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AN ELLIPTIC EQUATION WITH NO MONOTONICITY CONDITION ON THE NONLINEARITY

GREGORY S. SPRADLIN

Abstract. An elliptic PDE is studied which is a perturbation of an autonomous equation. The existence of a nontrivial solution is proven via variational methods. The domain of the equation is unbounded, which imposes a lack of compactness on the variational problem. In addition, a popular monotonicity condition on the nonlinearity is not assumed. In an earlier paper with this assumption, a solution was obtained using a simple application of topological (Brouwer) degree. Here, a more subtle degree theory argument must be used.

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1. INTRODUCTION

In this paper we consider an elliptic equation of the form

\[-\Delta u + u = f(x, u), \quad x \in \mathbb{R}^N,\]

where \( f \) is a “superlinear” function of \( u \). For large \( |x| \), the equation resembles an autonomous equation

\[-\Delta u + u = f_0(u), \quad x \in \mathbb{R}^N.\]

Under weak assumptions on \( f \) and \( f_0 \), we prove the existence of a nontrivial solution \( u \) of (1.1) with \( |u(x)| \to 0 \) as \( |x| \to \infty \).

Let \( f \) satisfy

(\( f_1 \)) \( f \in C^2(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}) \).

(\( f_2 \)) \( f(x, 0) = 0 = f_q(x, 0) \) for all \( x \in \mathbb{R}^N \), where \( f \equiv f(x, q) \).

(\( f_3 \)) If \( N > 2 \), there exist \( a_1, a_2 > 0 \), \( s \in (1, (N + 2)/(N - 2)) \) with \( |f_q(x, q)| \leq a_1 + a_2|q|^s - 1 \) for all \( q \in \mathbb{R} \), \( x \in \mathbb{R}^N \). If \( N = 2 \), there exist \( a_1 > 0 \) and a function \( \varphi : \mathbb{R}^+ \to \mathbb{R} \) with \( |f_q(x, q)| \leq a_1 \exp(\varphi(|q|)) \) for all \( q \in \mathbb{R} \), \( x \in \mathbb{R}^N \) and \( \varphi(t)/t^2 \to 0 \) as \( t \to \infty \).

Keywords and phrases. Mountain-pass theorem, variational methods, Nehari manifold, Brouwer degree, concentration-compactness.

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(f_4) There exists \( \mu > 2 \) such that

\[
0 < \mu F(x, q) \equiv \mu \int_0^q f(x, s) \, ds \leq f(x, q) q
\]

for all \( q \in \mathbb{R}, x \in \mathbb{R}^N \).

Let \( f_0 \in C^2(\mathbb{R}, \mathbb{R}) \) with satisfy (f_1)-(f_4) (except there is no dependence on \( x \)). Let \( f \) also satisfy (f_5) \( (f(x, q) - f_0(q))/f_0(q) \to 0 \) as \( |x| \to \infty \), uniformly in \( q \in \mathbb{R}^N \setminus \{0\} \).

In order to state the theorem, we need to outline the variational framework of the problem. Define functionals \( I_0, I \in C^2(W^{1,2}(\mathbb{R}^N, \mathbb{R}), \mathbb{R}) \) by

\[
I_0(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F_0(u(x)) \, dx,
\]

\[
I(u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^N} F(x, u(x)) \, dx,
\]

where \( \|u\| \) is the standard norm on \( W^{1,2}(\mathbb{R}^N, \mathbb{R}) \) given by

\[
\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u(x)|^2 + u(x)^2 \, dx.
\]

Critical points of \( I_0 \) correspond exactly to solutions \( u \) of (1.2) with \( u(x) \to 0 \) as \( |x| \to \infty \), and critical points of \( I \) correspond exactly to solutions \( u \) of (1.1) with \( u(x) \to 0 \) as \( |x| \to \infty \).

