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The Development and Application of a Trapezoidal Shear Panel for Use in Finite Element Codes

Glenn P. Greiner
Embry-Riddle Aeronautical University - Daytona Beach

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THE DEVELOPMENT AND APPLICATION OF A TRAPEZOIDAL SHEAR PANEL FOR USE IN FINITE ELEMENT CODES

by

GLENN P. GREINER

Thesis Submitted to the School of Graduate Studies and Research in Partial Fulfillment of the Requirements for the Degree of Master of Science in Aeronautical Engineering

Embry-Riddle Aeronautical University
Daytona Beach, Florida

August 1990
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THE DEVELOPMENT AND APPLICATION OF A TRAPEZOIDAL SHEAR PANEL FOR USE IN FINITE ELEMENT CODES

by

Glenn P. Greiner

This thesis was prepared under the direction of the candidate's thesis committee chairman, Dr. Howard D. Curtis, Department of Aerospace Engineering, and has been approved by the members of his thesis committee. It was submitted to the School of Graduate Studies and Research and was accepted in partial fulfillment of the requirements for the degree of Master of Science in Aeronautical Engineering.

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NOMENCLATURE

MATRIX SYMBOLS

[ ] rectangular or square matrix
[ ] rectangular matrix for independent stress parameters
{ }, [ ] column vector, row vector
[ ], { } element local system
[ ], { } elements edge local system
[ ]^T matrix transpose
[ ]^{-1} matrix inversion
, partial derivative ( \frac{\partial w}{\partial x} = w_x )

LATIN SYMBOLS

a constant of general displacement field
A area
[A] polynomial terms evaluated at B.C.'s
b body forces per unit volume
B.C.'s boundary conditions
[B] relates strain to nodal displacement
{B} B.C.'s vector
C constants
{C} Stress polynomial vector
d.o.f.  degree of freedom

[D]  stress-strain matrix

E  modulus of elasticity

[E]  strain-stress matrix

f, g, h  polynomial stress functions

F  force

[F]  relates load vector to stress parameters

G  modulus of rigidity

[H]  natural flexibility matrix

J  determinate of Jacobian matrix

[J]  Jacobian matrix

[K]  stiffness matrix

i, j, k  unit vectors in direction of xyz

l, m, n  directional cosines

L  length

[L]  relates surface tractions to stress parameters

[M]  stress polynomial terms with respect to \{C\}

n, t  normal and tangential coordinate system

N  bilinear transformation function

N  normal unit vector

[N], \{N\}  interior shape functions

\{\bar{N}\}, \{\bar{N}\}  boundary shape functions

n_g  number of Gauss points

n_n  number of nodes
\( n_q \) number of d.o.f. displacements

\( n_p \) degree of polynomial

\( n_s \) number of stress terms

\( p \) perpendicular distance to baseline PQ

\( P \) baseline point

\( [P] \) stress polynomial terms with respect to \( \{\beta\} \)

\( \hat{P} \) independent stress polynomial terms

\( \{q\} \) nodal d.o.f. displacement vector

\( Q \) baseline point, corner d.o.f. load

\( \textbf{Q} \) corner diagonal vector

\( \{Q\} \text{ load vector} \)

\( \mathbf{r}^* \) position vector

\( s \) shear flow

\( \text{sgn} \) sign of line node

\( S \) surface

\( [S] \) partition of stiffness matrix

\( t \) thickness of panel

\( t \) tangential unit vector

\( \{T\} \) surface traction

\( [T] \) relates \( \{Q'\} \) to \( \{Q''\} \)

\( u \) general displacements

\( U \) internal strain energy

\( \delta U, \delta U^* \) internal virtual strain energy, complementary

\( V \) volume, shear force
V arbitrary vector
W weighting factors, warping loads
δW, δW* external virtual energy, complementary
xyz global cartesian coordinate system
x'y'z' local cartesian coordinate system
[X] solution vector for [H]^{-1}[F]^T

GREEK SYMBOLS

α, β, γ, δ angles of quadrilateral panel with respect to baseline
α, β, γ warped panel coordinate system
{β} stress polynomial coefficients, stress parameters
δ virtual
ε strain
[λ] directional cosines
[Λ] transformation matrix
ζ, η isoparametric coordinate system
Φ Airy stress function, relates \{β_1\} to \{β_2\}
ϕ, ψ angles of infinitesimal parallelogram to baseline
ν Poisson’s ratio
σ stress
θ interior acute angle of parallelogram
τ shear stress
This thesis documents the efforts of the writer and his colleagues over the past several years to improve the theoretical foundation of the arbitrary quadrilateral shear panel used in structural analysis codes to model aircraft structures. An equilibrium stress-based element with pure shear resultants on its sides was developed using the principle of complementary virtual work. The internal stress field was derived from a complete polynomial Airy stress function. The element was numerically tested as a pure stress element and a hybrid element to assess the
deflection properties for highly distorted planar panels. Linear-stress and quadratic-displacement rods were used, as appropriate, to model the stiffeners required to surround the shear panels. Panel displacements were compared with other well-known shear panels as well as with a finite element model of the shear panel.

The pure-stress element, based on a third degree stress polynomial, was finally chosen because it gave displacements in agreement with the other shear panels (but usually on the order of twice the magnitude of the displacement-based finite element model) and panel performance was essentially unchanged with choice of higher-order stress polynomials.

Performance of the hybrid version of the panel was spurious and further study is required to understand its behavior.
To my father who encouraged me to become an Aeronautical Engineer.
CHAPTER 1

INTRODUCTION

One of the challenges faced by an aircraft structural design engineer is that of defining the load paths, i.e., determining a basic layout of a structure that will distribute externally applied loads efficiently throughout that structure. The wing of an aircraft or a fuselage section are such structures. Aircraft structures typically consist of ribs, spars, stringers and bulkheads which are surrounded by or attached to a thin skin. The basic configuration of the wing structure allows the preliminary design engineer to model it using flat panels to simulate the ribs, skin and the webs of spars. The spar flanges and the stringers can be modeled using rods. It is assumed, for simplification of the analysis, that the panels carry only shear forces applied along their edges; therefore they are called “shear panels.” The rods are assumed to transmit only normal loads directed along their axes. The shear forces around the edges of the panels are converted to average shear flows by dividing each one by the length of its edge. Clearly, shear flow is the shear force intensity (shear force per unit length).

If the structure is simple and statically determinate, the shear flows in the panels and forces in the rods are straightforward. As the structure gets more elaborate and statically
indeterminate, simple calculations become impractical. A computer can then be used to aid the engineer in analyzing these complex structures. A structural finite element computer code is a means of handling such problems. These codes and their capabilities vary considerably, depending on the complexity of the structure and the objective of the analysis. They range from simple static structural analysis to complex multi-structured dynamic analysis. With a shear panel in its element library, a basic finite element code can be used to solve aircraft structures problems, yielding displacements, forces in the rods, and shear flows within the panels. Obtaining these shear flows is therefore, a principal concern of preliminary design activity.

The actual application of a finite element program involves the basic steps of modeling the structure, defining the boundary conditions, and executing the analysis. The modeling of a typical aircraft structure entails specifying the global coordinates of the nodes and then interconnecting them with rod and panel elements. The individual element stiffness matrices are then assembled to form the overall structural stiffness matrix. Rigid body displacements are suppressed, and loads are applied to the nodes of the structure. The determination of the primary unknowns—the displacements—can be accomplished by means of a Gauss elimination routine. It is these displacements, multiplied by the stiffness matrix of each element, that enables the solution of each element's forces and stresses. The preliminary design engineer can then easily visualize the load paths for the particular structure under analysis.
Of course, before the structure can be properly modeled and the computer output properly interpreted, the engineer must have a knowledge of fundamental structural concepts.

Students of aircraft structures are exposed to the procedures for analyzing box beams, such as wing-type structures modeled as rods and shear panels. They calculate precisely how shear, bending and torsional loadings are distributed through a structure as axial forces in the rods and shear flows in the panels. The experience of having analyzed representative but relatively simple structures by hand provides the insight needed for dealing with more complicated problems requiring the aid of a computer. Using the computer as an analysis tool gives students hands-on experience in the art of modeling and analyzing structures. This is an essential part of their education for a career in the computer-oriented aerospace industry.

Embry-Riddle Aerospace Engineering students have access to several finite element codes, but none of these codes have a shear panel. This thesis addresses that deficiency.

The Shear Panel as a Finite Element

The shear panel is approached here as a finite element and developed on that basis, but first some characterizing initial assumptions are made. Its shape is an arbitrarily quadrilateral of uniform thickness. It is a stress-based, rather than displacement-based, element, and the stress field is assumed to be such that it yields a set of four self-equilibrating shear forces directed tangentially along the element’s edges.
The nodal degrees of freedom (d.o.f.) associated with this panel can be chosen to lie at the corners (point nodes) and/or along the edges (line nodes) of the panel. In either case, upon the analysis of a structure, the computed displacements of the nodes are used to determine the average shear flow on each edge of the panel. It is these shear flows that are of interest. They provide the information needed to visualize load paths and, for example, to determine rivet spacing and size. The shear panel can cover a much larger area than a typical displacement-based finite element. It is this feature that places the shear panel in the category of a rod (truss element) or beam (frame element). The shear panel is thus better referred to as a structural element and not a finite element.

The rod elements, or stiffeners, which surround and transmit direct loads to the shear panels, must also have the ability of taking a lengthwise shear flow (line load).

Suppose the shear flow in a panel of an aircraft wing is to be determined using a structural analysis computer code. Suppose also that a typical displacement-based membrane finite element, capable of resisting normal and shear stresses, is used. The wing panel must then be grided with a relatively large number of these finite elements if an accurate solution is to be obtained. The computer output of the element stresses must be cataloged to determine which elements are along the edge of the panel. These elements' shear stresses must then be averaged to determine the shear flow. It would be much easier and faster if the engineer could model the wing panel using one element, the shear panel, and obtain as output
the average shear flows around its periphery. This would save the engineer valuable time not having to grid the panel and average the output stresses.

The rational derivation of a family of reliable shear panels which can be readily incorporated into displacement-based structural finite element computer codes is the main focus of this undertaking. Other investigators [1, 2, 3, 4, 5] have proposed shear panels, but most of their findings are not based on firm physical or mathematical grounds. Several of these shear panels will be compared with those derived herein by subjecting each of them to the same loads and boundary conditions and comparing the computed shear flows and displacements. An outline of their theoretical bases is included so that the assumptions can be compared and contrasted. Hopefully, future investigations of the shear panel will benefit from having this information summarized under one cover.
CHAPTER 2

ENERGY METHODS

Energy methods based on the principles of virtual work have been used throughout the development of finite element theory to formulate the governing equations for a large variety of finite elements. These methods provide the means of implementing two of the more common approaches to determining element stiffness matrices.

One approach is the “displacement method,” in which the functional form (commonly a polynomial) of the displacement field within an element is prescribed using its nodal displacements as parameters. In a structure composed of this type of element, the nodal displacements become the primary unknowns in the analysis. These displacements are ultimately used to determine the stresses within each element. The strains in a displacement-based element are automatically compatible, but there is no guarantee that the corresponding stress field is self-equilibrating.

Another approach is to start by assuming a functional form for the stress field within the element such that it is self-equilibrating. This stress field may be coupled with an assumed boundary displacement field to produce what is referred to as a hybrid element. If the displacements on the boundary are not
prescribed, the formulation of the stiffness matrix produces what is called a stress element.

To proceed from either of these assumptions requires using the basic equations of the theory of elasticity. The strain-displacement relations provide the compatibility equations. The generalized Hooke's Law [6] provides the stress-strain relationships. Using the Airy stress function [7] in its polynomial form provides a systematic approach to formulating the stress elements. The transformation from the element's local coordinate system to the structure's global coordinate system is performed quite frequently. The components of this transformation matrix can easily be determined. The fundamental groundwork will be reviewed next, after which the formulations of the various shear panel stiffness matrices and the accompanying rod elements will be easily understood.

**Stress Relations**

If a differential plane area element of specified orientation is passed through a point within an object the differential force vector acting upon the element. The stress vector is the ratio of the differential force vector and the differential oriented area. If we imagine the point to be surrounded by a differential cube oriented along the 
\(xyz\) coordinate axes, then the stress vector on each face or the "state of stress" can be resolved into three stress components: one component normal to the face and two orthogonal
"shear" components in the plane of the face. On the three faces of the cube oriented in the positive coordinate directions, there are a total of three normal and six shear stress components. (The stress components on each of the remaining three faces are equal but opposite in direction to those on the opposite plane, from Newton's action-reaction principle.) The state of stress is simpler in the case of plane stress, such as exists in thin sheets like a shear panel. The three-dimensional cube can then be viewed as a two-dimensional square plane, because the stress vector on any point of the plane is assumed to be zero, and the in-plane stresses are constant throughout the sheet thickness. Taking the z-axis perpendicular to the plane, the two-dimensional differential element has two normal stress components (σ_xx, σ_yy) and two shear stress components (σ_xy, σ_yx) Summing moments about any point on the element reveals that σ_xy = σ_yx in order for the element to be in equilibrium. The normal stress is defined as positive if directed away from the edge (tension) and negative if directed towards the edge (compression). Likewise, the shear stress has a positive value if it acts in a positive coordinate direction on an edge whose outward normal points in a positive coordinate direction. A state of plane stress with positive stress components is represented in Fig. 2.1a.

To account for the variation of the stress field within the sheet, the differential element is considered to be a free body acted on by stresses, which differ slightly on opposite edges, and body forces as well, as shown in Fig. 2.1b.
Figure 2.1 Two-dimensional plain stress element. (a) positive normal and shear stresses, (b) differential stresses and body forces.

Setting the net force in the \( x \) and \( y \) directions equal to zero yields the two-dimensional, differential equations of equilibrium, which, using subscript notation, can be written

\[
\begin{align*}
\sigma_{xx,x} + \sigma_{xy,y} + b_x &= 0 \\
\sigma_{xy,x} + \sigma_{yy,y} + b_y &= 0
\end{align*}
\] (2.1)

where \( b_x \) and \( b_y \) are the body forces per unit volume, which will hereafter be assumed zero.

The components of the stress vector \( \{T\} \) (the surface traction) at a point on a surface, can be expressed in terms of the stresses in the \( xy \) coordinate system and the components of the unit surface outward normal \( n \) as
\[ T_x = \sigma_{xx} n_x + \sigma_{xy} n_y \quad \text{or} \quad \begin{bmatrix} T_x \\ T_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} \] (2.2)

where the normal direction cosines (as shown in Fig. 2.2b) may be written as

\[ n_x = \frac{dy}{ds} = \cos(\theta) \]
\[ n_y = \frac{dx}{ds} = \sin(\theta) \] (2.3)

Figure 2.2 Surface tractions. (a) x-y coordinate system, (b) n-t coordinate system.

To transform the surface tractions from the x-y coordinate system to the normal and tangential n-t coordinate system (as shown in Fig. 2.2b) a simple two-dimensional rotation must be applied to obtain
Upon substituting Eq. 2.2 into Eq. 2.4, the normal and tangential surface tractions may be expressed either in terms of the direction cosines of the unit normal \( \mathbf{n} \) (Eq. 2.5) or in terms of the direction cosines of the unit tangent \( \mathbf{t} \) (Eq. 2.6).

\[
\begin{align*}
T_n & = \sigma_{xx} n_x^2 + \sigma_{yy} n_y^2 + 2 \sigma_{xy} n_x n_y \\
T_t & = (\sigma_{yy} - \sigma_{xx}) n_x n_y + \sigma_{xy} (n_x^2 - n_y^2)
\end{align*}
\]

(2.5)

\[
\begin{align*}
T_n & = \sigma_{xx} t_y^2 + \sigma_{yy} t_x^2 - 2 \sigma_{xy} t_x t_y \\
T_t & = (\sigma_{xx} - \sigma_{yy}) t_x t_y + \sigma_{xy} (t_y^2 - t_x^2)
\end{align*}
\]

(2.6)

Consider a straight line drawn in the plane from point \( i \) to point \( j \) in Fig. 2.2. The endpoint coordinates of this line are \((x_i, y_i)\) and \((x_j, y_j)\), respectively, and the length of the line is \( L_{ij} \). In terms of this data, the components of the unit tangent and unit normal to the line can be calculated as follows:

\[
\begin{align*}
t_x & = -n_y = \frac{(x_j - x_i)}{L_{ij}} \\
t_y & = n_x = \frac{(y_j - y_i)}{L_{ij}}
\end{align*}
\]

where \( L_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \) (2.7)
The two normal principal stresses, \( \sigma_{\text{max}} \) and \( \sigma_{\text{min}} \), the in-plane shear principal stress, \( \tau_{\text{max}} \), and the angle to the principal plane, \( \theta_p \), can be determined using the equations

\[
\sigma_{\text{max}} - \sigma_{\text{min}} = \frac{(\sigma_{xx} + \sigma_{yy})}{2} = \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2} \quad (2.8)
\]

\[
\tau_{\text{max}} = \frac{1}{2} \left| \sigma_{\text{max}} - \sigma_{\text{min}} \right| \quad (2.9)
\]

\[
2\theta_p = \tan^{-1}\left(\frac{2\sigma_{xy}}{\sigma_{xx} - \sigma_{yy}}\right) \quad (2.10)
\]

where the orientation of the normal principal planes and shear principal planes are shown in Fig. 2.3a and Fig. 2.3b, respectively.

Figure 2.3 Principal planes. (a) normal, (b) shear.
Strain Relations

The differential displacement of two initially orthogonal line segments is shown in Fig. 2.4. Normal strain can be defined as the ratio of change in length to a reference length. The normal strain in the $x$ direction, $\varepsilon_{xx}$, and the normal strain in the $y$ direction, $\varepsilon_{yy}$, can be expressed in terms of the displacements using the fundamental definition of strain:

$$
\varepsilon_{xx} = \frac{(u_x + u_{x,x}dx) - u_x}{dx} = u_{x,x} \\
\varepsilon_{yy} = \frac{(u_y + u_{y,y}dy) - u_y}{dy} = u_{y,y} 
$$

(2.11a)

Figure 2.4 Normal displacements and shear distortion.
The shear strain is defined as the total decrease in the initially 90 degree angle between the line segments in Fig. 2.4:

\[ \varepsilon_{xy} = \tan \beta_1 + \tan \beta_2. \]

Since the deformations are assumed to be small, \( \tan \beta_1 \approx \beta_1 \) and \( \tan \beta_2 \approx \beta_2 \); therefore the shearing strain becomes

\[ \varepsilon_{xy} = \frac{(u_x + u_{x,y} dy) - u_x}{dy} + \frac{(u_y + u_{y,x} dx) - u_y}{dx} = u_{x,y} + u_{y,x} \quad (2.11b) \]

Eqs. 2.11 show that the three strains are a function of the two displacements.

Differentiating the normal strains Eq. 2.11a

\[
\begin{align*}
\varepsilon_{xx,yy} &= u_{x,xy} \\
\varepsilon_{yy,xx} &= u_{y,xy}
\end{align*}
\quad (2.12a)
\]

and differentiating the shear strain in Eq. 2.11b

\[ \varepsilon_{xy,xy} = u_{x,xy} + u_{y,xy} \quad (2.12b) \]

then substituting Eqs. 2.12a into Eq. 2.12b, results in an equation relating the normal strains to the shear strains which is called the strain compatibility equation,

\[ \varepsilon_{xy,xy} = \varepsilon_{xx,yy} + \varepsilon_{yy,xx} \quad (2.13) \]


Stress-Strain Relationship

Plotting the variation of normal stress $\sigma$ and normal strain $\varepsilon$ for a one dimensional linear elastic rod produces a typical normal stress-strain diagram shown in Fig. 2.5. This shows that the normal stress is directly proportional to the normal strain up to the proportional limit. This proportionality constant is called the Modulus of Elasticity, $E$.

\[ \sigma = E \varepsilon \]  

(2.14)

which can be generalized into a two-dimensional relationship by introducing Poisson's ratio $\nu$, defined as the ratio of lateral strain
to longitudinal strain. (The "Poisson effect" is illustrated in the right of Fig. 2.5.) The normal strain-normal stress relations can then be expressed as

\begin{align*}
\epsilon_{xx} &= \frac{1}{E} \left[ \sigma_{xx} + v \sigma_{yy} \right] \\
\epsilon_{yy} &= \frac{1}{E} \left[ \sigma_{yy} + v \sigma_{xx} \right]
\end{align*}

(2.15)

and the inverse, normal stress-normal strain equations are

\begin{align*}
\sigma_{xx} &= \frac{E}{(1 - v^2)} \left[ \epsilon_{xx} + v \epsilon_{yy} \right] \\
\sigma_{yy} &= \frac{E}{(1 - v^2)} \left[ \epsilon_{yy} + v \epsilon_{xx} \right]
\end{align*}

(2.16)

To find the relationship between the shear stress and the shear strain, consider an element subjected to a positive shear force. As pointed out above, for small displacements the change in right angle (cf. Fig. 2.6) is defined as the shearing strain \( \epsilon_{xy} \). If the variation of shear stress with shear strain is plotted, a curve like that shown in Fig. 2.6 is produced. This shear stress-strain diagram, like the normal stress-strain diagram, is linear up to the proportional limit, and the linearity constant is defined as the Modulus of Rigidity, \( G \). The version of Hooke's Law relating shear stress to shear strain is

\[ \sigma_{xy} = G \, \epsilon_{xy} \]

(2.17)
For isotropic materials, there is a relationship between the material properties $E$, $G$ and $\nu$ which states:

$$G = \frac{E}{2(1 + \nu)}$$  \hspace{1cm} (2.18)

Using this equation, the complete two-dimensional strain-stress relationship can now be written in matrix form as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix} = [D] \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}$$  \hspace{1cm} (2.19a)

i.e.,

$$\begin{align*}
\varepsilon_{xx} &= \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1 + \nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \\
\varepsilon_{yy} &= \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1 + \nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \\
\varepsilon_{xy} &= \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1 + \nu) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}
\end{align*}$$  \hspace{1cm} (2.19b)
and the two-dimensional stress-strain relationship is

\[ \{\sigma\} = [E] \{\varepsilon\} \]  

i.e.,

\[ \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{E}{(1 - \nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1 - \nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} \]  

Airy Stress Function

It may be possible to express a plane stress field in terms of a scalar stress potential called the Airy stress function. This can take the form of a polynomial, a Fourier series or some other continuous function. The stresses are defined in terms of the Airy stress function as follows:

\[ \begin{align*}
\sigma_{xx} &= \Phi_{,yy} \\
\sigma_{yy} &= \Phi_{,xx} \\
\sigma_{xy} &= -\Phi_{,xy}
\end{align*} \]  

Upon substituting Eq. 2.21 into Eq. 2.1, assuming the body forces are zero, it is easily seen that, regardless of the particular form of the Airy stress function, the stresses automatically satisfy the equations of equilibrium.

The strain compatibility equation can be written in terms of stresses by substituting Hooke's law (Eq. 2.19) into Eq. 2.13 and applying the equations of equilibrium (Eq. 2.1). Assuming that the
body forces are zero, this gives the stress compatibility equation:

\[ \sigma_{xy} - 2\sigma_{xy,xy} + \sigma_{yy,xx} = 0 \]  \hspace{1cm} (2.22)

After substituting Eqs. 2.21, the compatibility equation takes the form

\[ \Phi_{,xxxx} + 2\Phi_{,xy} + \Phi_{,yyyy} = 0 \]  \hspace{1cm} (2.23).

Eq. 2.23 implies that the Airy stress function cannot be chosen arbitrarily; it must satisfy the \textit{biharmonic equation}.

In this thesis, polynomial Airy stress functions will be used to define the stress field within a shear panel. The degree of the polynomial to be chosen is part of the research. The terms which appear in a complete polynomial of a given degree are taken from the Pascal triangle [8], the first four rows of which are shown in Fig. 2.7. The corresponding 10-term cubic polynomial is shown in Eq. 2.24.

\[
\begin{align*}
1 \\
x \\
x^2 \\
x^3 \\
0 \text{ degree (constant)} \\
1 \text{st degree (linear)} \\
2 \text{nd degree (quadratic)} \\
3 \text{rd degree (cubic)} \\
\vdots \\
1 \text{ term} \\
3 \text{ terms} \\
6 \text{ terms} \\
10 \text{ terms} \\
\end{align*}
\]

\text{Figure 2.7 Two-dimensional Pascal triangle.}

\[ \Phi = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 \\
+ c_7 x^3 + c_8 x^2 y + c_9 xy^2 + c_{10} y^3 \]  \hspace{1cm} (2.24)
A complete polynomial should always be selected for the Airy stress function. If terms are arbitrarily removed for one reason or another, unpredictable results may occur and the solution may not be exact. Since the plane stresses are second derivatives of the Airy stress function, the stress polynomials will be two degrees lower. The cubic Airy stress function in Eq. 2.24 will therefore yield a linear stress distribution, i.e., using Eq. 2.21, we find

\[
\begin{align*}
\Phi_{,yy} &= \sigma_{xx} = 2c_6 + 2c_9 x + 6c_{10} y \\
\Phi_{,xx} &= \sigma_{yy} = 2c_4 + 2c_7 y + 6c_8 x \\
-\Phi_{,xy} &= \sigma_{xy} = -c_5 - 2c_8 x - 2c_9 y
\end{align*}
\] (2.25)

These stresses can be easily shown to satisfy compatibility (Eq. 2.22) and equilibrium (Eq. 2.1). In this way Table 2.1 can be formed, relating the polynomial degree of the Airy stress function to the degree of the stresses.

<table>
<thead>
<tr>
<th>Airy stress function polynomial form</th>
<th>Stresses satisfying the biharmonic equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree</td>
<td># of terms</td>
</tr>
<tr>
<td>------</td>
<td>----------</td>
</tr>
<tr>
<td>0th</td>
<td>1</td>
</tr>
<tr>
<td>1st</td>
<td>3</td>
</tr>
<tr>
<td>2nd</td>
<td>6</td>
</tr>
<tr>
<td>3rd</td>
<td>10</td>
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<tr>
<td>4th</td>
<td>15</td>
</tr>
<tr>
<td>5th</td>
<td>21</td>
</tr>
<tr>
<td>6th</td>
<td>28</td>
</tr>
<tr>
<td>7th</td>
<td>36</td>
</tr>
<tr>
<td>8th</td>
<td>45</td>
</tr>
</tbody>
</table>
**Transformation Matrix**

The transformation of vectors from an element-based local coordinate system to the common global frame, and vice versa, is often required in solving structural problems. The local transformation or direction cosine-matrix $[\lambda]$ is formed by determining the orientation of the element's local coordinate axis system in the structure's global coordinate axis system. This is achieved by finding the components of the local coordinate unit vectors $i'$, $j'$, $k'$ along the global axes, whose unit vectors are $i$, $j$, $k$. The components of $i'$ are $(l_1, m_1, n_1)$, those of $j'$ are $(l_2, m_2, n_2)$, and for $k'$ they are $(l_3, m_3, n_3)$.

![Figure 2.8 Vector V in the global xyz and local x'y'z' system.](image-url)
If the components of a vector \( \mathbf{V} \) in the global coordinate system are \( v_x, v_y, v_z \), then

\[
\mathbf{V} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}
\]  

(2.26)

The component \( v_x' \) of \( \mathbf{V} \) along the local \( x' \) axis is \( \mathbf{V} \cdot \mathbf{i}' \), the projection of \( \mathbf{V} \) onto \( \mathbf{i}' \). Likewise, \( v_y' = \mathbf{V} \cdot \mathbf{j}' \) and \( v_z' = \mathbf{V} \cdot \mathbf{k}' \).

Substituting Eq. 2.26 into each of these dot products and noting that \( i \cdot i' = l_1, \ j \cdot i' = m_1, \ k \cdot i' = n_1, \ i \cdot j' = l_2, \ldots, \ k \cdot k' = n_3 \), we find

\[
\begin{align*}
  v_x' &= v_x l_1 + v_y m_1 + v_z n_1 \\
  v_y' &= v_x l_2 + v_y m_2 + v_z n_2 \\
  v_z' &= v_x l_3 + v_y m_3 + v_z n_3
\end{align*}
\]  

(2.27)

so that, relative to the local system,

\[
\mathbf{V} = v_x' \mathbf{i}' + v_y' \mathbf{j}' + v_z' \mathbf{k}'
\]  

(2.28)

Eq. 2.27 is conveniently expressed in matrix form as

\[
\{V'\} = [\lambda] \{V\}
\]  

(2.29a)

i.e.,

\[
\begin{bmatrix}
  v_x' \\
  v_y' \\
  v_z'
\end{bmatrix} =
\begin{bmatrix}
  l_1 & m_1 & n_1 \\
  l_2 & m_2 & n_2 \\
  l_3 & m_3 & n_3
\end{bmatrix}
\begin{bmatrix}
  v_x \\
  v_y \\
  v_z
\end{bmatrix}
\]  

(2.29b)

This is called the global to local transformation equation.
As indicated above, the direction cosines are defined as

\[
\begin{align*}
I_1 &= \cos(\theta_{x \rightarrow x'}) \\
I_2 &= \cos(\theta_{x \rightarrow y'}) \\
I_3 &= \cos(\theta_{x \rightarrow z'}) \\
M_1 &= \cos(\theta_{y \rightarrow x'}) \\
M_2 &= \cos(\theta_{y \rightarrow y'}) \\
M_3 &= \cos(\theta_{y \rightarrow z'}) \\
N_1 &= \cos(\theta_{z \rightarrow x'}) \\
N_2 &= \cos(\theta_{z \rightarrow y'}) \\
N_3 &= \cos(\theta_{z \rightarrow z'})
\end{align*}
\] (2.30)

If the axes of both coordinate systems, \(xyz\) and \(x'y'z'\), are mutually perpendicular, then it can be shown that the direction cosine matrix \([\lambda]\) is an orthogonal matrix. A property of an orthogonal matrix is that its inverse equals its transpose:

\[
[\lambda]^{-1} = [\lambda]^T
\] (2.31)

Taking advantage of this fact, the local to global transformation equation—the inverse of Eq. 2.29a—can be expressed as

\[
\{V\} = [\lambda]^T \{V'\}
\] (2.32a)

i.e.,

\[
\begin{bmatrix}
V_x \\
V_y \\
V_z
\end{bmatrix} =
\begin{bmatrix}
I_1 & I_2 & I_3 \\
M_1 & M_2 & M_3 \\
N_1 & N_2 & N_3
\end{bmatrix}
\begin{bmatrix}
V'_x \\
V'_y \\
V'_z
\end{bmatrix}
\] (2.32b)

If a number of vectors \(\{v_1\}, \{v_2\}, \ldots, \{v_n\}\) are to be simultaneously transformed, the operation can be compactly represented as follows. An extended column vector \(\{V\}\) is formed by stacking the vectors one onto the other.
Then a transformation matrix $[\Lambda]$ is fashioned by placing the direction cosine matrix $[\lambda]$ as many times along the principal diagonal of $[\Lambda]$ as there are partitioned vectors in the column vector $\{V\}$. $[\Lambda]$ will then have the following form, with all entries outside of those occupied by $[\lambda]$ being zero:

$$
[\Lambda] = \begin{bmatrix}
[\lambda] & [0] & \ldots & [0] \\
[0] & [\lambda] & \ldots & [0] \\
\vdots & \vdots & \ddots & \vdots \\
[0] & [0] & \ldots & [\lambda]
\end{bmatrix} \quad (2.34)
$$

Therefore the general transformation equation for the global to local conversion can be written as

$$
\{V\} = [\Lambda] \{V\} \quad \text{ - global to local} \quad (2.35)
$$

and to transform from the local to global system

$$
\{V\} = [\Lambda]^T \{V\} \quad \text{ - local to global} \quad (2.36)
$$
**Displacement-Based Finite Elements**

The displacement-based finite element is the most common type found in finite element computer code libraries. Their theoretical formulation begins by assuming the simplest possible form of the displacement field (usually polynomials) consistent with the physical and geometrical complexity of the element itself. One then applies the Principle of Virtual Work [9, 10, 11] (also known as the Principle of Virtual Displacements) or an allied energy principle (Minimum Potential Energy [8, 11] or Castigliano’s First Theorem [7]) to find the stiffness matrix.

