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Titchmarsh–Weyl Theory for Vector-Valued Discrete Schrödinger Operators

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Titchmarsh–Weyl theory for vector-valued discrete Schrödinger operators

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Abstract
We develop the Titchmarsh–Weyl theory for vector-valued discrete Schrödinger operators. We show that the Weyl m functions associated with these operators are matrix valued Herglotz functions that map complex upper half plane to the Siegel upper half space. We discuss about the Weyl disk and Weyl circle corresponding to these operators by defining these functions on a bounded interval. We also discuss the geometric properties of Weyl disk and find the center and radius of the Weyl disk explicitly in terms of matrices.

1 Introduction

The theory of Titchmarsh–Weyl m functions has been an important tool in the spectral theory of Jacobi and Schrödinger operators. In order to study the asymptotic behavior of solutions of Jacobi and Schrödinger equations, one needs to study these m functions. Moreover, the absolutely continuous, singular continuous and essential spectrum of the operators associated with these equations are well explained in terms of m functions. These m functions were first introduced in 1910 by Weyl [16] for Sturn–Liouville differential equations for finding a square integrable solution. These were further studied by Titchmarsh [15] where he established the connection between the analyticity of solutions and the spectrum of the operator associated to Sturn–Liouville differential equations. For further history of m function, one can see [7]. The theory of m functions in one dimensional space has been widely studied, some of the studies can be found in the papers [2,8,12–14], and the literature therein.

In this paper, we extend the Titchmarsh–Weyl theory for vector-valued discrete Schrödinger operators. Consider d-dimensional discrete Schrödinger equations of the form,

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$$y(n+1)+y(n-1)+B(n)y(n)=zy(n), \quad z\in\mathbb{C} \tag{1}$$

where $y(n)\in\mathbb{C}^d$, and the potential $B(n)$ is a $d\times d$ matrix.

Equation (1) can be generalized to a $d-$dimensional Jacobi equation.

$$A(n)y(n+1)+A(n-1)y(n-1)+B(n)y(n)=zy(n), \quad z\in\mathbb{C} \tag{2}$$

with $A(n)$, $B(n)$ are sequences of $d\times d$ matrices bounded in l^2 norm. An operator induced by (2) can be expressed as a tri-diagonal block matrix, called block Jacobi matrix, [10]. Notice that (1) is a particular case of Jacobi equation with $A(n)\equiv I$, a $d\times d$ identity matrix.

Let $l^2(\mathcal{I},\mathbb{C}^d)$ is a Hilbert space of square summable vector-valued sequences with the inner product

$$\langle u,v\rangle=\sum_{n\in I}u(n)^*v(n),$$

where * stands for conjugate transpose, and denote the space of all $d\times d$ complex matrices by $\mathbb{C}^{d\times d}$.

Equation (1) induces a vector-valued discrete Schrödinger operator J on $l^2(\mathcal{I},\mathbb{C}^d)$ as

$$J\,y(n)=y(n+1)+y(n-1)+B(n)y(n).$$

Usually $\mathcal{I}=\mathbb{Z}$ or $\mathbb{N}$. When $\mathcal{I}=\mathbb{N}$, we need to modify the operator J for $n=1$ as

$$Jy(1)=y(2)+B(1)y(1).$$

It can be easily observed that if $B(n)$ is a Hermitian matrix, $B(n)^*=B(n)$, then J is a self-adjoint operator on $l^2(\mathbb{N},\mathbb{C}^d)$. The spectrum of J is then a set of real numbers $\sigma(J)\subset\mathbb{R}$.

To get a solution $u(n)$ of the Eq. (1), for fixed $z\in\mathbb{C}$, we chose any two vectors $c,d\in\mathbb{C}^d$ and fix the values $u(k)=c, u(k+1)=d$ and evolve according to (1). In particular, we fix $u(0)$ and $u(1)$ then $u(n)$ is obtained by solving the difference equation (1) using transfer matrices

$$T(m;z)=\begin{pmatrix} zI-B(m) & -I\\ I & 0\end{pmatrix}, \quad m=1,2,\ldots,n. \tag{3}$$

Let

$$\mathcal{A}(n;z)=T(n;z)T(n-1,z)\ldots T(1,z), \tag{4}$$

then, u solves (1) for every n if and only if

$$\begin{pmatrix} u(n+1)\\ u(n)\end{pmatrix}=\mathcal{A}(n;z)\begin{pmatrix} u(1)\\ u(0)\end{pmatrix}. \tag{5}$$

Similarly, $\mathcal{A}(n, m; z) = T(n; z)T(n-1, z)\dots T(m; z)$, for $m < n$ can be used to get a solution $u(n)$ from $u(m)$ by the following equation

$$\begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix} = \mathcal{A}(n, m; z) \begin{pmatrix} u(m) \\ u(m-1) \end{pmatrix}. \tag{6}$$

For every pair of vectors $u_i(n), v_i(n) \in \mathbb{C}^d$, there exists a solution of (1), therefore, the space of solutions is a $2d$-dimensional vector space. It is shown in [1], that there are exactly d linearly independent solutions of (1) that are in $l^2(\mathbb{N}, \mathbb{C}^d)$.