By (f_4), \( F_0 \) and \( F \) are “superquadratic” functions of \( q \), with, for example, \( F(x, q)/q^2 \to 0 \) as \( q \to 0 \) and \( F(x, q)/q^2 \to \infty \) as \( |q| \to \infty \) for all \( x \in \mathbb{R}^N \), uniformly in \( x \). Therefore \( I(0) = I_0(0) = 0 \), and there exists \( r_0 > 0 \) with \( I(u) \geq \|u\|^2/3 \) and \( I_0(u) \geq \|u\|^2/3 \) for all \( u \in W^{1,2}(\mathbb{R}^N) \) with \( \|u\| < r_0 \), and there also exist \( u, u_0 \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \) with \( I_0(u_0) < 0 \) and \( I(u) < 0 \). So the sets of “mountain-pass curves” for \( I_0 \) and \( I \),

\[
\Gamma_0 = \{ \gamma \in C([0,1], W^{1,2}(\mathbb{R}^N, \mathbb{R})) \mid \gamma(0) = 0, \ I_0(\gamma(1)) < 0 \},
\]

\[
\Gamma = \{ \gamma \in C([0,1], W^{1,2}(\mathbb{R}^N, \mathbb{R})) \mid \gamma(0) = 0, \ I(\gamma(1)) < 0 \},
\]

are nonempty, and the mountain-pass values

\[
c_0 = \inf_{\gamma \in \Gamma_0} \max_{\theta \in [0,1]} I_0(\gamma(\theta))
\]

\[
c = \inf_{\gamma \in \Gamma} \max_{\theta \in [0,1]} I(\gamma(\theta))
\]

are positive.

We are now ready to state the theorem.

**Theorem 1.1.** If \( f_0 \) and \( f \) satisfy (f_1)-(f_4) and \( f \) satisfies (f_5), and if there exists \( \alpha > 0 \) such that

\[
I_0 \text{ has no critical values in the interval } [c_0, c_0 + \alpha)
\]

then there exists \( c_0 = c_0(f_0) > 0 \) with the following property: if \( f \) satisfies

\[
|f(x, q) - f_0(q)| < c_0|f_0(q)|
\]

for all \( x \in \mathbb{R}^N, q \in \mathbb{R} \), then (1.2) has a nontrivial solution \( u \neq 0 \) with \( u(x) \to 0 \) as \( |x| \to \infty \).

As shown in [9], (1.12) holds in a wide variety of situations.
The missing monotonicity assumption

One interesting aspect of Theorem 1.1 is a condition that is not assumed. We do not assume

\[ F_0(q)/q^2 \text{ is a nondecreasing function of } q \text{ for } q > 0; \]
\[ F_0(q)/q^2 \text{ is a nonincreasing function of } q \text{ for } q < 0; \]
\[ F(x, q)/q^2 \text{ is a nondecreasing function of } q \text{ for } q > 0; \]
\[ F(x, q)/q^2 \text{ is a nonincreasing function of } q \text{ for } q < 0. \]  \hspace{1cm} (1.13)

This condition holds in the power case, \( F_0(q) = |q|^{\alpha}/\alpha, \alpha > 2 \). The condition is due to Nehari.

If (1.13) were case, then for any \( u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \), the mapping \( s \mapsto I(su) \) would begin at \( s = 0 \), increase to a positive maximum, then decrease to \(-\infty\) as \( s \to \infty \). Defining

\[
S = \{ u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \mid I'(u)u = 0 \},
\]  \hspace{1cm} (1.14)

\( S \) would be a codimension-one submanifold of \( E \), homeomorphic to the unit sphere in \( W^{1,2}(\mathbb{R}^N, \mathbb{R}) \) via radial projection. \( S \) is known as the Nehari manifold in the literature. Any ray of the form \( \{ su \mid s > 0 \} \) (\( u \neq 0 \)) intersects \( S \) exactly once. All nonzero critical points of \( I \) are on \( S \). Conversely, under suitable smoothness assumptions on \( F \), any critical point of \( I \) constrained to \( S \) would be a critical point of \( I \) (in the large) (see [17]). Therefore, one could work with \( S \) instead of \( W^{1,2}(\mathbb{R}^N, \mathbb{R}) \), and look for, say, a local minimum of \( I \) constrained to \( S \) (which may be easier than looking for a saddle point of \( I \)). There is another way to use (1.13): for any \( u \in S \), the ray from 0 passing through \( u \) can be used (after rescaling in \( \theta \)) as a mountain-pass curve along which the maximum value of \( I \) is \( I(u) \). Conversely, any mountain-pass curve \( \gamma \in \Gamma \) intersects \( S \) at least once [6]. Therefore, one may work with points on \( S \) instead of paths in \( \Gamma \). Since assumption (1.13) is so helpful, it is found in many papers, such as [1,5,20], and [18].