The Principle of Virtual Work (PVW) states that a body is in equilibrium if and only if, for any arbitrary virtual deformation, the internal virtual work \( \delta U_i \) of the actual stresses acting through the virtual strains is equal to the virtual work \( \delta W_e \) of the external loads acting through the virtual boundary displacements:

\[
\delta U_i = \delta W_e \quad (2.37a)
\]

where the volume integral defining the internal virtual work is

\[
\delta U_i = \int_V \{\delta \varepsilon\}^T \{\sigma\} \, dV \quad (2.37b)
\]

and the nodal point loads, the surface integral and the body force volume integral defining the external virtual work is

\[
\delta W_e = [\delta q]^T \{Q_r\} + \int_S \{\delta u\}^T \{T_S\} \, dS + \int_V \{\delta u\}^T \{b\} \, dV \quad (2.37c)
\]
where

\[
\begin{align*}
\{\delta \varepsilon\} &= \text{Virtual strains} \\
\{\sigma\} &= \text{Actual stresses} \\
\{\delta q\} &= \text{Virtual nodal displacements (one per each nodal d.o.f.)} \\
\{Q_n\} &= \text{Nodal loads in the direction of the nodal displacements} \\
\{\delta u\} &= \text{Virtual displacement field} \\
\{T_S\} &= \text{Surface tractions} \\
\{b\} &= \text{Body forces per unit volume} \\
dS &= \text{Surface integral} \\
dV &= \text{Volume integral}
\end{align*}
\]

Thus the PVW becomes

\[
\int_V \{\delta \varepsilon\}^T \{\sigma\} dV = \{\delta q\}^T \{Q_n\} + \int_S \{\delta u\}^T \{T_S\} dS + \int_V \{\delta u\}^T \{b\} dV \quad (2.38)
\]

If the body forces are neglected this simplifies to

\[
\int_V \{\delta \varepsilon\}^T \{\sigma\} dV = \{\delta q\}^T \{Q_n\} + \int_S \{\delta u\}^T \{T_S\} dS \quad (2.39)
\]

Consider a two-dimensional element in its own local $x'$-$y'$ coordinate system having $n_n$ nodes, with each node having $n_q$ degrees of freedom. The two-component displacement vector $\{u\}$ at any point in the element is interpolated from the nodal displacements $\{q\}$ by means of complete polynomial \textit{shape functions},
\{u\} = [N] \{q\} \quad (2.40a)

i.e.,

\[
\begin{bmatrix}
\{u_x\} \\
\{u_y\}
\end{bmatrix} =
\begin{bmatrix}
N_{11} & N_{12} & \cdots & N_{1n_q} \\
N_{21} & N_{22} & \cdots & N_{2n_q}
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
\vdots \\
q_{n_q}
\end{bmatrix} \quad (2.40b)
\]

where the components of \([N]\) are the shape functions, each one being a polynomial of the same degree in \(x'\) and \(y'\). The degree of the polynomial depends on the number of nodal degrees of freedom.

The matrix form of the strain-displacement relations (Eqs. 2.11 and 2.12) is

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{xy}
\end{bmatrix} =
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 \\
0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & \frac{\partial}{\partial x}
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y
\end{bmatrix} \quad (2.41)
\]

Upon substituting Eq. 2.40b this becomes

\[
\{\varepsilon\} = [B] \{q\} \quad (2.42a)
\]

i.e.,

\[
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{yy} \\
\varepsilon_{xy}
\end{bmatrix} =
\begin{bmatrix}
N_{11'}x & N_{12'}x & \cdots & N_{1n_q'}x \\
N_{21'}y & N_{22'}y & \cdots & N_{2n_q'}y \\
(N_{11'y} + N_{21'x})(N_{12'y} + N_{22'x}) & \cdots & (N_{1n_q'y} + N_{2n_q'x})
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
\vdots \\
q_{n_q}
\end{bmatrix} \quad (2.42b)
\]
Assuming linear elastic isotropic material properties, Eq. 2.42 can be substituted into Eq. 2.20a to obtain the stresses in terms of the nodal displacements:

\[ \{\sigma\} = [E] [B] \{q\} \]  

(2.43)

Now that the element displacement field (Eq. 2.40a) and strains (Eq. 2.42a) are expressed in terms of the nodal displacements of the element, the virtual displacement field \( \{\delta u\} \) and virtual strain field \( \{\delta e\} \) can be written in terms of the virtual nodal displacements \( \{\delta q\} \), as

\[ \{\delta u\} = [N]\{\delta q\} \]
\[ \{\delta e\} = [B]\{\delta q\} \]

(2.44)

Upon applying the *reversal rule*, which states that

if \( [A] = [B][C] \) then \( [A]^T = [C]^T[B]^T \)  
(2.45a)

or if \( [A] = [B][C][D] \) then \( [A]^T = [D]^T[C]^T[B]^T \)  
(2.45b)

and after substituting Eq. 2.44 into Eq. 2.39 we obtain the PVW equation in the following form:

\[ \int_V \{\delta q\}^T E [B] \{q\} dV = \{\delta q\}^T \{Q_n\} + \int_S \{\delta q\}^T \{N\}^T \{T_S\} dS \]  

(2.46a)
Since this equality must hold for arbitrary choice of \( \{\delta q\} \), it follows that the coefficients of the virtual displacements on each side of the equation must be equal. That is,

\[
\left[ \int_{V} [\mathbf{B}]^{T} \mathbf{B} dV \right] \{q\} = \{Q_n\} + \int_{S} [N]^{T} \{T\}_g dS \quad (2.46b)
\]

Defining the left-side integral as the local stiffness matrix \([K']\) and the entire right hand side as the load vector \(\{Q'\}\), which consists of point loads \(\{Q_n\}\) and the point-load equivalents of the surface tractions \(\{Q_s\}\) (where the prime is used to denote quantities defined in the element’s local coordinate system), Eq. 2.46b can be written as

\[
[K'] \{q'\} = \{Q'\} \quad (2.47)
\]

where

\[
[K] = \int_{V} [\mathbf{B}]^{T} \mathbf{B} dV \quad (2.48)
\]

and

\[
\{Q_s\} = \int_{S} [N]^{T} \{T\}_g dS \quad (2.49)
\]

so that

\[
\{Q'\} = \{Q_n\} + \{Q_s\} \quad (2.50)
\]

Eq. 2.47 will be referred to as the standard form of the local stiffness equations. The global stiffness equations have the same form, without the primes. The global stiffness matrix \([K]\) is formed from the element stiffness matrices \([K']\), each of which
must be transformed into the global frame before they can be assembled into \([K]\).

The element's local stiffness equation (Eq. 2.47) is transformed into the global system by using Eq. 2.35. Applying Eq. 2.35 to \(\{q\}\), the element's displacement vector in global coordinates, we have

\[
\{q\}' = [\Lambda] \{q\} \tag{2.51}
\]

Similarly, the element's load vector \(\{Q\}'\) in the element coordinate system is related to its global counterpart by

\[
\{Q\}' = [\Lambda] \{Q\} \tag{2.52}
\]

By substituting Eqs. 2.51 and 2.52 into the local stiffness equations (Eq. 2.47) produces

\[
[K'] [\Lambda] \{q\} = [\Lambda] \{Q\} \tag{2.53}
\]

Multiplying this equation through from the left by \([\Lambda]^T\) and using the fact that \([\Lambda]\) is an orthogonal matrix yields

\[
[\Lambda]^T [K'] [\Lambda] \{q\} = \{Q\} \tag{2.54}
\]
This global stiffness formula can be written as

$$[K] \{q\} = \{Q\}$$  \hspace{1cm} (2.55)

where the element global stiffness matrix is defined as

$$[K] = [\Lambda]^T [K'] [\Lambda]$$  \hspace{1cm} (2.56)

Once all of the elements' stiffness matrices have been transformed into the global frame, they are assembled to form the structure's global array $[K_{global}]$ by matching the nodal degrees of freedom of each element with those of the structure. The element loads are likewise assembled into the global load vector $\{Q_{global}\}$. Then the structure's stiffness equations are solved for the global displacements $\{q_{global}\}$ using a standard linear system solver, such as a Gauss elimination routine:

$$\{q_{global}\} = [K_{global}]^{-1} \{Q_{global}\}$$  \hspace{1cm} (2.57)

Finally, each element's global displacement vector is picked out of $\{q_{global}\}$ and transformed back into the local coordinate system using Eq. 2.35. The local displacement vector $\{q'\}$ is then multiplied by the local stiffness matrix $[K']$ to solve for the element's local load vector $\{Q'\}$ (cf. Eq. 2.47), and Eq. 2.43 is used to find the stresses in the element.
The number of nodes an element has along with the nature of the nodal displacements determines the degree of the polynomial shape functions that can be used. For example, the axial displacement in a two-node rod element can only be interpolated by a linear polynomial whose coefficients are the axial displacements of the nodes. The axial displacement in a three-node rod element must be described using a quadratic polynomial. The transverse displacement in a two-node beam element requires a cubic shape function to interpolate both the deflection and slope from the four nodal values.

Displacement-based elements are inherently stiff because the element response is restricted to just the displacement modes allowed by the shape functions. With added nodes, the degree of the polynomial shape functions can be increased, and the additional displacement modes increase the flexibility of the element. Instead of devising higher-order elements to model a structure more accurately, the exact solution can be approached by using a larger number of low-order elements. This increases the total number of degrees of freedom which leads to a more flexible model. Of course the increased number of nodes increases the size of the stiffness matrix and the CPU time required to do the structural analysis. The alternative use of the higher-order displacement elements seems attractive because fewer elements would be needed and the size of the stiffness matrix \[ K_{\text{global}} \] could be smaller, yielding faster execution time. However, use of high order elements incurs computational overhead at the element formation level, due to the
more complex polynomials which have to be integrated to form \( [K'] \) (cf. Eqs. 2.48 and 2.42). The displacement-based elements are the most common finite elements used today. Popular computer codes which use these elements are ABAQUS, ANSYS, GIFTS, NASTRAN, SAP, to mention only a few.

**Stress-Based Finite Elements**

The stress-based elements differ from the displacement-based elements in that these elements have an assumed, self-equilibrating stress field instead of an assumed displacement field. If the stresses are assumed to be polynomials, the coefficients of the polynomial terms are the *stress parameters*. If the load vector is formulated using an assumed displacement field on the element boundary, the element is said to be a hybrid element. If the load vector is expressed solely in terms of the stress parameters, the element is called a stress element.

The derivation of the the stiffness matrix of a stress-based element is accomplished using the Principle of Complementary Virtual Work [9, 10, 11] (also known as the Principle of Virtual Forces) or one of its allied energy forms (the Principle of Minimum Complementary Energy [8,11] or Castigliano's Second Theorem [7]).

The Principle of Complementary Virtual Work (PCVW) states that the strain field within a body is compatible with the displacements if and only if for any arbitrary self-equilibrating virtual stress field, the internal complementary virtual work \( \delta U^* \) of the virtual stresses acting through the actual strains equals the
external complementary virtual work $\delta W_e^*$ of the corresponding virtual external loads acting through the actual boundary displacements:

\[ \delta U_i^* = \delta W_e^* \quad (2.58a) \]

where the volume integral defining the complementary internal virtual work is

\[ \delta U_i^* = \int_{V} \{\delta a\}^T \{\varepsilon\} dV \quad (2.58b) \]

and the nodal point loads, the surface integral and the body force volume integral defining the complementary external virtual work is

\[ \delta W_e^* = \{\delta Q_n\}^T \{q\} + \int_{S} \{\delta T_S\}^T \{u\} dS + \int_{V} \{\delta b\}^T \{u\} dV \quad (2.58c) \]

where

- $\{\delta a\} = \text{Virtual stresses}$
- $\{\varepsilon\} = \text{Actual strains}$
- $\{\delta Q_n\} = \text{Virtual nodal loads (one per each nodal d.o.f.)}$
- $\{q\} = \text{Actual nodal displacements in the direction of the nodal loads}$
- $\{\delta T_S\} = \text{Virtual surface tractions}$
- $\{u\} = \text{Boundary displacement field}$
- $\{\delta b\} = \text{Virtual body forces per unit volume}$
- $dS = \text{Surface integral}$
- $dV = \text{Volume integral}$
Thus, the PCVW can be expressed as

\[ \int_V \{\delta \sigma\}^T \{\varepsilon\} \, dV = \{\delta Q_n\}^T \{q\} + \int_S \{\delta T_s\}^T \{u\} \, dS + \int_V \{\delta b\}^T \{u\} \, dV \]  

(2.59)

If the body forces are neglected, the PCVW equation simplifies to

\[ \int_V \{\delta \sigma\}^T \{\varepsilon\} \, dV = \{\delta Q_n\}^T \{q\} + \int_S \{\delta T_s\}^T \{u\} \, dS \]  

(2.60)

The virtual stresses and the strains need to be expressed in terms of nodal displacements \( \{q\} \). Assume a two-dimensional element in a local \( x'-y' \) coordinate system which has \( n_n \) nodes and each node has \( n_q \) d.o.f.'s. The stress vector \( \{\sigma\} \), which for a two-dimensional element is composed of three stress components (two normal and one shear), must be chosen such that the internal stresses at any point within the element or on the boundary are in equilibrium and continuous throughout the element. If we assume that polynomials are used to describe this stress field, then

\[ \{\sigma\} = [P] \{\beta\} \]  

(2.61a)

i.e.,

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} =
\begin{bmatrix}
P_{11} & P_{12} & \cdots & P_{1n_s} \\
P_{21} & P_{22} & \cdots & P_{2n_s} \\
P_{31} & P_{32} & \cdots & P_{3n_s}
\end{bmatrix}
\begin{bmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_{n_s}
\end{bmatrix}
\]  

(2.61b)
where the matrix $[P]$ contains the stress polynomial terms. The size of the $[P]$ matrix depends on $n_s$, the number of polynomial terms chosen. The components of the stress coefficient vector $\{\beta\}$ are the stress parameters, which are to be at least partly determined from the imposed boundary conditions.

Upon substituting Eq. 2.61a into the strain-stress relation for linear elastic materials (Eq. 2.19a), the strains can be expressed in terms of the stress parameters:

$$\{\varepsilon\} = [D] [P] \{\beta\}$$  \hspace{1cm} (2.62)

The surface traction vector $\{T_S\}$ is linearly related to the internal stresses by Eq. 2.2. Since the internal stresses are linearly related to the stress parameters (Eq. 2.61a), it follows that the surface traction vector is also linearly related to the stress parameters,

$$\{T_S\} = [L] \{\beta\}$$  \hspace{1cm} (2.63)

where the form of the matrix $[L]$ will depend whether the components of surface traction are in the $x'$-$y'$ coordinate directions or taken normal and tangential to the element boundary (cf. Eqs. 2.2 and 2.4).

A relationship between the two components of the displacement field $\{u\}$ at any point on the edge of the element and the element's nodal displacements $\{q\}$ can be assumed in order to
distribute the surface tractions to the nodes and provide for interelement displacement compatibility along the element edges.

The boundary interpolation, or shape, functions \([\mathbf{N}]\) are chosen to be polynomials, the degree of which depends on the number of nodes on the edge and the nature of the nodal displacements. (The bar over the \(N\) symbolically distinguishes the boundary interpolation functions from those which apply throughout the element interior.) In terms of the boundary shape functions we have, as in the displacement-based method (Eq. 2.40),

\[
\{u\} = [\mathbf{N}] \{q\}
\]  \hspace{1cm} (2.64)

The relation between the nodal load vector \(\{Q_n\}\) and the stress parameters is determined by the matrix \([F_n]\) in the following equation:

\[
\{Q_n\} = [F_n] \{\beta\}
\]  \hspace{1cm} (2.65)

An arbitrary choice of virtual stress parameters \(\{\delta \beta\}\) yields a virtual stress field \(\{\delta \sigma\}\) by means of Eq. 2.61a. Likewise, the virtual surface traction vector \(\{\delta T_s\}\) and virtual nodal load vector \(\{\delta Q_n\}\) are obtained from Eq. 2.63 and Eq. 2.65, respectively, so that

\[
\{\delta \sigma\} = [P] \{\delta \beta\}
\]
\[
\{\delta T_s\} = [L] \{\delta \beta\}
\]  \hspace{1cm} (2.66)
\[
\{\delta Q_n\} = [F_n] \{\delta \beta\}\]
After substituting these expressions and the assumed displacement field of Eq. 2.64 into the PCVW formula (Eq. 2.60), and applying the reversal rule (Eq. 2.45) we find:

\[
\int_{V} \{\delta \beta\}^{T} [P]^{T} [D] [P] \{\beta\} dV = \{\delta \beta\}^{T} [F_{n}]^{T} \{q\} + \int_{S} \{\delta \beta\}^{T} [L]^{T} [N] \{q\} dS \quad (2.67)
\]

According to PCVW, this equation must hold for arbitrary choice of \( \{\delta \beta\}^{T} \). This implies that the coefficients of the virtual stress parameters on each side of the equation must be equal. That is,

\[
\left[ \int_{V} [P]^{T} [D] [P] dV \right] \{\beta\} = \left[ [F_{n}]^{T} + \int_{S} [L]^{T} [N] dS \right] \{q\} \quad (2.68)
\]

This equation can be put into the standard form, \( [K'] \{q'\} = \{Q'\} \), by letting

\[
[H] = \int_{V} [P]^{T} [D] [P] dV \quad (2.69)
\]

and

\[
[F]^{T} = [F_{n}]^{T} + \int_{S} [L]^{T} [N] dS \quad (2.70a)
\]

or

\[
[F] = [F_{n}] + \int_{S} [N]^{T} [L] dS \quad (2.70b)
\]

or simply

\[
[F] = [F_{n}] + [F_{S}] \quad (2.70c)
\]

where

\[
[F_{S}] = \int_{S} [N]^{T} [L] dS \quad (2.70d)
\]
Therefore the PCVW equation can be rewritten in the local system as

\[ [H] \{\beta\} = [F]^T \{q'\} \]  
(2.71)

The matrix \([H]\) is referred to as the natural flexibility matrix, and it relates the generalized loads (stress parameters) \(\{\beta\}\) to the generalized displacements \(\{d\} = [F]^T \{q'\}\). Eq. 2.71 is used to solve for the stress parameters \(\{\beta\}\) once the local nodal displacements are known:

\[ \{\beta\} = [X] \{q'\} \]  
(2.72a)

where \([X]\) is the solution of the linear system

\[ [H] [X] = [F]^T \]  
(2.72b)

To determine the local displacements, we multiply Eq. 2.70c by the stress parameter vector \(\{\beta\}\)

\[ [F] \{\beta\} = [F_n] \{\beta\} + [F_s] \{\beta\} \]  
(2.73)

From Eq. 2.65, \(\{Q_n\} = [F_n] \{\beta\}\), the nodal load vector. Using Eq. 2.70d and Eq. 2.63, we find that

\[ [F_s] \{\beta\} = \int_S [\overline{N}]^T [U] \{\beta\} dS = \int_S [\overline{N}]^T \{T_S\} dS \]  
(2.74)
The last integral is just the equivalent nodal load vector \( \{Q_S\} \) due to surface tractions:

\[
\{Q_S\} = \int_S [\mathbf{N}]^T \{T_S\} dS \quad (2.75)
\]

The sum of these two load vectors produces the element’s local load vector

\[
\{Q’\} = \{Q_n\} + \{Q_S\} \quad (2.76)
\]

so that Eq. 2.73 simplifies to

\[
[F] \{\beta\} = \{Q’\} \quad (2.77)
\]

Observe that the matrix \([F]\) establishes the relationship between the generalized loads and the nodal point load vector. By substituting Eq. 2.71 into Eq. 2.77 the PCVW formula can be written as

\[
[F] [H]^{-1} [F]^T \{q’\} = \{Q’\} \quad (2.78)
\]

from which we infer that the local stiffness matrix is given by

\[
[K’] = [F] [H]^{-1} [F]^T \quad (2.79)
\]

so that the standard form for the local stiffness equations is obtained:

\[
[K’] \{q’\} = \{Q’\} \quad (2.80)
\]
The element's local stiffness matrix and load vector must be transformed into the global system for assembly into the structural stiffness matrix \([K_{\text{global}}]\) and structural load vector \([Q_{\text{global}}]\). The solution for the structure's global displacements \([q_{\text{global}}]\) is then performed. These displacements are transformed back into each element's local system using the local displacement solution procedure outlined above for the displacement-based elements. With the local displacements in hand, the stress parameters \([\beta]\) can be determined using Eq. 2.72. The stresses at any point within the element or on the boundary of the element are found by substituting the stress parameters into Eq. 2.61. The stress parameters can be substituted into Eq. 2.65 to determine the element's point load vector \([Q_n]\) and into Eq. 2.75 to determine the point load vector \([Q_s]\) equivalent to the surface tractions.

The stress-based element inherently satisfies the conditions of equilibrium. It is generally more flexible than the displacement-based element. As the polynomial degree of the internal stress field increases the flexibility of the element decreases toward the exact solution. This provides the user the ability of using less elements to model a structure. However, the computation time to analyze this structure will increase. Stress-based elements are used when stresses are the primary unknowns of a problem and displacement-based elements are used when displacements are the primary unknowns, as in dynamic analysis.
CHAPTER 3

PREVIOUS WORKS

In the past, different researchers [1, 2, 3, 4, 5] approached the tapered shear panel using different methods and assumptions. Nearly all of the studies have assumed that purely tangential shear forces are applied along the four sides of the panel. Several of the methods will be outlined below, followed by a theoretical discussion. The local stiffness matrix $[K']$ for a constant thickness quadrilateral shear panel will be derived using each approach along with the appropriate assumptions. Each element’s load vector will also be presented to provide a full understanding of these assumptions.

Peery [1] presented an analysis of a trapezoidal shear panel by considering a tapered cantilever beam modeled as two non-parallel flanges separated by a thin web with a shear load applied at the free end (smallest section). The flanges were assumed to carry the loads normal to the section caused by bending while the panel absorbed only the shear load. The panel does not resist all of the shear, because part of it is taken up by the transverse component of the axial force in the inclined flanges. Peery showed that the uniform shear flow on a section of the panel is inversely proportional to the square of the distance of the section from the projected location.
intersection of the two flanges. Since the shear flow varies along the span, it is convenient to use the mean shear flow in extending this analysis to box beams. Peery provided his solution to the common tapered shear panel problem using only statics. Consideration of the beam's flexibility was not required since the problem is statically determinate and deflections were not required.

Garvey [2] also used statics to determine the shear flow in a flat arbitrarily-shaped quadrilateral panel ("trapezium") subjected to self-equilibrating shear tractions around its edges. He assumed that at any point in the panel there exists a differential parallelogram whose edges are in pure shear. He proceeded to show that this shear flow is inversely proportional to the square of the distance from a baseline defined by the geometry of the quadrilateral. Integrating the exact expression for the strain energy of a differential parallelogram in pure shear over the entire quadrilateral, Garvey derived an expression for the strain energy of a trapezium panel. The expression simplifies for the special case of a trapezoid. Garvey's results are not exact because his assumption on the stress field is only correct for a parallelogram with pure shear traction on its edges. From his strain energy formulae, the natural flexibility matrix \([H]\) can be inferred. The load vector consisting of mean shear flows is then used to define the matrix \([F]\). These two matrices are then combined to form the local stiffness matrix \([K']\).

Azar [3] approached the shear panel as a displacement-based finite element. Centering his local right-handed orthogonal
coordinate system at one of the corners of the panel, with the x-axis lying along an edge, he assumed a quadratic displacement field with the coefficients selected in such a way that the shear strain $\varepsilon_{xy}$ is constant. The accompanying normal strains are not zero, nor even constant. Nevertheless, Azar used the PVW to derive the stiffness matrix assuming only the shear strain contributed to the internal virtual work (or strain energy). Azar's approach can be justified only for a rectangular panel.

Robinson [4] used the PCVW to determine the local stiffness matrix of an eight-node warped quadrilateral shear panel. The panel's load vector is comprised of the four average shear flows on each edge plus the warping forces at each corner. He assumed, without justification, that the scalar natural flexibility of the rectangular shear panel is adequate for the warped shear panel as well. Robinson derived the matrix $[F]$ which is needed along with the element flexibility matrix to compute the stiffness matrix using Eq. 2.79. His derivation can be simplified for a plane shear panel, dropping the warping forces but retaining the average shear flows along the edges.

Nack [5] used the assumed-stress, assumed boundary-displacement approach and the PCVW to derive a flat hybrid shear panel. He assumed that the surface traction resultants on the boundary form a self-equilibrating set of four shear forces directed along the edges of the panel. Therefore, using statics, three of the shear forces can be expressed in terms of the fourth, which he used as the single stress parameter. Nack derived the stresses within
the panel from a fifth-order Airy stress polynomial. The polynomial coefficients were determined by enforcing the pure shear force condition on the boundary of the panel. Nack used a point collocation scheme wherein he required that the normal traction at two specified points on an edge be zero and the shear traction at three other specified points on the same edge equal the shear traction expressed in terms of the stress parameter. This provided twenty boundary conditions required to solve for the twenty independent coefficients Nack chose to retain in his modified Airy stress function. The natural scalar flexibility $H$ was computed by numerical integration. The matrix $[F]$ needed to determine the local stiffness matrix was found by assuming a linear shape function for the boundary displacements and carrying out the integral in Eq. 2.70d along each edge of the quadrilateral.

**Peery's Shear Panel**

Peery's book, although it became somewhat dated, was widely used as a text on aircraft structures before it went out of print. It remains a valuable reference on classical force-method analysis techniques. In his text, Peery describes how a cantilevered stiffened web structure composed of two flanges and a thin rectangular panel subjected to a shear load at its free end acts like a simplified beam. The concentrated flanges are assumed to resist all axial forces due to bending while the thin panel resists all of the shear load. The web, or "shear panel," is in a state of pure shear stress, $\sigma_{xy}$. It is convenient to represent this state of stress in
terms of the shear flow $s$, defined as $s = \sigma_{xy}t$, where $t$ is the uniform thickness of the shear panel.

Peery shows how this procedure can be extended to the tapered beam shown in Fig. 3.1a. Since the top and bottom flanges are inclined, the axial load directed along a flange can be resolved into components normal to and parallel to a spanwise section. The two equal but oppositely-directed components of flange load $P$ normal to the section form a couple which equilibrates the moment of the applied load. The sum of the vertical components of flange load $V_f$, parallel to this section equals part of the applied shear load. The rest of the shear, $V_w$, will be reacted by the panel and uniformly distributed along the vertical cut. Therefore, the total shear force $V$ on this spanwise section is the sum of the two vertical flange components and that of the shear panel:

$$V = V_w + V_f$$

(3.1)

Figure 3.1 Peery's tapered beam analysis. (a) cantilever beam, (b) spanwise station.
Each inclined flange's contribution to the transverse shear force (Fig. 3.1) is found by projecting its axial load onto the section. This leads to the following formula:

\[
V_f = \frac{V_b}{h} (\tan \alpha_1 + \tan \alpha_2) = \frac{V_b}{h} \left( \frac{h_1}{x} + \frac{h_2}{x} \right) = \frac{V_b}{h} \left( \frac{h}{x} \right) = \frac{V_b}{x} \quad (3.2)
\]

When this equation is substituted into Eq. 3.1 the shear force carried by the panel is found to be

\[
V_w = V - \frac{V_b}{x} = V \left( \frac{x - b}{x} \right) = V \frac{x_0}{x} = V \frac{h_0}{h} \quad (3.3)
\]

The shear flow \( s_b \) in the panel at spanwise station \( b \) is found by dividing the shear force on the panel by the depth of the section, so that

\[
s_b = \frac{V_w}{h} = \frac{V h_0}{h^2} = \frac{V \left( \frac{h_0}{h} \right)^2}{h_0} = s_0 \left( \frac{h_0}{h} \right)^2 = s_0 \left( \frac{x_0}{x} \right)^2 \quad (3.4)
\]

which is uniform along the boundary of the cut. The variation of the spanwise shear flow, along the flanges, as can be seen from Eq. 3.4, varies inversely as the square of the distance from the projected intersection of the upper and lower inclined flanges.

It is convenient to express this varying shear flow along the flanges as an average shear flow over the entire flange length. This can be accomplished in two ways. One way to find the average shear flow is by integrating Eq. 3.4 over the spanwise length and dividing by that length [12]:
The other way, as Peery suggests, is to calculate the horizontal component of the flange load at some spanwise location and divide it by the spanwise distance to obtain the average shear flow from the free end to that spanwise station.

\[ s_{av} = \frac{1}{(x - x_0) x_0} \int s_o \left( \frac{x_0}{x} \right)^2 \, dx = s_0 \left[ \frac{b}{x x_0} \right] = s_0 \frac{x_0}{x} \]  

(3.5)

This tapered planar beam analysis can be extended to tapered single-cell box beams modeled as thin webs and concentrated stiffeners. If there are more than three flanges, then the problem is statically indeterminate: there are not enough equations of equilibrium to solve for the flange loads and web shear flows. Therefore, as in any statically indeterminate situation, the deformation of the structure must be considered. This is most simply done by making the Bernoulli-Euler assumption for beams that plane sections remain plane and invoking the flexure formula [1, 6]. (This point is not clearly explained in [1]). By means of the flexure formula, the flange loads at a station can be found and used to determine their contribution to the vertical shear and, if the beam is doubly tapered, to the lateral shear as well. The part of the shear force resisted by the shear panels is determined from Eq. 3.1. This is used to compute the flange load increments which, together
with requiring moment equilibrium about a spanwise axis, yields the shear flows.