It is now convenient to fix a basis of the solution space of (1). An easier way is to prescribe a pair of initial conditions. For $z \in \mathbb{C}$, let

$$\begin{aligned} U(n, z) &= (u_1(n), u_2(n), \dots, u_d(n)), \\ u_i(n) &= (u_{1,i}(n), u_{2,i}(n), \dots, u_{d,i}(n))^T \\ V(n, z) &= (v_1(n), v_2(n), \dots, v_d(n)), \\ v_i(n) &= (v_{1,i}(n), v_{2,i}(n), \dots, v_{d,i}(n))^T \end{aligned}$$

be the sets of solutions of (1). Both $U(n, z)$ and $V(n, z)$ consist of d linearly independent solutions of (1). Suppose τ is the expression on left side of (1) then $U(n, z)$ and $V(n, z)$ are matrix valued solutions of

$$(\tau - z)u(n) = 0. \tag{7}$$

We further suppose that these solutions satisfy the following initial conditions

$$U(0, z) = -I, \quad V(0, z) = 0, \quad U(1, z) = 0, \quad V(1, z) = I. \tag{8}$$

By iterating (1), we see that for fixed $n \in \mathbb{N}$, $U(n, z)$, $V(n, z)$ are polynomial matrices in z of degree $n - 2$ over $\mathbb{C}^{d\times d}$. So $\overline{U(n, z)} = U(n, \bar{z})$ and $\overline{V(n, z)} = V(n, \bar{z})$.

Let

$$\mathbb{W}(n, z) = \begin{pmatrix} U(n+1, z) & V(n+1, z) \\ U(n, z) & V(n, z) \end{pmatrix}, \tag{9}$$

then (5) can be generalized for matrix valued solutions $U(n, z)$, $V(n, z)$ as

$$\mathbb{W}(n, z) = \mathcal{A}(n; z)\mathbb{W}(0, z) = \mathcal{A}(n; z)\mathbb{J}, \tag{10}$$

where $\mathbb{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

These solution sets satisfy the symplectic identity given by the following lemma.

Lemma 1 *For any* $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{W}(n, z)$ *satisfies*

$$\mathbb{W}(n, z)^T \mathbb{J}\mathbb{W}(n, z) = \mathbb{W}(n, z)\mathbb{J}\mathbb{W}(n, z)^T = \mathbb{J}. \tag{11}$$

Proof Notice that $T(n;z)^T \mathbb{J} T(n;z) = T(n;z)\mathbb{J}T(n;z)^T = \mathbb{J}$ for any n so that $\mathcal{A}(n;z)^T \mathbb{J}\mathcal{A}(n;z) = \mathcal{A}(n;z)\mathbb{J}\mathcal{A}(n;z)^T = \mathbb{J}$. Then

$$\begin{aligned}
\mathbb{W}^T \mathbb{J}\mathbb{W} &= (\mathcal{A}(n;z)\mathbb{J})^T \mathbb{J}\mathcal{A}(n;z) \\
&= \mathbb{J}^T \mathcal{A}(n;z)^T \mathbb{J}\mathcal{A}(n;z)\mathbb{J} \\
&= \mathbb{J}^T \mathbb{J}\mathbb{J} \\
&= \mathbb{J}.
\end{aligned}$$

Exactly in the same we also have $\mathbb{W}\mathbb{J}\mathbb{W}^T = \mathbb{J}$. □

We extend the definition of Wronskian for vector-valued sequences.

Definition 1 The Wronskian of any two sequences $f(n), g(n) \in l^2(\mathbb{N}, \mathbb{C}^d)$ is defined by

$$W_n(f, g) = f(n+1)^T g(n) - f(n)^T g(n+1). \tag{12}$$

In [1], it is shown that for fixed $z \in \mathbb{C}$, if $f(n,z)$, $g(n,z)$ are any two solutions to (1) then $W_n(f,g)$ is independent of n. Moreover, the Wronskian W_n is linear in both arguments.

For any two sequences $f(n), g(n) \in l^2(\mathbb{N}_0, \mathbb{C}^d)$ the Green's identity corresponding to Eq. (7) is given by

$$\sum_{j=0}^{n} \Big(f^*(\tau g) - (\tau f)^* g\Big)(j) = W_0(\bar{f}, g) - W_n(\bar{f}, g). \tag{13}$$

We extend the definition of Wronskian for matrix valued sequences $F(n), G(n)$, each contains d vector-valued sequence in $l^2(\mathbb{N}_0, \mathbb{C}^d)$. The Wronskian $W_n(F,G)$ is a $d \times d$ matrix valued function defined by

$$W_n(F, G) = F(n+1)^T G(n) - F(n)^T G(n+1). \tag{14}$$

A calculation shows that for fixed $z \in \mathbb{C}$, if $U(n,z)$ and $V(n,z)$ are any two matrix valued solutions of (1) then $W_n(U,V)$ is independent of $n \in \mathbb{N}$.

We also extend the Green's Identity for these matrix valued sequences and the proof of which is obtained by simple calculation.

$$\sum_{j=0}^{n} \Big(F(j,z)^*(\tau G(j,z)) - (\tau F(j,z))^* G(j,z)\Big) = W_0(\bar{F}, G) - W_n(\bar{F}, G). \tag{15}$$

2 Titchmarsh–Weyl *m* function

The study of Titchmarsh–Weyl theory for the second order difference equations was initiated by Hellinger [9] and Nevanlinna [11]. Their work focused on the existence of

$l^2(\mathbb{C})$ solutions and their properties. They also discussed on limit-point and limit-circle classification for the difference equations. Following their work, many scholars have worked since then on the theory of Titchmarsh–Weyl m functions in one dimension, which has been well developed. However, in higher dimensions, only few articles can be found in the continuous case [4–6] which consider a matrix-valued potential. In [4], the authors discussed the asymptotics of Titchmarsh–Weyl m functions corresponding to matrix valued Schrödinger operators and in [5], the Borg-type theorem for matrix valued Schrödinger operators has been proved. In [6], authors showed that the diagonal Green's matrix function and its derivative uniquely determine the matrix valued potential.