In the paper [17], a result similar to Theorem 1.1 was proven for the \( N = 1 \) (ODE) case. The proof of Theorem 1.1 is similar except that a simple connectivity argument must be replaced by a degree theory argument [18]. proves a version of Theorem 1.1 under the assumption (1.13). Without 1.13, the manifold \( S \) must be replaced by a set with similar properties.

Define \( B_1(0) = \{ x \in \mathbb{R}^N \mid |x| < 1 \} \), and \( \overline{\Omega} \) and \( \partial \Omega \) to be, respectively, the topological closure and topological boundary of \( \Omega \). It is a simple consequence of the Brouwer degree [7] that for any continuous function \( h : B_1(0) \to \mathbb{R}^N \) with \( h(x) = x \) for all \( x \in \partial B_1(0) \), there exists \( x \in B_1(0) \) with \( h(x) = 0 \). We will need the following generalization:

**Lemma 1.2.** Let \( h \in C(\overline{B_1(0)} \times [0,1], \mathbb{R}^N) \) with, for all \( x \in \overline{B_1(0)} \) and \( t \in [0,1] \),

\( (i) \) \( h(x, 0) = x = h(x, 1) \).
\( (ii) \) \( x \in \partial B_1(0) \Rightarrow h(x, t) = x \).

Then there exists a connected subset \( C_0 \subset \overline{B_1(0)} \times [0,1] \) with \( (0,0), (0,1) \in C_0 \) and \( h(x, t) = 0 \) for all \( (x, t) \in C_0 \).

Using the Brouwer degree, it is clear that under the hypotheses of Lemma 1.2, for each “horizontal slice” \( \overline{B_1(0)} \times \{t\} \) of the cylinder \( \overline{B_1(0)} \times [0,1] \), there exists \( x \in B_1(0) \) with \( h(x, t) = 0 \). The conclusion of Lemma 1.2 does not follow from this observation. A generalization of Lemma 1.2 is known [16]: however, the reference may be difficult to find, so a proof is given here.

This paper is organized as follows: Section 2 contains the proof of Theorem 1.1. The proof of Lemma 1.2 is deferred until Section 3.
2. Proof of Theorem 1.1

It is fairly easy to show that
\[ c \leq c_0, \]  
where \( c \) and \( c_0 \) are from (1.9)–(1.10): it is proven in [11] that there exists \( \gamma_1 \in \Gamma_0 \) with \( \max_{\theta \in [0,1]} I_0(\gamma_1(\theta)) = c_0 \). Define the translation operator \( \tau \) as follows: for a function \( u \) on \( \mathbb{R}^N \) and \( a \in \mathbb{R}^N \), define \( \tau u(x) = u(x - a) \). Let \( \epsilon > 0 \). Let \( \epsilon_1 = 1, 0, 0, \ldots, 0 > 0 \in \mathbb{R}^N \) and define \( \tau_\epsilon, \gamma_1 \) by \( (\tau_\epsilon, \gamma_1)(\theta) = \tau(\gamma_1(\theta)) \). Then for large \( R > 0 \), by \( (f_5) \), \( \tau_\epsilon, \gamma_1 \in \Gamma_0 \) and \( \max_{\theta \in [0,1]} I((\tau_\epsilon, \gamma_1(\theta)) < c_0 + \epsilon \). Since \( \epsilon > 0 \) was arbitrary, \( c \leq c_0 \).

A Palais-Smale sequence for \( I \) is a sequence \( (u_m) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R}) \) with \( I(u_m) \) convergent and \( \|I'(u_m)\| \to 0 \) as \( m \to \infty \). It is well-known that \( I \) fails the “Palais-Smale condition”. That is, a Palais-Smale sequence need not converge. However, the following proposition states that a Palais-Smale sequence “splits” into the sum of a critical point of \( I \) and translates of critical points of \( I_0(\theta) \).