In discussing the fact that his analysis of the tapered shear panel is based on the assumption that pure shear traction acts on the edges, Peery states that "pure shearing stresses may exist on only two planes, which must be at right angles to each other." This is not true, as may be seen from a quick sketch of Mohr's circle with its center offset less than one radius from the origin O of the normal stress-shear stress axes. Points on the circle directly above and below O represent non-orthogonal planes on which there is no normal stress. Peery goes on to say in the very next sentence, "Since the corners of the tapered panel do not form right angles, it is necessary for some normal stresses to act at the boundary of the web." This statement is true, but not for the reason given. It is true because there exists no stress field satisfying the differential equations of equilibrium (Eq. 2.1) and compatibility (Eq. 2.22) which yield pure shearing surface tractions on the boundary of a trapezoidal plane stress region.

As Peery points out, the closest we can come to an exact solution resembling the assumed stress field in the trapezoid panel is the state of pure shear stress in polar coordinates, given by
\[ \sigma_{rr} = \sigma_{\theta\theta} = 0 \text{ and } \sigma_{r\theta} = K/r^2, \]
where \( K \) is a constant [13]. If one substitutes this stress field into the polar coordinate form of the equilibrium equations and the compatibility equation, it is readily verified that all of them are satisfied. The pure shear stress on the boundary of a "curved trapezoid" panel whose edges lie along polar
coordinate lines, is shown in Fig 3.2a. The inverse square variation of the polar shear stress is reminiscent of that found for the shear flow (Eq. 3.4). However, $\sigma_r \theta$ is constant along concentric circular sections and, therefore, unlike the shear flow in a trapezoid, it varies over transverse straight-line sections, as shown in Fig. 3.2b. Thus, although the assumption of pure shear along the tapered edges of a trapezoid is correct, along the right and left boundaries there must exist some normal stress, as can be seen when a polar stress element is rotated into the cartesian system.

This normal stress distribution is similar to that of the bending stress of a beam, but its magnitude compared to the shear stress along this edge is negligible for panels with small taper ($\alpha < 5$ degrees).

\[
\sigma_r \theta = \frac{K}{r^2}
\]

Figure 3.2 Shear element. (a) Polar coordinate system, (b) Cartesian coordinate system.
Garvey’s Shear Panel

Garvey derived an expression for the strain energy of an arbitrary quadrilateral ("trapezium") flat panel in equilibrium under purely tangential forces acting on its edges. The panel was regarded as stable so that no buckling or warping was considered. Since the panel is in equilibrium, statics can be used to express any three of the shear loads in terms of the fourth. The quadrilateral panel is defined as ABCD in Fig. 3.3.

On a given side, the shear load divided by the length of the side gives the average shear flow, \( s_i \), \( i = 1, 2, 3, 4 \). These are shown in Fig. 3.3. The points of intersection, \( P \) and \( Q \), of opposite sides of the quadrilateral are key reference points through which the baseline \( PQ \) is drawn. The shear force on side 1 (edge AB) will be taken as independent of the others.

Garvey’s basic assumption was that pure shear traction acts on any section of the panel through \( P \) or \( Q \). Garvey derived a number of relationships among the quantities shown in Fig. 3.3 using just statics and trigonometry. One of these involves the mean shear flows along side 1 (edge AB) and side 3 (edge CD) and the perpendicular distances \( p_A \), \( p_B \), \( p_C \) and \( p_D \) from the corner nodes to the baseline \( PQ \):

\[
s_1 p_A p_B = s_3 p_C p_D \quad (3.7)
\]

Similarly, he found that:

\[
s_2 p_B p_C = s_4 p_A p_D \quad (3.8)
\]
Figure 3.3 Garvey's quadrilateral shear panel with the perpendicular distances to each corner from the baseline PQ shown. \(s_1, s_2, s_3\) and \(s_4\) are the mean shear flows.

A relationship between the mean shear flows along sides 2 and 3 can be determined by summing moments about point A using the enclosed area.

\[
\frac{s_2}{s_3} = \frac{\Delta ACD}{\Delta ACB} = \frac{DX}{BX}
\]  

(3.9a)

The distance BX and DX is defined from the corner nodes B and D to the intersection of the two diagonals of the panel, point X. If the line DXB is extended to intersect the baseline, PQ, then the ratio of DX and BX can be rewritten in terms of the perpendicular distances using similar triangles as
Garvey therefore established that the mean shear flow on each side of the quadrilateral can be found in terms of a constant \( C_S \), that is, from Eqs. 3.7, 3.8 and Eq. 3.9b we see that

\[
\frac{s_2}{s_3} = \frac{p_D}{p_B} \quad \text{or} \quad s_2 p_B p_C = s_3 p_C p_D \quad (3.9b)
\]

Garvey showed that this differential element is in equilibrium and that the constant shear flow \( s_z \) around its edges is inversely proportional to the square of its perpendicular distance \( p_Z \) from the baseline \( PQ \) (cf. Fig. 3.4):

\[
s_z = \frac{C_S}{p_Z^2} \quad (3.11)
\]

If the panel is sectioned using lines \( PL'M' \), \( PL'M' \) and \( QNR \) and \( QN'R' \) as shown in Fig. 3.4, the four points of intersection are the corners of an infinitesimal parallelogram centered at point \( Z \). Fig 3.5a shows a finite parallelogram with acute interior angle \( \theta \). It is in equilibrium under the constant shear flow \( s \) shown acting on its edges. This state of stress is constant throughout the panel.
Figure 3.4 Garvey's quadrilateral shear panel, showing the shear flow on a differential parallelogram at an arbitrary point \( Z \).

If the parallelogram is sectioned as shown in Fig. 3.5b, then using statics, the uniform internal stresses along the section can be determined and written in matrix form as follows:

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} = \begin{bmatrix}
2 \cot \theta \\
t \\
0 \\
1 \\
t
\end{bmatrix} \{s\}
\]  

(3.12)

Since these are constant, the stresses trivially satisfy the equations of equilibrium and the compatibility equation.
Eq. 3.12 is of the same form as Eq. 2.61a, \( \{\sigma\} = [P] \{\beta\} \), where the stress parameter vector here consists of the single shear flow \( s \).

The general form of the elastic strain energy [10] in terms of stresses is, making use of Eq. 2.19a,

\[
U = \frac{1}{2} \int_{V} \{\sigma\}^T \{\varepsilon\} dV = \frac{1}{2} \int_{V} \{\sigma\}^T [D] \{\sigma\} dV
\]

(3.13)

Upon substituting Eq. 2.61 into 3.13, using the the reversal rule (Eq. 2.45), substituting Eq 2.69 and using the fact that \( \{\beta\} \) has but one component (so that \( H \) is a scalar), the strain energy can be written as

\[
U = \frac{1}{2} \int_{V} \{\beta\}^T [P]^T [D][P] \{\beta\} dV = \frac{1}{2} \{\beta\}^T [H] \{\beta\} = \frac{1}{2} H \beta^2
\]

(3.14)
By substituting Eq. 3.12 into Eq. 3.13, utilizing Eq. 2.18, and noting that \( dV = tdA \), we see that the strain energy for the parallelogram has the form

\[
U = \frac{1}{2} \left[ \int_A \frac{1}{Gt} \left( 1 + \frac{2\cot^2 \theta}{(1 + \nu)} \right) dA \right] s^2 \quad (3.15)
\]

The term inside the brackets is the natural flexibility \( H \). The area of a quadrilateral panel can easily be computed as

\[
A = \frac{1}{2} |r_{AC} \times r_{DB}| \quad (3.16)
\]

which for a parallelogram simplifies to

\[
A = L_{AB} L_{BC} \sin \theta \quad (3.17)
\]

Therefore, the natural flexibility simplifies to

\[
H_p = \frac{L_{AB} L_{BC} \sin \theta}{Gt} \left( 1 + \frac{2\cot^2 \theta}{(1 + \nu)} \right) \quad (3.18)
\]

The principal stresses in the parallelogram under pure shear are found by substituting Eq. 3.12 into Eq. 2.8 and simplifying using trigonometric identities to obtain

\[
\sigma_{\text{max}} = \frac{s}{t} \cot \left( \frac{\theta}{2} \right); \quad \sigma_{\text{min}} = -\frac{s}{t} \tan \left( \frac{\theta}{2} \right) \quad (3.19)
\]
The direction of these principal stresses, the principal plane angle \( \theta_p \), is obtained from Eq. 2.10, which simplifies to \( \theta_p = \theta/2 \). This shows that the maximum principle stress within the parallelogram lies along the bisector of the interior angle \( \theta \).

Garvey proceeded to derive the strain energy for a general quadrilateral shear panel by integrating Eq 3.15 as applied to a differential parallelogram. The differential parallelogram in Fig 3.4 is defined by sectioning the panel at angles \( \phi \) and \( d\phi \) through point P and \( \psi \) and \( d\psi \) through point Q. The interior acute angle of the parallelogram is therefore \( \theta = \phi + \psi \). The lengths of the sides of the parallelogram can be expressed in terms of these angles and the length of baseline \( L_{PQ} \), so that the differential form of Eq. 3.17 becomes

\[
dA = \frac{L_{PQ} \sin \phi \, \delta \phi}{\sin^2(\phi + \psi)} \frac{L_{PQ} \sin \psi \, \delta \psi}{\sin^2(\phi + \psi)} \sin(\phi + \psi) \, d\phi \, d\psi \quad (3.20)
\]

Garvey simplified this formula using trigonometric identities to yield the expression

\[
dA = \frac{L_{PQ}^2 \delta \phi \delta \psi}{(\cot \phi + \cot \psi)^3 \sin^2 \phi \sin^2 \psi} \quad (3.21)
\]

It is convenient to express the shear flow on the infinitesimal parallelogram in terms of the average shear flow along side 1 of the quadrilateral. The combination of Eq. 3.10 and Eq. 3.11 produces the relationship...
Substituting Eq. 3.21 and Eq. 3.22 into the differential form of Eq. 3.15 and simplifying yields

\[
\frac{dU}{2L^2_{PQ}tG \sin^2 \phi \sin^2 \varphi} = \frac{(s_1 p_A p_B)^2}{2L^2_{PQ}tG} \left( \cot \phi + \cot \varphi \right)^2 \left( 1 + \frac{2 (\cot \phi \cot \varphi - 1)^2}{(1 + \nu)(\cot \phi + \cot \varphi)^2} \right) d\phi d\varphi \tag{3.23}
\]

The integral of this strain energy over the quadrilateral panel is

\[
U = \frac{(s_1 p_A p_B)^2}{2L^2_{PQ}tG} \int \int_{\alpha} \left( \cot \phi + \cot \varphi + \frac{2 (\cot \phi \cot \varphi - 1)^2}{(1 + \nu)(\cot \phi + \cot \varphi)^2} \right) d\phi d\varphi \sin^2 \phi \sin^2 \varphi \tag{3.24}
\]

Garvey made the substitutions \(\cot \phi = x\) and \(\cot \varphi = y\) so that \(d\phi = -\sin^2 \phi \, dx\) and \(d\varphi = -\sin^2 \varphi \, dy\). Changing the limits to \(a = \cot \alpha\), \(b = \cot \beta\), \(c = \cot \gamma\) and \(d = \cot \delta\), the integral simplifies to

\[
U = \frac{(s_1 p_A p_B)^2}{2L^2_{PQ}tG} \int \int_{ba} \left( (x + y) + \frac{2 (xy - 1)^2}{(1 + \nu)(x + y)} \right) dx dy \tag{3.25}
\]

According to Garvey, the integral evaluates to

\[
U = \frac{(s_1 p_A p_B)^2}{2L^2_{PQ}tG} \left( \frac{1}{2} (a - c)(b - d)(a + b + c + d) + \frac{2F}{(1 + \nu)} \right) \tag{3.26a}
\]
where

\[ F = \left[ (a + b) + \frac{2}{3}(a^3 + b^3) + \frac{1}{5}(a^5 + b^5) \right] \ln (a + b) + \left[ (c + d) + \frac{2}{3}(c^3 + d^3) + \frac{1}{5}(c^5 + d^5) \right] \ln (c + d) - \left[ (b + c) + \frac{2}{3}(b^3 + c^3) + \frac{1}{5}(b^5 + c^5) \right] \ln (b + c) - \left[ (d + a) + \frac{2}{3}(d^3 + a^3) + \frac{1}{5}(d^5 + a^5) \right] \ln (d + a) \]

\[ + \frac{1}{10}(a^2 - c^2)(b^3 - d^3) + \frac{1}{10}(b^2 - d^2)(a^3 - c^3) - \frac{1}{5}(a - c)(b^4 - d^4) - \frac{1}{5}(b - d)(a^4 - c^4) - \frac{2}{3}(a - c)(b - d)(a + b + c + d) \]  

(3.26b)

Using Eq. 3.16, the area of the quadrilateral panel can be written

\[ A = \frac{L_{PQ}}{2} \{ (P_A + P_C) - (P_B + P_D) \} \]  

(3.27)

where the perpendicular distances from the baseline PQ in terms of the panel angles are

\[ P_A = \frac{L_{PQ}}{(c + d)}; \quad P_B = \frac{L_{PQ}}{(a + d)} \]

\[ P_C = \frac{L_{PQ}}{(a + b)}; \quad P_D = \frac{L_{PQ}}{(b + c)} \]

(3.28)
Upon substituting Eq. 3.27 into Eq. 3.26 and using Eq. 3.28 the internal strain energy reduces to

\[
U = \frac{1}{2} \left[ \frac{A P A B}{t G p_c p_D} \left( 1 + \frac{4F}{(1+\nu)(a-c)(b-d)(a+b+c+d)} \right) \right] S_1^2 \tag{3.29}
\]

Comparing this with Eq. 3.14 reveals that the quantity inside the brackets is the scalar natural flexibility \(H_T\) for the trapezium shear panel,

\[
H_T = \frac{A P A B}{t G p_c p_D} \left( 1 + \frac{4F}{(1+\nu)(a-c)(b-d)(a+b+c+d)} \right) \tag{3.30}
\]

Garvey's assumption that there exists pure shear flow at any point within the panel violates the basic equations of elasticity except for the special case of a parallelogram [14]. Therefore, (Eq. 3.30) must be regarded as an approximation.

A special case of the trapezium is the trapezoid, which has only two non-parallel sides. Therefore, it has either point \(Q\) at infinity referred to as trapezoid 1 panel or point \(P\) at infinity referred to as trapezoid 2.

A trapezoid 1 panel is shown in Fig 3.6. Edges \(AB\) and \(CD\) are parallel to line \(PQ\), so the shear flows on these as well as all sections parallel to \(PQ\) are uniform, according to Eq. 3.11, since all points are equidistant from \(PQ\). The mean shear flows along edges \(BC\) and \(AD\) then equal each other, from Eq. 3.10.
From Fig. 3.6 it can be seen that as point Q tends to infinity, the angles $\beta$ and $\delta$ (cf. Fig. 3.4) tend to zero so that $\cot\beta = \frac{L_{PQ}}{p_C}$ and $\cot\delta = \frac{L_{PQ}}{p_B}$ approach infinity as the length $L_{PQ}$ approaches infinity. According to Garvey for the trapezoid 1 panel, in the limit Eq. 3.30 becomes

$$H_{T1} = \frac{A p_A^2}{t G p_C^2} \left(1 + \frac{2}{3(1+v)}(\cot^2\alpha + \cot\cot\gamma + \cot^2\gamma)\right)$$ (3.31)

Taking the limit is a tedious exercise. Eq. 3.31 is more easily found by integrating the strain energy (cf. Eq. 3.24) for the special case of a trapezoid.
For the trapezoid 2 panel shown in Fig 3.7, point P tends to infinity. Edges AD and BC become parallel to line PQ, and uniform shear flow exists on all sections parallel to PQ, as above. The mean shear flow along edges AB and CD then equal each other, according to Eq. 3.10. Similar to the above argument, as point P moves to infinity, the angles $\alpha$ and $\gamma$ tend to zero, but $\cot \alpha = \frac{L_{PQ}}{p_B}$ and $\cot \gamma = \frac{L_{PQ}}{p_A}$ tend to infinity as the length $L_{PQ}$ approaches infinity. Taking the limit as Garvey suggests or integrating Eq. 3.24 for this special case, the internal strain energy equation for the trapezoid 2 panel becomes

$$H_{T2} = \frac{A}{tG} \left(1 + \frac{2}{3(1+\nu)} \left( \cot^2 \beta + \cot \beta \cot \delta + \cot^2 \delta \right) \right)$$

(3.32)

Figure 3.7 Trapezoid 2, P at infinity.
Naturally, Eqs. (3.31) and (3.32) must be considered approximations, to be used with caution.

To use Eq. 2.79 for calculating the stiffness matrices of Garvey's panels requires coming up with the local load vector so that the matrix \([F]\) can be inferred from Eq. 2.77. The load vector \(\{Q'\}\) is here assumed to be comprised of the four average shear flows. Taking positive shear flow as counterclockwise around the panel, and referring to Fig. 3.3,

\[
\{Q\} = \begin{bmatrix}
-s_1 \\
\ \\ s_2 \\
\ \\ -s_3 \\
\ \\ s_4
\end{bmatrix} \tag{3.33a}
\]

The components of this load vector can be expressed in terms of the single stress parameter \(s_1\) by using the shear flows found in Eq. 3.10. Thus,

\[
\{Q\} = \begin{bmatrix}
-1 \\
\ \\ \frac{p_A}{p_C} \\
\ \\ \frac{p_A p_B}{p_C p_D} \\
\ \\ \frac{p_B}{p_D}
\end{bmatrix} \{s_1\} \tag{3.33b}
\]

Since \(\{Q'\} = [F] \{\beta\}\) (Eq. 2.77), it is clear that
\[
[F] = \begin{bmatrix}
-1 & \frac{p_A}{p_C} & \frac{p_A p_B}{p_C p_D} & -\frac{p_B}{p_D} \\
\frac{p_A}{p_B} & \frac{p_A^2}{p_C} & \frac{p_A^2 p_B}{p_C p_D} & -\frac{p_A p_B}{p_D} \\
-\frac{p_A}{p_B} & \frac{p_A}{p_C} & \frac{p_B^2}{p_C p_D} & -\frac{p_B}{p_D} \\
\frac{p_B}{p_D} & \frac{p_A p_B}{p_C p_D} & \frac{p_B^2 p_D}{p_C p_D} & \frac{p_B}{p_D}
\end{bmatrix}
\]

(3.33c)

(The panel edge forces can also be derived by integrating the local shear flow given by Eq. 3.11 along each edge and using Eq. 3.10' to express the result in terms of the shear flow along side 1. This is detailed in Appendix A.)

Now that \([F]\) and \(H\) have been determined, the local stiffness matrix can be computed using Eq. 2.79 which, since \(H\) is a scalar, simplifies to \([K'] = 1/H [F] [F]^T\). Thus,

\[
[K'] = \frac{1}{H} \begin{bmatrix}
1 & -\frac{p_A}{p_B} & \frac{p_A p_B}{p_C p_D} & -\frac{p_B}{p_D} \\
-\frac{p_A}{p_B} & \frac{p_A^2}{p_C} & \frac{p_A^2 p_B}{p_C p_D} & -\frac{p_A p_B}{p_D} \\
\frac{p_A}{p_B} & \frac{p_A}{p_C} & \frac{p_B^2}{p_C p_D} & -\frac{p_B}{p_D} \\
-\frac{p_B}{p_D} & \frac{p_A p_B}{p_C p_D} & \frac{p_B^2 p_D}{p_C p_D} & \frac{p_B}{p_D}
\end{bmatrix}
\]

(3.34)
The exact solution of the Garvey shear panel's scalar natural flexibility matrix $H$ for the parallelogram is Eq. 3.18. The approximate solution for the trapezium is Eq. 3.30, for the "trapezoid 1" it is Eq 3.31 and for the "trapezoid 2" it is Eq. 3.32. The appropriate $H$ is chosen for use in Eq. 3.34 to determine the local stiffness matrix.

NASTRAN is a large scale structural analysis computer program using finite elements which was developed by the National Aeronautics and Space Administration (NASA). The MacNeal-Schwendler Corporation (MSC) then enhanced this computer program and markets a version of NASA's NASTRAN called MSC/NASTRAN. This code uses a variety of structural finite elements derived using either the displacement method or the stress method. An element of particular interest is their shear panel developed using the stress method. In the MSC/NASTRAN theoretical manual [14] this shear panel is described in the usual way as a flat quadrilateral element which by itself can resist only tangential forces applied to its edges. The normal resisting properties of the panel are lumped to the stiffeners surrounding the panel. The natural flexibility matrix used is that of Garvey and the matrix $[F]$ relating the element load vector to the single stress parameter (shear flow) is obtained by arbitrarily dividing the shear force on each edge by two and applying it equally to the edge corner nodes, referred to as the "half-half" rule.

As Peery pointed out, for a trapezoidal shear panel, the assumption of pure shear stress on the boundaries is not correct, which is also the defect in Garvey's work. However, as long as the
distortion is kept reasonably small, the coupling effect of shear stress and normal stress on the panels edges may be neglected. It is MSC's opinion that the amount of work required to devise a better shear panel is not justified.

Azar's Shear Panel

Azar derived the stiffness matrix for a rectangular four-node displacement-based shear panel using the principle of virtual work. He starts by considering a constant thickness flat rectangular "constant shear flow" panel with a uniform shear flow on its boundary. The boundary shear forces are then lumped to the corner nodes using the "half-half" rule to form an equivalent set of nodal loads, as seen in Fig 3.8. Along each side, the sum of the two corner load components directed along that edge must be equal to the resultant of the average shear flow, \( s \).

![Diagram of Azar's rectangular shear panel]

Figure 3.8 Azar's rectangular shear panel. (a) shear flows, (b) equivalent nodal shear forces.
Thus, there are four relations:

\[
\begin{align*}
Q_{Ax} + Q_{Bx} & = -sL_x; \\
Q_{Ay} + Q_{Dy} & = -sL_y;
\end{align*}
\]

(3.35)

Since the shear flow resultants, on each side, is equally distributed to its end nodes using the "half-half" rule, then the corner nodal loads become

\[
\begin{align*}
-Q_{Ax} - & Q_{Bx} = Q_{Dx} = Q_{Cx} = \frac{sL_x}{2} \\
-Q_{Ay} - Q_{Dy} & = Q_{By} = Q_{Cy} = \frac{sL_y}{2}
\end{align*}
\]

(3.36)

With two translational degrees of freedom per node, this four-node panel has a total of 8 degrees of freedom. Using the displacement method requires assuming a complete polynomial displacement field with 8 undetermined coefficients, i.e., a bilinear form:

\[
\begin{align*}
u_x & = a_1 + a_2x + a_3y + a_4xy \\
u_y & = a_5 + a_6x + a_7y + a_8xy
\end{align*}
\]

(3.37)

The coefficients are normally found in terms of the nodal displacements by evaluating Eq. 3.37 at each node.

Azar, however, consistent with the notion of a constant shear flow panel, assumes that the shear strain throughout the panel is constant, so that from Eq. 2.11b,

\[
\varepsilon_{xy} = u_{x,y} + u_{y,x} = C_1
\]

(3.38)
He claims that this implies

\[
\begin{align*}
\varepsilon_{xx} &= u_{x,x} = C_2 + C_3 y \\
\varepsilon_{yy} &= u_{y,y} = C_4 + C_5 x \\
\end{align*}
\]  

(3.39)

These satisfy the strain compatibility equation, but do not satisfy the equations of equilibrium (typical of displacement-based elements). Integrating these equations yields

\[
\begin{align*}
u_x &= C_2 x + C_3 xy + f(y) \\
u_y &= C_4 y + C_5 xy + g(x) \\
\end{align*}
\]

(3.40)

where \(f(y)\) and \(g(x)\) are arbitrary functions. Substituting \(u_x\) and \(u_y\) into Eq. 3.38,

\[
C_3 x + f(y)_y + C_5 y + g(x)_x = C_1
\]

(3.41)

In order for this equality to hold for all \(x\) and \(y\),

\[
\begin{align*}
C_5 y + f(y)_y &= C_6 \\
C_3 x + g(x)_x &= C_9 \\
\end{align*}
\]

(3.42)

where,

\[
C_6 + C_9 = C_1
\]

(3.43)

Integrating Eqs. 3.42 yields the functions \(f\) and \(g\),

\[
\begin{align*}
f(y) &= -\frac{C_5}{2} y^2 + C_6 y + C_7 \\
g(x) &= -\frac{C_3}{2} x^2 + C_9 x + C_8 \\
\end{align*}
\]

(3.44)
Substituting Eq. 3.44 into Eq. 3.40 and using Eq. 3.43, displacement field components are found in terms of eight unknown coefficients:

\[ u_x = C_2x + C_3xy - \frac{C_5}{2}y^2 + C_6y + C_7 \]  
(3.45a)

\[ u_y = C_4y + C_5xy - \frac{C_3}{2}x^2 + (C_1 + C_6)x + C_8 \]  
(3.45b)

Thus, Azar's effort to represent constant shear strain in the panel leads to an incomplete quadratic displacement field instead of the bilinear one in Eq. 3.38. This will cause displacement incompatibility along interelement boundaries where two of these quadratic-displacement panels are joined, or where a two-node, linear-displacement rod element is attached to act as a stiffener.

By evaluating Eq. 3.45 at each of the nodes, Azar shows that, in terms of the nodal displacements, \( u_x \) and \( u_y \) take the form

\[ u_x = q_1 + (q_3 - q_4) \frac{x}{L_x} + \left[ \frac{(q_7 - q_9)}{L_y} + \frac{(q_2 + q_6 - q_4 - q_8)}{2L_x} \right] y \]

\[ + (q_1 + q_5 - q_3 - q_7) \frac{xy}{L_xL_y} + (q_4 + q_8 - q_2 - q_6) \frac{y^2}{2L_xL_y} \]  
(3.46a)

\[ u_y = q_2 + \left[ \frac{(q_4 - q_2)}{L_x} + \frac{(q_1 + q_5 - q_7 - q_3)}{2L_y} \right] x + (q_8 - q_2) \frac{y}{L_y} \]

\[ + (q_2 + q_6 - q_4 - q_8) \frac{xy}{L_xL_y} + (q_3 + q_7 - q_1 - q_5) \frac{x^2}{2L_xL_y} \]  
(3.46b)
Therefore, in terms of the nodal displacements, the shear strain \( \varepsilon_{xy} = u_{x,y} + u_{y,x} \) is

\[
\varepsilon_{xy} = [B]\{q\} \tag{3.47a}
\]

where

\[
[B] = \frac{1}{2} \left[ -\frac{1}{L_y} - \frac{1}{L_x} - \frac{1}{L_y} \frac{1}{L_x} \frac{1}{L_y} \frac{1}{L_x} \right] \tag{3.47b}
\]

and \( \{q\} \) is the element displacement vector, with components \( q_1, q_2, \ldots, q_8 \). Azar excludes the normal stresses from the internal virtual work integral, Eq. 2.37b, so that

\[
\delta U_i = \int_V \delta \varepsilon_{xy}^T \sigma_{xy} \, dV \tag{3.48a}
\]

\[
= \int_A \delta \varepsilon_{xy}^T G \varepsilon_{xy} \, t \, dA \tag{3.48b}
\]

\[
= Gt A \delta q^T [B] [B] \{q\} \tag{3.48c}
\]

and finally

\[
\delta U_i = \{\delta q\}^T [GtA [B] [B]] \{q\} \tag{3.49}
\]

The stiffness matrix is the bracketed term on the right:

\[
[K] = Gt A [B] [B]^T \tag{3.50}
\]
For a rectangular panel, \( A = L_x L_y \), so that after carrying out the matrix multiplication we have

\[
[K] = \frac{Gt}{4A} \begin{bmatrix} [S] & [-S] \\ [-S] & [S] \end{bmatrix}
\]

(3.51a)

where,

\[
[S] = \begin{bmatrix} r & 1 & r & 1 \\ 1 & 1 & 1 & 1 \\ r & r & 1 & r \\ -1 & 1 & 1 & r \end{bmatrix}
\]

and \( r = \frac{L_x}{L_y} \)  

(3.51b)

A stiffness matrix for the displacement-based trapezium shear panel shown in Fig. 3.9 was also presented by Azar, but he did not present the details nor even indicate clearly what his assumed form for the displacement field was. The result is

\[
[K] = \frac{Gt}{4A} \begin{bmatrix} [S] & [-S] \\ [-S] & [S] \end{bmatrix}
\]

(3.52a)

where,

\[
[S] = \begin{bmatrix} x_{DB}^2 & -y_{DA} x_{DB} & -x_{CA} x_{DB} & y_{CA} x_{DB} \\ -x_{DB} y_{DA} & y_{DA}^2 & x_{CA} y_{DA} & -y_{CA} y_{DA} \\ -x_{DB} x_{CA} & y_{DA} x_{CA} & x_{CA}^2 & -y_{CA} x_{CA} \\ x_{DB} y_{CA} & -y_{DA} y_{CA} & -x_{CA} y_{CA} & y_{CA}^2 \end{bmatrix}
\]

(3.52b)

and \( x_{DB} = x_D - x_B \), etc.

\( A = (x_{BA} y_{CA} + x_{CA} y_{DB} - x_{DA} y_{CA})/2 \) (the area)
Robinson's Shear Panel

Robinson presented a deceptively simple force method alternative to Garvey's for deriving the stiffness matrix of a quadrilateral shear panel. An interesting aspect of his approach was to provide for the case where the four nodes of the panel are not in the same plane, i.e., the panel is initially "warped." Since the edge shear forces are not coplanar, forces normal to the projected plane of the panel develop at the corners. These "warping forces" are included in Robinson's development.

Robinson's shear panel analysis begins with the derivation, using PCVW, of the stiffness matrix for a rectangular constant shear flow panel of area $A$ and thickness $t$. Such a panel is in equilibrium with pure tangential shear flow $s$ along its edges and a
state of constant pure shear stress $\sigma_{xy} = s/t$ throughout (where the
xy axes are aligned with the sides of the rectangle). This state of
stress obviously satisfies equilibrium and compatibility. Letting $q$
be the generalized displacement corresponding to $s$, the PCVW
(Eq. 2.59) can be written

$$\delta s q = \int_{V} \delta \sigma_{xy} \varepsilon_{xy} dV \quad (3.53a)$$

$$= \int_{A} (\frac{\delta s}{t})(\frac{s}{G})t dA \quad (3.53b)$$

so that, since $s$ is constant,

$$\delta s q = \left(\frac{A}{Gt}\right) s \delta s \quad (3.54a)$$

Cancelling $\delta s$ yields

$$q = H s \quad (3.54b)$$

where

$$H = \frac{A}{Gt} \quad (3.54c)$$

and $H$ is the well-known scalar natural flexibility matrix for the
rectangular constant shear flow panel.

