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# Stochastic Failure Models Based Upon Distribution of Stress Peaks

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## Summary

This paper summarizes the development of three stress-strength models of reliability in which the stress producing mechanism is a one dimensional random process evolving in time. The general reliability functions are given along with examples of special cases. The problem of modeling randomly deteriorating strength is briefly discussed and one model is discussed from the point of view of time series analysis.

## Introduction

Stress and strength are terms which are used rather loosely to describe physical properties of components and their operating environment. A component is said to "fail" if its physical state deteriorates to some condition in which it is inoperative, unsafe, or performs outside acceptable tolerance limits. Its failure may be "catastrophic", i.e., a near instantaneous transfer to the failed state or it may undergo "wear" which means roughly that its performance deteriorates more gradually although not necessarily continuously.

A component experiences failure as a result of usage in a "stress" environment which can include such stressors as heat, voltage, hydraulic pressure, radiation, vibration, shock, and acceleration. A measure of component strength is its ability to resist the collective effects of a set of stress forces, i.e., its ability to maintain a performance level under an environmental stress profile. The "total strength" of a component is an ill defined quantity but theoretically it represents the component's resistance to deterioration under the cumulative effect of all stress forces acting during its period of performance.

Earlier "stress-strength" models of component reliability assumed that the component possessed a certain strength X and when placed in operation under a stress Y that failure occurred whenever stress exceeded strength during the period of operation, i.e, whenever $Y > X$. In the literature one finds X and Y to be assigned normal distributions $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$, respectively so that

$$\text{Reliability} = P(Y < X) = 1 - \phi\left(\frac{\mu_y - \mu_x}{\sqrt{\sigma_x^2 + \sigma_y^2}}\right) \qquad (1.0)$$

where $\phi(Z)$ is the cumulative standard normal distribution function.

This model assumes that Y represents the intensity of the maximum stress occurring during the interval of operation and that failure occurs if and only if the peak stress exceeds the component strength which is assumed to be selected at random from a normal distribution but remains fixed during the period of operation once it is selected. The time parameter is suppressed as is also the mechanism generating stress peaks among which occurs the maximum. Whether or not X and Y represent a single environmental stress variable or a "total stress" variable which, in effect, means that they are transformations from some multidimensional stress-strength space, is left to the analyst to decide.

The model given by Equation (1.0) can be easily generalized to the case where Y and X have gamma distributions with parameters $(\lambda, \ell)$ and $(\lambda(1 + \mu), k)$, respectively $(\mu > 0)$, so that

$$\text{Reliability} = P(X > Y) = B\left(k\text{-}1;\ k+\ell\text{-}1,\ \frac{1 + \mu}{2 + \mu}\right) \qquad (2.0)$$

where $B(x; n, p)$ denotes the cumulative binomial distribution with parameters n and p summed from 0 through x.

Models (1.0) and (2.0) permit neither an explicit estimate of life length (unless time appears as a parameter) nor a controlled variation in the parameters of the underlying stress environment that actually produces the sequence of stress peaks or pulses that are assumed to cause the deterioration.

In what follows a stress environment is represented explicitly in terms of the distribution of occurrences of stress peaks and their intensities. Component strength is at first held constant and then assumed to behave in some time dependent but deterministic manner. Failure events are defined and their probabilities of occurrence are presented. Finally the more difficult case of random deterioration of strength is considered and a particular model is developed. Stress and strength are represented as single variables. The question of whether the models presented can be valid one dimensional representations of the combined effects of a multidimensional stress environment is not discussed. A final point made in the paper concerns the inclusion of the reliability functions developed herein within the class of IHRA (increasing hazard rate average) life distributions as defined by Birnbaum, Esary, and Marshall in Reference 1. Derivations of the models presented in this paper are contained in Reference 2.

## Constant Strength Models

### Model 1: Stress Process Stationary with Independent Increments

Components are assumed to function in a random environment characterized by a sequence of stress "shocks" or "pulses" having an arbitrary but fixed distribution $F(x)$ of stress intensity. Let $\{N(t);\ t \geq 0\}$ denote a stationary random process with independent increments in which $N(t)$ is the number of pulses occurring in the time interval $(0, t)$. Stress peaks are assumed to be mutually independent and independent of $N(t)$. Let $Y_n(t)$ denote the maximum stress intensity occurring in $(0, t)$ given that n peaks occurred.