**Proposition 2.1.** If \( (u_m) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R}) \) with \( I'(u_m) \to 0 \) and \( I(u_m) \to a > 0 \), then there exist \( k \geq 0, v_0, v_1, \ldots, v_k \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \), and sequences \( (t_m^k)_{m \geq 1} \subset \mathbb{R}^N \), such that

\begin{itemize}
  \item[(i)] \( I'(v_0) = 0 \);
  \item[(ii)] \( I'(v_i) = 0 \) for all \( i = 1, \ldots, k \),
  \item[(iii)] \( \|u_m - (v_0 + \sum_{i=1}^{k} t_m^k v_i)\| \to 0 \) as \( m \to \infty \);
  \item[(iv)] \( |x_m^i| \to \infty \) as \( m \to \infty \) for \( i = 1, \ldots, k \);
  \item[(v)] \( |x_m^i - x_m^j| \to \infty \) as \( m \to \infty \) for all \( i \neq j \);
  \item[(vi)] \( I(t_0) + \sum_{i=1}^{k} I_0(v_i) = a \).
\end{itemize}

A proof for the case of \( x \)-periodic \( F \) is found in [6], and essentially the same proof works here. Similar propositions for nonperiodic coefficient functions, for both ODE and PDE, are found in [1,5], and [19], for example. All are inspired by the “concentration-compactness” theorems of P.-L. Lions [12].

If \( c < c_0 \), then by standard deformation arguments [15], there exists a Palais-Smale sequence \( (u_m) \) with \( I(u_m) \to c \). By [11], the smallest nonzero critical value of \( I_0 \) is \( c_0 \). Applying Proposition 2.1, we obtain \( k = 0 \), and \( (u_m) \) has a convergent subsequence, proving Theorem 1.1. So assume from now on that
\[ c = c_0. \]  
(2.2)

For \( u \in L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \) and \( i \in \{1, \ldots, N\} \), define \( L_i \), the \( i \)th component of the “location” of \( u \), by
\[ \int_{\mathbb{R}^N} u^2 \tan^{-1}(x_i - L_i(u)) \, dx = 0 \]  
(2.3)
and the “location” of \( u \) by
\[ L(u) = (L_1(u), \ldots, L_N(u)) \in \mathbb{R}^N. \]  
(2.4)
The following lemma establishes the existence and continuity of \( L \).

**Lemma 2.2.** \( L \) is well-defined and continuous on \( L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \).

**Proof.** It suffices to show that \( L_1 \) is well-defined and continuous on \( L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \). Let \( u \in L^2(\mathbb{R}^N, \mathbb{R}) \setminus \{0\} \). By Leibniz’s Theorem, the mapping \( \phi : s \mapsto \int_{\mathbb{R}^N} u^2 \tan^{-1}(x_1 - s) \, dx \) is continuous, differentiable, and strictly decreasing, with
\[ \phi'(s) = -\int_{\mathbb{R}^N} u^2(x)/((x_1 - s)^2 + 1) \, dx < 0. \]  
(2.5)
\( \phi(s) \to -\infty \) as \( s \to \pm \infty \). Therefore \( L_1(u) \) is unique and well-defined. Let \( \epsilon > 0 \) and \( u_m \to u \). Now \( \int_{\mathbb{R}^N} u^2 \tan^{-1}(x_1 - (L_1(u) + \epsilon)) \, dx < 0 \). Since \( u_m^2 \to u^2 \) in \( L^1(\mathbb{R}^N, \mathbb{R}) \), \( \int_{\mathbb{R}^N} u_m^2 \tan^{-1}(x_1 - (L(u) + \epsilon)) \, dx < 0 \) for
large $m$, so for large $m$, $\mathcal{L}_1(u_m) < \mathcal{L}_1(u) + \epsilon$. Similarly, for large $m$, $\mathcal{L}_1(u_m) > \mathcal{L}_1(u) - \epsilon$. Since $\epsilon$ is arbitrary, $\mathcal{L}_1(u_m) \to \mathcal{L}_1(u)$.