In moving on to the arbitrary quadrilateral shear panel,
Robinson—without apology—assumes that, in spite of the increased
complexity of the the state of stress which must exist, the formula
for $H$ remains the same as that for the basic rectangular panel,
Eq. 3.54 (with the area of the quadrilateral used, of course). The behavior of the panel is characterized by a fictitious, average shear flow $s$ which is used as the stress parameter. The shear flow along the edges of the panel must then be considered as an average shear flow (see Fig.3.10a).

![Figure 3.10](image)

Figure 3.10 Robinson's warped quadrilateral shear panel. (a) shear flows, (b) shear flow resultants.

The average shear flows are then distributed to the corner nodes in such a way that the corner force resultants lie along the diagonals of the panel pointing outward as shown in Fig 3.10b (and therefore are self-equilibrating). The magnitude of these corner forces is then assumed proportional to the length of the diagonal and dependent on the magnitude of the stress parameter $s$: 
Robinson assumed that the shear force on a side is distributed to the end nodes, but not by the "half-half" rule used in the NASTRAN implementation of Garvey's panel. If, for example, $F_{12}$ is the resultant shear force on edge 1-2, then $F_{12,1}$ and $F_{12,2}$ denote the portions of this force distributed to nodes 1 and 2, respectively. Using similar notation for the remaining sides, we write (cf. Fig. 3.11)

\[ F_{12} = F_{12,1} + F_{12,2} \]
\[ F_{23} = F_{23,2} + F_{23,3} \]
\[ F_{34} = F_{34,3} + F_{34,4} \]
\[ F_{41} = F_{41,4} + F_{41,1} \]

(3.56)

The two shear resultants $F_{ij,j}$ and $F_{jk,j}$ at corner $j$ and the warping force $W_j$ must add up vectorially to (i.e., be statically equivalent to) the corner load vector $Q_j$. That is, at corner $j$,

\[ F_{ij,j} t_{ij} + F_{jk,j} t_{jk} + W_j \gamma = Q_j \]

(3.57)
where \( t_{ij} \) is the unit tangent vector in the direction of edge \( ij \). The warping force is directed normal to the plane of the equivalent flat panel, in the direction of \( \gamma \), which is the unit vector in the direction of \( (\alpha \times \beta) \) (cf. Fig. 3.12). Robinson points out that the unit vectors \( \alpha \), \( \beta \) and \( \gamma \) are not, in general, an orthogonal set for warped panels.

\[
F_{41,1}t_{41} + F_{12,1}t_{12} + W_1\gamma = Q_1 \\
F_{12,2}t_{12} + F_{23,2}t_{23} + W_2\gamma = Q_2 \\
F_{23,3}t_{23} + F_{34,3}t_{34} + W_3\gamma = Q_3 \\
F_{34,4}t_{34} + F_{41,4}t_{41} + W_4\gamma = Q_4
\] (3.58)

Figure 3.11 Robinson's warped quadrilateral shear panel showing corner shear force components. (The warping vectors at each corner are perpendicular to the panel.)
These four vector equations are uncoupled. Therefore the three unknowns on the left side of each of them can be found by taking the dot product of the equation with each of the three unit vectors on the left and solving the three resulting scalar equations. At corner \( j \) this yields

\[
\begin{align*}
F_{ij}\mathbf{t}_i \cdot \mathbf{t}_j + F_{jk}\mathbf{t}_k \cdot \mathbf{t}_j + W_j \gamma \cdot \mathbf{t}_j &= Q_j \cdot \mathbf{t}_j \\
F_{ij}\mathbf{t}_i \cdot \mathbf{t}_k + F_{jk}\mathbf{t}_k \cdot \mathbf{t}_j + W_j \gamma \cdot \mathbf{t}_k &= Q_j \cdot \mathbf{t}_k \\
F_{ij}\mathbf{t}_i \cdot \gamma + F_{jk}\mathbf{t}_j \cdot \gamma + W_j \gamma \cdot \gamma &= Q_j \cdot \gamma 
\end{align*}
\]  

(3.59)

According to Robinson, the dot product \( Q_j \cdot \gamma \) is zero, meaning that the diagonally-directed corner loads are perpendicular to the warping forces.

The third of Eqs. 3.59 can thus be solved for the warping force:

\[
W_j = -F_{ij}\mathbf{t}_i \cdot \gamma - F_{jk}\mathbf{t}_j \cdot \gamma
\]  

(3.60)

Substituting Eq. 3.60 into the first two of Eqs. 3.59, simplifying, and using matrix notation yields

\[
\begin{bmatrix}
1 - (\mathbf{t}_i \cdot \gamma)^2 & (\mathbf{t}_i \cdot \mathbf{t}_j) - (\mathbf{t}_i \cdot \gamma)(\mathbf{t}_j \cdot \gamma) \\
(\mathbf{t}_i \cdot \mathbf{t}_j) - (\mathbf{t}_i \cdot \gamma)(\mathbf{t}_j \cdot \gamma) & 1 - (\mathbf{t}_j \cdot \gamma)^2
\end{bmatrix}
\begin{bmatrix}
F_{ij} \\
F_{jk}
\end{bmatrix}
= \begin{bmatrix}
Q_j \cdot \mathbf{t}_i \\
Q_j \cdot \mathbf{t}_k
\end{bmatrix}
\]

(3.61)

Solving this for \( F_{ij,i} \) and \( F_{jk,j} \), and using Robinson's notation
we get

\[
a = 1 - (t_{ij} \cdot \gamma)^2 \\
b = (t_{ij} \cdot t_{jk}) - (t_{ij} \cdot \gamma)(t_{jk} \cdot \gamma) \\
c = 1 - (t_{jk} \cdot \gamma)^2
\]

(3.62)

\[
F_{ij} = \frac{1}{(ac - b^2)} [c (Q_j \cdot t_{ij}) - b (Q_j \cdot t_{jk})] \\
F_{jk} = \frac{1}{(ac - b^2)} [-b (Q_j \cdot t_{ij}) + a (Q_j \cdot t_{jk})]
\]

(3.63)

Figure 3.12 Robinson’s warped quadrilateral shear panel showing the warping vectors at the corners.
As shown in Fig 3.12, the warping force is located at the corner nodes whereas the shear flow resultants are assumed to act at the mid-point of each side.

Using Eqs. 3.60, 3.62 and 3.63 the shear forces which are distributed to each corner node and the accompanying warping force can be detailed.

Node 1 \((i = 4, j = 1, k = 2)\):

\[
F_{41} = \frac{[\theta - (t_{12} \cdot \gamma)^2(Q_1 \cdot t_{41}) - ((t_{41} \cdot t_{12}) - (t_{41} \cdot \gamma)(t_{12} \cdot \gamma))(Q_1 \cdot t_{12})]}{[\theta - (t_{41} \cdot \gamma)^2(t_{12} \cdot \gamma)^2 - (t_{41} \cdot t_{12}) - (t_{41} \cdot \gamma)(t_{12} \cdot \gamma)]^2}
\]

\[
F_{12} = \frac{\left[-\{t_{41} \cdot t_{12}\} - (t_{41} \cdot \gamma)(t_{12} \cdot \gamma)\right] \left(Q_1 \cdot t_{41}\right) + \left(1 - (t_{41} \cdot \gamma)^2\right) \left(Q_1 \cdot t_{12}\right)}{[\theta - (t_{41} \cdot \gamma)^2(t_{12} \cdot \gamma)^2 - (t_{41} \cdot t_{12}) - (t_{41} \cdot \gamma)(t_{12} \cdot \gamma)]^2}
\]

\[
W = -F_{41}(t_{41} \cdot \gamma) - F_{12}(t_{12} \cdot \gamma)
\]

(3.64a)

Node 2 \((i = 1, j = 2, k = 3)\):

\[
F_{12} = \frac{\left(1 - (t_{23} \cdot \gamma)^2\right)(Q_2 \cdot t_{12}) - \left((t_{12} \cdot t_{23}) - (t_{12} \cdot \gamma)(t_{23} \cdot \gamma)\right) \left(Q_2 \cdot t_{23}\right)}{\left(1 - (t_{12} \cdot \gamma)^2\right)(1 - (t_{23} \cdot \gamma)^2) - \left((t_{12} \cdot t_{23}) - (t_{12} \cdot \gamma)(t_{23} \cdot \gamma)\right) \left(t_{23} \cdot \gamma\right)}^2}
\]

\[
F_{23} = \frac{\left[-\{t_{12} \cdot t_{23}\} - (t_{12} \cdot \gamma)(t_{23} \cdot \gamma)\right] \left(Q_2 \cdot t_{12}\right) + \left(1 - (t_{12} \cdot \gamma)^2\right) \left(Q_2 \cdot t_{23}\right)}{[\theta - (t_{12} \cdot \gamma)^2(t_{23} \cdot \gamma)^2 - (t_{12} \cdot t_{23}) - (t_{12} \cdot \gamma)(t_{23} \cdot \gamma)]^2}
\]

\[
W = -F_{12}(t_{12} \cdot \gamma) - F_{23}(t_{23} \cdot \gamma)
\]

(3.64b)
80

Node 3 (i = 2, j = 3 , k = 4):

\[
F_{23,3} = \frac{[(1 - (t_{34} \cdot \gamma)^2)(Q_3 \cdot t_{23}) - \{(t_{23} \cdot t_{34}) - (t_{23} \cdot \gamma)(t_{34} \cdot \gamma)\}(Q_3 \cdot t_{34})]}{[(1 - (t_{23} \cdot \gamma)^2)(1 - (t_{34} \cdot \gamma)^2) - \{(t_{23} \cdot t_{34}) - (t_{23} \cdot \gamma)(t_{34} \cdot \gamma)\}^2]}
\]

\[
F_{34,3} = \frac{[-\{(t_{23} \cdot t_{34}) - (t_{23} \cdot \gamma)(t_{34} \cdot \gamma)\}(Q_3 \cdot t_{23}) + (1 - (t_{23} \cdot \gamma)^2)(Q_3 \cdot t_{34})]}{[(1 - (t_{23} \cdot \gamma)^2)(1 - (t_{34} \cdot \gamma)^2) - \{(t_{23} \cdot t_{34}) - (t_{23} \cdot \gamma)(t_{34} \cdot \gamma)\}^2]}
\]

\[W_3 = -F_{23,3}(t_{23} \cdot \gamma) - F_{34,3}(t_{34} \cdot \gamma)
\]

(3.64c)

Node 4 (i = 3, j = 4 , k = 1):

\[
F_{34,4} = \frac{[(1 - (t_{41} \cdot \gamma)^2)(Q_4 \cdot t_{34}) - \{(t_{34} \cdot t_{41}) - (t_{34} \cdot \gamma)(t_{41} \cdot \gamma)\}(Q_4 \cdot t_{41})]}{[(1 - (t_{34} \cdot \gamma)^2)(1 - (t_{41} \cdot \gamma)^2) - \{(t_{34} \cdot t_{41}) - (t_{34} \cdot \gamma)(t_{41} \cdot \gamma)\}^2]}
\]

\[
F_{41,4} = \frac{[-\{(t_{34} \cdot t_{41}) - (t_{34} \cdot \gamma)(t_{41} \cdot \gamma)\}(Q_4 \cdot t_{34}) + (1 - (t_{34} \cdot \gamma)^2)(Q_4 \cdot t_{41})]}{[(1 - (t_{34} \cdot \gamma)^2)(1 - (t_{41} \cdot \gamma)^2) - \{(t_{34} \cdot t_{41}) - (t_{34} \cdot \gamma)(t_{41} \cdot \gamma)\}^2]}
\]

\[W_4 = -F_{34,4}(t_{34} \cdot \gamma) - F_{41,4}(t_{41} \cdot \gamma)
\]

(3.64d)

The preceding analysis is simplified if the corners of the quadrilateral panel are coplanar, so that there are no warping forces perpendicular to the panel shown in Fig. 3.11. In that case the normal vector \( \gamma \) is perpendicular to the edges of the panel, meaning all dot products in Eqs. 3.64 involving this vector vanish, so that they simplify to the following set:
Let the element load vector for the flat quadrilateral be comprised of the four average shear flows along the edges of the panel,
Using Eqs 3.55 and 3.65 this vector can be expressed in terms of the stress parameter \( s \) as

\[
\{ Q' \} = [F] s \quad (3.67)
\]

from which \([F]\) is inferred.

The local stiffness matrix \([K']\) is found from Eq. 2.79, using the scalar flexibility \( H \) (Eq. 3.54c), and the matrix \([F]\):

\[
[K'] = [F][H]^{-1}[F]^T = (Gt/A) [F] [F]^T \quad (3.68)
\]

**Nack's Shear Panel**

Nack described his effort to place the quadrilateral shear panel on firmer theoretical ground by approaching it as a hybrid element wherein the stress field was derived from an Airy stress function. This would ensure the panel has a compatible equilibrium stress field. The shear panel was considered to be a flat, four-node, eight d.o.f. element. Nack assumed the panel is in equilibrium while
acted on by just four shear forces directed along the sides. Using statics, he obtained expressions relating three of the shear forces to the fourth, which he used as the stress parameter.

The Airy stress function was chosen to be a complete seventh-order, bivariate polynomial with thirty-six coefficients:

\[
\Phi = a_1 + (a_2 x + a_3 y) + (a_4 x^2 + a_5 xy + a_6 y^2)
+ (a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3)
+ (a_{11} x^4 + a_{12} x^3 y + a_{13} x^2 y^2 + a_{14} xy^3 + a_{15} y^4)
+ (a_{16} x^5 + a_{17} x^4 y + a_{18} x^3 y^2 + a_{19} x^2 y^3 + a_{20} xy^4 + a_{21} y^5)
+ (a_{22} x^6 + a_{23} x^5 y + a_{24} x^4 y^2 + a_{25} x^3 y^3 + a_{26} x^2 y^4 + a_{27} xy^5 + a_{28} y^6)
+ (a_{29} x^7 + a_{30} x^6 y + a_{31} x^5 y^2 + a_{32} x^4 y^3 + a_{33} x^3 y^4 + a_{34} x^2 y^5 + a_{35} xy^6 + a_{36} y^7)
\]

(3.69)

Requiring this polynomial to satisfy the biharmonic equation (Eq. 2.23) reduces the number of unknowns as follows. Taking the appropriate derivatives,

\[
\Phi_{xxxx} = 24a_{11}
+ 120a_{16} x + 24a_{17} y
+ 360a_{22} x^2 + 120a_{23} xy + 24a_{24} y^2
+ 840a_{29} x^3 + 360a_{30} x^2 y + 120a_{31} xy^2 + 24a_{32} y^3
\]

(3.70a)
substituting them into the biharmonic equation (Eq. 2.23) and requiring that the coefficients of each polynomial term vanish yields a set of ten relations among the 36 coefficients, so that only 26 independent coefficients remain. The three coefficients of the zeroth and first degree terms of $\Phi$ can be dropped since the stresses are found as second derivatives of $\Phi$. This leaves 23 independent coefficients, in terms of which $\Phi$ can be written

$$\Phi = c_1 x^2 + c_2 xy + c_3 y^2$$

$$+ c_4 x^3 + c_5 x^2 y + c_6 xy^2 + c_7 y^3$$

$$+ c_8 (x^4 - 3x^2 y^2) + c_9 x^3 y + c_{10} xy^3 + c_{11} (y^4 - 3x^2 y^2)$$

$$+ c_{12} (x^5 - 5x^3 y^2) + c_{13} (x^4 y - x^2 y^3) + c_{14} (xy^4 - x^3 y^2) + c_{15} (y^5 - 5x^2 y^3)$$

$$+ c_{16} (x^5 y - \frac{5}{3} x^3 y^3) + c_{17} (x^4 y^2 - \frac{1}{15} y^6 - \frac{2}{15} x^6) + c_{18} (x^2 y^4 - \frac{2}{15} y^6 - \frac{1}{15} x^6)$$

$$+ c_{19} (x^5 y^2 - \frac{5}{3} x^3 y^3)$$

$$+ c_{20} \left( -\frac{2}{21} x^7 + x^5 y^2 - \frac{1}{3} x y^6 \right) + c_{21} \left( -\frac{2}{5} x^6 y + x^4 y^3 - \frac{1}{3} y^7 \right)$$

$$+ c_{22} \left( -\frac{1}{35} x^7 + x^3 y^4 - \frac{2}{5} x y^6 \right) + c_{23} \left( -\frac{1}{3} x^6 y + x^2 y^5 - \frac{2}{21} y^7 \right)$$

(3.71)
In an effort to enforce the pure shear force condition on the panel boundary, Nack proposed that the normal tractions at the one-quarter and three-quarter points along the four edges equal zero and that the shear tractions at the one-eighth, one-half and seven-eighth points equal the shear traction, expressed in terms of the independent stress parameter. There are five such collocation points per side for a total of 20 points for the entire shear panel. The Airy stress function in Eq. 3.71 contains 23 coefficients, which Nack reduced to 20 by setting $c_{20} = c_{23} = 0$ and $c_{21} = c_{22}$, producing a symmetric coefficient from the two asymmetric coefficients. The normal and shear stresses are derived from this modified Airy stress function using

$$\sigma_{xx} = \phi_{,yy}$$
$$\sigma_{yy} = \phi_{,xx}$$
$$\sigma_{xy} = -\phi_{,xy}$$

(3.72)

$$\sigma_{xx} = c_3(2) + c_6(2x) + c_7(6y) + c_8(-6x^2) + c_{10}(6xy) + c_{11}(12y^2 - 6x^2) + c_{12}(-10x^3) + c_{13}(-6x^2y) + c_{14}(12xy^2 - 2x^3) + c_{15}(20y^3 - 30x^2y) + c_{16}(-10x^3y) + c_{17}(2x^4 - 2y^4) + c_{18}(12x^2y^2 - 4y^4) + c_{19}(20xy^3 - 10x^3y) + c_{20}(6x^4y + 12x^3y^2 - 12xy^4 - \frac{6}{5}y^5)$$

(3.73a)
\[ \sigma_{yy} = c_1(2) \\
+ c_4(6x) + c_5(2y) \\
+ c_8(12x^2 - 6y^2) + c_9(6xy) + c_{11}(-6y^2) \\
+ c_{12}(20x^3 - 30xy^2) + c_{13}(12x^2y - 2y^3) + c_{14}(-6xy^2) + c_{15}(-10y^3) \\
+ c_{16}(20x^3y - 10xy^3) + c_{17}(12x^2y^2 - 4x^4) + c_{18}(2y^4 - 2x^4) + c_{19}(-10xy^3) \\
+ c_{20}(\frac{-6}{5}x^5 - 12x^4y + 12x^2y + 6xy^4) \\
\] 

\[ \sigma_{xy} = c_2(-1) \\
+ c_5(-2x) + c_6(-2y) \\
+ c_8(12xy) + c_9(-3x^2) + c_{10}(-3y^2) + c_{11}(12xy) \\
+ c_{12}(30x^2y) + c_{13}(-4x^3 + 6xy^2) + c_{14}(-4y^3 + 6x^2y) + c_{15}(30xy^2) \\
+ c_{16}(-5x^4 + 15x^2y^2) + c_{17}(-8x^3y) + c_{18}(-8xy^3) + c_{19}(-5y^4 + 15x^2y^2) \\
+ c_{20}(\frac{12}{5}x^5 - 12x^3y^2 - 12x^2y^3 + \frac{12}{5}y^5) \\
\] 

which in matrix form can be written

\[ \{\sigma\} = [M] \{C\} \] 

The matrix \([M]\) is a 3 by 20 matrix containing the stress polynomials in terms of \(x\) and \(y\) evaluated at the collocation points. The vector \(\{C\}\) consists of the 20 unknown constant stress coefficients to be determined by evaluating the stress field at each collocation point.
Substituting Eqs. 3.73 into Eq. 2.5 (or, alternatively, Eq. 2.6) and evaluating the surface tractions at the collocation points yields a set of twenty equations, expressed here in matrix form:

\[
[A] \{C\} = \{B\}
\] (3.75)

Each row of the square matrix \([A]\) consists of the stress polynomial terms evaluated at the same collocation point, and the vector \(\{C\}\) contains the twenty unknown stress polynomial coefficients. Each row of the \(\{B\}\) vector, called the boundary condition vector, contains the prescribed value of either the shear or normal traction at a collocation point.

As pointed out above, the shear forces on the panel illustrated in Fig. 3.13 form a self-equilibrating set, with \(F_{23}\), \(F_{34}\) and \(F_{41}\) considered dependent upon the independent shear \(F_{12}\). These dependencies are easily established by requiring moment equilibrium about three of the four corners of the panel. Choosing corners 1, 3 and 4 and denoting the included angle at corner \(i\) by \(\theta_i\), we find

Figure 3.13 Nack’s quadrilateral shear panel with shear forces.
\[ F_{23} = - \left[ \frac{L_{41}}{L_{34}} \sin \theta_1 \right] F_{12} \]  
(3.76a)

\[ F_{34} = \left[ \frac{L_{12}}{L_{34}} \sin \theta_1 \sin \theta_1 \right] F_{12} \]  
(3.76b)

\[ F_{41} = - \left[ \frac{L_{23}}{L_{34}} \sin \theta_2 \right] F_{12} \]  
(3.76c)

Having expressed the three dependent shear forces in terms of \( F_{12} \), it can be factored out of the boundary condition vector so that Eq. 3.75 can be rewritten in terms of this independent shear force as

\[ [A] \{Q - \{B\} F_{12} \} \]  
(3.77)

The stress polynomial coefficients are found using a standard linear equation solver:

\[ \{C\} = [A]^{-1} [B] F_{12} \]  
(3.78)

Substituting \( \{C\} \) into Eq. 3.74 gives

\[ \{\sigma\} = [P] \beta \]  
(3.79a)

where

\[ [P] = \{\bar{\sigma}\} = [M] [A]^{-1} [B] = [M] \{\bar{C}\} \]  
(3.79b)

and the single stress parameter \( \beta \) is the shear force \( F_{12} \).
Once the stress coefficients \( \{\bar{C}\} \) are known, the natural flexibility matrix \([H]\) (in this case a scalar) can be determined by means of Eq. 2.69, which, since the matrix \([P]\) can be thought of as a column vector of three stresses \( \{\bar{\sigma}\} \), can be written

\[
H = \int_V [D]\{\bar{\sigma}\}dV \quad (3.80)
\]

The matrix \([D]\) for isotropic materials is found in Eq. 2.19, so the natural flexibility matrix can be written as

\[
H = \frac{1}{E} \int_A h(x,y) dxdy \quad (3.81a)
\]

where

\[
h(x,y) = (\bar{\sigma}_{xx}^2 - 2\nu \bar{\sigma}_{xx} \bar{\sigma}_{yy} + \bar{\sigma}_{yy}^2 + 2(1 + \nu) \bar{\sigma}_{xy}^2) \quad (3.81b)
\]

Since the stresses vary throughout the panel and the shape of the panel is arbitrary, the integral in Eq. 3.80 must be done numerically. Nack chose Gaussian quadrature [9, 11]. To apply this method, a non-dimensional, square orthogonal grid whose coordinates \( \xi \) and \( \eta \) have extreme values of \( \pm 1 \) must be mapped onto the quadrilateral, such that the corners of the grid are mapped onto the corners of the panel. This is accomplished using the bilinear transformations [9]

\[
x_{\xi\eta} = \sum_{i=1}^{4} N_i x_i \quad \text{and} \quad y_{\xi\eta} = \sum_{i=1}^{4} N_i y_i \quad (3.82)
\]
\[ N_1 = \frac{1}{4} (1 - \xi) (1 - \eta) \quad N_2 = \frac{1}{4} (1 + \xi) (1 - \eta) \]
\[ N_3 = \frac{1}{4} (1 + \xi) (1 + \eta) \quad N_4 = \frac{1}{4} (1 - \xi) (1 + \eta) \] (3.83)

The limits of the integral in Eq. 3.80 must be changed to \( \xi - \eta \) limits. The differential area \( dx \, dy \) must then be changed to the \( \xi - \eta \) differential area using the determinate of the Jacobian matrix \( [J] \), so that \( dx \, dy = J \, d\xi \, d\eta \). The Jacobian matrix \([8]\) for a bilinear transformation is

\[
[J] = \begin{bmatrix}
N_{i, \xi} x_{i, \xi} & N_{i, \xi} y_{i, \xi} \\
N_{i, \eta} x_{i, \eta} & N_{i, \eta} y_{i, \eta}
\end{bmatrix} = \begin{bmatrix}
\sum_{i=1}^{4} N_{i, \xi} x_{i, \xi} & \sum_{i=1}^{4} N_{i, \xi} y_{i, \xi} \\
\sum_{i=1}^{4} N_{i, \eta} x_{i, \eta} & \sum_{i=1}^{4} N_{i, \eta} y_{i, \eta}
\end{bmatrix}
\] (3.84)

and the determinant of the Jacobian matrix, \( J \), is

\[ J = x_{\xi, \eta} y_{\xi, \eta} - y_{\xi, \eta} x_{\xi, \eta} \] (3.85)

Once in this form, application of the Gaussian quadrature formula can be used to turn the integrals:

\[
H \rightarrow \frac{1}{E} \int_{-1}^{1} \int_{-1}^{1} h(\xi, \eta) J(\xi, \eta) \, d\xi \, d\eta \] (3.86a)

into summations:

\[
H = \frac{1}{E} \sum_{i=1}^{ng} \sum_{j=1}^{ng} W_i W_j h(\xi, \eta) J(\xi, \eta) \] (3.86b)
The number of Gauss points \( n_g \) needed to exactly integrate a polynomial of degree \( n_p \) is determined by the formula \( n_p = 2n_g - 1 \) evaluated in Table 3.1. The Gauss weighting factors \( W_g \) and their corresponding sampling points \( \xi_g \) are also given in Table 3.1 for polynomial degrees 2 through 13. To evaluate \( H \) in Eq. 3.86 the stresses in Eq. 3.81 and the Jacobian determinant must be computed at each sampling point in the \( \xi-\eta \) coordinate system. The fifth degree stress functions in Eqs. 3.81 make \( h(\xi,\eta) \) a tenth degree polynomial and the determinant of the Jacobian will add one degree to that, making the integrand of Eq. 3.86 an eleventh degree polynomial. From Table 3.1, to exactly integrate this polynomial using Gaussian quadrature means that six Gauss points will be needed.

The determination of the stiffness matrix using the PCVW also requires that the matrix \([F]\), relating the element load vector to the stress parameters, be defined. Nack chose to place the element’s degrees of freedom at the corners, where, by the very nature of the shear panel, point loads are not applied. So the boundary surface tractions must be distributed to the corners as equivalent point loads by assuming a shape function \([\bar{N}]\) for the displacements on the boundary and using Eq. 2.75. The matrix \([F]\) relating the equivalent nodal loads to the stress parameter is then obtained from Eq. 2.70d,

\[
[F] = \int_{S} [\bar{N}]^T [L] dS
\]  
(3.87a)
Table 3.1 Gauss quadrature sampling points and corresponding weights.

<table>
<thead>
<tr>
<th>( n_p )</th>
<th>( n_{g_{\min}} )</th>
<th>( \xi_g )</th>
<th>( W_g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2,3</td>
<td>2</td>
<td>( \pm 0.57735 ) 02691 89626</td>
<td>1.00000 00000 00000</td>
</tr>
</tbody>
</table>
| 4,5      | 3              | \( \pm 0.77459 \) 6692 41483 | 0.55555 55555 55555  
0.00000 00000 00000 0.88888 88888 88888 |
| 6,7      | 4              | \( \pm 0.86113 \) 63115 94053 | 0.34785 48451 37454  
\( \pm 0.33998 \) 10435 84856 | 0.65214 51548 62546 |
| 8,9      | 5              | \( \pm 0.90617 \) 98459 38664 | 0.23692 68850 56189  
\( \pm 0.53846 \) 93101 05683 | 0.47862 86704 99366  
0.00000 00000 00000 | 0.56888 88888 88889 |
| 10,11    | 6              | \( \pm 0.93246 \) 95142 03152 | 0.17132 44923 79170  
\( \pm 0.66120 \) 93864 66265 | 0.36076 15730 48139  
\( \pm 0.23661 \) 91860 83197 | 0.46791 39345 72691 |
| 12,13    | 7              | \( \pm 0.94910 \) 79123 42759 | 0.12948 49661 68870  
\( \pm 0.74153 \) 11855 99394 | 0.27970 53914 89277  
\( \pm 0.40584 \) 51513 77397 | 0.38183 00505 05119  
0.00000 00000 00000 | 0.41795 91836 73469 |

This shear panel is a "hybrid" element, because an assumption about the boundary displacement field has been added to that of the internal stress field.

The integral in Eq. 3.87a is around the surface of the panel, and it must be written as the sum of four integrals, one along each edge of the panel,
The integral for a particular edge \( i-j \) is

\[
[F]_{ij} = \int [N]_{ij}^T [L]_{ij} t ds
\]

where \( ds \) is the incremental length along the element's boundary.

The components of surface traction \( \{T\} \) along the edge of the panel can be taken along the axes of the panel's local x-y coordinate system, as shown in Fig. 2.2a, and written in terms of the stresses as in Eq. 2.2. The stresses are expressed in polynomial form in Eq. 3.73 so that the matrix \([L]\) defined in Eq. 2.63, can be written as

\[
[L] = \begin{bmatrix}
\bar{\sigma}_{xx} n_x + \bar{\sigma}_{xy} n_y \\
\bar{\sigma}_{xy} n_x + \bar{\sigma}_{yy} n_y
\end{bmatrix}
\]

The shape functions \([N]_{ij}\) relate the components of the displacement of a point along edge \( i-j \) of the panel to the displacements of the two corner nodes defining that edge. The interpolation polynomial must be linear. If \( u(s) \) is the x-component of boundary displacement a distance \( s \) from point \( i \), then

\[
u(s) = c_1 + c_2 s
\]
We evaluate this at the end-points, letting $L_{ij}$ represent the length of the side:

\begin{align*}
\text{at } s = 0 & \quad u(0) = u_i \quad c_1 = u_i \\
\text{at } s = L_{ij} & \quad u(L_{ij}) = u_j \quad c_2 = (u_j - u_i)/L_{ij}
\end{align*}

(3.91) (3.92)

Then $u(s)$ in terms of the end-point displacements is

$$u(s) = (1 - \frac{s}{L_{ij}})u_1 + \frac{s}{L_{ij}}u_2$$

(3.93)

A similar relation holds for the $y$-component of displacement, $v(s)$.