In this paper, we focus on developing the theory of Titchmarsh–Weyl m functions for vector-valued discrete Schrödinger operators. First, the Weyl m function is expressed in terms of resolvent operator similar to the one found in [1] and then show that such Weyl m functions are matrix valued Herglotz functions that map the complex upper half plane to the Siegel upper half space.

The Titchmarsh–Weyl m function for the vector-valued discrete Schrödinger operators associated to (1) is defined in terms of solutions as follows.

Definition 2 Let $z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The Titchmarsh–Weyl m function is defined as the unique complex matrix $M(z) \in \mathbb{C}^{d\times d}$ such that

$$F(n, z) = U(n, z) + V(n, z)M(z) \tag{16}$$

where $U(n, z)$, $V(n, z)$ are matrix valued solutions consisting of d linearly independent solutions with initial values (8) and the matrix valued solution $F(n, z)$ is a set of d linearly independent solutions of (1) that are in $l^2(\mathbb{N}, \mathbb{C}^d)$.

This definition, is in fact well defined. As we mentioned above that there are only d linearly independent solutions in $l^2(\mathbb{N}_0, \mathbb{C}^d)$, if there is another $M(z)$ satisfying (8), then the solutions from both $U(n, z)$ and $V(n, z)$ will be in $l^2(\mathbb{N}_0, \mathbb{C}^d)$. The solution $V(n, z)$ is such that $V(0, z) = 0$ which implies that $V(n, z)$ is the set of eigenfunctions for the self adjoint operator J. This contradicts that the spectrum of J is a set of real numbers. In [1], $M(z)$ is solved explicitly in terms of a solution and in terms of resolvent operator, the proof of which is worth presenting here for completeness of the paper.

Theorem 3 *Let $z \in \mathbb{C}^+$. If $(\tau - z)F = 0$ and F is a $d \times d$ matrix valued solution whose d columns are linearly independent solutions of* (1) *that are in $l^2(\mathbb{N}_0, \mathbb{C}^d)$. Then*

$$M(z) = -F(1, z)F(0, z)^{-1}. \tag{17}$$

Moreover,

$$M(z) = (m_{ij}(z))_{d\times d} \in \mathbb{C}^{d\times d}, \quad m_{ij}(z) = \langle \delta_j, (J - zI)^{-1}\delta_i \rangle, \tag{18}$$

where $\delta_i(n) \in l^2(\mathbb{N}, \mathbb{C}^d)$ is such that $\delta_i(n) = (0, \ldots, 0)^T$ if $n > 1$ and $\delta_i(1) = (0, \ldots, 1, \ldots 0)^T$ for $i = 1, 2, \ldots, d$.

Proof If the matrix valued solution F is given by (16) then $F(0,z) = -I$ and $F(1,z) = M(z)$. So (17) holds. Suppose $G(n,z)$ is any $d \times d$ matrix valued solution then it is a constant (matrix) multiple of the solution set $F(n,z)$ from (16) because (16) is a set of d linearly independent solutions. That is,

$$G(n,z) = F(n,z)C \tag{19}$$

where C is a $d \times d$ scalar invertible matrix.

$$F(n,z) = G(n,z)C^{-1}$$

so that

$$\begin{aligned} -G(1,z)G(0,z)^{-1} &= -F(1,z)CC^{-1}F(0,z)^{-1} \\ &= -F(1,z)F(0,z)^{-1} \\ &= M(z). \end{aligned}$$

Let $F(n,z)$ as in (17) and let

$$g_i = (J - zI)^{-1}\delta_i.$$

Then $(J - zI)g_i = \delta_i$. So $(\tau - zI)g_i(n) = 0$ for $n \geq 2$. Moreover $g_i \in l^2$ for all $i = 1, 2, \ldots\ldots., d$. Let

$$G(n,z) = (g_1(n), g_2(n), \ldots\ldots., g_d(n)).$$

Then $G(n,z) = F(n,z)C$, $C \in \mathbb{C}^{d\times d}$. By comparing values at $n = 1$,

$$G(1,z) = (g_1(1), g_2(1), \ldots\ldots\ldots, g_d(1))$$

where

$$g_1(1) = (J - zI)^{-1}\delta_1(1),$$

and

$$g_1(1) = (g_{11}, g_{21}, \ldots, \ldots\ldots, \ldots, g_{d1})^T, \quad g_{i1} = \langle \delta_i, g_1(1)\rangle, i = 1, 2, \ldots., d.$$

Then $M(z) = G(1,z)C^{-1}$ and $M(z) = (m_{ij}(z))$ is a $d \times d$ matrix such that $(m_{ij}(z)) = (\langle \delta_j, (J - zI)^{-1}\delta_i\rangle)C^{-1}$. To find the value of C, we compare values at $n = 2$. First $(J - z)G(1,z) = (\delta_1, \delta_2, \ldots\ldots, \delta_d)$ so

$$(J - zI)G(1,z) = I.$$

It follows that,

$$G(2,z) = (z - B(1))G(1,z) + I. \tag{20}$$

Since $F(n,z)$ is a solution to (1) we have

$$F(2,z) = (z - B(1))F(1,z) - F(0,z). \tag{21}$$

From (19) for $n = 2$ and from (21) we obtain

$$G(2,z) = (z - B(1))F(1,z)C - F(0,z)C. \tag{22}$$

Comparing (20) and (22), we obtain $-F(0,z)C = I$, and since $-F(0,z) = I$, $C = I$. Hence (18) holds. That is

$$M(z) = (\langle \delta_j, (J - zI)^{-1}\delta_i \rangle).$$

□

Theorem 3 connects $M(z)$ with a matrix valued Borel measure using functional calculus for the resolvent operators $\langle \delta_j, (J - z)^{-1}\delta_i \rangle$.