We are ready to begin the minimax argument. First we construct a mountain-pass curve $\gamma_0$ with some special properties:

**Lemma 2.3.** There exists $\gamma_0 \in \Gamma_0$ such that for all $\theta \in [0,1]$,

(i) $I_0(\gamma_0(\theta)) \leq c_0$.
(ii) $\theta > 0 \Rightarrow \gamma_0(\theta) \neq 0$.
(iii) $\theta \leq 1/2 \Rightarrow I_0(\gamma_0(\theta)) \leq c_0/2$.
(iv) $\theta > 0 \Rightarrow L(\gamma(\theta)) = 0$.

**Proof.** By [10], there exists $\gamma_1 \in \Gamma_0$ with $\max_{\theta \in [0,1]} I_0(\gamma_1(\theta)) = c_0$. Assume without loss of generality that $\gamma_1(\theta) \neq 0$ for $\theta > 0$. By rescaling in $\theta$ if necessary, assume that $I_0(\gamma_1(\theta)) \leq c_0/2$ for $\theta \leq 1/2$. Finally, define $\gamma_0$ by $\gamma_0(0) = 0$, $\gamma_0(\theta) = \tau_{-L(\gamma_1(\theta))} \gamma_1(\theta)$ for $\theta > 0$.

Assume $\epsilon_0$ in (1.12) is small enough so that for all $x \in \mathbb{R}^N$ and $\theta \in [0,1]$,

$$I(\tau_x(\gamma_0(\theta))) < \min(2c_0, c_0 + \alpha) \text{ and } I(\tau_x(\gamma_0(1))) < 0,$$

where $\alpha$ is from (1.11).

**A substitute for $S$**

Using the mountain-pass geometry of $I$ and the fact that Palais-Smale sequences of $I$ are bounded in norm [6], we construct a set which has similar properties to $\mathcal{S}$, described in Section 1. Let $\nabla I$ denote the gradient of $I$, that is, $(\nabla I(u), w) = I'(u)w$ for all $u, w \in W^{1,2}(\mathbb{R}^N, \mathbb{R})$. Here, $(\cdot, \cdot)$ is the usual inner product defined by $(u, w) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla w + uw \, dx$. Let $\varphi : W^{1,2}(\mathbb{R}^N, \mathbb{R}) \to \mathbb{R}$ be locally Lipschitz, with $I(u) \geq -1 \Rightarrow \varphi(u) = 1$ and $I(u) \leq -2 \Rightarrow \varphi(u) = 0$. Let $\eta$ be the solution of the initial value problem

$$\frac{d\eta}{ds} = -\varphi(\eta) \nabla I(u), \quad \eta(0, u) = u. \quad (2.7)$$

In [19] it is proven that $\eta$ is well-defined on $\mathbb{R}^+ \times W^{1,2}(\mathbb{R}^N)$. Let $\mathcal{B}$ be the basin of attraction of 0 under the flow $\eta$, that is,

$$\mathcal{B} = \left\{ u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) \mid \eta(s, u) \to 0 \text{ as } s \to \infty \right\} \quad (2.8)$$

$\mathcal{B}$ is an open neighborhood of 0 [19]. Let $\partial \mathcal{B}$ be the topological boundary of $\mathcal{B}$ in $W^{1,2}(\mathbb{R}^N, \mathbb{R})$. $\partial \mathcal{B}$ has some properties in common with $\mathcal{S}$. For example, for any $\gamma \in \Gamma$, $\gamma([0,1])$ intersects $\partial \mathcal{B}$ at least once.

A pseudo-gradient vector field for $I'$ may be used in place of $\nabla I$, in which case $\mathcal{B}$ and $\partial \mathcal{B}$ would be different, but the ensuing arguments would be the same.

Let

$$c^+ = \inf \{ I(u) \mid u \in \partial \mathcal{B}, \, |\mathcal{L}(u)| \leq 1 \}. \quad (2.9)$$

The reason for the label "$c^+$" will become apparent in a moment. From now on, let us assume

$$I \text{ has no critical values in } (0, c_0] = (0, c]. \quad (2.10)$$

This will lead to the conclusion that $I$ has a critical value greater than $c_0$.