Therefore, $x$-$y$ components of the general displacement vector can be expressed in terms of the position $s$ along an edge as

$$\begin{pmatrix} u(s) \\ v(s) \end{pmatrix}_{i,j} = [N]_{i,j} \begin{pmatrix} u_i \\ v_i \\ u_j \\ v_j \end{pmatrix}$$

(3.94a)

where

$$[N]_{i,j} = \begin{bmatrix}
1 - \frac{s}{L_{ij}} & 0 & \frac{s}{L_{ij}} & 0 \\
0 & 1 - \frac{s}{L_{ij}} & 0 & \frac{s}{L_{ij}}
\end{bmatrix}$$

(3.94b)

Since the integrand in Eq. 3.88 is a complicated polynomial function of position along the edge of the panel, a numerical integration is in order. Nack chose the Newton-Cotes quadrature
formula [9], according to which if \( f(s) \) is a polynomial of degree \( n_p \), then its integral is computed exactly as

\[
\int_a^b f(s) \, ds = \left( \frac{b - a}{n_p} \right) C_{n_p} \sum_{k=0}^{n_p} W_k f(s_k), \quad s_k = a + k \left( \frac{b - a}{n_p} \right)
\]

where values of \( C_{n_p} \) and the weighting factors \( W_k \) for polynomials through degree 6 are given in Table 3.2.

Table 3.2  Newton-Cotes quadrature constants A and \( W_k \).

<table>
<thead>
<tr>
<th>( n_p )</th>
<th>( C_{n_p} )</th>
<th>( W_0 )</th>
<th>( W_1 )</th>
<th>( W_2 )</th>
<th>( W_3 )</th>
<th>( W_4 )</th>
<th>( W_5 )</th>
<th>( W_6 )</th>
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<td>1</td>
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<td>3</td>
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<td>12</td>
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<td>50</td>
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<tr>
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<td>216</td>
<td>27</td>
<td>272</td>
<td>27</td>
<td>216</td>
</tr>
</tbody>
</table>

Each component of the matrix integrand in Eq. 3.88 is a function of \( x \) and \( y \), evaluated on the boundary segment \( i-j \). \( x \) and \( y \) at any point on the line a distance \( s \) from point \( i \) are found in terms of the coordinates of the end-points by

\[
x(s) = \left( 1 - \frac{s}{L_{ij}} \right) x_i + \left( \frac{s}{L_{ij}} \right) x_j \quad \text{and} \quad y(s) = \left( 1 - \frac{s}{L_{ij}} \right) y_i + \left( \frac{s}{L_{ij}} \right) y_j
\]
Thus, a polynomial of degree \( n_p \) in \( x \) and \( y \), when evaluated along an edge, becomes a polynomial of degree \( n_p \) in \( s \). This means each component \( F(x,y) \) of \([F]_{ij}\) can be found using Eq. 3.95 by setting \( f(s) = F(x(s),y(s)) \), \( a = 0 \) and \( b = L_{ij} \).

After computing the 4 by 1 matrix \([F]_{ij}\) for an edge, it is expanded to an 8 by 1 matrix by placing zeros in those four positions representing element degrees of freedom not associated with that edge. Then the complete matrix for the element is

\[
[F] = [F]_{12} + [F]_{23} + [F]_{34} + [F]_{41} \tag{3.97}
\]

Having computed the scalar natural flexibility \( H \) and the matrix \([F]\), Nack found the local stiffness matrix using Eq 2.79:

\[
[K'] = \frac{1}{H}[F][F]^T \tag{3.98}
\]

At the conclusion of the global displacement analysis, the panel element's displacements are available, and Eq. 2.72 is used to evaluate the stress parameter,

\[
F_{12} = \left(\frac{1}{H}\right)[F]^T q \tag{3.99}
\]

where the shear force \( F_{12} \) is not to be confused with the components of the matrix \([F]\) on the right. The shear forces on the other edges can then be found using Eqs. 3.76. The mean shear flows are shear forces divided by the appropriate edge lengths.
If the stress at any point within the panel or on its boundary is desired, Eq. 3.79 may be used. The principal stresses and directions are found from Eqs 2.8, 2.9 and 2.10.

Nack used his shear panel to solve some plane stiffened-web problems involving rectangular shear panels and showed that his shear flow solutions compared nicely to those obtained by Peery. He did not use his element to solve a tapered beam problem. Although Nack used the two-node, linear-displacement rod as the stiffener element in his paper, he pointed out that using linear force rod elements would help improve the panel’s "equilibrium characteristics." Verifying this contention and correcting some problems with Nack’s panel is the major thrust of this research.
CHAPTER 4

RESEARCH

For several years the writer and his faculty colleague Howard D. Curtis have, as a team, done much heretofore unreported work towards carrying the development of a theoretically-sound shear panel beyond the point reported [5] by their former colleague and now frequent consultant, Wayne V. Nack. The object was simply to provide Embry-Riddle engineering students with the option of using a computer-base structural element which they had become thoroughly familiar with in their aircraft structures courses. The element had to be robust and reliable and give answers to textbook problems that agreed reasonably well with hand calculations. Hopefully, it would also perform passably against finite element models of the same structure.

The effort began with Nack's shear panel since the computer code was available. Some subroutines were modified to improve the panel's reliability when it was tapered. A modified "Nack-Curtis" panel was then coded, and Robinson's line-node concept [15] was adopted to create the stress-based Curtis 4-DOF shear panel. As usual, this panel must be surrounded by stiffeners, and the familiar [1] linear-force rod was chosen because of its ability to carry

1Presently with the MacNeal-Schwendler Corporation.
constant shear flow along its length. Using the same stress field, but assuming a quadratic boundary displacement field, gave rise to a hybrid panel, the Curtis 12-DOF panel. The rods surrounding this panel are quadratic-displacement elements with a mid-side node.

The Garvey shear panel was compared with the Curtis panels. For the comparison to be informative, the Garvey panel was coded to incorporate the line node concept using the linear-stress rod element. As such, it will be referred to as the Garvey 4-DOF panel. The Garvey 12-DOF hybrid panel was also coded using the same boundary displacements and three-node rod stiffeners as the Curtis 12-DOF shear panel.

The displacement-based Azar panel was examined. For a rectangular shape, it agrees with the stress-based panels. However, no serious comparisons were made with his quadrilateral panel, since the theoretical assumptions leading to the stiffness matrix he presented are not given.

The appealingly simple (but theoretically unsound) Robinson four d.o.f. flat quadrilateral shear panel was also compared with the Curtis and Garvey panels. Its load vector consists of the four shear flows, and the stiffening element is the linear-stress rod.

Since the uniform state of stress in a parallelogram with pure shear on its boundaries is exact, it provided a test shape for comparing all of the panels. Tests of panel performance as trapezoids were also done. Comparison among the Curtis equilibrium stress-based panel, Robinson's assumed natural flexibility panel and Garvey's pure shear stress panel proved of interest.
The Modified Nack Shear Panel

Nack claimed his panel (Chapter 3) could be used for tapered elements, but he presented no results. Before Curtis and the writer could do any comparisons, they found the panel had to be modified. Some corrections to Nack's computer code were made and the collocation scheme used to enforce the pure surface traction condition on the edges was refined. The condition was better prescribed by requiring that, on each edge, the integrated resultant of normal surface tractions be set to zero and the integrated resultant of the shear surface tractions be set equal to the shear force computed from overall panel equilibrium (Eqs. 3.76). Gauss quadrature was chosen for the integration of the tractions.

Nack stated in that a fifth degree Airy polynomial stress function (Chapter 2), with several conditions arbitrarily placed on the coefficients to reduce their number to 20 (cf. Eq. 3.71), was needed for sufficient accuracy in determining the shear flows. In this investigation it was considered worthwhile to try to obtain reasonably accurate solutions with the lowest-degree polynomial possible. Therefore, the degree of the polynomial stress function became a variable to investigate the effects it had on the panel's shear flows.

With the above in mind, let the panel stresses, derived from an Airy stress function with \( n_s \) terms, be written as follows

\[
\sigma_{xx} = \sum_{i=1}^{n_s} f_i(x,y) C_i
\]  

(4.1a)
The polynomial stress terms $f_i$, $g_i$ and $h_i$ common to the same Airy stress function coefficient $C_i$ are listed in Table 4.1 for polynomials through degree six. The stress coefficients $C_i$ are the undetermined constants to be obtained from the boundary conditions.

By means of Eq. 2.6, the normal and shear surface tractions at a given point along the edge of the panel can be expressed in terms of the stress terms, $f_i$, $g_i$ and $h_i$, and the unit tangent direction cosines:

\[ T_n = \sum_{i=1}^{n_s} S_{nli} C_i \quad (4.2a) \]

where

\[ S_{nli} = f_i t_y^2 + g_i t_x^2 - 2 h_i t_x t_y \quad (4.2b) \]

and

\[ T_t = \sum_{i=1}^{n_s} S_{tli} C_i \quad (4.3a) \]

where

\[ S_{tli} = (f_i - g_i) t_x t_y + h_i (t_y^2 - t_x^2) \quad (4.3b) \]
The condition that the normal surface traction resultant along edge $i-j$ vanish is

$$\int_0^{L_{ij}} T_n(s) t \, ds = 0, \quad T_n(s) = T_n(x(s), y(s))$$  \hspace{1cm} (4.4a)

where $x(s)$ and $y(s)$ are given by Eq. 3.96, and $t$ is the uniform panel thickness. Using Gauss quadrature [8, 9], we can write

$$\int_0^{L_{ij}} T_n(s) \, ds = \frac{L_{ij}}{2} \sum_{i=1}^{n_g} W_i T_n(s_i), \quad s_i = \frac{L_{ij}}{2}(1 + \xi_i)$$  \hspace{1cm} (4.4b)

The weights $W_i$ and corresponding sampling points ("Gauss points") $\xi_i$ are found in Table 3.1, which also shows the minimum number of Gauss points $n_g$ required to exactly integrate a polynomial of degree $n_p$. Requiring this integral in Eq. 4.4b to vanish is accomplished by simply setting $T_n(s)$ equal to zero at the Gauss points:

On each of the four sides: $T_n(s_i) = 0, \quad i = 1, \cdots, n_g$  \hspace{1cm} (4.4c)

The additional boundary condition that the resultant shear traction along an edge $k-l$ equals the shear force $F_{kl}$ along that edge is

$$\int_0^{L_{kl}} T_t(s) t \, ds = F_{kl}, \quad T_t(s) = T_t(x(s), y(s))$$  \hspace{1cm} (4.5a)
<table>
<thead>
<tr>
<th>n_p</th>
<th>i</th>
<th>f_i</th>
<th>g_i</th>
<th>h_i</th>
</tr>
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<td>1</td>
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<td>0</td>
<td>0</td>
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<td>0</td>
<td>6x</td>
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<td>0</td>
<td>-2y</td>
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<td>12x²-6y²</td>
<td>12xy</td>
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<td>9</td>
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<td>-6y²</td>
<td>12xy</td>
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<td>20x³·30xy²</td>
<td>30x²y</td>
</tr>
<tr>
<td></td>
<td>13</td>
<td>-6x²y</td>
<td>12x²y-2y³</td>
<td>-4x³+6xy²</td>
</tr>
<tr>
<td></td>
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<td>12xy²-2x³</td>
<td>-6xy²</td>
<td>-4y³+6x²y</td>
</tr>
<tr>
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<td>15</td>
<td>20y³-30x²y</td>
<td>-10y³</td>
<td>30xy²</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>-10x³</td>
<td>20x³y-10xy³</td>
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</tr>
<tr>
<td></td>
<td>17</td>
<td>2x⁴-2y⁴</td>
<td>12x²y²-4x⁴</td>
<td>-8x³y</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>12xy⁴-2y⁴</td>
<td>2y⁴-2x⁴</td>
<td>-8xy³</td>
</tr>
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<td></td>
<td>19</td>
<td>20xy³-10x³y</td>
<td>-10xy³</td>
<td>-5y⁴+15x²y²</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>2x⁵-10xy⁴</td>
<td>-4x⁵+20x³y²</td>
<td>-10x⁴y+2y⁵</td>
</tr>
<tr>
<td></td>
<td>21</td>
<td>6x⁴y·6y⁵</td>
<td>-12x⁴y+12x²y³</td>
<td>12x⁵·12x³y²</td>
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<td>12x³y²-12xy⁴</td>
<td>6x⁵+6xy⁴</td>
<td>-12x²y³+12y⁵</td>
</tr>
<tr>
<td></td>
<td>23</td>
<td>20x²y³-12xy⁴</td>
<td>-10xy³+2y⁵</td>
<td>2x⁵-10xy⁴</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>-840x⁴y²+1680x²y⁴</td>
<td>56x⁶-840x²y⁴+112y⁶</td>
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<td>42x⁵y·42xy⁵</td>
<td>-7x⁹+105x²y⁴-14y⁶</td>
</tr>
<tr>
<td></td>
<td>26</td>
<td>2x⁶-60x⁴y²+90x²y⁴-8y⁶</td>
<td>30x⁴y²-60x²y⁴+6y⁶</td>
<td>12x⁵y+80x³y³36xy⁵</td>
</tr>
<tr>
<td></td>
<td>27</td>
<td>6x⁵y-40x³y³+18xy⁵</td>
<td>20x³y³-12xy⁵</td>
<td>15x⁴y²+30x²y⁶</td>
</tr>
</tbody>
</table>

Table 4.1 Polynomial stress terms.
Applying the quadrature formula in Eq. 4.4b yields

\[
\text{On edge } k-l: \quad \sum_{i=1}^{n_g} W_i T_i(s_i) = \frac{2F_{k,l}}{L_{k,l}} \quad (4.5b)
\]

Eq. 4.4c places \(4 \times n_g\) conditions on the stress coefficients and Eq. 4.5b contains four more, for a total of \(4 \times (n_g + 1)\) conditions on the \(n_s\) \(C_i\)'s. To determine the \(C_i\)'s from the boundary conditions requires that

\[
4 \times (n_g + 1) = n_s \quad (4.5c)
\]

For this formula to yield an integer for the number of required Gauss points along each edge, \(n_s\) must be divisible by 4. As can be seen from Table 4.1, the number of terms in a complete Airy stress polynomial of any order is not divisible by 4. However, if the polynomial is of degree 1 or higher, it is clearly possible to make the total number of terms in the polynomial divisible by 4 by reducing the number of coefficients by 3.

To do so, early on in this project as the effect of polynomial degree on the solution was being studied, the first and last of the four stress coefficients (cf. Table 4.1) of the highest order terms of a stress polynomial of degree \(n_p\) were dropped and the two remaining stress coefficients were combined. This left only one unknown coefficient multiplying all of the highest order terms. This procedure is shown in Table 4.2 for stress polynomials of degree 3 through 6. (This scheme will not work for the second degree stress
polynomial, which would be reduced to 8 terms. The minimum number of Gauss points required to exactly integrate the normal surface tractions along an edge to zero is two [cf. Table 3.1]. This alone provides eight boundary conditions (Eq. 4.4c), using up the eight coefficients and leaving the four shear stress boundary conditions (Eq. 4.5a) unsatisfied.)

Table 4.2  Nack-Curtis boundary conditions for modified polynomials.

<table>
<thead>
<tr>
<th>$n_p$</th>
<th>$n_{g_{min}}$</th>
<th>$n_g$</th>
<th>$n_s$</th>
<th>modifications</th>
<th>$n_{T_n}$ + $n_{T_t}$ = $n_s'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>15</td>
<td>$C_{12} = C_{15} = 0$ $C_{13} + C_{14} = C_{12}'$</td>
<td>8 4 12</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
<td>19</td>
<td>$C_{16} = C_{19} = 0$ $C_{17} + C_{18} = C_{16}'$</td>
<td>12 4 16</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>4</td>
<td>23</td>
<td>$C_{20} = C_{23} = 0$ $C_{21} + C_{22} = C_{20}'$</td>
<td>16 4 20</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>5</td>
<td>27</td>
<td>$C_{24} = C_{27} = 0$ $C_{25} + C_{26} = C_{24}'$</td>
<td>20 4 24</td>
</tr>
</tbody>
</table>

$n_p = \text{degree of stress polynomial.}$

$n_{g_{min}} = \text{minimum number of Gauss points for exact quadrature.}$

$n_g = \text{number of Gauss points used to produce boundary conditions.}$

$n_s = \text{number of terms in complete stress polynomial.}$

$n_{T_n} = \text{number of conditions placed by zero normal traction requirement.}$

$n_{T_t} = \text{number of conditions placed by shear stress boundary condition.}$

$n_{s'} = n_s - 3$
To solve for the unknown stress coefficients requires Eq. 3.77:

$$[A] \{C\} = \{B\} F_{12} \quad (4.6)$$

Setting $T_n = 0$ at each of the $4n_g$ Gauss points, listed in order counterclockwise around the panel starting with edge 1-2, we get

$$\sum_{j=1}^{n_g} S_{n_j}(i) C_j = 0, \quad i = 1, \cdots, 4n_g \quad (4.7)$$

where $S_{n_j}(i)$ is $S_{n_j}$ (Eq. 4.2b) evaluated at the $i$th Gauss point around the panel. The shear boundary condition for each of the four edges (Eq. 4.5a) can be written

$$\sum_{j=1}^{n_g} \left[ \sum_{i=1}^{n_g} W_i S_{t_j}(s_i) \right] C_j = \frac{2F_{kl}}{L_{kl}} \quad (4.8)$$

where $F_{12}$ is independent, and the remaining three shears are found from Eqs. 3.76. Eqs. 4.7 and 4.8 together can be written in matrix form as Eq. 4.6. The stress coefficients $\{C\}$ are then found in terms of the independent shear force. The solution of the shear panel then continues the same as Nack's (Chapter 3).

A computer code was written to implement the above procedure and provide output of intermediate steps. The output consisted of the input file, degree of the polynomial used, edge lengths with the cosines of their normal and tangential unit vectors,
the shear forces along each side in terms of that on side 1, and the matrix \([A]\) at the users request. The boundary condition vector \{B\} and the solution vector \{C\}, obtained using a professionally-written Gauss elimination routine incorporating the pivoting technique, were output next. The accuracy of the solution was then tested by multiplying the solution vector \{C\} by the saved matrix \([A]\) to produce a new boundary condition vector \{B'\}. This new vector was then subtracted from the original one to obtain the residual vector \{R\} = \{B\} - \{B'\}. If the residuals were zero the exact solution was presumably obtained. The two vectors \{B'\} and \{R\} were output. The stresses along the edges of the panel were computed at 10% increments and plotted in order to visualize the surface traction distribution. The last output information consisted of the nodal deflections, the axial forces in the rods, and the shear flows along the edge of the panel.

The test case used to verify the code was a rectangular cantilever beam consisting of one shear panel surrounded by four two-node, linear-displacement rods. The solution for the coefficients \{C\} produced small residuals \{R\} and a constant shear flow along the panel's edges. The displacements agreed with hand calculations and other already existing rectangular shear panel code [10]. The results of this particular analysis provided confidence in the computer code.

The next case investigated was the parallelogram. As the panel was swept away from a rectangle by a small interior acute angle \(\theta\), (see Fig. 3.5a), the residuals \{R\} became large, producing a slightly different boundary vector \{B'\}. This indicated that the
inversion routine was not generating the correct solution for the stress coefficients \( \{C\} \). Examining the output of the stress term matrix \([A]\) revealed that the magnitude of its components varied considerably. A scaling technique was incorporated which scaled the components of the \([A]\) matrix according to the power of the corresponding stress term. This was done by nondimensionalizing the x-y coordinates in the stress polynomials using the longest edge length of the panel. Each stress coefficient was multiplied by an appropriate power of the length in order not to alter the original equations. The scaled stress coefficients \( \{\bar{C}\} \) were found by solving the system \([A]\{\bar{C}\} = \{B\}\). The coefficient vector \( \{C\} \) was recovered by dividing each component of \( \{\bar{C}\} \) by the appropriate power of the length.

This did not improve the solution for the coefficients; the residuals were still not zero. A procedure was then incorporated to take the computed residuals and use them as the solution to determine a set of error stress coefficients \([16,17]\). These were then added to the original set of stress coefficients and their residuals were checked. If these residuals were not zero they were used to determine another set of error stress coefficients and added to the previous summed solution of stress coefficients. This iterative improvement was to be continued until the magnitude of all the residuals was zero. (This iterative procedure is sometimes needed to converge to the exact solution due to the fixed length of a computer's double precision numbers.) Unfortunately, the iterative computation of the stress coefficients did not converge to the exact
solution or any other solution where all coefficient residuals approached zero at the same time. This was tried for the scaled and unscaled \([A]\) matrix.

The nature of the stresses along the edges of the parallelogram shear panel were then investigated. The stresses that were output and plotted revealed that the normal surface tractions at the locations of the Gauss points were zero, but their resultants were not zero. The resultant shear traction did not agree with the shear force computed from Eqs. 3.76 along any of the panel's edges. From the plots of the edge stress distributions it was apparent that the normal surface tractions were resulting in a moment. It was later revealed that for the third degree stress polynomial this moment was of considerable magnitude.

Figure 4.1 Cubic normal surface traction distribution with zero force resultant but non-zero moment resultant.
The reason for this can be inferred from Fig. 4.1, which shows a typical third degree polynomial surface traction distribution over an edge, forced to pass through the two Gauss points given in Table 3.1 such that the resultant normal force is zero: \( R_1 = R_2 + R_3 \). Since the traction distribution is not symmetric over the edge, \( d_1 \neq d_2 \), and the net couple is not zero \((R_2 d_1 \neq R_3 d_2)\).

The boundary conditions had to be augmented to include the requirement that the resultant moment of the normal surface traction (taken about the starting node of each edge) is zero, i.e.,

\[
L_{ij} = \int_0^{s} s T_n(s) t ds = 0 \tag{4.9}
\]

If the degree of the stress polynomials is \( n_p \), the integrand of Eq. 4.9 is a polynomial of degree \( n_p + 1 \). For an exact Gauss quadrature, the number of Gauss points is inferred from Table 3.1, and will be greater than or equal to the number required for an exact calculation of Eq. 4.4. For a polynomial, using more than the minimum number of Gauss points required for an exact quadrature cannot improve the answer. Thus, setting \( T_n = 0 \) at those Gauss points required to make the integral in Eq. 4.9 zero ensures as well that the normal force resultant will vanish. For example, Fig. 4.2 shows how adding the required third additional Gauss point for the cubic normal traction and setting \( T_n = 0 \) there as well yields a symmetric traction distribution for which the resultant moment must clearly be zero \((R_1 d_1 = R_2 d_2)\).
The next task was to determine, for a given stress polynomial of degree $n_s$, the minimum number of Gauss points needed to calculate the resultant of the normal tractions, the moment resultant of the normal tractions and shear surface tractions over a given edge. If the stress polynomials from Table 4.1 were used, reducing the number of terms by three as described in Table 4.2, the following is observed. For a sixth degree stress polynomial, the integrand in Eq. 4.9 is a seventh degree polynomial requiring, according to Table 3.1, a minimum of four Gauss points per edge, so that five points can be used as well. For all four edges, that yields 20 boundary conditions forcing the normal and moment resultants to zero. Together with the four shear boundary conditions, there are 24 conditions for the 24 stress coefficients. A similar analysis of the modified fifth and fourth degree stress polynomials shows there are precisely enough boundary conditions to solve for the stress coefficients. The modified cubic stress polynomial has 12 terms.
Three Gauss point per edge are required to exactly enforce the zero normal force/zero moment on each edge. Hence, the boundary conditions on the shear traction cannot be enforced, or if they are, then the zero moment condition cannot. The same holds for the modified, eight-term quadratic stress polynomial. Hence, the problem was not that the moment condition was not being satisfied but that the modifications of the stress polynomial function were producing erroneous results. Upon recognizing this, the use of modified stress polynomials was abandoned in favor of retaining all terms of the complete stress polynomials.

The task at hand then became to determine, for a given degree of complete stress polynomial, the minimum number of Gauss points needed to satisfy all of the boundary conditions. From Table 4.1 it was easy to see that, for complete polynomials of degree 1 or more, the relation between the degree of the polynomial $n_p$ and the number of terms $n_s$ in the polynomial is $n_s = 4n_p + 3$. If $n_g$ is the number of Gauss points per edge required to enforce the zero force/zero moment conditions, then to solve for the $n_s$ stress coefficients requires that $4n_p + 3 = 4n_g + n_{T_s}$, where $n_{T_s}$ is the number of boundary shear force conditions. From Table 3.1 it was clear that choosing $n_g = n_p$ would provide (usually many more than) enough points for setting $T_n = 0$ to yield the zero force and moment resultants. Therefore, if $n_{T_s}$ could be set to 3, then there would always be enough boundary conditions to solve for the stress coefficient vector $\{C\}$, regardless of the degree of the stress polynomial. It was therefore decided to apply the shear force
boundary condition (Eq. 4.5a) only to sides 2, 3 and 4, which, along with Eqs. 3.76, forces those three shear resultants to be in equilibrium with the one on side 1, namely, $F_{12}$, the stress parameter. Since the solution for $F_{12}$ follows from the PCVW and is obtained during the course of the analysis after the stress coefficients have been found, for side 1 Eq. 4.5a should be used only to check on the solution for the panel shear flows.

The computer code was modified to incorporate the changes, and the previous tests were redone. The rectangular shear panel yielded small residuals and a set of stress coefficient vector $\{C\}$ corresponding to the exact, pure shear condition (all but $C_2$ equal zero). All degree of stress polynomials, from 2 through 6, gave with the same result.

The parallelogram panel also produced small residuals, and when the solution $\{C\} = [A]^{-1}\{B\}$ was backsubstituted into the product $[A]\{C\}$, it yielded the very same boundary vector $\{B\}$. The resultant normals and moments along each edge were checked and found to be zero, and the resultant shears satisfied Eq. 3.76. At this point, there arose some confidence that the stress coefficient matrix was being computed correctly. Other configurations were tested, the trapezoid 1, the trapezoid 2 and the trapezium (cf. Garvey's Panel in Chapter 3), all providing small residuals and giving edge resultants that matched the boundary conditions.

Nevertheless, there remained concern over the stability of the solution for $\{C\}$, which was explored in two ways. The solution of the set of simultaneous equations
\[ [A]\{C\} = \{B\} \quad (4.10a) \]

was solved using standard direct elimination procedures to obtain, symbolically,

\[ \{C\} = [A]^{-1}\{B\} \quad (4.10b) \]

If both sides of Eq. 4.10a are multiplied by the transpose of the matrix \( [A] \),

\[ [A]^{T}[A]\{C\} - [A]^{T}\{B\} \quad (4.11a) \]

then the solution for \( \{C\} \) should not be affected,

\[ \{C\} = [[A]^{T}[A]]^{-1}[A]^{T}\{B\} \quad (4.11b) \]

However, it turned out that the solution represented by Eq. 4.11b was different than that represented in Eq. 4.10b, even though both cases yielded small residuals and resultants which matched the boundary conditions. Concern over the condition of \( [A] \) grew as further investigation revealed that solution (Eq. 4.10b) was sensitive to the inversion software used. The Gaussian elimination routine using partial pivoting was then modified so that the determinate could be calculated. It was found to be very small and a condition number \[16, 17\] of the matrix \( [A] \) was also calculated using the formula
\[
C = \frac{\text{ABS}(\text{Determinate}[A])}{\text{Norm}[A]} \geq 0 \tag{4.12}
\]

which, when applied to the test case, was very small, suggesting that \([A]\) was indeed ill-conditioned. Various inversion routines were then incorporated, each producing their own unique set of coefficients \([C]\). These sets of stress coefficients varied by an order of magnitude. The residual iterative improvement method described above was then tried to possibly help force these different inversion routines to converge to an agreeable solution. For each inversion technique, the coefficient vector \([C]\) remained unchanged over repeated iterations, and the residuals were acceptably small. This meant that for a particular inversion routine the solution was exact.

It was noticed throughout this testing sequence that the stress coefficients \([C]\) found using different inversion routines produced different panel displacement vectors, which in turn produced different axial forces in the rod elements. The internal stress distribution in the panel also differed, but the resultant shear flow along the edges of the panel remained the same, regardless of the choice of inversion routine. Attempts to explain this puzzling and unacceptable phenomenon failed.

In one last attempt to obtain the exact solution for the stress coefficients, the origin of the local coordinate system was moved to the center of the panel. It was hoped this would improve the stability of the matrix \([A]\) by providing the stress polynomial terms
(of which the matrix consists) x-y coordinates of equal magnitude and symmetric about the origin. This did not improve the solution for the stress coefficients \{C\}. After this attempt failed it was clearly time to evaluate the whole approach to the shear panel problem.

**The Curtis 4-DOF Shear Panel**

It was decided to retain as the stress field within the arbitrary quadrilateral one which is derived from a complete Airy stress polynomial (cf. Table 4.1). The variables are the panel's local x-y coordinates. The origin of this coordinate system (cf. Fig. 4.3) is located at the node arbitrarily labeled "1," and the positive x-axis lies along edge 1-2, where the nodes are numbered counterclockwise around the panel. The positive y-axis is perpendicular to the x-axis and points towards side 3-4, so that the z-axis is normal to the plane of the panel, forming a positive right-hand coordinate triad.

The boundary conditions required of this panel are that each edge is subjected to pure shear stress. This means that the resultant normal surface tractions and their associated moments on each edge must vanish. The shear traction resultants on the other hand must equal the shear forces which equilibrium, via Eqs. 3.76, requires to exist along the edges of the panel. As discussed in Chapter 2, stresses derived from an Airy stress function automatically satisfy the equations of equilibrium.
Since the stresses are in equilibrium, the stress resultants computed around the boundary of the panel must form a self-equilibrating set. Since the normal force and couple on each edge are required to be zero, the only remaining stress resultants, namely, the four edge shear forces, must be self-equilibrating and therefore must satisfy Eqs. 3.76 \textit{a priori}. In the previous section, like in Nack's work, these equilibrium conditions were erroneously forced onto an already equilibrium stress field in forming Eq. 4.6. It became clear that this now-obvious redundancy was the reason for the ill-conditioning problems described above.
Thus, the only boundary conditions to be applied to the stress field are those which eliminate normal force and moment on the edges, and that can be done by setting $T_n = 0$ at an appropriate number of points, depending on the degree of the stress polynomials. Table 4.3 summarizes the systematic procedure decided on for choosing these "collocation" points. The number of Gauss points per edge is to be set equal to the degree of the polynomial. This provides enough collocation points to ensure that the resultant normal force and moment on the edges can be forced to be zero. This scheme always yields a number $n_{T_n}$ of collocation points that is always smaller than the number $n_s$ of stress polynomial coefficients. In fact, it turns out that, invariably, $n_s - n_{T_n} = 3$. Therefore, the point collocation scheme will always leave three stress coefficients undetermined, and they will be considered as the stress parameters to be found using the PCVW. Let the shear panel based on this approach be called a "Curtis" panel owing to the suggestions made by H. D. Curtis that lead to its development.