By functional calculus, for each i, j,

$$\begin{aligned} m_{ij}(z) &= \langle \delta_j, (J - z)^{-1}\delta_i \rangle \\ &= \int_{\mathbb{R}} \frac{1}{t - z} d\mu_{ij} \end{aligned} \tag{23}$$

where μ_{ij} is a spectral measure for the vectors δ_j and δ_i. Therefore,

$$M(z) = \int_{\mathbb{R}} \frac{1}{t - z} d\mu, \quad \mu = (\mu_{ij})_{d\times d},$$

and

$$M(z) = \int_{\mathbb{R}} \frac{1}{t - z} d\mu = \left(\int_{\mathbb{R}} \frac{1}{t - z} d\mu_{ij} \right)_{d\times d}.$$

The matrix valued measure μ is a spectral measure of the operator J.

For each i, j and $z \in \mathbb{C}^+$,

$$\operatorname{Im} m_{ij}(z) = \frac{1}{2i}(m_{ij}(z) - m_{ij}(\bar{z})) = \int_{\mathbb{R}} \frac{y}{|t - z|^2} d\mu_{ij} > 0. \tag{24}$$

Therefore, $m_{i,j}(z)$ maps the complex upper half plane to itself.

Suppose $\overline{M(z)}$ denotes the complex conjugate of $M(z)$. Then by (23), we have $m_{ij}(z) = m_{ij}(\bar{z})$ so that $\overline{M(z)} = M(\bar{z})$. Also, $M(z) = (\langle \delta_j, (J - zI)^{-1}\delta_i \rangle)$ so that

$$\begin{aligned} m_{ij}(z) &= \langle \delta_j, (J - zI)^{-1}\delta_i \rangle \\ &= \langle (J - \bar{z}I)(J - \bar{z}I)^{-1}\delta_j, (J - zI)^{-1}\delta_i \rangle \\ &= \langle (J - \bar{z}I)^{-1}\delta_j, (J - \bar{z}I)^{*}(J - z)^{-1}\delta_i \rangle \end{aligned} \tag{25}$$

Since J is self adjoint, $(J - \bar{z}I)^{*} = (J - zI)$. Then (25) becomes

$$\begin{aligned} m_{ij}(z) &= \langle (J - \bar{z}I)^{-1}\delta_j, \delta_i \rangle \\ &= \overline{\langle \delta_i, (J - \bar{z}I)^{-1}\delta_j \rangle} \\ &= \overline{m_{ji}(\bar{z})} \\ &= m_{ji}(z) \end{aligned} \tag{26}$$

for all i, j. It follows that $M(z)$ is symmetric.

Let $\mathcal{S}$ be a subspace of $\mathbb{C}^{d\times d}$, consisting of all symmetric matrices with positive definite imaginary part. That is,

$$\mathcal{S} = \left\{ M \in \mathbb{C}^{d\times d} : \frac{1}{2i}(M - M^{*}) > 0 \right\}.$$

The space $\mathcal{S}$ is so called the Seigel upper half space and can be considered as a generalization of complex upper half plane. The following theorem shows the Weyl m functions associated with the Schrödinger equations in higher dimensions are the matrices in $\mathcal{S}$.

Theorem 4 *For $z \in \mathbb{C}^{+}$, the map $z \mapsto M(z)$ maps $\mathbb{C}^{+}$ to $\mathcal{S}$.*

Proof Equation (26) shows that $M(z)$ is symmetric. Moreover, in (24) we see that every entry of $M(z)$ has positive imaginary part. Therefore, $\operatorname{Im} M(z) = \frac{1}{2i}(M(z) - M(z)^{*}) > 0$. □

As shown above, the entries of $M(z)$ are Herglotz functions mapping complex upper half plane to itself, therefore $M(z)$ is a matrix valued Herglotz function mapping complex upper half plane to Siegel upper half space.

3 Titchmarsh–Weyl circles and disks

In this section, we define the Titchmarsh–Weyl circles and disks. We consider (1) on $\mathbb{N}_{-} = \{0, 1, 2, \ldots\ldots, N\}$. Suppose $U(n, z)$, $V(n, z)$ are the matrix valued solutions of (1) with initial values (8). For $z \in \mathbb{C}^{+}$, define

$$F(n, z) = U(n, z) + V(n, z)M_N^{\beta}(z) \tag{27}$$

satisfying a boundary condition

$$\beta_2 F(N, z) + \beta_1 F(N+1, z) = 0 \tag{28}$$

where $\beta = (\beta_1, \beta_2) \in \mathbb{R}^{d\times 2d}$ satisfying

$$\beta^T \beta = I, \quad \beta J \beta^T = 0. \tag{29}$$

The unique coefficient $M_N^\beta(z)$ is called the Weyl m function on $\mathbb{N}_-$.