We claim that under assumptions (2.2) and (2.10),

$$c^+ > c_0. \quad (2.11)$$

We use arguments that are sketched here and found in more detail in [19] and [5].
To prove the claim, suppose first that \( c^+ < c_0 \). Then there exists \( u_0 \in \partial B \) with \( I(u_0) < c_0 \). By arguments in [19], there exists a large positive constant \( P \) with
\[
I(u) \leq c_0 \text{ and } \|u\| \geq 2P \Rightarrow I(\eta(s, u)) < 0 \text{ for some } s > 0, \text{ and } \|\eta(s, u)\| > P
\] (2.12)
for all \( s > 0 \). Suppose \( a > 0 \) and \( \|I'(\eta(s_m, U_0))\| \geq a \) for some sequence \( (s_m) \) with \( s_m \to \infty \). Since \( u_0 \in \partial B \), \( \|\eta(U_0)\| < 2P \) for all \( s > 0 \). \( I'' \) is bounded on bounded subsets of \( W^{1,2}(\mathbb{R}) \), so \( I' \) is Lipschitz on bounded subsets of \( W^{1,2}(\mathbb{R}) \). Therefore \( I(\eta(s, u_0)) < 0 \) for some \( s > 0 \). This is impossible since \( u_0 \in \partial B \). Therefore \( I'(\eta(s, u_0)) \to 0 \) as \( s \to \infty \).

Define \( u_n = \eta(n, u_0) \). Since \( I'(u_n) \to 0 \) and \( u_n \in \partial B \), there exists \( b \in (0, c_0) \) with \( I(u_n) \to b \). By [11], \( I_0 \) has no critical values between 0 and \( c_0 \). Therefore, Proposition 2.1, with \( k = 0 \), implies that \( (u_n) \) converges along a subsequence to a critical point \( w \) of \( I \) with \( 0 < I(w) < c_0 \). This contradicts assumption (2.10).

Next, suppose that \( c^+ = c_0 \). Then there exists a sequence \( (u_n) \subset \partial B \) with \( \|L(u_n)\| \leq 1 \) for all \( n \) and \( I(u_n) \to c_0 \) as \( n \to \infty \). As above, \( I'(u_n) \to 0 \) as \( n \to \infty \); to prove, suppose otherwise. Then there exist \( a > 0 \) and a subsequence of \( (u_n) \) (also called \( (u_n) \)) along which \( \|I'(u_n)\| > a \). Since \( \partial B \) is forward-\( \eta \)-invariant [19], \( \eta(1, u_n) \in \partial B \) for all \( n \). Since \( (\eta(1, u_n))_{n \geq 1} \) is bounded and \( I' \) is Lipschitz on bounded subsets of \( W^{1,2}(\mathbb{R}^N, \mathbb{R}) \), for large \( n \), \( \eta(1, u_n) \in \partial B \) with \( I'(\eta(1, u_n)) < c_0 \). By the argument above, this implies that \( I \) has a critical value in \((0, c_0)\), contradicting assumption (2.2). Thus \( I'(u_n) \to 0 \) as \( n \to \infty \). Applying Proposition 2.1 and using the fact that \( |L(u_n)| \leq 1 \) for all \( n \), \( (u_n) \) converges along a subsequence to a critical point of \( I \), contradicting assumption (2.10). (2.11) is proven.

Let \( R > 0 \) be big enough so that for all \( x \in \partial B_R(0) \subset \mathbb{R}^N \) and \( \theta \in [0, 1] \),
\[
I(\tau_x \gamma_0(\theta)) < c^+.
\] (2.13)
This is possible by (1.12), (2.11), and Lemma 2.3(i). Define the minimax class
\[
\mathcal{H} = \{ h \in C(B_R(0) \times [0, 1], W^{1,2}(\mathbb{R}^N, \mathbb{R})) \mid \text{ for all } x \in \overline{B_R(0)} \text{ and } t \in [0, 1], \\
t > 0 \Rightarrow h(x, t) \neq 0 \\
0 \leq t \leq 1/2 \Rightarrow h(x, t) = \tau_x \gamma_0(t) \\
x \in \partial B_R(0) \Rightarrow h(x, t) = \tau_x \gamma_0(t) \\
h(x, 1) = \tau_x \gamma_0(1) \}
\]
and the minimax value
\[
h_0 = \inf_{h \in \mathcal{H}} \max_{(x, t) \in \overline{B_R(0) \times [0, 1]}} I(h(x, t)).
\] (2.14)
We claim
\[
c_0 < c^+ \leq h_0 < \min(2c_0, c_0 + \alpha).
\] (2.15)