Table 4.3 Curtis boundary conditions for complete polynomial.

<table>
<thead>
<tr>
<th>$n_p$</th>
<th>$n_{g_{min}}$</th>
<th>$n_p = n_g$</th>
<th>$n_s$</th>
<th>$n_{T_n}$</th>
<th>$\beta_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>11</td>
<td>8</td>
<td>3</td>
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<tr>
<td>3</td>
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<td>12</td>
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<td>19</td>
<td>16</td>
<td>3</td>
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<tr>
<td>5</td>
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<td>23</td>
<td>20</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>6</td>
<td>27</td>
<td>24</td>
<td>3</td>
</tr>
</tbody>
</table>
To facilitate setting up the system analogous to Eq. (4.6), let the 3 by \( n_s \) matrix \([P]\) consist of the stress polynomial terms \(f_i, g_i\) and \(h_i\) from Table 4.1 such that

\[
[P] = \begin{bmatrix}
[P_1] \\
[P_2] \\
[P_3]
\end{bmatrix} = \begin{bmatrix}
f_1 & f_2 & f_3 & \cdots & f_{n_s} \\
g_1 & g_2 & g_3 & \cdots & g_{n_s} \\
h_1 & h_2 & h_3 & \cdots & h_{n_s}
\end{bmatrix}
\] (4.13)

This is in the form of Eq. 2.61, with the stress polynomial coefficients playing the role of stress parameters \(\{\beta\}\):

\[
\{\sigma\} = [P] \{\beta\}
\] (4.14)

Substituting the stresses so-defined into the expressions for normal and tangential surface tractions (Eq. 2.6), yields

\[
\{T\} = \begin{bmatrix}
T_n \\
T_t
\end{bmatrix} = [L]\{\beta\}
\] (4.15a)

where

\[
[L] = \begin{bmatrix}
[L_n] \\
[L_t]
\end{bmatrix} = \begin{bmatrix}
t_y^2[P_1] + t_x^2[P_2] - 2t_xt_y[P_3] \\
t_x t_y ([P_1] - [P_2]) + (t_y^2 - t_x^2)[P_3]
\end{bmatrix}
\] (4.15b)
Since \( T_n = \{L_n\} \{\beta\} \), setting \( T_n \) equal to zero at the \( n_{T_n} \) collocation points yields the equation

\[
[A] \{\beta\} = 0 \quad (4.16a)
\]

where \([A]\) is \( n_{T_n} \) by \( n_s \), and its row vectors are the row vectors \([L_n]\) evaluated at successive collocation points around the boundary. The matrix \([A]\) can be partitioned into two submatrices by grouping the stress parameters \( \{\beta\} \) into a dependent set \( \{\beta_2\} \) and an independent set \( \{\beta_1\} \). It was decided that the independent set of stress parameters should consist of the coefficients of the first three terms in Table 4.2, since they define the uniform state of stress and should be present in stress polynomials of any degree. Therefore, the matrix \([A]\) is partitioned columnwise into two submatrices \([A_1]\) and \([A_2]\), so that Eq. 4.15 can be written

\[
[A_1]\{\beta_1\} + [A_2]\{\beta_2\} = 0 \quad (4.16b)
\]

\[
[A_2]\{\beta_2\} = -[A_1]\{\beta_1\} \quad (4.16c)
\]

Solving Eq. 4.16b for \( \{\beta_2\} \) yields the relation between the dependent and independent stress parameters:

\[
\{\beta_2\} = [\Phi]\{\beta_1\} \quad (4.17a)
\]

where,

\[
\{\beta_2\} = \begin{bmatrix} C_4 \\ C_5 \\ \vdots \\ C_{n_s} \end{bmatrix} \quad \text{and} \quad \{\beta_1\} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \quad (4.17b)
\]
and \([\Phi]\) is found using the standard inversion method

\[
[A_2] [\Phi] = -[A_1] \\
[\Phi] = -[A_2]^{-1} [A_1]
\]

(4.18a)
(4.18b)

The stress polynomial matrix \([P]\) in Eq. 4.14 can then be partitioned in terms of the independent stress parameters by utilizing Eq. 4.17 to obtain

\[
\{\sigma\} = [P_1] \{\beta_1\} + [P_2] \{\beta_2\} = [P_1] \{\beta_1\} + [P_2] [\Phi] \{\beta_1\}
\]

(4.19)

so that

\[
\{\sigma\} = [\hat{P}] \{\beta_1\}
\]

(4.20)

where

\[
[\hat{P}] = [P_1] + [P_2] [\Phi]
\]

(4.21)

\([\hat{P}]\) is the reduced matrix of stress polynomials corresponding to the reduced set of independent stress parameters \(\{\beta_1\}\). The natural flexibility matrix \([H]\) can then be determined using Eq. 2.69, written in terms of the reduced stress function matrix:

\[
[H] = \int [\hat{P}]^T [D] [\hat{P}] \, dV
\]

(4.22)

Note that \([H]\) is a 3 by 3 symmetric matrix using the Curtis approach, whereas in all the other stress-based shear panel methods discussed herein the flexibility has been a scalar. The integral of
each component of the matrix integrand in Eq. 4.22 is carried out using two-dimensional Gauss quadrature discussed in Chapter 3 (cf. Eq. 3.86).

The shear surface traction must now be integrated along each edge to obtain the resultant shear forces. From above, we have

\[ T_t = [L_t] \{\beta\} \quad (4.23) \]

This equation can then be partitioned just as \([A]\) was to express the tangential stress in terms of the independent stress parameters \(\{\beta_1\}\).

\[ T_t = [L_{t_1}] \{\beta_1\} + [L_{t_2}] \{\beta_2\} = [L_{t_1}] \{\beta_1\} + [L_{t_2}] [\Phi] \{\beta_1\} \quad (4.24) \]

so that

\[ T_t = [\hat{L}_t] \{\beta_1\} \quad (4.25) \]

where

\[ [\hat{L}_t] = [L_{t_1}] + [L_{t_2}] [\Phi] \quad (4.26) \]

From Eq. 4.5b the shear force per unit length on a given edge is

\[ F_{k_1} = \frac{t}{2} \sum_{i=1}^{n_g} W_i \left[ \hat{L}_t(s_i) \right] \{\beta_1\} \quad (4.27) \]

The shear force per unit length is, of course, the average shear flow. The four average shear flows for sides 1, 2, 3 and 4 form the panel element’s 4 d.o.f. load vector \(\{Q’\}\).\(^2\) Thus, writing the expression

\(^2\)Hence the name, “Curtis 4-DOF panel.”
in Eq. (4.27) for each side in turn yields

\[ \{Q\}' = [F]\{\beta_1\} \]  \hspace{1cm} (4.28a)

where

\[
\begin{align*}
\{Q\} &= \begin{bmatrix} F_{12} \\ L_{12} \\ F_{23} \\ L_{23} \\ F_{34} \\ L_{34} \\ F_{41} \\ L_{41} \end{bmatrix} \\
\end{align*}
\]

and

\[
[F] = \left( \frac{t}{2} \right) \\
\begin{bmatrix}
\sum_{i=1}^{n_g} W_i \hat{L}_t (s_i) & \sum_{i=1}^{n_g} W_i \hat{L}_t (s_i) \\
\sum_{i=1}^{n_g} W_i \hat{L}_t (s_i) & \sum_{i=1}^{n_g} W_i \hat{L}_t (s_i) \\
\end{bmatrix}
\]  \hspace{1cm} (4.28b)

The edges of the element are considered "line nodes" [15]. The local stiffness matrix \([K']\) relating the shear flows to the line node displacements is found in the usual way from Eq. 2.79:

\[
[K'] = [F][H]^{-1}[F]^T = \\
\begin{bmatrix}
K'_{11} & K'_{12} & K'_{13} & K'_{14} \\
K'_{21} & K'_{22} & K'_{23} & K'_{24} \\
K'_{31} & K'_{32} & K'_{33} & K'_{34} \\
K'_{41} & K'_{42} & K'_{43} & K'_{44} \\
\end{bmatrix}
\]  \hspace{1cm} (4.29)

and transformed into the global system utilizing Eq. 2.56

\[
[K] = [\Lambda]^T[K'][\Lambda] \]  \hspace{1cm} (4.30)
The transformation matrix $[\Lambda]$, however, is not as defined by Eq. 2.34. Instead, $[\Lambda]$ consists simply of plus or minus 1's along the diagonal and zeroes everywhere else. The reason is that no matter how the panel is oriented, shear flow is viewed as acting along an edge of the panel and the only question is whether it is positive or negative. The positive direction of a global line node joining two global point nodes is from the lower numbered node to the higher numbered node. Locally, the direction of a line node is, like shear flow, counterclockwise around the boundary of the panel. Each of the element's line nodes is given a sign: positive if it is in the direction of the global line node, negative otherwise. Then $[\Lambda]$ has the form

$$
[\Lambda] = [\Lambda] = \begin{bmatrix}
\text{sgn}(1) & 0 & 0 & 0 \\
0 & \text{sgn}(2) & 0 & 0 \\
0 & 0 & \text{sgn}(3) & 0 \\
0 & 0 & 0 & \text{sgn}(4)
\end{bmatrix}
$$

(4.31)

where $\text{sgn}(i)$ is the sign of the local line node for side $i$, as determined by the global node numbers that side connects. Substituting Eq. 4.31 into Eq. 4.30 yields a global stiffness matrix the same size as the local stiffness matrix (4 by 4), and the components of the two matrices differ by at most a sign.
The local load vector (Eq. 4.28) is also transformed into the global system using Eq. 2.36 where the transformation matrix is that in Eq. 4.31.

\[
\{Q\} = \begin{pmatrix}
\text{sgn}(1) & F_{12} \\
\text{sgn}(2) & L_{12} \\
\text{sgn}(3) & F_{23} \\
\text{sgn}(4) & L_{23} \\
\end{pmatrix}
\]  \hspace{1cm} (4.33)

Once the global line node displacements have been found, they are transformed into the local system, \{q'\}, using the global to local transformation equation (Eq. 2.35) and Eq. 4.31.

\[
\{q'\} = [\Lambda]\{q\}
\]  \hspace{1cm} (4.34)
Eq. 2.72 is employed to find the independent stress parameters from the element displacements:

\[
\{\beta_1\} = [H]^{-1}[F]^T \{q\}'
\]  

(4.35)

The remaining stress coefficients can then be determined using Eq. 4.17. The shear flows \(\{Q'\}\) are then found by substituting the independent stress parameters into Eq. 4.28. (They could also be found from the local stiffness equations, \(\{Q'\} = [K']\{q'\}\), if the local stiffness matrix is saved after assembly.) Multiplying the shear flows by the lengths of the sides gives the corresponding shear forces, which can then be checked by substituting them into Eqs. 3.76. The stresses at any point within or along the edges of the shear panel can be found by substituting \(\{\beta_1\}\) into Eq. 4.20. These stresses can then be used to determine the principal stresses from Eqs. 2.8, 2.9 and 2.10.

This panel was coded and numerical testing resumed. These tests were conducted using a linear-stress rod element, derived using the PCVW in the following section. The rectangular panel provided the assurance that the newly derived panel gave the exact solution and that the stress coefficients were all zero except for \(C_2\), showing that the panel was in uniform pure shear. The edge shear forces were also verified.

The parallelogram was tested next. The residuals were small and the solution for the stress coefficients (all of them zero except \(C_2\) and \(C_3\)) produced the requisite constant shear flow (Fig. 3.5a)
and the exact uniform state of stress given by Eq. 3.12.

Several different linear equation solvers were then used to determine the stability of the solution for the stress coefficients, and all produced the same result. All of the residuals were very small and scaling the \([A_2]\) matrix had no effect on the solution for \(\{\beta_1\}\). Modifying Eq. 4.18a the manner of Eq. 4.11 also had no effect on the \(\{\beta_1\}\). Other panel configurations were tested, including the trapezoid and the trapezium, and all showed the same stable behavior as did the parallelogram panel. Since the Curtis 4-DOF panel was giving good numbers and seemed to be free of the numerical/mathematical problems which plagued the Nack panel, further testing of it continued.

**The Stress-Based Rod Element**

The two-node linear-displacement rod element could not be used as a stiffener with the 4-DOF panel since it cannot attach to the line nodes on the panel. Neither can the three-node quadratic-displacement rod, whose mid-side point node has no counterpart on the edge of the 4-DOF panel to which it can be attached. A rod element with two point nodes joined by a line node was required. The line node would provide the means of the rod's transferring direct loads applied at its ends to the webs of attached shear panels.
Let the uniaxial stress distribution in the rod of Fig. 4.4 be represented by a one-dimensional polynomial in \( s \), such that the stress within the rod can be expressed as

\[
\sigma = [P] \{\beta\} \tag{4.36}
\]

where the stress terms in \([P]\) can be of any order \( n_s \)

\[
[P] = [1 \ s \ s^2 \ s^3 \ s^4 \ \ldots \ s^{n_s}] \tag{4.37}
\]

and the stress parameter vector consists \( \{\beta\} \) of the polynomial coefficients,

\[
\{\beta\} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{n_s} \end{bmatrix} \tag{4.38}
\]
The natural flexibility matrix is determined using Eq. 2.69,

\[
[H] = \int [P]^T [D] [P] \, dV = \int_0^L [P]^T [D] [P] \, ds
\] (4.49)

Substituting Eq. 4.37 and noting that the \([D]\) matrix (Eq. 2.19) is in this case \(1/E\), the matrix \([H]\) becomes

\[
[H] = \frac{A}{E} \int_0^L \begin{bmatrix}
1 & s & s^2 & \ldots & s^{n_s+1} \\
& s^2 & s^3 & \ldots & s^{n_s+1} \\
& & \ddots & \vdots & \ddots \\
& & & s^{n_s+2} & \ddots \\
& & & & \ddots & \ddots \\
& & & & & s^{n_s+1}
\end{bmatrix} \, ds = \frac{A}{E} \begin{bmatrix}
L & \frac{L^2}{2} & \frac{L^3}{3} & \ldots & \frac{L^{n_s+1}}{n_s+1} \\
& \frac{L^3}{3} & \frac{L^4}{4} & \ldots & \frac{L^{n_s+2}}{n_s+2} \\
& & \ddots & \vdots & \ddots \\
& & & \frac{L^{n_s+3}}{n_s+3} & \ddots \\
& & & & \ddots & \ddots \\
& & & & & \frac{L^{n_s+1}}{n_s+1}
\end{bmatrix}
\] (4.40)

The local element load vector \(\{Q'\}\) is found in terms of the stress parameters by multiplying the stresses evaluated at the end points of the rod, \(Q'_1\) and \(Q'_2\), by the constant area,

\[
Q_1 = -A \sigma(0) = -A[P(0)]\{\beta\} = -A[1 \ 0 \ 0 \ \ldots \ 0]\{\beta\}
\] (4.41)

\[
Q_2 = A \sigma(L) = A[P(L)]\{\beta\} = A[1 \ L \ L^2 \ \ldots \ L^{n_s}]\{\beta\}
\] (4.42)
The overall equilibrium of the rod can be expressed as

\[ Q_1' + Q_2' + Q_3' L = 0 \]  \hspace{1cm} (4.43)

from which the shear force \( Q_3' \) is obtained:

\[ Q_3' = \frac{1}{L} (Q_1' + Q_2') = \frac{A}{L} \left( [1 \, 0 \, 0 \, \ldots \, 0] \{\beta\} + [1 \, L \, L^2 \, \ldots \, L^{n_s}] \{\beta\} \right) \]  \hspace{1cm} (4.44)

This simplifies to

\[ Q_3' = -A [0 \, 1 \, L \, \ldots \, L^{n_s-1}] \{\beta\} \]  \hspace{1cm} (4.45)

Thus, the local load vector becomes

\[ \{Q'\} = [F] \{\beta\} \]  \hspace{1cm} (4.46a)

or

\[ \begin{bmatrix} Q_1' \\ Q_2' \\ Q_3' \end{bmatrix} = -A \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ -1 & -L & -L^2 & \ldots & -L^{n_s} \\ 0 & 1 & L & \ldots & L^{n_s-1} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{n_s} \end{bmatrix} \]  \hspace{1cm} (4.46b)

The local stiffness matrix is found using Eq. 2.79, by substituting Eqs. 4.40 and 4.46 into Eq. 2.79,

\[ [K'] = [F][H]^{-1}[F]^T \]  \hspace{1cm} (4.47)
The global stiffness matrix $[K]$ is obtained from Eq. 2.56 where the transformation matrix $[\Lambda]$ consists of the element’s directional cosines $l$, $m$ and $n$, and the element’s line node sign, $\text{sgn}(l)$. In this case the $[\Lambda]$ matrix is written as

$$
[\Lambda] = \begin{bmatrix}
1 & m & n & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & m & n & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \text{sgn}(l)
\end{bmatrix}
$$

and when substituted into Eq. 2.56 the global stiffness matrix is obtained,

$$
[K] = [\Lambda][K'][\Lambda]^T
$$

After the global displacements of the structure have been computed, the appropriate components are projected onto the element by means of $[\Lambda]$ to obtain the local displacements, $\{q'\}$:

$$
\{q'\} = [\Lambda]\{q\}
$$

These local displacements and the local stiffness matrix are then used in Eq. 2.80 to determine the local element load vector, $\{Q'\}$:

$$
\{Q'\} = [K]\{q'\}
$$

This load vector consists of the two nodal point loads, $Q'_1$ and $Q'_2$ and the average shear flow, $Q'_3$. 
The Curtis 12-DOF Shear Panel

A hybrid element using the PCVW was derived using two independent assumptions. These assumptions include an equilibrium stress field within the element and an assumed displacement field along the boundary of the element. The internal stress field is that of the Curtis 4-DOF element, so the natural flexibility matrices \([H]\) obtained from Eq. 4.22 are identical. The difference between the Curtis 4-DOF and 12-DOF panels is the nature of the load vector associated with the element. The 4-DOF element is a pure stress element in which absolutely no assumptions were made about the element's displacement field. The corner nodes have no d.o.f.'s. In the 12-DOF element on the other hand, an assumption is made about the boundary displacement field so that the boundary tractions are lumped as point loads to the mid-side and corner nodes, rather than being smeared out, as it were, over the sides (line nodes) of the 4-DOF panel.

The displacements along the edges of the panel were assumed to lie in the plane of the panel and to be quadratic functions of position. In coming up with the specific form of Eq. 2.64 for the case at hand, the displacement vector \(\{u\}\) at any point along a side of the panel is resolved into components normal \(\{u_n\}\) and tangential \(\{u_t\}\) to the edge. The position coordinate \(s\) is the distance to a point from the starting corner node of the edge. Therefore, on edge \(ijk\),

\[
\begin{align*}
u_n(s) &= [\bar{N}(s)] \{q_n\} \quad (4.55a) \\
u_t(s) &= [\bar{N}(s)] \{q_t\} \quad (4.55b)
\end{align*}
\]
where the quadratic shape function vector \([ \vec{N}(s) ]\) is

\[
[ \vec{N}(s) ] = \left( \begin{array}{c}
1 - \frac{3s}{L_{ijk}} + \frac{2s^2}{L_{ijk}^2} \\
\frac{4s}{L_{ijk}} - \frac{4s^2}{L_{ijk}^2} \\
-\frac{s}{L_{ijk}} + \frac{2s^2}{L_{ijk}^2}
\end{array} \right)
\] (4.56)

and \(L_{ijk}\) is the length of the edge. This quadratic displacement field will be derived in the next section.

The edge nodal displacement vector is \(\{q\} = \{q_n\} \}

\[
\begin{align*}
\{q_n\} &= \begin{Bmatrix}
q_{n_i} \\
q_{n_j} \\
q_{n_k}
\end{Bmatrix} \\
\{t\} &= \begin{Bmatrix}
q_{t_i} \\
q_{t_j} \\
q_{t_k}
\end{Bmatrix}
\end{align*}
\] (4.57)

Figure 4.5 Displacement components along a three-node edge.
and the three components of these vectors are, respectively, the normal and tangential nodal displacements at the beginning node $i$, the mid-point node $j$, and the end node $k$, as shown in Fig. 4.5.

The equivalent load vector $\{Q''\}$ (Eq. 2.75) is also resolved into normal and tangential components. These two nodal load components can be written as

$$\{Q''_n\} = t \int_0^L [\bar{N}]^T T_n \, ds \quad (4.58a)$$

$$\{Q''_t\} = t \int_0^L [\bar{N}]^T T_t \, ds \quad (4.58b)$$

where the shape function is that given in Eq. 4.56, and the normal and tangential surface tractions, $T_n$ and $T_t$, are given by Eqs. 4.2 and Eq. 4.3. The load vector components $\{Q''_n\}$ and $\{Q''_t\}$, consisting, respectively, of the normal and tangential equivalent loads at the three nodes along an edge of the panel, are

$$\{Q''_n\} = \begin{bmatrix} Q''_{n_i} \\ Q''_{n_j} \\ Q''_{n_k} \end{bmatrix} \quad \text{and} \quad \{Q''_t\} = \begin{bmatrix} Q''_{t_i} \\ Q''_{t_j} \\ Q''_{t_k} \end{bmatrix} \quad (4.59)$$

A fundamental characteristic of a shear panel is that the point load equivalents to the surface tractions should lie totally along the
element edge; that is, on each edge of the panel the condition
\( \{Q''_n\} = \{0\} \) should be automatically satisfied by the stress field.
Since the shape functions are quadratic in \( s \), \( \{Q''_n\} \) will vanish if, on each side of the panel, the normal surface traction has previously been required to be zero at those Gauss points for which a Gauss quadrature yields an exact integral of \( s^2 T_n(s) \) (and therefore as well of \( s T_n(s) \) and \( T_n(s) \)). A study of Tables 4.1 and 3.1 shows that for stress polynomials of degree three\(^3\) or more this can be achieved by choosing the number of Gauss points \( n_g \) on each edge equal to the degree \( n_p \) of the stress polynomial. This is precisely the strategy employed above to ensure that \( \{\beta_1\} \) would always consist of just the three coefficients of the zeroth order terms of the of the stress polynomial. It follows that stress function polynomials of degree greater than two should be used in the formulation of the Curtis 12-DOF panel.

With the normal load vector \( \{Q''_n\} \) automatically equal to zero, the local panel load vector \( \{Q''\} \) consists of just three tangential loads per side: one at the starting node, one at the mid-side node and one at the ending node. These 12 loads are shown in Fig. 4.6. The sum of these three load components per edge will be statically equivalent to the shear force \( F_{ijk} \) along that edge,

\[
F_{ijk} = Q''_i + Q''_j + Q''_k
\]  

(4.60)

\(^3\)For stress polynomials of degree two, the number of polynomial terms is 11, which is one less than the minimum number of Gauss collocation points required to ensure that \( \int s^2 T_n ds = 0 \) on each edge.
The complete 12-component element load vector \( \{Q''\} \) is found by substituting Eq. 4.25 into Eq. 4.58b, and carrying out the integrals along each side to obtain

\[
\begin{bmatrix}
Q''_1 \\
Q''_2 \\
Q''_3 \\
\vdots \\
Q''_{10} \\
Q''_{11} \\
Q''_{12}
\end{bmatrix} =
\begin{bmatrix}
L_{12}^T \\
0 \\
L_{23}^T \\
0 \\
L_{34}^T \\
0 \\
L_{41}^T \\
0
\end{bmatrix}
\begin{bmatrix}
\int [N]_{12} [\hat{\mathbf{L}}_t]_{12} ds \\
\int [N]_{23} [\hat{\mathbf{L}}_t]_{23} ds \\
\int [N]_{34} [\hat{\mathbf{L}}_t]_{34} ds \\
\int [N]_{41} [\hat{\mathbf{L}}_t]_{41} ds
\end{bmatrix}
\]  

(4.61a)
In terms of the matrix $[F']$, the local edge load vector $\{Q''\}$ is written as (Eq. 2.77)

$$\{Q''\} = [F'] \{\beta_1\}$$ (4.61)

Performing the integrations in Eq. 4.61 by using a one-dimensional Gaussian quadrature with $n_p + 2$ Gauss points, we identify $[F']$ as

$$[F'] = \begin{bmatrix}
\frac{t}{2} \sum_{i=1}^{n_p+2} W_i [N(s)]_{12}^T \left[ \hat{L}_t(s_i) \right]_{12} \\
\frac{t}{2} \sum_{i=1}^{n_p+2} W_i [N(s)]_{23}^T \left[ \hat{L}_t(s_i) \right]_{23} \\
\frac{t}{2} \sum_{i=1}^{n_p+2} W_i [N(s)]_{34}^T \left[ \hat{L}_t(s_i) \right]_{34} \\
\frac{t}{2} \sum_{i=1}^{n_p+2} W_i [N(s)]_{41}^T \left[ \hat{L}_t(s_i) \right]_{41}
\end{bmatrix}$$ (4.62)

In order to transform the panel's local stiffness matrix into the global rectangular coordinate system, it is convenient to resolve the corner tangential loads in Fig. 4.6 into orthogonal components in the element's coordinate system (see Fig 4.7). Setting the two alternative decompositions of the same vector equal at each corner,

$$\begin{align*}
Q'_1 + Q'_2 &= Q''_{12} t_{12} + Q''_{12} t_{41} \\
Q'_3 + Q'_4 &= Q''_{32} t_{12} + Q''_{42} t_{23} \\
Q'_5 + Q'_6 &= Q''_{62} t_{23} + Q''_{72} t_{34} \\
Q'_7 + Q'_8 &= Q''_{92} t_{34} + Q''_{102} t_{41}
\end{align*}$$ (4.63)
Taking the dot product of each of these equations with \( i \) and \( j \), respectively, it follows that

\[
Q_1 = Q''_{12}t_{12} + Q''_{12}t_{41} = Q''_{1} + Q''_{12}t_{41}
\]

\[
Q_2 = Q''_{12}t_{12} + Q''_{12}t_{41} = Q''_{12}t_{41}
\] (4.64)

\[
Q_3 = Q''_{3}t_{12} + Q''_{4}t_{23} = Q''_{3} + Q''_{4}t_{4}t
\]

\[
Q_4 = Q''_{3}t_{12} + Q''_{4}t_{23} = Q''_{4}t_{23}
\] (4.65)

\[
Q_5 = Q''_{6}t_{23} + Q''_{7}t_{34}
\]

\[
Q_6 = Q''_{6}t_{23} + Q''_{7}t_{34}
\] (4.66)
\[ Q_7 = Q''_9 t_{34x} + Q''_{10} t_{41x} \]
\[ Q_8 = Q''_9 t_{34y} + Q''_{10} t_{41y} \]  

(4.67)

In matrix form Eqs. 4.64 through 4.67 can be written

\[ \{Q'\} = [\mathbf{T}] \{Q''\} \]  

(4.68)

where

\[ [\mathbf{T}] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{41x} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{41y} \\
0 & 0 & 0 & 0 & 1 & t_{12x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t_{12y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t_{23x} & t_{34x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t_{23y} & t_{34y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{34x} & t_{41x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{34y} & t_{41y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]  

(4.69)

Substituting Eq. 4.61 into Eq. 4.68 yields

\[ \{Q'\} = [\mathbf{F}] \{\beta_1\} \]  

(4.70a)

in which

\[ [\mathbf{F}] = [\mathbf{T}] [\mathbf{F'}] \]  

(4.70b)
The local 12-by-12 stiffness matrix is determined from Eq. 2.79:

\[
[K'] = [F][H]^{-1}[F]^T
\]  \hspace{1cm} (4.71)

where the natural flexibility matrix [H] is identical to that of the Curtis 4-DOF panel (Eq. 4.22) and the matrix [F] is given by Eq. 4.70. As usual, the global stiffness matrix is obtained using Eq. 2.56,

\[
[K] = [A]^T[K'][A]
\]  \hspace{1cm} (4.72)

after the form of the transformation matrix [A] for this particular element is determined. Since the loads at the mid-side nodes are directed along the edges of the panel, it makes sense to assign those nodes one d.o.f. in both the local and global systems. This means that the local and global descriptions of a mid-side load can differ only in sign. Therefore, the mid-side nodes can be treated logically as line nodes, in the fashion of the 4-DOF panel. If the element's twelve local d.o.f.'s are ordered such that those at the four point nodes are listed first, followed by those of the four line nodes, then

\[
[A] = \\
\begin{bmatrix}
\lambda & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & \lambda \\
\end{bmatrix}
\]  \hspace{1cm} (4.73)
where $[\lambda]$ consists of the direction cosines of the element's $x'-y'$ axes relative to the global system, and $[\lambda']$ contains the signs of the line-nodes, (Eq. 4.31):

$$[\lambda] = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix} \quad (4.74a)$$

and

$$[\lambda'] = \begin{bmatrix} \text{sgn}(5) & 0 & 0 & 0 \\ 0 & \text{sgn}(6) & 0 & 0 \\ 0 & 0 & \text{sgn}(7) & 0 \\ 0 & 0 & 0 & \text{sgn}(8) \end{bmatrix} \quad (4.74b)$$

After the panel's global displacements have been found, they are transformed as usual back into the local coordinate system by means of Eq. 2.35. The independent stress parameters $\{\beta_1\}$ are then determined from Eq. 4.35, after which Eq. 4.17 is used to find the remaining ones. Eq. 4.20 yields the stresses at any point within the panel or along the boundary, and the principal stresses result from Eqs. 2.8, 2.9 and 2.10. Substituting $\{\beta_1\}$ into Eq. 4.61 gives the twelve tangential load components, and the four average shear flows may then be determined using Eq. 4.60 and then checked against Eqs. 3.76.

This panel was coded and tested. The stiffener used was the quadratic displacement rod element, discussed in the following section. Running the panel as a rectangle produced the exact, constant shear flow solution. For a parallelogram in pure shear, this panel gave the exact solution (Eq. 3.12) at the centroid, but unlike
the Curtis 4-DOF panel, this state of stress was not uniform throughout the element. But the performance was stable and numerical testing went on to the trapezoid cases for comparison with the solutions given by the previously-tested elements.

The Three-Node Quadratic Displacement Rod Element

The sides of the Curtis 12-DOF panel have three nodes. This makes it possible to fit a quadratic displacement field to each edge, which requires that the edge stiffeners be three-node quadratic displacement rods in order to ensure interelement displacement compatibility along the panel/stiffener interface.