Finding $M_N^\beta(z)$ and using the boundary conditions (28), we have

$$M_N^\beta(z) = -\big(\beta_2 V(N, z) + \beta_1 V(N+1, z)\big)^{-1}\big(\beta_2 U(N, z) + \beta_1 U(N+1, z)\big). \tag{30}$$

Note that $\beta_2 V(N, z) + \beta_1 V(N+1, z)$ is invertible. Since z, N, β varies, $M_N^\beta(z)$ becomes a function of these arguments.

Lemma 2 *The Weyl m function $M_N^\beta(z)$ is symmetric.*

Proof Let $\mathbb{U}(z) = \begin{pmatrix} U(N+1) \\ U(N) \end{pmatrix}$ and $\mathbb{V}(z) = \begin{pmatrix} V(N+1) \\ V(N) \end{pmatrix}$, then

$$\mathbb{U}(z) = \mathcal{A}(N; z)\begin{pmatrix} U(1) \\ U(0) \end{pmatrix} = \mathcal{A}(N; z)\begin{pmatrix} 0 \\ -I \end{pmatrix},$$

and

$$\mathbb{V}(z) = \mathcal{A}(N; z)\begin{pmatrix} V(1) \\ V(0) \end{pmatrix} = \mathcal{A}(N; z)\begin{pmatrix} I \\ 0 \end{pmatrix}.$$

Using (30), the Weyl m function can be written as $M_N^\beta(z) = -(\beta\mathbb{V}(z))^{-1}(\beta\mathbb{U}(z))$. Suppose $E = \beta\mathbb{U}(z)$ and $F = \beta\mathbb{V}(z)$ so that $M_N^\beta(z) = -F^{-1}E$. Now,

$$\begin{aligned}
&M_N^\beta(z)^T - M_N^\beta(z) \\
&= (-F^{-1}E)^T - (-F^{-1}E) \\
&= F^{-1}E - E^T F^{-T} \\
&= F^{-1}(EF^T - FE^T)F^{-T} \\
&= F^{-1}\Big(\beta\mathbb{U}(\beta\mathbb{V})^T - \beta\mathbb{V}(\beta\mathbb{U})^T\Big)F^{-T} \\
&= F^{-1}\beta\Big(\mathbb{U}\mathbb{V}^T - \mathbb{V}\mathbb{U}^T\Big)\beta^T F^{-T} \\
&= F^{-1}\beta\Bigg(A(N; z)\begin{pmatrix} 0 \\ -I \end{pmatrix}\left(\mathcal{A}(N; z)\begin{pmatrix} I \\ 0 \end{pmatrix}\right)^T \\
&\quad - \mathcal{A}(N; z)\begin{pmatrix} I \\ 0 \end{pmatrix}\left(\mathcal{A}(N; z)\begin{pmatrix} 0 \\ -I \end{pmatrix}\right)^T\Bigg)\beta^T F^{-T}
\end{aligned}$$

$$
\begin{aligned}
&= -F^{-1}\beta\Big(\mathcal{A}(N;z)\Big(\begin{pmatrix}0\\-I\end{pmatrix}\begin{pmatrix}I & 0\end{pmatrix} - \begin{pmatrix}I\\0\end{pmatrix}\begin{pmatrix}0 & -I\end{pmatrix}\Big)\mathcal{A}(N;z)^T\Big)\beta^T F^{-T}\\
&= -F^{-1}\beta\Big(\mathcal{A}(N;z)J\mathcal{A}(N;z)^T\Big)\beta^T F^{-T}\\
&= -F^{-1}\beta J\beta^T F^{-T}\\
&= 0,
\end{aligned}
$$

since $\beta J\beta^T = 0$, according to (29). □

Lemma 3 *For a matrix valued solution* $F(n,z) = U(n,z) + M_N^{\beta}(z)V(n,z)$ *of* (1) *we have*

$$
W_N(\bar{F}, F) = 2i\,\mathrm{Im}\,M - 2i\,\mathrm{Im}\,z\sum_{j=0}^{N} F(j,z)^* F(j,z). \tag{31}
$$

Proof We use the Green's identity (15) with $G = F$.

$$
\sum_{j=0}^{N}\Big(F(j,z)^*(\tau F(j,z)) - (\tau F(j,z))^* F(j,z)\Big) = W_0(\bar{F}, F) - W_N(\bar{F}, F).
$$

It follows that,

$$
(z-\bar{z})\sum_{j=0}^{N} F(j,z)^* F(j,z) = W_0(\bar{F}, F) - W_N(\bar{F}, F). \tag{32}
$$

Using the linearity of the Wronskian given by (14) for $F(n,z) = U(n,z) + M_N^{\beta}(z)V(n,z)$ we have

$$
\begin{aligned}
(z-\bar{z})\sum_{j=0}^{N} F(j,z)^* F(j,z) &= W_0(\bar{F}, F) - W_N(\bar{F}, F)\\
&= W_0(\overline{U+VM}, U+VM) - W_N(\bar{F}, F)\\
&= W_0(\overline{U}, U) + W_0(\overline{U}, VM) + W_0(\overline{VM}, U)\\
&\quad + W_0(\overline{VM}, VM) - W_N(\bar{F}, F).
\end{aligned} \tag{33}
$$