**Proof of Claim.** Define \( h \in \mathcal{H} \) by \( h(x, t) = \tau_x (\gamma_0(t)) \). Then \( h \in \mathcal{H} \) and by (2.6),
\[
\max_{(x, t) \in \overline{B_R(0) \times [0, 1]}} \beta(h(x, t)) < \min(2c_0, c_0 + \alpha).
\]
Therefore \( h_0 < \min(2c_0, c_0 + \alpha) \).

Next, let \( h \in \mathcal{H} \). By Lemma 1.2, and a suitable rescaling of \( x \) and \( t \), there exists a connected set \( C_2 \subset B_R(0) \times [1/2, 1] \) with \((0, 1/2), (0, 1) \in C_2 \) and along which for all \((x, t) \in C_2 \),
\[
\mathcal{L}(h(x, t)) = 0.
\] (2.16)
Joining \( C_2 \) with the segment \( \{0\} \times [0, 1/2] \), we obtain a connected set \( C_3 \subset B_R(0) \times [0, 1] \) such that \((0, 0), (0, 1) \in C_3 \) and for all \((x, t) \in C_3 \), \( \mathcal{L}(h(x, t)) = 0 \). \( C_3 \) is not necessarily path-connected, so let \( r > 0 \) be small enough so
that for all
\[(x, t) \in N_r(C_3) \equiv \{(y, s) \in B_R(0) \times [0, 1] | \exists (x', t') \in B_R(0) \times [0, 1] \text{ with } |y - x'|^2 + |s - t'|^2 < r^2\}, \]
(2.17)
\[|\mathcal{L}(h(x, t))| < 1. \]

\(N_r(C_3)\) is path-connected [21], so there exists a path \(g \in C([0, 1], N_r(C_3))\) with \(g(0) = (0, 0), g(1) = (0, 1),\) and
\[g(\theta) \in N_r(C_3) \text{ for all } \theta \in [0, 1].\]
If we define \(\gamma \in \Gamma\) by \(\gamma(\theta) = h(g(\theta)),\) then \(|\mathcal{L}(\gamma(\theta))| < 1\) for all \(\theta \in [0, 1].\) Since \(\gamma(0) = 0\) and \(I(\gamma(1)) < 0,\) there exists \(\theta^* \in [0, 1] \text{ with } \gamma(\theta^*) \in \partial B.\) By the definition of \(c^+ (2.9), I(\gamma(\theta^*)) \geq c^+.\)

Since \(h\) was an arbitrary element of \(\mathcal{H}, h_0 \geq c^+.\)

By standard deformation arguments, such as described in [15], there exists a Palais-Smale sequence \((u_n) \subset W^{1,2}(\mathbb{R}^N, \mathbb{R})\) with \(I'(u_n) \to 0\) and \(I(u_n) \to h_0\) as \(n \to \infty, c_0 < h_0 < \min(2c_0, c_0 + \alpha).\) Apply Proposition 2.1 to \((u_n).\) Since \(I_0\) has no positive critical values smaller than \(c_0 [11], k \leq 1.\) By (2.10), \((u_n)\) converges along a subsequence to a critical point \(z\) of \(I, \text{ with } I(z) = h_0.\) Theorem 1.1 is proven.

3. A DEGREE-THEORETIC LEMMA

Here, we prove Lemma 1.2. Let \(h\) be as in the hypotheses of the lemma. For \(l > 0,\) define \(A_l \subset \overline{B_1(0)} \times [0, 1]\) by
\[A_l = \{(x, t) \in \overline{B_1(0)} \times [0, 1] | |f(x, t)| < l\}. \]
(3.1)
\(A_l\) is an open neighborhood of \((0, 0).\) Let \(C_l\) be the component of \(A_l\) containing \((0, 0).\) We will prove the following claim:

For all \(\epsilon > 0, (0, 1) \in C_\epsilon\).
(3.2)

Then we will use the \(C_\epsilon's\) to construct \(C_0.\) For \(l > 0,\) define
\[C_l = \{x \in \overline{B_1(0)} | (x, t) \in C_l\}. \]
(3.3)

Fix \(\epsilon \in (0, 1).\) Define \(\phi: [0, 1] \to \mathbb{Z}\) by
\[\phi(t) = d(h(\cdot, t), C_l^\epsilon, 0), \]
(3.4)
where \(d\) is the topological Brouwer degree [7]. We will prove \(\phi(t) = 1\) for all \(t \in [0, 1],\) in particular \(\phi(1) = 1,\) so (3.2) is satisfied.