![Figure 4.8 Three-node quadratic displacement rod element.](image)

The rod element in Fig. 4.8 has a mid-side node in addition to those at the ends. Each node has one local, axial d.o.f. If a quadratic displacement field is assumed,

\[
  u(s) = c_1 + c_2 s + c_3 s^2
\]  

(4.75)
then by satisfying the nodal boundary conditions, we get

\begin{align*}
\text{at } s = 0 & \quad u(0) = u_1 \quad c_1 = u_1 \quad (4.76) \\
\text{at } s = L/2 & \quad u(L/2) = u_2 \quad c_2(L/2) + c_3(L^2/4) = u_2 - u_1 \quad (4.77) \\
\text{at } s = L & \quad u(L) = u_3 \quad c_2L + c_3L^2 = u_3 - u_1 \quad (4.78)
\end{align*}

Solving Eqs. 4.76 through 4.79 yields the coefficients in terms of the nodal displacements. \( c_1 \) is given by Eq. 4.76, and the remaining two are given by

\begin{align*}
\frac{1}{3L}(-3u_1 + 4u_2 - u_3) \quad (4.79) \\
\frac{2}{L^2}(u_1 - 2u_2 + u_3) \quad (4.80)
\end{align*}

Substituting these three coefficients into Eq. 4.75 yields

\begin{align*}
\{u\} &= [N] \{q\} \quad (4.81a) \\
i.e., \quad u(s) &= \begin{bmatrix}
1 - \frac{3s}{L} + \frac{2s^2}{L^2} \\
-\frac{s}{L} + \frac{2s^2}{L^2} \\
\frac{4s}{L} - \frac{4s^2}{L^2}
\end{bmatrix} \begin{bmatrix}
q_1 \\
q_2 \\
q_3
\end{bmatrix} \quad (4.81b)
\end{align*}
From this the strain-displacement matrix \([B]\) (Eq. 2.42) is easily shown to be

\[
[B] = \left[ \left( -\frac{3}{L} + \frac{4s^2}{L^2} \right), \left( -\frac{1}{L} + \frac{4s}{L^2} \right), \left( \frac{4}{L} - \frac{8s}{L^2} \right) \right] \tag{4.82}
\]

Then the local stiffness matrix is found from Eq. 2.48, where the matrix \([E]\) for this case reduces to just the scalar Modulus of Elasticity \(E\), and the volume integral becomes a line integral in terms of \(s\). If the cross-sectional area is constant, then

\[
[K] = \int_{V}^{L} [B]^T [B] dV = \int_{0}^{L} AE [B]^T [B] ds \tag{4.83a}
\]

which yields

\[
[K] = \frac{AE}{3L} \left[ \begin{array}{ccc}
7 & 1 & -8 \\
1 & 7 & -8 \\
-8 & -8 & 16 \\
\end{array} \right] \tag{4.83b}
\]

Eq. 2.56 is used to determine the rod's global stiffness matrix. The mid-side node can be treated formally as a line node for the reasons explained for the 12-DOF panel in the previous section. Therefore, the transformation matrix \([\Lambda]\) is the same as that of the two-node stress-based rod element (Eq. 4.48). If \(l\), \(m\), and \(n\) are the rod's direction cosines and \(\text{sgn}\) is the sign of its line node, then substituting Eqs. 4.48 and 4.83b into Eq. 2.56 yields the 7 by 7 global stiffness matrix,
After the element’s global displacements are determined, they are substituted into the global-to-local transformation (Eq. 2.35) along with the transformation matrix $[\Lambda]$ (Eq. 4.48) to obtain the local displacements, $\{q\}$:

$$\{q\} = [\Lambda]\{q\} \quad (4.85)$$

These local displacements and the local stiffness matrix (Eq. 4.84) are then used in Eq. 2.47 to determine the local element load vector $\{Q\}$:

$$\{Q\} = [K']\{q\} \quad (4.86)$$

**The Garvey 4-DOF Shear Panel**

Garvey’s well-known panel was implemented as a 4-DOF element so that its deflections could be compared with those of the Curtis 4-DOF panel. Recall that both panels yield the exact stress
solution for the rectangle and parallelogram in pure shear, and since both are equilibrium models, they should yield identical results for the mean boundary shear flows for any shape. Their deflections, though not expected to be identical, were expected to compare favorably.

The local stiffness matrix for the Garvey 4-DOF panel was developed in Chapter 3, and it is given in Eq. 3.34. The matrix shown is of the same form for any quadrilateral, but evaluation of the scalar flexibility $H$ is given by different formulas, depending on whether the panel is a parallelogram [or rectangle] (Eq. 3.18), trapezoid (Eq. 3.31 or 3.32) or trapezium (Eq. 3.30). Therefore, it was important in the coding of the stiffness matrix to include a procedure to determine whether the sides of the panel intersect to calculated the coordinates of the points of intersection $P$ and $Q$ (see Fig. 3.3).

Since edges $AD$ and $BC$ of the parallelogram in Fig. 4.9 are parallel to each other, their slopes are equal, so that

$$\frac{y_D}{x_D} = \frac{y_C}{x_C - x_B}$$  \hspace{1cm} (4.87)

which can be written as $(x_C - x_B) y_D - x_D y_C = 0$. Let $c_1$ ("condition 1") be defined as

$$c_1 = \left| (x_C - x_B) (y_D) - x_D y_C \right|$$  \hspace{1cm} (4.88)
Figure 4.9 Conditions for a parallelogram.

Clearly, for the parallelogram, \( c_1 = 0 \). An additional requirement for the parallelogram is that \( y_c = y_D \). Let \( c_2 \) ("condition 2") be defined as

\[
c_2 = |y_c - y_D|
\]

Therefore, \( c_1 = 0 \) and \( c_2 = 0 \) are the conditions for a parallelogram.

The interior acute angle \( \theta \) can easily be determined by computing the cross product of the unit vector along edge \( AB \) into the unit vector along edge \( AD \), the magnitude of which equals \( \sin \theta \). This enables the scalar natural flexibility matrix for the parallelogram, \( H_p \), to be calculated using Eq. 3.18. The matrix \([F]\) from Eq. 3.33c simplifies to

\[
[F_p] = \begin{bmatrix}
-1 \\
1 \\
-1 \\
1
\end{bmatrix}
\]

(4.90)
since the perpendicular distance ratios tend towards 1 as both $P$ and $Q$ tend towards infinity (cf. Fig. 3.3). The matrix in Eq. 3.34 simplifies in a similar fashion, so that for the parallelogram shear panel we get

$$[K_p] = \frac{1}{H_P} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$ (4.91)

If $c_1 > 0$ and $c_2 = 0$, then we have a "trapezoid 1" panel, which is a trapezium in which the base point $Q$ is at infinity (cf. Fig. 4.10). The intersection of edges $AD$ and $BC$ determines the coordinates of point $P$, after which the perpendicular distances $p_A = p_B$ and $p_C = p_D$ are easily determined.

![Trapezoid 1, Q @ infinity](image)  

Figure 4.10 Conditions for a "trapezoid 1."
The angles $\alpha$ and $\gamma$ can then be determined by, respectively, taking the cross product of edge AB into edge BC and edge CD into edge DA. The scalar natural flexibility $H_{T_1}$ for the trapezoid 1 is found using Eq. 3.31. The $[F]$ matrix from Eq. 3.33c reduces to

$$
[F_{T_1}] = \begin{bmatrix}
-1 \\
1 \\
p_A^2 \\
p_C^2 \\
p_B \\
p_D
\end{bmatrix}
$$

(4.92)

since $p_A/p_B \to 1$ as $Q$ tends towards infinity, and, clearly, $p_A = p_B$ and $p_C = p_D$. Likewise the the stiffness matrix in Eq. 3.34 becomes

$$
[K_{T_1}] = \frac{1}{H_{T_1}} \begin{bmatrix}
1 & -1 & \frac{p_A^2}{p_C} & -\frac{p_B}{p_D} \\
-1 & \frac{p_A}{p_C} & \frac{p_A^3}{p_C} & -\frac{p_A^2}{p_C} \\
\frac{p_A^2}{p_C} & \frac{p_A}{p_C} & \frac{p_A}{p_C} & -\frac{p_A}{p_C} \\
-\frac{p_B}{p_D} & \frac{p_A}{p_C} & \frac{p_A}{p_C} & -\frac{p_B}{p_D}
\end{bmatrix}
$$

(4.93)
If \( c_1 = 0 \) and \( c_2 > 0 \), then we have the "trapezoid 2" panel shown in Fig. 4.11. It is a trapezium in which the base point \( P \) is at infinity.

![Figure 4.11 Conditions for a "trapezoid 2."](image)

The intersection of edges \( AB \) and \( CD \) will enable the coordinates of point \( Q \) to be determined. The perpendicular distances \( p_B = p_C \) and \( p_A = p_D \) from the baseline \( PQ \) can then be calculated. The angles \( \beta \) and \( \delta \) can be determined by taking the cross product of edge \( CD \) into edge \( DA \) and edge \( AB \) into edge \( BC \). The scalar natural flexibility \( H_{T2} \) for the trapezoid 2 is found using Eq. 3.32.

The \([F]\) matrix for this case reduces from Eq. 3.33c to

\[
[F_{T2}] = \begin{bmatrix}
-1 \\
\frac{p_A}{p_B} \\
\frac{-1}{p_B} \\
\frac{p_B}{p_D}
\end{bmatrix}
\]

(4.94)
and the local stiffness matrix in Eq. 3.34 simplifies for the trapezoid 2 to the form

\[
[K_{T_2}] = \frac{1}{H_{T_2}} \begin{bmatrix}
1 & -\frac{p_A}{p_B} & 1 & -\frac{p_B}{p_D} \\
-\frac{p_A}{p_B} & p_A^2 & -\frac{p_A}{p_C} & 1 \\
1 & -\frac{p_A}{p_C} & (\frac{p_A p_B}{p_C})^2 & -\frac{p_B}{p_D} \\
-\frac{p_B}{p_D} & 1 & -\frac{p_B}{p_D} & \frac{p_B^2}{p_D^2}
\end{bmatrix}
\]

(4.95)

Finally, if \( c_1 > 0 \) and \( c_2 > 0 \) we have the trapezium shown in Fig. 4.12.

![Figure 4.12 Conditions for a trapezium.](image-url)
Computing the points of intersection of opposite sides of the panel determine the coordinates of the baseline points \( P \) and \( Q \). The procedure is straightforward, and for \( P \) it yields

\[
P_x = -\frac{x_4 (y_3 x_2)}{[y_4 (x_3 - x_2) - y_3 x_4]}
\]

\[
P_y = -\frac{y_4 (y_3 x_2)}{[y_4 (x_3 - x_2) - y_3 x_4]}
\]

and the coordinates of point \( Q \) are

\[
Q_x = -\frac{(x_4 y_3 - y_4 x_3)}{(y_3 - y_4)}
\]

\[
Q_y = 0
\]

The length of the baseline \( PQ \) is easily calculated. The angles \( \alpha \), \( \beta \), \( \gamma \) and \( \delta \) between the baseline and the four sides of the quadrilateral can be determined by taking the appropriate cross products of the baseline \( PQ \) into the edges of the panel. The angles must be properly labeled since the specific shape of the panel determines the location of \( P \) and \( Q \) relative to the panel. Therefore the following notation was adhered to:

If \( P_y > 0 \) and \( Q_x > 0 \) then \( a = \cot \alpha; b = \cot \beta; c = \cot \gamma; d = \cot \delta \)
If \( P_y > 0 \) and \( Q_x < 0 \) then \( a = \cot \gamma; b = \cot \beta; c = \cot \alpha; d = \cot \delta \)
If \( P_y < 0 \) and \( Q_x < 0 \) then \( a = \cot \gamma; b = \cot \delta; c = \cot \alpha; d = \cot \beta \)
If \( P_y < 0 \) and \( Q_x > 0 \) then \( a = \cot \alpha; b = \cot \delta; c = \cot \gamma; d = \cot \beta \)
These angles are substituted into Eq. 3.28 to determine the perpendicular distances $p_A$, $p_B$, $p_C$ and $p_D$, which are then used to calculate the area of the panel by means of Eq. 3.27. All of this information is brought to bear on the calculation of the natural flexibility $H_T$ in Eq. 3.30 and the matrix $[F]$ in Eq. 3.33. Finally, Eq. 3.34 yields the stiffness matrix of the trapezium panel.

Once the solution for the line node displacements has been obtained in the course of a structural analysis, the shear flows are found in the same way as in the Curtis 4-DOF panel.

The Garvey 12-DOF Shear Panel

A hybrid Garvey panel was derived for comparisons with the Curtis 12-DOF panel. The natural flexibility matrix is the same as that of the Garvey 4-DOF panel. The 12-component element point load vector $\{Q'\}$ is defined in the same way as the Curtis 12-DOF panel. The shape functions of an assumed quadratic tangential boundary displacement are multiplied by the shear flow and integrated along the edge of the panel.

The shear flow $s'_z$ at a point on the edge of the Garvey panel is found from Eqs. 3.10 and 3.11:

$$s'_z = \frac{s_1 p_A p_B}{p_z^2} \quad \text{(4.98)}$$

where the terms in this formula are illustrated in Figs. 3.3 and 3.4. The tangential surface traction on an edge of the panel, defined as
directed positive counterclockwise around the boundary, is just 
±(s'Z/t) . From Eq. 2.75,

\[ \{Q'\} = \int_S \left( \pm \frac{s'Z}{t} \right) [N]^T dS - [F]\{\beta\} - [F]s_1 \quad (4.99) \]

The edge loads are expressed in terms of the single stress parameter \( s_1 \), which is the mean shear flow on side AB. As can be seen in Fig. 4.5, \( \{Q'\} \) consists of three nodal loads per side directed tangentially along the edge. On a given side of the panel the three edge loads can be written in terms of the starting node \( i \), the mid-side node \( j \), and the ending node \( k \) as

\[
\begin{bmatrix}
Q'_i \\
Q'_j \\
Q'_k
\end{bmatrix} = \pm P_A P_B \begin{bmatrix}
L_{ijk}N_1(s) \\
\int_0^{p_Z(s)} ds \\
L_{ijk}N_2(s) \\
\int_0^{p_Z(s)} ds \\
L_{ijk}N_3(s) \\
\int_0^{p_Z(s)} ds
\end{bmatrix} s_1 \quad (4.100)
\]

The quadratic shape functions \( N_1 \), \( N_2 \) and \( N_3 \) are found from Eq. 4.81 as

\[
N_1 = \left( 1 - \frac{3s}{L_{ijk}} + \frac{2s^2}{L_{ijk}^2} \right) ; \quad N_2 = \left( \frac{4s}{L_{ijk}} - \frac{4s^2}{L_{ijk}^2} \right) ; \quad N_3 = \left( -\frac{s}{L_{ijk}} + \frac{2s^2}{L_{ijk}^2} \right) \quad (4.101)
\]

where \( s \) is the distance from the starting node \( i \).
An expression for $p_z$ in terms of $s$ is required in order to do the integrals in Eq. 4.100. An expression for each of the four sides can be written with the aid of Fig. 3.4. Let the distance $P_n$ and $Q_n$ be the distances base points $P$ and $Q$, respectively, to the corner node $n$ of the panel. Then

side 1, edge AB  
$p_z = [QA - \text{sgn}(Q_x)s] \sin \delta$

side 2, edge BC  
$p_z = [PB - \text{sgn}(P_y)s] \sin \alpha$

side 3, edge CD  
$p_z = [QC + \text{sgn}(Q_x)s] \sin \beta$  \hspace{1cm} (4.102)

side 4, edge EF  
$p_z = [PD + \text{sgn}(P_y)s] \sin \gamma$

where the angles are those shown in Fig. 3.4, and $P_y$ and $Q_x$ are found in Eqs. 4.96 and 4.97.

Upon substituting the above shape functions (Eq. 4.101) and the perpendicular distances (Eq. 4.102) into the edge local load vector formula (Eq. 4.100), we find for side 1 (edge AB),

$$\begin{bmatrix} Q''_1 \\ Q''_2 \\ Q''_3 \end{bmatrix} = \frac{p_A p_B}{\sin^2 \delta} \begin{bmatrix} e_1 - \frac{3}{L_1} e_2 + \frac{2}{L_1^2} e_3 \\ \frac{4}{L_1^2} e_2 - \frac{4}{L_1 e_3} \\ - \frac{1}{L_1} e_2 + \frac{2}{L_1^2} e_3 \end{bmatrix} s_1$$  \hspace{1cm} (4.103)

where $L_1$ is the length of the edge and
\[ e_1 = \frac{L_1}{Q_A (Q_A - \text{sgn}(Q_x) L_1)} \]
\[ e_2 = \ln \left( \frac{Q_A - \text{sgn}(Q_x) L_1}{Q_A} \right) + \frac{L_1}{Q_A - \text{sgn}(Q_x) L_1} \]
\[ e_3 = L_1 + 2Q_A \ln \left( \frac{Q_A - \text{sgn}(Q_x) L_1}{Q_A} \right) + \frac{Q_A L_1}{Q_A - \text{sgn}(Q_x) L_1} \]

(4.104)

For side 2 (edge BC), the equations are the same as Eqs. 4.103 and 4.104, except for the following changes:

- **Load vector**: \( Q''_1, Q''_2, Q''_3 \) \( \Rightarrow \) \(-Q''_4, -Q''_5, -Q''_6\)
- **Angle**: \( \delta \) \( \Rightarrow \) \( \alpha \)
- **Length**: \( L_1 \) \( \Rightarrow \) \( L_2 \)
- **Distance**: \( \overline{QA} \) \( \Rightarrow \) \( \overline{PB} \)
- **Sign**: \( \text{sgn}(Q_x) \) \( \Rightarrow \) \( \text{sgn}(P_y) \)

On side 3 (edge CD) the Eqs. 4.103 and 4.104 is slightly different, but essentially has the same form. Eq. 4.103 requires the following modifications:

- **Load vector**: \( Q''_1, Q''_2, Q''_3 \) \( \Rightarrow \) \( Q''_7, Q''_8, Q''_9 \)
- **Angle**: \( \delta \) \( \Rightarrow \) \( \beta \)
- **Length**: \( L_1 \) \( \Rightarrow \) \( L_3 \)
and Eq. 4.104 becomes

\[
\begin{align*}
e_1 &= \frac{L_3}{QC(QC + \text{sgn}(Q_x) L_3)} \\
e_2 &= \ln\left(\frac{QC + \text{sgn}(Q_x) L_3}{QC}\right) - \frac{L_3}{QC + \text{sgn}(Q_x) L_3} \\
e_3 &= L_3 - 2QC \ln\left(\frac{QC + \text{sgn}(Q_x) L_3}{QC}\right) - \frac{QC L_3}{QC + \text{sgn}(Q_x) L_3}
\end{align*}
\]  
(4.105)

Side 4 (edge DA) uses Eq. 4.103, with the following changes:

- load vector \( Q''_1, Q''_2, Q''_3 \) \( \Rightarrow \) \(-Q''_{10}, -Q''_{11}, -Q''_{12}\)
- angle \( \delta \) \( \Rightarrow \) \( \gamma \)
- length \( L_1 \) \( \Rightarrow \) \( L_4 \)

and Eq. 4.105 is used with these changes:

- distance \( QC \) \( \Rightarrow \) \( PD \)
- sign \( \text{sgn}(Q_x) \) \( \Rightarrow \) \( \text{sgn}(P_y) \)
- length \( L_3 \) \( \Rightarrow \) \( L_4 \)

For the "trapezoid 1", the local edge nodal loads of the shear panel can be found by applying the same procedure as the trapezium, noting that sides 1 and 2 each have a constant perpendicular distance from the baseline. Therefore, for side 1, the local edge nodal loads become
\[
\begin{pmatrix}
Q'_1 \\
Q'_2 \\
Q'_3
\end{pmatrix} = \frac{p_A p_B L_1}{(PB \sin \alpha)^2} \begin{pmatrix}
\frac{1}{6} \\
\frac{2}{3} \\
\frac{1}{6}
\end{pmatrix} s_1
\tag{4.106}
\]

Side 2 is the same as for the trapezium panel above. Side 3 is the same as side 1 (Eq. 4.106), except for these changes:

- Load vector: \(Q'_1, Q'_2, Q'_3 \Rightarrow Q''_7, Q''_8, Q''_9\)
- Length: \(L_1 \Rightarrow L_3\)
- Distance: \(PB \Rightarrow PC\)

and side 4 is the same as the trapezium side 4 above.

The "trapezoid 2" panel's local edge nodal loads for side 1 are the same as the trapezium panel side 1 (Eq. 4.103 and Eq. 4.104) and side 2 of the trapezoid 2 panel is the same as Eq. 4.106 with the following changes:

- Load vector: \(Q'_1, Q'_2, Q'_3 \Rightarrow -Q''_4, -Q''_5, -Q''_6\)
- Angle: \(\alpha \Rightarrow \delta\)
- Length: \(L_1 \Rightarrow L_2\)
- Distance: \(PB \Rightarrow \overline{CC}\)

Side 3 of the trapezoid 2 panel is integrated the same way as side 3 of the trapezium panel. The load vector for side 4 is determined using Eq. 4.106 providing the following are changed:
load vector \( Q''_1, Q''_2, Q''_3 \) \( \Rightarrow \) \(-Q''_{10}, -Q''_{11}, -Q''_{12}\)
angle \( \alpha \) \( \Rightarrow \) \( \delta \)
length \( L_1 \) \( \Rightarrow \) \( L_4 \)
distance \( \overline{PB} \) \( \Rightarrow \) \( \overline{CD} \)

Note that \( p_A = p_D \) and \( p_B = p_C \) for this panel.

For the parallelogram, the local nodal loads reduce to the following for each side of the panel, because the shear flow is constant:

\[
\begin{bmatrix}
Q''_i \\
Q''_j \\
Q''_k
\end{bmatrix}
= \begin{bmatrix}
\frac{L_{ijk}}{6} \\
\frac{2L_{ijk}}{3} \\
\frac{L_{ijk}}{6}
\end{bmatrix}s_1 \tag{4.107}
\]

Sides 2 and 4 must in addition be multiplied by \(-1\) to maintain the correct sign convention for the shear flows.

Once these equivalent shear flows have been distributed to the local edge nodes, they can be transformed into the element’s local load coordinate system the same way as in the Curtis 12-DOF shear panel, using Eq. 4.68. From this point, the analysis of the shear panel proceeds the same as outlined in Curtis’ 12-DOF shear panel.
The Robinson 4-DOF Shear Panel

The Robinson 4-DOF shear panel discussed in Chapter 3 was coded to determine its performance against the Curtis and Garvey 4-DOF elements. The local stiffness matrix for the Robinson panel is found in Eq. 3.68:

\[
[K'] = (Gt/A)[F][F]^T
\]  

(4.108)

where \( G \) is the Modulus of Rigidity, \( t \) is the constant thickness of the panel and \( A \) is the area of the panel. The matrix \( [F] \) is given by Eq. 3.67. The global stiffness matrix is found by transforming the local stiffness matrix according to Eq. 2.79 and using the line-node transformation matrix \([\Lambda]\) defined in Eq. 4.31.

The local load vector, \( \{Q'\} \), consists of the average shear flows along each edge, and by applying local to global transformation (Eq. 2.36) the global load vector can be written as

\[
\begin{bmatrix}
Q_1 \\
Q_2 \\
Q_3 \\
Q_4
\end{bmatrix} =
\begin{bmatrix}
\text{sgn}(1) Q'_1 \\
\text{sgn}(2) Q'_2 \\
\text{sgn}(3) Q'_3 \\
\text{sgn}(4) Q'_4
\end{bmatrix}
\]  

(4.109)

When the global displacements have been determined, they are transformed into the local displacements by means of Eq. 2.35. The average shear flows are then determined by multiplying the local displacements by the local stiffness matrix: \( \{Q'\} = [K']\{q'\} \).

A hybrid Robinson 12-DOF panel cannot be obtained because, as discussed in Chapter 3, the stress distribution in the Robinson panel is not specified.
CHAPTER 5

TEST DATA

The following shear panels were programmed in order to investigate the behavior of their displacements and the accuracy of their shear flows for a sequence of simple structures subjected to the same boundary conditions:

- Curtis 4-DOF
- Curtis 12-DOF
- Garvey 4-DOF
- Garvey 12-DOF
- Robinson 4-DOF

It was anticipated that each method should produce displacements of approximately the same order of magnitude but slightly different according to the method and assumptions. The displacements of these stress-based shear panels were compared to a displacement-based finite element model of the same structure using GIFTS [18].

The displacements of the shear panels from above will be compared to and nondimensionalized to the Garvey 4-DOF shear panel. This is due to the fact that the Garvey solution for the parallelogram shear panel is exact and is currently used by MSC/NASTRAN.
The structure in which the panels were tested was a thin flat quadrilateral shear web surrounded by stiffeners on all four sides. This stiffened web was simply-supported on the left edge and was otherwise free of constraints (except, of course, the out-of-plane displacements were suppressed). Throughout the tests, the length $h$ of the vertical side of the web at the wall remained constant. The perpendicular distance $L$ of the parallel free end of the web from the wall, i.e., the span, was varied to investigate the effect of aspect ratio, $AR$, defined as

$$\text{AR} = \frac{L}{h}$$

(5.1)

A vertical upward point load $P$ was applied to the bottom of the vertical stiffener at the free end, causing a vertical tip displacement $\delta_t$ at that point.

For each shape of the structure, the web was modeled in turn by each of the stress-based shear panels, using just one panel for the entire web. The stiffeners used were linear-stress rod elements or quadratic-displacement rod elements, depending on the shear panel. The GIFTS finite element model of the structure consisted of a 10 by 10 grid of "QM9" [18] quadratic-displacement membrane finite elements for the web. Ten "ROD3" quadratic-displacement rod elements per side surrounded this grid. (The number of finite elements was determined by selecting one of the stiffened web shapes and running the problem on GIFTS using successive mesh

\footnote{No general trapezium shapes were tested.}
refinement. It was found that with 100 elements $\delta_t$ was at 99.5% of the value computed with a 20 by 20 grid, so that using more than 100 was clearly unnecessary.) The elements used for each shape of the structure are summarized in Table 5.1.

Table 5.1 Structural models used for each stiffened web shape.

<table>
<thead>
<tr>
<th>No. of Web Elements</th>
<th>Web Element Type</th>
<th>No. of Stiffener Elements per Edge</th>
<th>Stiffener Element Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>QM9, 9 nodes, 18 DOF</td>
<td>9</td>
<td>3 node, quadratic-displacement</td>
</tr>
<tr>
<td>1</td>
<td>Curtis, 4 nodes, 4-DOF</td>
<td>1</td>
<td>2 node, linear-stress</td>
</tr>
<tr>
<td>1</td>
<td>Curtis, 8 nodes, 12-DOF</td>
<td>1</td>
<td>3 node, quadratic-displacement</td>
</tr>
<tr>
<td>1</td>
<td>Garvey, 4 nodes, 4-DOF</td>
<td>1</td>
<td>2 node, linear-stress</td>
</tr>
<tr>
<td>1</td>
<td>Garvey, 8 nodes, 12-DOF</td>
<td>1</td>
<td>3 node, quadratic-displacement</td>
</tr>
<tr>
<td>1</td>
<td>Robinson, 4 nodes, 4-DOF</td>
<td>1</td>
<td>2 node, linear-stress</td>
</tr>
</tbody>
</table>

The web shapes investigated were the rectangle, the parallelogram, an unsymmetric trapezoid and a symmetric trapezoid. These are common shaped panels found in aircraft structures. The material properties were:

- Modulus of Elasticity: 70 GPa (10E6 psi)
- Poisson's ratio: 0.25
- Shear modulus (Eq. 2.18): 28 GPa (4E6 psi).

The stiffeners had a cross-sectional area of 0.0001 m$^2$ (0.16 in$^2$), and the web's thickness was 0.001 m (0.04 in). The aspect ratio varied from 1 to 4. The degree of stress polynomial for the Curtis
panels was varied from 2 (s2) to 6 (s6) to determine the simplest possible polynomial that can be used to produce acceptable results.

The upward point load at the free end was 10,000 N (2250 lb). The displacement $\delta_t$ was plotted as a function of aspect ratio. For comparison purposes, plots of a non-dimensional displacement ratio $\tilde{\delta}_t$ were also made. $\tilde{\delta}_t$ is defined as the ratio of the tip displacement $\delta_t$ for a given model of the structure to that obtained using the Garvey 4-DOF panel. Plots of $\delta_t$ and $\tilde{\delta}_t$ for all of the panels include the stress-based Garvey 4-DOF solution and the displacement-based GIFTS solution. This provided a convenient means of comparison between the different panels tested.

The testing procedure excluded the patch test [8, 9, 19, 20] because the shear panel is not intended to be used as a finite element. A finite element is subjected to the patch test to determine whether in a given situation the solution towards which a finite element model converges with successive mesh refinement will be the correct one.

Consider the stiffened web structure in Fig. 5.1a. If the web is modeled as a shear panel so that it takes no bending load, then the shear force applied at the free end will produce a state of uniform pure shear in the web. That is the exact solution and the Curtis 4-DOF panel, for example, yields precisely that solution when used to model this structure. Suppose the same structure is modeled by the mesh of five arbitrary-shaped quadrilateral Curtis 4-DOF shear panels shown in Fig. 5.1b. If Curtis 4-DOF is to pass the patch test, then this mesh must also yield the constant shear flow solution.
That is, in each of the five quadrilaterals the state of stress must be pure shear on sections parallel to the sides of the rectangle. But that is impossible, because by design, each element of the mesh is required to have pure shear along its sides. So the shear panel definitely fails the patch test.

![Figure 5.1 The patch test. a) single panel, b) multiple panels.](image)

**The Parallelogram Shear Panel**

In this case, the top and bottom edges of the web were inclined downward at an angle $\alpha$ ranging from 0 degrees (a rectangle) to 50 degrees. The inclination of the top edge to the horizontal is referred to as the sweep of the panel. The free edge remained vertical and parallel with the simply-supported edge whose length was fixed at 1 meter for all of the tests. The panel was swept in 10 degree increments at each of four aspect ratios (1 through 4). Fig. 5.2 shows the general configuration of the parallelogram structure, and the results of the tests are plotted in Fig. 5.3 through Fig. 5.26.
Figure 5.2 Stiffened parallelogram shear panel.

The parallelogram displacements for the Curtis 4-DOF shear panel are plotted in Figs. 5.3 through 5.10 for stress polynomial degrees 2 (s2) through 6 (s6) and aspect ratios 1 through 4. The figures show that the displacements of the Curtis and Garvey 4-DOF panels are almost the same and about 70% greater than the GIFTS finite element solution.

The displacements of the Curtis 4-DOF panel using odd degree stress polynomials agree exactly with the Garvey 4-DOF solution \((\delta_t = 1)\), whereas those for the Curtis 4-DOF with even polynomials are slightly less than Garvey's. As the degree of the even polynomials increases from 2 (s2) to 6 (s6), the displacements
converge towards the Garvey solution. The tendency to converge with increasing complexity of the stress field is typical of stress-based elements [8]. It is also clear that for a given sweep angle, with increasing aspect ratio, the displacements of Curtis 4-DOF even polynomial panel converge towards the Garvey solution. It was concluded that the distortion of the parallelogram due to sweep and aspect ratio has little, if any, effect on the displacements computed from the Curtis 4-DOF model.

