresponding to the canonical system* (2.5) *and* (2.8) *respectively then $m_{c_+} = \frac{-1}{m_{c_-}}$.*

Proof. Notice that $-JH_+J = H_-$. Here u is a solution of $Ju' = zH_+u$ if and only if Ju is a solution of $Ju' = zH_-u$. Let $T_{c_+}(x)$ and $T_{c_-}(x)$ be the transfer matrices and m_{c_+} and m_{c_-} are the Weyl m- functions of the canonical systems with the Hamiltonians H_+ and H_- respectively. Then $T_{c_-}(x) = -JT_{c_+}(x)J$ and

$$\begin{aligned}
&\int_0^\infty (1, \bar{m}_{c_-}) T_{c_-}^*(x) H_- T_{c_-}(x) \begin{pmatrix} 1 \\ m_{c_-} \end{pmatrix} dx < \infty \\
&\Rightarrow \int_0^\infty (1, \bar{m}_{c_-})(-JT_{c_+}(x)J)^* H_- (-JT_{c_+}(x)J) \begin{pmatrix} 1 \\ m_{c_-} \end{pmatrix} dx < \infty \\
&\Rightarrow \int_0^\infty \begin{pmatrix} 1, & \frac{-1}{\bar{m}_{c_-}} \end{pmatrix} T_{c_+}^*(x) H_+ T_{c_+}(x) \begin{pmatrix} 1 \\ -1/\bar{m}_{c_-} \end{pmatrix} dx < \infty.
\end{aligned}$$

Since m_{c_+} is the unique coefficient such that

$$\int_0^\infty (1, \bar{m}_{c_+}) T_{c_+}^*(x) H_+ T_{c_+}(x) \begin{pmatrix} 1 \\ m_{c_+} \end{pmatrix} dx < \infty,$$

we have $m_{c_+} = \frac{-1}{m_{c_-}}$. □

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