Since $W_0(\overline{U}, U) = W_0(\overline{VM}, VM) = 0$, $W_0(\overline{VM}, U) = -\bar{M}$, $W_0(\overline{U}, VM) = M$, then (33) becomes

$$
(z-\bar{z})\sum_{j=0}^{N} F(j,z)^* F(j,z) = M - \bar{M} - W_N(\bar{F}, F)
$$

which implies

$$W_N(\bar{F}, F) = 2i \operatorname{Im} M - 2i \operatorname{Im} z \sum_{j=0}^{N} F(j,z)^* F(j,z).$$

□

Let $\mathcal{W}(n, z, M) = \mathbb{W}(n, z) \begin{pmatrix} I \\ M \end{pmatrix}$ and define a matrix function

$$E(M, N) = -i\mathcal{W}(N, z, M)^* J \mathcal{W}(N, z, M) \tag{34}$$

then

$$\begin{aligned} E(M, N) &= -i\big(F(N+1, z)^*, F(N, z)^*\big) J \begin{pmatrix} F(N+1, z) \\ F(N, z) \end{pmatrix} \\ &= -i W_N(\bar{F}, F) \\ &= -2 \operatorname{Im} M + 2 \operatorname{Im} z \sum_{j=0}^{N} F(j,z)^* F(j,z). \end{aligned} \tag{35}$$

Definition 5 Let $z \in \mathbb{C}^+$. The sets

$\mathcal{D}(N, z) = \{M \in \mathbb{C}^{d\times d} | E(M, N) \leq 0\}$ and $\mathcal{C}(N, z) = \{M \in \mathbb{C}^{d\times d} | E(M, N) = 0\}$

are respectively called the Weyl disk and Weyl circle.

Clearly, $\mathcal{C}(N, z) = \{M_N^\beta(z) : \beta \in \mathbb{R}^{d\times d},$ for some β satisfying (29)}.

Theorem 6 *The map $z \mapsto M_N^\beta(z)$ maps complex upper half plane $\mathbb{C}^+$ to Seigel half space $\mathcal{S}$.*

Proof By Lemma 2, $M_N^\beta(z)$ is symmetric. Since $M_N^\beta(z) \in C(N, z)$, $E(M, N) = 0$. It follows that $-iW_N(\bar{F}, F) = 0$. By Lemma 3, we have

$$2 \operatorname{Im} M - 2 \operatorname{Im} z \sum_{j=0}^{N} F(j,z)^* F(j,z) = 0.$$

That is

$$\frac{\operatorname{Im} M}{\operatorname{Im} z} = \sum_{j=0}^{N} F(j,z)^* F(j,z) > 0,$$

which implies that Im M is positive definite. □

Lemma 4 (Nesting property of Weyl disks) *Let* $z \in \mathbb{C}^+$. *Then*

$$\mathcal{D}(N+1, z) \subset \mathcal{D}(N, z), \quad N \in \mathbb{N}_0$$

Proof Let $M \in \mathcal{D}(N+1, z)$. From (35) we have

$$\begin{aligned} E(M, N) &= -2 \operatorname{Im} M + 2 \operatorname{Im} z \sum_{j=0}^{N} F(j, z)^* F(j, z) \\ &\leq -2 \operatorname{Im} M + 2 \operatorname{Im} z \sum_{j=0}^{N+1} F(j, z)^* F(j, z) \\ &= E(M, N+1) \leq 0. \end{aligned}$$

This shows that $M \in \mathcal{D}(N, z)$. Hence the result. □

From (34) we have

$$\begin{aligned} &E(M, N) \\ &= -i(I, M^*) \begin{pmatrix} U(N+1, z)^* & U(N, z)^* \\ V(N+1, z)^* & V(N, z)^* \end{pmatrix} J \begin{pmatrix} U(N+1, z) & V(N+1, z) \\ U(N, z) & V(N, z) \end{pmatrix} \begin{pmatrix} I \\ M \end{pmatrix} \\ &= -i(I, M^*) \begin{pmatrix} W_N(\bar{U}, U) & W_N(\bar{U}, V) \\ W_N(\bar{V}, U) & W_N(\bar{V}, V) \end{pmatrix} \begin{pmatrix} I \\ M \end{pmatrix} \\ &= -i\big(W_N(\bar{U}, U) + W_N(\bar{U}, V)M + M^* W_N(\bar{V}, U) + M^* W_N(\bar{V}, V)M\big) \end{aligned} \tag{36}$$

Note that

$$\begin{aligned} W_N(\bar{U}, U)^* &= -W_N(\bar{U}, U), \\ W_N(\bar{V}, V)^* &= -W_N(\bar{V}, V), \\ W_N(\bar{V}, U)^* &= -W_N(\bar{U}, V). \end{aligned} \tag{37}$$

Using (37) in (36), $E(M, N)$ can be written as

$$\begin{aligned} E(M, N) = -i\Big(&\big(M - W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)^*\big)^* W_N(\bar{V}, V) \\ &\big(M - W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)^*\big) \\ &+ W_N(\bar{U}, U) + W_N(\bar{U}, V) W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)^*\Big) \end{aligned} \tag{38}$$

Lemma 5 *For* $z \in \mathbb{C}^+$,

$$W_N(\bar{U}, V) W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)^* + W_N(\bar{U}, U) = -W_N(V, \bar{V})^{-1}. \tag{39}$$