Then

$$P(Y_n(t) \leq x) = (F(x))^n \quad (n = 0,1,2, \ldots)$$

and the probability of occurrence of the joint event that n stress peaks occur in $(0, t)$ and the maximum does not exceed x in intensity, is

$$P(Y_n(t) \leq x)\, P(N(t) = n)$$

$$= (F(x))^n\, P(N(t) = n)$$

$$(n = 0,1,2, \ldots)$$

The distribution function $P(Y(t) \leq x)$ of the maximum stress intensity occurring in $(0, t)$ is therefore

$$P(Y(t) \leq x) = \sum_{n=0}^{\infty} P(N(t) = n)\, (F(x))^n \tag{3.0}$$

Example. Let $\{N(t);\ t \geq 0\}$ be a stationary Poisson process of intensity $\lambda$. Then for $x > 0$ the number $M(t,x)$ of stress pulses occurring in $(0, t)$ and exceeding x in intensity is Poisson distributed with parameter $\lambda(1 - F(x))$, i.e.,

$$P(M(t, x) = n)$$

$$= \frac{(\lambda(1 - F(x))t)^n}{n!} e^{-\lambda(1 - F(x))t} \tag{3.1}$$

$$(n = 0,1,2, \ldots)$$

Therefore if component failure is defined to be the event that m or more stress pulses exceed x in intensity during $(0, t)$ for some predetermined m and x, then

$$\text{Reliability} = 1 - \sum_{i=m}^{\infty} P(M(t, x) = i)$$

$$= \sum_{i=0}^{m-1} \frac{(\lambda(1 - F(x))t)^i}{i!} e^{-\lambda(1 - F(x))t} \tag{3.2}$$

This model can be used to estimate the increase in component life length that can be achieved by increasing component strength x or by decreasing the occurrence rate $\lambda$ of stress shocks impinging upon the component. For instance, suppose that the distribution of stress peak intensity is negative exponential with mean $1/\theta$. The number of stress peaks occurring in $(0, t)$ is Poisson with mean $\lambda t$. Failure occurs if at least one pulse exceeds x in intensity. The mean time to failure of a component is therefore $\frac{1}{\lambda(1 - F(x))} = \frac{1}{\lambda} e^{x/\theta}$ which shows the relative effect of stress peak intensity and the occurrence rate of stress peaks upon component life length.

Stress Process Non-homogeneous Poisson with Independent Increments

The generalization here concerns the instantaneous rate of occurrence of stress pulses. In the present case, let $h(t)dt$ = instantaneous probability of occurrence of a stress pulse in $(t, t + dt)$. All other assumptions remain the same.

Let $P_n(t, x)$ denote the probability that n stress pulses exceed level x in intensity during $(0, t)$.

Thus, one can write

$$P_n(t + \Delta t, x) = P_n(t, x)[1-h(t)\Delta t(1-F(x))]$$

$$+ P_{n-1}(t, x)h(t)\Delta t(1-F(x))$$

$$+ O(\Delta t); \quad (n = 1,2, \ldots)$$

$$P_o(t + \Delta t, x) = P_o(t, x)[1-h(t)\Delta t(1-F(x))];$$

$$(n = 0)$$

By forming the difference quotient on the left hand side with respect to t and taking limits, one arrives at a system of differential difference equations having the solution

$$P_n(t, x) = \frac{\left[(1 - F(x)) \int_0^t h(\alpha)d\alpha\right]^n}{n!} \cdot e^{-(1 - F(x)) \int_0^t h(\alpha)d\alpha} \tag{4.0}$$

As before failure may be defined as the event that m or more pulses exceed level x in $(0, t)$ and the reliability function is the corresponding generalization of Equation (3.2).

Model 2: Stress Peaks Defined by a Renewal Process

A renewal process is defined to be a sequence of non negative identically distributed and mutually independent random variables $\{X_i;\ i = 1, 2, \ldots\}$. In the present context $X_i$ represents the length of the time interval separating the (i-1)st and i-th stress peak. Denote the common distribution function of the $X_i$ as $G(t)$ with density $g(t)$. As before the common distribution function of the intensity of each stress peak is $F(x)$. The assumption concerning $G(t)$ is more general in this case than in the two previous models since the number of stress pulses occurring in $(0, t)$ need not be Poisson. When failure is defined to be the event that at least one pulse exceed level x in intensity during $(0, t)$, it is shown in Reference 2 that the Laplace transform of the reliability function $R(t, x)$ is

$$\mathscr{L}(R(t, x)) = R^*(s, x)$$

$$= \frac{1 - g^*(s)}{s[1 - F(x)\, g^*(s)]} \tag{5.0}$$

where
$g^*(s)$ is the Laplace transform of $g(t)$.