\(f\) is continuous on a compact domain, so \(f\) is uniformly continuous. Let \(\rho > 0\) be small enough so that for all \(x \in \overline{B_1(0)}\) and \(t_1, t_2 \in [0, 1],\)
\[|t_1 - t_2| < \rho \Rightarrow |h(x, t_1) - h(x, t_2)| < \epsilon/4. \]
(3.5)

Clearly
\[\phi(0) = d(id, B_1(0), 0) = 1. \]
(3.6)

Let \(0 \leq t_1 < t_2 \leq 1\) with \(t_2 - t_1 < \rho.\) We will show \(\phi(t_1) = \phi(t_2),\) proving that \(\phi\) is constant, which by (3.6), implies (3.2).

\(\Omega\) is nonempty. For all \(x \in \partial C^\epsilon_{t_1}, |h(x, t_1)| = \epsilon,\) so by (3.5),
\[x \in \partial C^\epsilon_{t_1} \Rightarrow |h(x, t_1)| \geq \frac{3}{4} \epsilon. \]
(3.7)

By the additivity property of \(d [7],\)
\[\phi(t_2) \equiv d(f(\cdot, t_2), C^\epsilon_{t_2}, 0) \]
\[= d(f(\cdot, t_2), C^\epsilon_{t_2} \setminus C^\epsilon_{t_1}, 0) + d(f(\cdot, t_2), C^\epsilon_{t_1} \cap C^\epsilon_{t_2}, 0). \]
(3.8)
We will show:

There does not exist $x \in C_t^{2} \setminus \overline{C_t^{1}}$ with $h(x,t_2) = 0$. (3.9)

Suppose such an $x$ exists. Then by (3.5), $|h| < \epsilon/4$ on the set $\{x\} \times [t_1, t_2]$. $x \in C_t^{2}$, so $(x,t_2) \in C_t$, and by the definition of $C_t$, $(x,t_1) \in C_t$, and $x \in C_t^{1}$, contradicting $x \in C_t^{2} \setminus \overline{C_t^{1}}$. So (3.9) is true. Therefore by (3.8),

$$\phi(t_2) = d(f(\cdot, t_2), C_t^{1} \cap C_t^{2}, 0).$$

By the same argument, switching the roles of $t_1$ and $t_2$,

$$\phi(t_1) = d(f(\cdot, t_1), C_t^{1} \cap C_t^{2}, 0).$$

For all $t \in [t_1, t_2]$ and $x \in \partial C_t^{1} \cup \partial C_t^{2}$, (3.5) gives $|h(x, t_1)| > 3\epsilon/4$ and $|h(x, t) - h(x, t_1)| < \epsilon/4$. Therefore by the homotopy invariance property of the degree [7],

$$\phi(t_1) = d(f(\cdot, t_1), C_t^{1} \cap C_t^{2}, 0) = d(f(\cdot, t_2), C_t^{1} \cap C_t^{2}, 0) = \phi(t_2).$$

$\phi(0) = 1$ and $\phi(t_1) = \phi(t_2)$ for any $t_1 < t_2$ with $t_1, t_2 \in [0, 1]$ and $t_2 - t_1 < \rho$. Therefore $\phi$ is constant, and $\phi(1) = 1$. Therefore $(0, 1) \in C_t$.

Now let

$$C_0 = \bigcap_{\epsilon > 0} C_{\epsilon}. \quad (3.13)$$

Each $C_t$ is a connected set containing $(0, 0)$ and $(0, 1)$, so it is easy to show that $C_0$ is a connected set containing $(0, 0)$ and $(0, 1)$, and clearly for all $(x, t) \in C_0$, $h(x,t) = 0$.

References