Proof Let $\Omega = \mathbb{W}^* J \mathbb{W}$. Notice that $\mathbb{W}^* J \mathbb{W} = \begin{pmatrix} W_N(\bar{U}, U) & W_N(\bar{U}, V) \\ W_N(\bar{V}, U) & W_N(\bar{V}, V) \end{pmatrix}$. From Lemma 1 we see that

$$\mathbb{W}^* J \bar{\mathbb{W}} = J.$$

Then,

$$\begin{aligned} \Omega^T J \Omega &= (\mathbb{W}^* J \mathbb{W})^T J (\mathbb{W}^* J \mathbb{W}) \\ &= \mathbb{W}^T J^T \mathbb{W}^{*^T} J \mathbb{W}^* J \mathbb{W} \\ &= -\mathbb{W}^T J \mathbb{W} \\ &= J. \end{aligned} \tag{40}$$

On the other hand,

$$\begin{aligned} \Omega^T J \Omega &= \begin{pmatrix} W_N(\bar{U}, U)^T & W_N(\bar{V}, U)^T \\ W_N(\bar{U}, V)^T & W_N(\bar{V}, V)^T \end{pmatrix} J \begin{pmatrix} W_N(\bar{U}, U) & W_N(\bar{U}, V) \\ W_N(\bar{V}, U) & W_N(\bar{V}, V) \end{pmatrix} \\ &= \begin{pmatrix} W_N(U, \bar{U})^* & W_N(V, \bar{U})^* \\ W_N(U, \bar{V})^* & W_N(V, \bar{V})^* \end{pmatrix} J \begin{pmatrix} W_N(\bar{U}, U) & W_N(\bar{U}, V) \\ W_N(\bar{V}, U) & W_N(\bar{V}, V) \end{pmatrix}. \end{aligned} \tag{41}$$

Using (40) and (41) we have

$$- W_N(V, \bar{V})^* W_N(\bar{U}, U) + W_N(U, \bar{V})^* W_N(\bar{V}, U) = -I, \tag{42}$$

and

$$- W_N(V, \bar{V})^* W_N(\bar{U}, V) + W_N(U, \bar{V})^* W_N(\bar{V}, V) = 0. \tag{43}$$

From (43) we obtain

$$\begin{aligned} W_N(\bar{U}, V)^* &= W_N(\bar{V}, V)^* W_N(U, \bar{V}) W_N(V, \bar{V})^{-1} \\ &= -W_N(\bar{V}, V) W_N(U, \bar{V}) W_N(V, \bar{V})^{-1}. \end{aligned} \tag{44}$$

Using (44) on the left side of (39) we obtain

$$\begin{aligned} & W_N(\bar{U}, V) W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)^* + W_N(\bar{U}, U) \\ &\quad = W_N(\bar{U}, V) W_N(\bar{V}, V)^{-1} \big(- W_N(\bar{V}, V) W_N(U, \bar{V}) W_N(V, \bar{V})^{-1} \big) \\ &\qquad + W_N(\bar{U}, U) \\ &\quad = -W_N(\bar{U}, V) W_N(U, \bar{V}) W_N(V, \bar{V})^{-1} + W_N(\bar{U}, U). \end{aligned} \tag{45}$$

It follows from (42) that

$$W_N(U, \bar{V})^* W_N(\bar{V}, U) = -I + W_N(V, \bar{V})^* W_N(\bar{U}, U),$$

and by using (37), it yields

$$-W_N(U, \bar{V})^* W_N(\bar{U}, V)^* = -I + W_N(V, \bar{V})^* W_N(\bar{U}, U). \tag{46}$$

Taking the complex conjugate on both sides of (46) we have

$$W_N(\bar{U}, V) W_N(U, \bar{V}) = I - W_N(\bar{U}, U)^* W_N(V, \bar{V})$$

which, by (37), becomes

$$W_N(\bar{U}, V) W_N(U, \bar{V}) = I + W_N(\bar{U}, U) W_N(V, \bar{V}).$$

Then the right side of (45) becomes

$$\begin{aligned} &- W_N(\bar{U}, V) W_N(U, \bar{V}) W_N(V, \bar{V})^{-1} + W_N(\bar{U}, U) \\ &\quad = -(I + W_N(\bar{U}, U) W_N(V, \bar{V})) W_N(V, \bar{V})^{-1} + W_N(\bar{U}, U) \\ &\quad = -W_N(V, \bar{V})^{-1}. \end{aligned}$$

Thus, we have

$$W_N(\bar{U}, V) W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)^* + W_N(\bar{U}, U) = -W_N(V, \bar{V})^{-1}.$$

□

Using Lemma 5 and (38) we can express $E(M, N)$ in the form

$$\begin{aligned} E(M, N) = -i\Big(&\big(M - W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)^*\big)^* W_N(\bar{V}, V) \\ &\big(M - W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)^*\big) - W_N(V, \bar{V})^{-1}\Big). \end{aligned} \tag{47}$$

Thus, (47) is

$$E(M, N) = -\big((M - C_N(z))^* R(N, z)^{-2} (M - C_N(z)) - R(N, \bar{z})^2\big) \tag{48}$$

where $C_N(z) = W_N(\bar{V}, V)^{-1} W_N(\bar{U}, V)^*$ and $R(N, z) = (i W_N(\bar{V}, V))^{-1/2}$.