Example. Let

$$g(t) = \frac{\lambda(\lambda t)}{2!} e^{-\lambda t}$$

so that $$g^*(s) = \left(\frac{\lambda}{s + \lambda}\right)^2 .$$

Then

$$R^*(s, x) = \frac{1 - \left(\frac{\lambda}{s + \lambda}\right)^2}{s[1 - F(x)\left(\frac{\lambda}{s + \lambda}\right)^2]} .$$

Upon inversion,

$$R(t, x) = \frac{1}{2\sqrt{F(x)}} [(1+\sqrt{F(x)})e^{-\lambda(1-\sqrt{F(x)})t}$$

$$- (1-\sqrt{F(x)})e^{-\lambda(1+\sqrt{F(x)})t}] \quad (5.1)$$

Using the facts

(a) $sR^*(s, x) - R(0+, x) = R'^*(s, x)$

and

(b) $- \frac{d}{ds} R'^*(s, x)\Big|_{s=0} =$ mean life length

one can compute the mean life length of a component whose reliability is given by Equation (5.0). In this particular case the mean life length is computed to be

$$\frac{2}{\lambda(1 - F(x))} .$$

The precision of the estimate of life length can be computed working with the first and second derivatives of the transform (5.0) in the usual way.

## Variable Strength Models

The assumption concerning component strength differs basically in that strength x is assumed to be a time dependent function

$$x = m(t)$$

where m(t) is defined at those non negative values of t for which $x \geq 0$. For what follows it is assumed that m(t) is continuous in t.

### Model 3: Stress Peaks Defined by a Non-homogeneous Poisson Process

Let strength x be time dependent so that $x = m(t)$, satisfying the conditions stated above. Let the instantaneous probability of occurrence of a stress peak in $(t, t + \Delta t)$ be $h(t)\Delta t$ $(h(t) \geq 0)$ and independent of the instants at which peaks occurred previously. The distribution function of the stress peak intensity is F(x). Then

Prob [n stress peaks exceed strength

$x = m(t)$ in (0, t)]

$$= \frac{(c(t))^n}{n!} e^{-c(t)} \quad p_n(t) \quad (6.0)$$

where

$$c(t) = \int_0^t h(x)[1 - F(m(x))]dx$$

If failure in (0, t) is defined as the event that k or more stress pulses exceed strength in (0, t), then component reliability is

$$P_{k-1}(t) \equiv \sum_{n=0}^{k-1} \frac{(c(t))^n}{n!} e^{-c(t)} \quad (6.1)$$

Example. Let

$$x = m(t) = b - at^2 \quad (0 \leq t \leq \sqrt{b/a})$$
$$= 0, \quad (t > \sqrt{b/a})$$
$$h(t) = \lambda t,$$
$$F(t) = 1 - e^{-\theta t},$$
$$k = 1.$$

Then for

$$0 \leq t \leq \sqrt{b/a},$$

$$P_o(t) = \exp \{-[(\frac{\lambda e^{-\theta b}}{2\theta a})(e^{\theta at^2}-1)]\} ;$$

for

$$t > \sqrt{b/a},$$

$$P_o(t) = \exp \{-\lambda[\frac{1 - e^{-\theta b}}{2\theta a} + \frac{1}{2}(t^2 - \frac{b}{a})]\} .$$

Example. Let

$$x = m(t) = a,$$
$$h(t) = \lambda t,$$
$$k = 1$$

$$F(t) = \begin{cases} 1 & : t > a \\ t/a & : 0 \leq t \leq a \\ 0 & : t \leq 0 \end{cases}$$

Then $F(m(t)) = F(a) = 1$ for all $t \geq 0$ and

$$P_o(t) = 1 \quad \text{for all } t \geq 0.$$

Example. Let

$$h(x) = a/b \left(\frac{x}{b}\right)^{a-1} \quad (a, b > 0),$$

$$1 - F(m(x)) = 1 - e^{-(x/b)^a},$$

$$k = 1.$$

Then

$$P_o(t) = \exp \{-[(t/b)^a - 1 + e^{-(t/b)^a}]\}$$

which has the approximate Weibull distribution for large t. Additional examples are given in Reference 2. Equation (6.0) assumes that the number of stress peaks occurring in (0, t) is independent of strength x. While such an assumption would not be valid in many situations, Equation (6.0) may still be a useful model inasmuch as m(t) may be a slowly decreasing function representing the effects of, say, wear and the stress pulses occurring at random are independent of wear and are the cause of catastrophic failure. There is no mathematical requirement that m(t) be monotonic.

