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Analysis of a Partial Differential Equation and Real World Applications Regarding Water Flow in the State of Florida

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Abstract

In the article "Exact solutions of a nonlinear diffusion-convection equation" a partial differential equation is presented and analyzed. Namely, we analyze the behavior and time evolution of the phenomenon as the speed of the wave solution is adjusted. The real solutions are plotted and a hypothesis involving water flow is mentioned, as the original partial differential equation arises from a current problem of interest among engineers in Florida studying the water flow in the aquifer. Namely, the equation is a governing equation for the flow of water under gravity through a homogeneous isotropic porous medium [1].

1. Introduction

The partial differential equation (PDE) under study, which illustrates the flow of water under gravity through a homogeneous isotropic porous medium, is stated as:

\[ \frac{\partial u}{\partial t} = \frac{\partial (u^n)}{\partial x} + \frac{\partial^2 (u^m)}{\partial x^2}, n \geq m > 1 \quad (1.1) \]

Equation (1.1) was reduced to solvable form, through a change of variables, to become an ordinary differential equation (ODE) represented below, which is where our research began.

\[ (c + nu^{n-1} - 2\mu u^{m-1}y) \frac{dU}{dY} + m(m - 1)u^{m-2}\mu(1 - Y^2)\left(\frac{dU}{dY}\right)^2 \]

\[ + mU^{m-1}\mu(1 - Y^2) \frac{d^2 U}{dY^2} = 0 \quad (1.2) \]

The ODE (1.2) was solved by hand to arrive at exact solutions. The purpose of this research is to analyze the exact solutions, while adjusting the speed of the traveling wave and the interval of
time over which it changes, and thus to observe the behavior of the wave solution's graphical representations. Finally, our analysis of the exact solutions leads to a hypothesis involving water flow, which we apply to the real world problem of water flow in the Floridan Aquifer.

1.1. Floridan Aquifer

The Floridan Aquifer is one of the primary sources of ground-water in the United States, and most of Florida's public water systems utilize this ground water system. The aquifer underlies Florida, South Georgia, Alabama, and the lower regions of South Carolina, which is shown in Figure 1 [2]. In total it covers over 100,000 square miles of land, and a total of three-billion gallons of water is drawn up from it each day. The aquifer plays a vital role in the drinkable water that the U.S. consumes and should be maintained carefully for future use.

![Figure 1: Coverage of the Floridan Aquifer](https://commons.erau.edu/mcnair/vol1/iss1/8)

The aquifer's underground water system is defined by three levels: the upper confining unit, middle semi-confining unit, and lower semi-confining unit. It is divided by a section of low porosity to create two sections, which are known as the Upper and Lower aquifer [2]. The aquifer receives replenishing from bodies of water such as lakes and springs here is a look at the make-up of the aquifer.
1.2. Water Scarcity

Water is a renewable resource, but it needs to be dealt with carefully. Salt water accounts for 97 percent of water on earth, while freshwater only accounts for three percent [3]. As most of the freshwater comes from rain, only a small percentage of annual rainfall is actually transferred to domestic use. Water scarcity is a relevant global issue, but for this research project we will be honing in on water scarcity in the local Floridan Aquifer system. Here is a look at one of the aquifer's largest springs:

First magnitude springs provide the largest volume of potable, or drinkable, water. In March of 2010 due to increasing rainfall the water level in the Floridan Aquifer's wells rose to the 75th percentile, yet in only a span of four months the rainfall decreased and the water level declined to the 50th percentile [3]. This dramatic difference seems to show the unreliable grounds for which we depend on our drinking water. The water levels of the aquifer need to be maintained in order to keep the sustainability of procuring potable water for domestic utilization. This report explores the characteristics of wave-like solutions of a partial differential equation and analyzes the behavior and time evolution of the phenomenon as the speed of the wave solution is adjusted.
2. Methods

The partial differential equation (1.1) was first simplified to an ordinary differential equation by a change of variables. The new variable is:

\[ Y = \tanh(\mu \Psi) \]

where,

\[ \Psi = x - ct \]

Therefore, \( u(x,t) \) becomes \( U(\Psi) \). Through a change of variables we get our ordinary differential equation (1.2).