So the equation of Weyl circle can be written as

$$(M - C_N(z))^* R(N, z)^{-2} (M - C_N(z)) = R(N, \bar{z})^2. \tag{49}$$

Theorem 7 *For all* $z \in \mathbb{C}^+$, $\lim_{N \to \infty} R(N, z)$ *exists and* $\lim_{N \to \infty} R(N, z) \geq 0$.

Proof By Green's identity we have

$$2 \operatorname{Im} z \sum_{j=0}^{N} V(j, z)^* V(j, z) = i W_N(\bar{V}, V) = R(N, z)^{-2} > 0.$$

Thus, $R(N, z)$ is non increasing and $\lim_{N\to\infty} R(N, z)$ exists. □

Theorem 8 *For all* $z \in \mathbb{C}^+$, $\lim_{N\to\infty} C_N(z)$ *exists.*

Proof Let $M \in \mathcal{C}(N, Z)$, then using (49) we get

$$(M - C_N(z))^* R(N, z)^{-2}(M - C_N(z)) = R(N, \bar{z})^2.$$

It follows that

$$\left(R(N, z)^{-1}(M - C_N(z))R(N, \bar{z})^{-1}\right)^* \left(R(N, z)^{-1}(M - C_N(z))R(N, \bar{z})^{-1}\right) = I.$$

Suppose $U = \left(R(N, z)^{-1}(M - C_N(z))R(N, \bar{z})^{-1}\right)$ so that $U^*U = I$ that is U is unitary. Also,

$$M = C_N(z) + R(N, z)UR(N, \bar{z}).$$

Suppose $M \in C_{N+1}(z) \subset C_N(z)$ then we have

$$M = C_{N+1}(z) + R(N+1, z)U_{N+1}R(N+1, \bar{z}), \tag{50}$$

and

$$M = C_N(z) + R(N, z)U_N R(N, \bar{z}). \tag{51}$$

Equating (50) and (51) and taking the norm we obtain

$$\begin{aligned}
&\|C_{N+1}(z) - C_N(z)\| \\
&\quad = \|R(N+1, z)U_{N+1}R(N+1, \bar{z}) - R(N, z)U_N R(N, \bar{z})\| \\
&\quad \le \|R(N+1, z)U_{N+1}R(N+1, \bar{z}) - R(N, z)U_{N+1}R(N+1, \bar{z})\| \\
&\qquad + \|R(N, z)U_{N+1}R(N+1, \bar{z}) - R(N, z)U_N R(N+1, \bar{z})\| \\
&\qquad + \|R(N, z)U_N R(N+1, \bar{z}) - R(N, z)U_N R(N, \bar{z})\| \\
&\quad \le \|R(N+1, z) - R(N, z)\|\|U_{N+1}\|\|R(N+1, \bar{z})\| \\
&\qquad + \|R(N, z)\|U_{N+1} - U_N\|\|R(N+1, \bar{z})\| \\
&\qquad + \|R(N, z)\|\|U_N\|\|R(N+1, \bar{z}) - R(N, z)\|.
\end{aligned}$$

This shows that $C_N(z)$ is a Cauchy sequence, hence converges. □

Let $C_0(z) = \lim_{N\to\infty} C_N(z)$ and $R_0(z) = \lim_{N\to\infty} R(N, z)$.
Define

$$D_0(z) = \{M \in \mathbb{C}^{d\times d} : (M - C_0(z))^* R_0(z)^{-2}(M - C_0(z)) \le R_0(\bar{z})^2\} \tag{52}$$

then

$$D_0(z) = \cap_{N\ge 1} D(N, z). \tag{53}$$

Theorem 9 *Let* $z \in \mathbb{C}^+$ *and* $M \in \mathbb{C}^{d\times d}$. *Then for* $F(N,z) = U(N,z) + V(N,z)M$, $M \in \mathcal{D}_0(z)$ *if and only if*

$$\sum_{N=1}^{\infty} F(N,z)^* F(N,z) \le \frac{\operatorname{Im} M}{\operatorname{Im} z}.$$

Proof Let $M \in \mathcal{D}_0(z)$. Then by (53), $M \in \mathcal{D}(N,z)$ for all N, and from (35) we have

$$E(M,N) = -2\operatorname{Im} M + 2\operatorname{Im} z \sum_{j=0}^{N} F(j,z)^* F(j,z) \le 0$$

which yields

$$\sum_{j=0}^{N} F(j,z)^* F(j,z) \le \frac{\operatorname{Im} M}{\operatorname{Im} z}.$$

Taking the limit as $N \to \infty$ we get

$$\sum_{N=1}^{\infty} F(N,z)^* F(N,z) \le \frac{\operatorname{Im} M}{\operatorname{Im} z}.$$

Conversely, for any N we have

$$\sum_{j=1}^{N} F(j,z)^* F(j,z) \le \sum_{j=1}^{\infty} F(j,z)^* F(j,z) \le \frac{\operatorname{Im} M}{\operatorname{Im} z}.$$

So $E(M,N) \le 0$ for all N and hence $M \in \mathcal{D}_0(z)$. □

4 Conclusion

This paper generalized the classical Titchmarsh–Weyl theory of Schrödinger operators from one dimension to higher dimensions and it provided a foundation for studying spectral theory of multi valued discrete Schrödinger operators. The spectrum of these operators can be described in terms of Weyl m functions as such we can study the asymptotic behavior of solutions. Moreover, these Weyl m functions are the building blocks for extending the Breimesser–Pearson's results [3], and Remling's result [12], for vector-valued Schrödinger operators.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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