## Models of Randomly Decreasing Strength

Component strength is now assumed to deteriorate in random decrements which may occur at instants in time separated by intervals of random length. Whereas the previous models have been rather simple mathematically, the representation of strength as a random process which may deteriorate in a monotonic fashion is by no means simple. (The possibility of a component increasing in strength will not be considered here.) The difficulty is of two types. First there is the problem of representing the size of the strength decrement in terms of the intensity of not only the most recent stress but previous stresses as well. Birnbaum and Saunders[3] have considered some alternative assumptions that might lead to useful models. The other problem is one of how to design an experiment in which strength deterioration can be adequately measured so that model parameters can be estimated. The point of view taken here is that strength can be represented as a time series which is monotonic non increasing with time. The mathematical difficulties that one encounters depend upon the assumptions concerning the frequency with which decrements occur, their magnitude, and the degree of dependence of these two variables upon the strength history. Embedded within these assumptions are the parameters of the underlying stress mechanism. To illustrate the time series approach one model will be developed with the following assumptions:

a) strength is monotonic non increasing in time;

b) strength at time t, Z(t) is the difference between some initial value Z(0) and a random sum

$$Y(t) = \sum_{i=1}^{N(t)} x_i, \text{ i.e.,}$$

$$Z(t) = Z(0) - (X_1 + \ldots + X_{N(t)}) = Z(0) - Y(t) \qquad (7.0)$$

c) $\{X_i;\ i = 1,2, \ldots\}$ is a sequence of mutually independent and identically distributed random variables having a distribution with mean E(X) and variance Var(X).

d) $\{N(t);\ t \geq 0\}$ is a random process defined upon the non negative integers having a distribution with mean E(N(t)) and variance Var (N(t)).

e) $\{N(t);\ t \geq 0\}$ is independent of $\{X_i;\ i = 1,2, \ldots\}$ .

The decrements in strength are represented in terms of the $X_i$'s which are presumably transformations of stress but independent of past stress history. The distribution of Z(t) is then known whenever the distribution of Y(t) can be determined.

Example. Let the $X_i$'s have the negative exponential distribution with parameter $\lambda$ and let $\{N(t)\}$ be a homogeneous Poisson process with parameter p. Then the density F'(y, t) of Y(t) is[2]

$$F'(y, t) = \delta(y)e^{-\rho t} + e^{-(\rho t + \lambda y)} \sqrt{\frac{\rho t \lambda}{y}} \cdot I_1(2\sqrt{\rho \lambda t y})$$

where

$$I_j(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+j}}{n!(n+j)!}$$

is the modified Bessel function of index j and $\delta(y)$ is the Dirac delta function.

It may be possible, as in Model 5 given by Equation (7.0) together with assumptions (a) through (e), to use the theory of first passage times to advantage in studying the process of strength deterioration. In this regard Model 5 is particularly simple and the mean and variance of the time $T_y$ at which Y(t) first reaches a predetermined level y is the following:

$$E(T_y) = \mu_T\{1 + \frac{y}{E(X)} + \frac{Var(X) - (E(X))^2}{2(E(X))^2} + O(\tfrac{1}{t})\} \qquad (7.1)$$

and

$$Var(T_y) = Var(T)[1 + \frac{y}{E(X)} + \frac{Var(X) - (E(X))^2}{2\,E(X)^2}] + \mu_T^2\ [\frac{y\ Var(X)}{(E(X))^3} + \left(\frac{1}{12} + \frac{5(Var\ X)^4}{4(E(X))^4} - \frac{2\mu_3}{3(E(X))^3}\right) + O(\tfrac{1}{t})] \qquad (7.2)$$

where $\mu_T$ and Var(t) are the mean and variance of the common distribution governing the lengths of the time intervals separating instants at which strength decrements occur,

and $\mu_3$ is the third moment of the distribution governing the $X_i$'s.

If failure is taken to be synonymous with the event that Y(t) first reaches level y, then Equation (7.1) gives the mean life length of the component and Equation (7.2) is a measure of the variability of its life length. E(X) and Var(X) are related to the applied stress and may themselves be functions of other parameters.

## IHRA Reliability Functions

Birnbaum, et. al., defines the increasing hazard rate average (IHRA) class of functions by their property that for any member F(t) the "survival" distribution $\overline{F}(t) = 1 - F(t)$ satisfies the condition

$$- \ln \frac{\overline{F}(t)}{t} \geq - \ln \frac{\overline{F}(\tau)}{\tau}$$

whenever $t \geq \tau$.

IHRA distributions are useful in the theory of system reliability as Birnbaum's paper shows and a short calculation demonstrates that the survival functions given by Equations (3.2), (4.0), and (6.1) are IHRA.

## References


1. Birnbaum, Z. W., Esary, J. D., and Marshall, A. W., "A Stochastic Characterization of Wear-Out for Components and Systems," Annals of Mathematical Statistics, 37 (1966) pp. 816-825.

2. Patterson, R. L., "Stochastic Failure Models Based upon Distributions of Stress Peaks," Technical Report No. 3, Project THEMIS, Department of Industrial and Systems Engineering, University of Florida.

3. Birnbaum, Z. W. and Saunders, S. C., "A Probabilistic Interpretation of Miners Rule," SIAM Journal of Applied Mathematics, Vol. 16, No. 3, May 1968, pp. 637, 652.