2.1. Power Series Method

Next, the ODE was solved by hand using the power series method. This method required that the solution be in the form:

\[ u(x,t) = U(\varphi) = S(Y) = \sum_{k=0}^{R_1} a_k Y^k + \sum_{l=1}^{R_2} b_l Y^{-l} \quad (2.1) \]

After solving the expansion of the power series, the highest and lowest degree terms in \( Y \) were balanced to obtain:

\[ u(x,t) = U(\varphi) = S(Y) = a_0 + a_1 + \frac{b_1}{Y} \quad (2.2) \]

The sum of these terms is an approximation of \( u(x,t) \) [5]. Next, we input the values of \( m \) and \( n \) into the ODE (1.2), where \( m = 2 \) and \( n = 3 \):

\[ (c + 3S^2 - 4\mu SY) \frac{dS}{dY} + 2\mu(1 - Y^2) \left( \frac{dS}{dY} \right)^2 + 2\mu(1 - Y^2)S \frac{d^2S}{dY^2} = 0 \quad (2.3) \]

The expansion (2.2) was then substituted into (2.3) whereby we obtained a system of algebraic equations and were able to solve for the exact solutions. Once the process of solving the system of equations was complete we arrived at four solutions, although for this research project we were only interested in one of the solutions (3.1). Out of the four, it is the only solution that can be used for real applications, since it is bounded on the total real line [1].

3. Problem Statement and Solution

3.1. Problem

The problem of this research project is to solve the partial differential equation (1.1), which models the flow of water under gravity through a homogeneous isotropic porous medium...
and to observe the physical changes of the traveling wave solutions when the speed of the wave and time is increased incrementally [1].

3.2. Solution

The solution results suggest a physical interpretation of how the wave solutions behave under specific conditions imposed on the equation. There are four solutions to the ODE, and we arrive at them using the tangent hyperbolic method, which is based on the assumption that a traveling-wave solution can be expressed in terms of tangent x or tangent hyperbolic x [1]. Of the four solutions we only use the tangent hyperbolic solution, the reason being that it is the only solution of the four which is usable for real applications, to obtain significant physical solutions. The solution we use here is:

\[ u(x, t) = \sqrt{-c} \tanh \left( \frac{1}{2} \sqrt{-c}(x - ct) \right), \quad c < 0 \]  

(3.1)

One of the reasons that this solution is the only one out of the four solutions that is valid for real application is given below:

\[ \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \approx \lim_{x \to \infty} 1 \]  

(3.2)

While the other three solutions contain discontinuities.

3.3. Substituting values for c and t

Here we substitute the values of speed: c = -0.01, -0.02, and -0.03 and time: t = 0, 500, 1000, and 1500 seconds into solution (3.1).

When c = -0.01 the solution becomes:

\[ u(x, t) = \sqrt{0.01} \tanh \left( \frac{1}{2} \sqrt{0.01}(x + 0.01t) \right), \quad c = -0.01 \]  

(3.1)

Which simplifies to:

\[ u(x, t) = 0.1 \tanh \left( \frac{0.1}{2} (x + 0.01t) \right), \quad c = -0.01 \]  

(3.2)

When c = -0.02 the solution becomes:

\[ u(x, t) = \sqrt{0.02} \tanh \left( \frac{1}{2} \sqrt{0.02}(x + 0.02t) \right), \quad c = -0.02 \]  

(3.3)

When c = -0.03 the solution becomes:
\[ u(x, t) = \sqrt{0.03} \tanh \left( \left( \frac{1}{2} \right) \sqrt{0.03} (x + 0.03t) \right), c = -0.03 \]  

(3.4)

3. Results

All three solutions were carefully plotted in MATLAB and were graphed as time increases from zero to 1500 seconds. They each display an accurate representation of how their wave-like solutions behave and potentially provide important information on water flow behavior.

4.1. Plot of equation (3.4)

The results of solution (3.4) show absolute convergence and that as time increases the amplitude of the wave also increases.

4.2. Plot of equation (3.5)

The results of solution (3.5) show absolute convergence, and as time increases the amplitude of the wave also increases.
4.3. Plot of equation (3.6)

The results of solution (3.6) show absolute convergence, and again as time increases the amplitude of the wave also increases.
5. Discussion and Conclusion

From this work, and based on the convergence in all three of the graphs, we infer that as the speed of the tangent hyperbolic wave-like solution increases the greater the amplitude becomes. From this research we hypothesize that in the real world problem, pertaining to the Floridan Aquifer, as the flow of water into the aquifer reduces the amount of water will not decay linearly; rather, we will see exponential decay. We suggest further investigation into this problem in the future because as water scarcity becomes an issue the need for sustaining the water level in the aquifer becomes extremely important.

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6. References