The discrepancy in the displacements given by the even and odd stress polynomials in the Curtis 4-DOF panel was thought to be caused by interelement incompatibility of the boundary stresses. The stress-based rod element was then modified to incorporate higher order stress functions. Perhaps matching the order of the stress polynomial in the panel to the order of the stress function in the rod element would eliminate the discrepancy. However, when this conjecture was tested, it was found that no matter what combination of rod and panel degree stress function was used, the oscillation of displacements from the lower to the higher degree polynomials remained. It was also noticed that using a higher degree stress function for the rod element decreased the displacements by approximately 50%, moving them away from the Garvey solution and towards the displacement-based GIFTS solution. However, since the Garvey model yields displacements computed by the PCVW using the exact stresses, it was logical to retain Garvey’s displacements as the “exact” values to which those of the other stress-based panels would be compared. From that point of view,
the investigation into the use of the higher-order rod elements led to the conclusion that the linear-stress rod element should be retained as the stiffener element.
Figure 5.3 Tip displacement of a parallelogram stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 1.

Figure 5.4 Tip displacement ratio of a parallelogram stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 1.
Figure 5.5 Tip displacement of a parallelogram stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 2.

Figure 5.6 Tip displacement ratio of a parallelogram stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 2.
Figure 5.7 Tip displacement of a parallelogram stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 3.

Figure 5.8 Tip displacement ratio of a parallelogram stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 3.
Figure 5.9 Tip displacement of a parallelogram stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 4.

Figure 5.10 Tip displacement ratio of a parallelogram stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 4.
The tip displacements obtained using the Curtis 12-DOF hybrid shear panel with stress polynomials of degree three (s3) through six (s6) are shown in Figs. 5.11 through 5.18. The reason for excluding the second degree stress polynomial (s2) from this panel is discussed in Chapter 4. As with the Curtis 4-DOF panel, all the displacements increase as the sweep angle increases. The magnitude of the Curtis 12-DOF displacements are all less than Garvey’s. As the sweep angle increases, these displacements split into two distinct groups. For the even stress polynomials, there is convergence towards the Garvey solution, while displacements from the odd polynomials follow a path similar to the Garvey solution but remain less than Garvey’s. This is just the opposite of the trends observed for the Curtis 4-DOF panel.

As the aspect ratio increases, the Curtis 12-DOF even-order polynomial solution continues to adhere to the Garvey 4-DOF solution for all sweep angles. The odd-degree polynomial displacements, however, tend towards the GIFTS solution for small sweep angles then increase toward the Garvey 4-DOF solution as the sweep angle increases. The difference between the displacements given by the odd polynomials increases with increasing aspect ratio, with the fifth degree (s5) yielding a more rapid convergence towards the Garvey 4-DOF solution than the third degree (s3) for increasing sweep angles. In any case, as the degree of the polynomial increases, the displacement comes close to Garvey’s.
The displacement ratio plots distinctly show the divergence of the even and odd-order stress polynomial displacements. The solution of the square panel for all degree polynomials produced displacements 15% less than Garvey's. This is probably due to the hybrid nature of the panel. As the sweep angle increases, the even-order polynomials converge to the exact solution and, as the aspect ratio increases, the even-order polynomials levels off to the exact solution for all angles of sweep. For the rectangle, the odd-order polynomial solutions give displacement ratios that decrease towards the GIFTS solution as the aspect ratio increases. As the aspect ratio and sweep angle increases, these odd-order polynomial solutions tend towards the Garvey 4-DOF solution.

It was hoped that the Curtis 12-DOF panel would perform better than it did so that the implementation of this shear panel into existing computer code would be easier. The three-node quadratic displacement rod element is found in most commercially available finite element structural analysis codes.
Figure 5.11 Tip displacement of a parallelogram stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 1.

Figure 5.12 Tip displacement ratio of a parallelogram stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 1.
Figure 5.13 Tip displacement of a parallelogram stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 2.

Figure 5.14 Tip displacement ratio of a parallelogram stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 2.
Figure 5.15 Tip displacement of a parallelogram stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 3.

Figure 5.16 Tip displacement ratio of a parallelogram stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 3.
Figure 5.17 Tip displacement of a parallelogram stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 4.

Figure 5.18 Tip displacement ratio of a parallelogram stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 4.
The Garvey 12-DOF and the Robinson shear panels were plotted together since neither panel has selectable stress functions like the polynomial-based Curtis panel. The displacements and displacement ratios can be seen in Figs. 5.19 through 5.26. The displacements of the Garvey 12-DOF hybrid element agree with those of the Garvey 4-DOF stress element for all angles of sweep and aspect ratios.

The Robinson panel displacements agree with Garvey's for the rectangular panel, but the magnitude falls away slightly as the sweep angle increases. This is due to the fact that, unlike Garvey, Robinson does not adjust the panel's scalar flexibility $H$ to account for the sweep (cf. Chapter 3). However, the divergence between the two solutions is small and gets even smaller as the aspect ratio increases.
Figure 5.19 Tip displacement of a parallelogram stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 1.

Figure 5.20 Tip displacement ratio of a parallelogram stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 1.
Figure 5.21 Tip displacement of a parallelogram stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 2.

Figure 5.22 Tip displacement ratio of a parallelogram stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 2.
Figure 5.23 Tip displacement of a parallelogram stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 2.

Figure 5.24 Tip displacement ratio of a parallelogram stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 3.
Figure 5.25 Tip displacement of a parallelogram stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 4.

Figure 5.26 Tip displacement ratio of a parallelogram stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 4.
The Right Trapezoidal Shear Panel

A "right trapezoid" is one with two adjacent included right angles. A stiffened web structure of that shape is illustrated in Fig. 5.27. Only the top edge of the structure is inclined, swept toward the bottom edge. The bottom edge remains perpendicular to the two vertical sides which keeps the local coordinate system aligned to the global coordinate system. The sweep angle starts at 0 degrees and increases in increments of 5 degrees until the panel is almost distorted into a triangle. The aspect ratio will vary from 1 to 3 only, due to the limited number of data points at an aspect ratio of 4, at which point the trapezoid becomes a triangle for a sweep angle of 14 degrees.

As can be seen in Figs. 5.28, 5.30 and 5.32, the Curtis 4-DOF trapezoid panel produced displacements equal to those of the Garvey 4-DOF panel as the sweep angle increased. However, as the shape of the trapezoid panel approached a triangle, the displacements diverged markedly from the Garvey 4-DOF solution. It appears that for a right trapezoid of aspect ratio of 1 the sweep angle for the Curtis 4-DOF panel should be limited to 35 degrees after which the displacement becomes very dependent on the degree of the stress polynomial. It can also be seen that as the sweep of the top edge increases, the Curtis panel becomes more rigid whereas the GIFTS finite element model becomes more flexible. Interestingly, the GIFTS and Garvey solutions intersect at roughly the sweep angle at which the Curtis panel becomes unstable with respect to the stress polynomial degree.
As the aspect ratio increased, the sweep of the top edge transformed the trapezoid into a triangle much faster, limiting the degrees of sweep possible per case. For an aspect ratio of 2, the trapezoid becomes a triangle at a sweep of about 26 degrees. As Figs. 5.30 and 5.31 show, the limiting sweep for acceptable displacement agreement with Garvey 4-DOF is 20 degrees for stress polynomials of degree three or more. The second degree stress polynomial was consistent up to 15 degrees. As the aspect ratio increased to 3, the number of data points was limited to 4. The displacements at this aspect ratio agreed with the Garvey displacements for all angles of sweep possible to test.

An unusual observation for aspect ratios of 2 and greater is the displacements for the Curtis 4-DOF and Garvey 4-DOF panel remained essentially constant as the sweep angle increases. This
could be the competing effects of the shear panel's decreasing area producing more flexibility and the increasing distortion producing more stiffness. The displacement ratio plots in Figs. 5.29, 5.31 and 5.33 provide another means of interpreting the observations from above. In summary, for modest sweep angle in a right trapezoid, the magnitude of the tip displacement given by the Curtis panel coincides with Garvey's and the Curtis solution is not influenced by the order of the stress polynomial used to formulate the 4-DOF panel.
Figure 5.28 Tip displacement of a right trapezoid stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 1.

Figure 5.29 Tip displacement ratio of a right trapezoid stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 1.
Figure 5.30 Tip displacement of a right trapezoid stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 2.

Figure 5.31 Tip displacement ratio of a right trapezoid stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 2.
Figure 5.32 Tip displacement of a right trapezoid stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 3.

Figure 5.33 Tip displacement ratio of a right trapezoid stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 3.
The displacements in Figs. 5.34 through 5.39 given by Curtis
12-DOF shear panel model of the right trapezoidal structure caused
concern as to whether the derivation of the panel is sound or an
error in the programming caused such disagreement with respect the
Garvey 4-DOF solution. For increasing sweep angles, the Garvey
displacements seem to act as the dividing line between the even
polynomial solutions' greater displacements and the odd polynomial
solutions' lower displacements. Neither the odd nor the even
polynomials agree with the Garvey 4-DOF solution. That in itself is
not a cause for alarm, since the Curtis 12-DOF panel is a hybrid.
What is alarming is the marked discrepancy between the odd and
even polynomial solutions. It is interesting that the odd polynomials
agree somewhat with the GIFTS solution, especially with increasing
aspect ratio, making those formulations of the Curtis 12-DOF panel
stiffer than those yielded by the even-order polynomials, which
seem to produce a slightly more flexible panel than Garvey's 4-DOF.
These trends are also made clear in the displacement ratio plots for
all aspect ratios tested. The displacements and displacement ratios
for aspect ratios between 2 and 25 degrees sweep show that the
panel is very unstable with respect to choice of stress polynomial
degree, due apparently to the nearly triangular shape of the
structure. The divergence among the displacement solutions at the
limiting sweep angle is much larger than it is for the Curtis 4-DOF
panel.
Figure 5.34 Tip displacement of a right trapezoid stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 1.

Figure 5.35 Tip displacement ratio of a right trapezoid stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 1.
Figure 5.36 Tip displacement of a right trapezoid stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 2.

Figure 5.37 Tip displacement ratio of a right trapezoid stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 2.
Figure 5.38 Tip displacement of a right trapezoid stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 3.

Figure 5.39 Tip displacement ratio of a right trapezoid stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 3.
As can be seen in Figs. 5.40 through 5.45, even with Robinson's assumption that the shape of the shear panel has no effect on the natural flexibility $H$ (cf. Chapter 3), his panel's displacements agree with the Garvey 4-DOF solution for all sweep angles and for all aspect ratios tested. The only exception found is for aspect ratio of 1, where the Robinson panel becomes only very slightly stiffer than the Garvey 4-DOF panel as the sweep angle increases. This is the same trend as for the Curtis 4-DOF trapezoid panel.

For all aspect ratios the Garvey 12-DOF panel stiffens dramatically with increasing sweep, and diverges markedly from the Garvey 4-DOF solution. This is entirely different behavior than for the parallelogram, where the two panels behaved identically for all sweep angles. It can be seen through a comparison of these plots that the quadratic displacement panels and rods can handle constant shear flow, as with the parallelogram, but not a quadratic shear flow, as with the trapezoid panel. The dividing of the edge shear flow into concentrated forces applied to three points along the panels and rod edge does not seem to work. The line-node concept is the only method that seems to accurately determine the displacement of both the parallelogram and trapezoid panel.
Figure 5.40  Tip displacement of a right trapezoid stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 1.

Figure 5.41  Tip displacement ratio of a right trapezoid stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 1.
Figure 5.42 Tip displacement of a right trapezoid stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 2.

Figure 5.43 Tip displacement ratio of a right trapezoid stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 2.
Figure 5.44 Tip displacement of a right trapezoid stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 3.

Figure 5.45 Tip displacement ratio of a right trapezoid stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 3.
The Symmetric Trapezoidal Shear Panel

A stiffened web whose top and bottom edges incline at the same angle and in opposite directions (Fig. 5.46) to produce a symmetric trapezoid was also tested. Due to the limited amount of symmetric sweep possible at large aspect ratios, only the case of aspect ratio of 1 is presented, for which the trapezoid becomes a triangle at approximately over 26 degrees of sweep.

A comparison between Figs 5.47. through 5.52 with those for the corresponding tests of the right trapezoid shows that the behavior of the different elements and trends of the displacements and displacement ratios are nearly identical.

![Figure 5.46. Stiffened symmetric trapezoidal shear panel.](image-url)
Figure 5.47 Tip displacement of a symmetric trapezoid stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 1.

Figure 5.48 Tip displacement ratio of a symmetric trapezoid stiffened web using the Curtis 4-DOF shear panel. Aspect ratio = 1.
Figure 5.49 Tip displacement of a symmetric trapezoid stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 1.

Figure 5.50 Tip displacement ratio of a symmetric trapezoid stiffened web using the Curtis 12-DOF shear panel. Aspect ratio = 1.
Figure 5.51 Tip displacement of a symmetric trapezoid stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 1.

Figure 5.52 Tip displacement ratio of a symmetric trapezoid stiffened web using the Robinson and Garvey shear panel. Aspect ratio = 1.
The Practical Applications of the Curtis 4-DOF Shear Panel

After analyzing the numerical test results, it was concluded that the Curtis 4-DOF panel using a third degree stress polynomial produced displacements comparable to the Garvey 4-DOF shear panel for the cases run. The fifth order polynomial produces identical results, so the increased computational time required to incorporate it is not justified. The Curtis 12-DOF shear panel is under investigation as to the abnormalities of its displacements as are the Curtis 4-DOF panels using even-order polynomials. The simplistic Robinson shear panel also showed excellent comparison to the Garvey 4-DOF panel throughout the full range of the tests. This panel will be used for the comparison of warped panels in future research.

To conclude this chapter, the Curtis 4-DOF panel will be applied to the solution of several examples of the types of practical problems it was intended to be used for.

Plane Tapered Beam

The first problem, adopted from Peery, is the simple tapered cantilever beam shown in Fig. 5.53. The structure consists of a single shear panel with stiffeners along each side. Using one Curtis 4-DOF panel and four linear-stress rods, the computed resultant shear flows matched Peery's results exactly at the free and the fixed ends. The shear flow along the inclined rods matched Peery's average shear flow over the span of the beam. Peery also provided shear flows at equally spaced intervals along
Figure 5.53 Single tapered stiffened web, modelled with a shear panel.

In order to compare these with the computer solution, one can either model the beam using multiple panels, as shown in Fig 5.54, or use the computer to obtain the shear flow at a given spanwise section of the single panel by integrating the computed shear stresses across the section.

The easiest approach to this problem, requiring no additional coding, is to remodel the structure using multiple panels (Fig. 5.54). The fact that these panels must be surrounded by rod elements requires one to decide what cross sectional area to use for the intermediate vertical rods. It seemed logical to use the panel's thickness squared, which was done, and the shear flows at the intermediate stations matched Peery's results exactly.
Figure 5.54 A single, tapered stiffened web modeled with several shear panels.

It must be remembered that statics alone dictates that the shear flows along the upper and lower rods are inverse quadratic functions as shown in Eq. 3.4, from which an average shear flow can be calculated using Eqs. 3.5 or 3.6. It is the rods' average shear flows that are calculated in the computer analysis. As the tapered beam is sliced into separate tapered substructures, the inverse quadratic shear flow over each substructure will be represented in the computer model by a localized average shear flow. The sum of the resultant shear forces over each section, divided by the total span yields the average shear flow over the entire length of the tapered beam. By breaking the web down into several separate shear panels, the structure is altered and the deflections of the structure in Fig. 5.54 should not be expected to be the same as that in Fig. 5.53. In fact, the tip displacement of the the multi-panel beam was 33% greater.
The second approach to obtaining intermediate internal loads along the span was to augment the computer code to output the resultant forces and moments within the single panel (Fig. 5.53) at selected intervals along the span of the beam. The normal stresses and tangential stresses along a cut were evaluated and numerically integrated using Gauss quadrature to determine the resultant normal force, bending moment and shear force. The normal traction integrated to zero. The normal traction varied nearly linearly and symmetrically across a section (as in beams with webs active in bending), was zero at the mid-height, and was maximum at the top and bottom of the cut, as illustrated in Fig. 5.55a. This showed that the resultant moment would not integrate to zero. (The moment resultant is required to be zero only at the boundary of the panel.) The resultant shear force was slightly less than what Peery determined and showed a slightly parabolic shape across the cut section shown in Fig. 5.55b.

Figure 5.55. Internal stress distribution diagrams. (a) normal stress (b) shear stress.
The flange forces in the rods were determined from the computer output of the uniform shear flow calculated for the linear-stress rods, and they proved to be smaller than the flange loads Peery calculated. The reason is that Peery assumed that there are no normal stresses on the web at any section and therefore the flanges absorbed all normal loads caused by bending. The internal stress field in the Curtis shear panel, however, can absorb a portion of the bending loads, thereby relieving the flanges of some of that responsibility, which will reduce the flange loads, which in return will throw more shear load into the panel.

Modeling the tapered beam using multiple panels (Fig. 5.54) imposes the condition of zero moment resultant on the sections between adjacent panels and the structure becomes more flexible. Hence, the more panels used the more flexible the structure becomes and the better the comparison becomes between the shear flow along the inclined rods and Peery's shear flows.

Tapered Box Beam

The next problem solved on the computer using the Curtis shear panel was a single cell tapered box beam like the one shown in Fig. 5.56. The hand calculation approach described by Peery makes use of the beam flexure formula, which is founded upon the well-known assumption "plane sections remain plane."
The beam was modeled using the linear stress rod elements to represent the flanges and the shear panel for the outer skins. The ends of the box beam where the point load was applied was capped with an additional shear web to provide the structure with torsional stiffness. This end cap, or "rib," provided the means of transmitting the torsional moment of the applied load to all of the spanwise panels. Rigid body displacements were suppressed at the fixed end. In texts such as Peery's the actual means by which the external loads are applied to a beam section are rarely specified; the loads are just "there." To model this loading for a computer structural analysis code requires coming up with a statically equivalent loading system that can be applied to the nodes at the end.
To determine the flange loads and shear flows at an intermediate spanwise location requires a transverse cut through the structure, which, for the computer analysis, required additional point nodes to be introduced, as shown at the mid-span in Fig. 5.56. The continuous flanges became two rod elements instead of one. The four webs each became two shear panels, and four rods were added around the periphery of the cut since shear panels must be surrounded by rods. The areas specified for these “fictitious” rods are arbitrary since the rod is placed in the structure solely to provide for a properly conditioned overall structural stiffness matrix. The rod area chosen was square of the panel’s thickness.

The computer analysis of the structure in Fig. 5.56 gave flange loads that were on an average 92% of those obtained using beam theory. The shear flows along the edges of the panels at the center section were 98% of the hand-calculated values for this section.

This model of the box beam consisted of 12 nodes, 20 rod elements and 10 shear panels. Modifications of the model were tested to determine if the hand-calculated solution could be approached more closely, and to determine the simplest computer model which yielded an acceptable comparison.

Two additional ribs are added to the middle of the fore and aft bays, increasing the number of nodes from 12 to 20, the number of rods from 20 to 28, and the number of shear panels from 10 to 21. The effort resulted in an average flange load increase of only 2%, while the shear flows remained the same. This shows that to provide sufficient torsional rigidity, only the end rib was needed.
The additional ribs complicated the modeling procedure and increased the time required to analyze the structure with no appreciable payback in increased agreement with the hand calculation approach.

Returning to the model in fig. 5.56, an additional rib was added at the midspan, leaving the model unchanged, except for the additional shear panel representing the rib web. In so doing, the average flange loads at this section increased to 96% of Peery's solution. With the rib present, the shear flows at this section needed to be interpreted as follows. If the shear flow ahead of the cut (towards the free end) is considered the forward shear flow $q_f$, then the shear flow behind the internal rib, the aft shear flow $q_a$, was less than the forward shear flow. The difference between these shear flows was carried by the added rib. The rib shear flow, labeled $q_r$, was constant around the rectangular rib. If this rib shear flow is considered as required to help enforce the condition that plane sections remain plane and the average of the forward and aft shear flows is considered "the" shear flow at that section, then the average shear flow came to 96% of Peery's results.

Adding additional ribs as before at the middle of the fore and aft bays did not affect the values of the average flange loads, but the shear flow in the rib at midspan decreased. This was due to the fact that the additional ribs aided in enforcing the condition of plane sections remaining plane. The average shear flow at the section of interest (mid-span) using the average of the forward and aft shear flow increased 3%. Again, the time required to remodel this
structure and reanalyze was not merited by the marginally closer agreement with hand calculations. It is therefore recommended that, in academic settings, a structure like that in Fig. 5.56 be modeled as it would for hand computation, with the simple addition of ribs at each end and the four nodes and four rods at the spanwise location where shear flows are to be compared. The computer-produced shear flows should be within 10% of the hand computation. *Statically Indeterminate Wing-Like Structure*

The final example problem is that of the swept, triply-tapered, four-bay, two-cell wing-like box beam shown in Fig. 5.57. This problem is statically indeterminate and would require a lengthy hand calculation. The purpose of the example is not to illustrate the procedure for properly simulating air loads on a wing but to demonstrate the ability of derived shear panel element to provide a visualization of load paths through a complex structure. Therefore a single point load was applied at the free end.

The cantilevered structure consists of 30 nodes, 59 rods and 30 panels. The first attempt at analyzing the model revealed that the stiffness matrix had a very large bandwidth [8]. Bandwidth is a measure of the amount of time required to solve for the displacements \( \{q\} \) in the standard stiffness equation \( [K]\{q\} = \{Q\} \). This time increases as the square of the bandwidth [8]. It is therefore advantageous to keep the bandwidth as small as possible and practical. The size of the bandwidth is determined by the global node numbering scheme. Using structural elements such as the linear-stress rod (cf. Chapter 4) having a line node as well as point nodes posed problems in keeping the bandwidth narrow. The
computer algorithm used during this research was to automatically assign the line node numbers and d.o.f.'s after the point nodes had been numbered in an arbitrary fashion by the user. This made it easy to prepare input data, but it did produce large bandwidths in large structures such as the one under consideration.

![Figure 5.57 Wing structure tested modeled with rods and shear panels.](image)

To reduce the bandwidth, the nodal d.o.f. were rearranged using an topological optimization scheme called the Cuthill-McKee method [20], which was incorporated into the computer program. The renumbered nodes will be referred to as system nodes (as opposed to "user nodes"). The original user-defined node numbers were not changed.
The procedure begins by defining a starting node as user-defined node number 1. This node is labeled internally as system node 1 and assigned the appropriate global d.o.f. The list of rod elements is then scanned to determine which rods are connected to this starting node. The line node of the first bar found is labeled system line node number 1 and it is assigned the next available d.o.f. The ending node of this rod is assigned system node number 2 and the next available d.o.f.'s. The search of the list of rod elements is continued to find the next rod connected to system node 1 and its line nodes and ending nodes are assigned their appropriate system node numbers and the next available d.o.f.'s. Each time a rod is assigned its complete set of d.o.f.'s, that rod is checked off and not used again. Once all rods have been searched, system node 2 becomes the new starting node, and above procedure is repeated. All rods connected to this node are accounted for and assigned d.o.f.'s as described above. Subsequent system node are treated in a similar fashion until all nodes and rods are renumbered and assigned appropriate d.o.f.'s. As this procedure marches through the rods, the system-user node list must be checked to ensure that a system node is not assigned d.o.f. twice. The d.o.f. reordering scheme reduces the bandwidth of the stiffness matrix. The stiffness matrix assembly subroutine must then be coded to "look up" each element's global d.o.f. from the cataloged system nodes to properly assemble its stiffness matrix into the structure's matrix. A listing of the computer code in BASIC for this optimization scheme appears in Appendix B.
The structure in Fig. 5.57 was reanalyzed with the bandwidth optimization subroutine in place, and the bandwidth was reduced considerably. Figs. 5.58 through 5.67 show the computed shear flows throughout the structure. The flange loads can be determined from these shear flows using statics.
Figure 5.58 Top of wing with shear flows.

Figure 5.59 Bottom of wing with shear flows.
Figure 5.60  Front spar of wing with shear flows.

Figure 5.61  Middle spar of wing with shear flows.

Figure 5.62  Rear spar of wing with shear flows.
Figure 5.63 Root rib of wing with shear flows.

Figure 5.64 One-quarter semispan rib of wing with shear flows.

Figure 5.65 One-half semispan rib of wing with shear flows.
Figure 5.66 Three-quarter semispan rib of wing with shear flows.

Figure 5.67 Tip rib of wing with shear flows.
CHAPTER 6

CONCLUSION

Several unique approaches to formulating the properties of the shear panel provide a basis for comparing the performance of the Curtis shear panel, which is an outgrowth of Nack's panel. The assumptions and simplifications that have been made in the past to produce workable shear panels for structural analysis codes were reviewed and, in the numerical tests, assessed in terms of each panels' deflection characteristics. It was hoped that the Curtis panel would do at least as well as any of the panels in computing displacements because, unlike Garvey's and Robinson's, it is based upon rational assumptions and satisfies all of the equations of elasticity. All of the pure stress-based elements (the Robinson, the 4-DOF Garvey and the 4-DOF Curtis) will yield the same shear flow results when used to model a structure. Slight differences may be found among the results presented by the hybrid versions of the Garvey and Curtis panels because of the coupling of the very different stress fields into the load vector through the boundary displacement assumption.
Summary of Test Results

The numerical tests revealed that the Curtis 4-DOF panel yields displacements that are, for all practical purposes, identical with the other 4-DOF equilibrium panels (Garvey and Robinson). In the Curtis panel, the influence of the degree of the stress polynomial on the displacements becomes significant only for extremely distorted panels. The lowest degree stress polynomial yielding displacements consistent with the higher degree polynomials throughout the widest range of panel distortion in all of the tests is the third degree polynomial. Therefore, the optimum stress polynomial was the third order polynomial.

The tests revealed a curious and as-yet unexplained dependence of the panel's displacements on whether the stress polynomials were odd or even. For example, for the right trapezoidal panel of aspect ratio 1 (Figs. 5.28 and 5.29), significant displacement divergence with sweep occurs with the fifth degree polynomial panel. At aspect ratio 2 (Figs. 5.30 and 5.31), it first occurs with the second degree polynomial. On the other hand, for the symmetric trapezoid and aspect ratio 1 (Figs. 5.47 and 5.48), the displacement divergence with increased distortion occurred first for the sixth degree polynomial panel.

This "odd versus even" phenomenon shows up much more dramatically in all of the test results for the Curtis 12-DOF hybrid panel. The computed displacements using the even polynomial stress functions are consistently and markedly greater than for the odd polynomials. It was hoped that this panel could be used in structural
analyses where accurate displacements are required as well as shear flows. However, the spurious behavior seen in the test data leads to the conclusion the 12-DOF panel cannot be put into practical use as it stands. There is a question of whether it is the stress field within the panel or in the rods attached to the panel that causes the unacceptable behavior.

This difference in displacement response between a pure stress element and its associated hybrid element is revealed as well in the tests of the Garvey panel. For a parallelogram, in which the shear flow is uniform on the boundary, the variation of tip displacement with sweep angle can be seen from the plots to be identical for the Garvey 4-DOF and 12-DOF panels. For the trapezoids tested, the displacements of the 4-DOF and 12-DOF panels diverge dramatically, with those of the hybrid panel becoming much larger than the 4-DOF panel with increased distortion. It may be that the more complex inverse quadratic shear flow variation along the edge of the trapezoid interacting with the linearly varying stress within the quadratic displacement rod element causes this disagreement between the displacements.

The numerical testing validated Robinson's assumption that the natural flexibility of the shear panel depends only on its area and not on its shape. The displacements of the Robinson 4-DOF panel are in excellent, consistent agreement with the 4-DOF Garvey panel, for which Garvey made a supreme effort to account for the shape. The fact that the simple Robinson panel performed as well as the Garvey and Curtis 4-DOF panels in the tests supports the contention
in the MSC/NASTRAN theoretical manual that the labor required to place the shear panel on firm theoretical ground is probably not justified.

Warped Panels

The computer code generated for this project needs to be extended to include warped panels. The corners of a quadrilateral panel may not lie in precisely the same plane. Moderate warping can be detected when the normal vectors of two adjacent edges at a corner do not agree with the other corners of the panel. Robinson recognized this and derived a stiffness matrix for warped quadrilaterals (see Chapter 3). Given the stable behavior of his flat panel in the tests conducted herein, his warped quadrilaterals may serve as benchmarks against which to test the results of extending warping capability to the Curtis 4-DOF panel. Another method that produces an equivalent flat panel from the warped panel was presented in a NASA document [21]. Although derived in the context of predicting aerodynamic loads, the method can be applied to producing an equivalent warped shear panel. It is achieved by projecting the warped panel onto a flat surface and determining the stiffness matrix from this projected panel. The MCS/NASTRAN approach to warped panels [19] is very similar to Robinson's.
Stability of Shear Panels

The elastic stability of thin shear panels in structures such as the ones discussed in Chapter 5 above is prevalent. However, the problem should not be ignored in the analysis of thin-walled, aircraft-type structures. The prediction of the critical shear flow for a flat rectangular panel can be found from commonly available graphs and formulas [1,22]. The mathematical method of calculating the buckling load of flat plates is well-known [23], but requires simple geometries. For non-rectangular quadrilateral shapes for which formulas cannot be found, it may be necessary to appeal to numerical methods and use a finite difference or finite element model together with an eigenvalue extraction routine to find the buckling loads.

Usefulness of Panel

The Curtis 4-DOF panel (or the Garvey or Robinson 4-DOF panels) can be added to a standard displacement-based finite element computer code and used to analyze the load paths in models of three-dimensional semimonocoque structures, without having to deal with massive number of grided elements. It should prove useful to students in their aircraft structures courses, who, after mastering the hand-computation methods, can use the computer to solve realistically-complex structures for which hand calculations would be impractical. Students' being able to explain the reasons for discrepancies between the hand and computer calculations would
deepen their understanding of structural theory. Computer implementation of the shear panel may also be useful in the aircraft design courses, making the recalculation effort associated with design iterations less painful. Aircraft structure instructors will find the code useful in generating example problems or homework sets, checking student solutions to design problems, etc. The Curtis 4-DOF panel is recommended, because its mathematical formulation rests on solid ground and can be presented to students without recourse to any ad hoc assumptions.

This is not to say that the finite element analysis is no longer necessary. Once the critical stress areas have been found using a shear panel model, one can zero-in on regions of high stress and use finite element grids to do localized stress analysis. The loads applied to the grid are those computed from the shear panel model.

**Future Work**

The Curtis 4-DOF panel with its equilibrium stress field, can be used to determine the stress at a point within the panel. It might be helpful to provide the analyst with the stresses at specified points. These can be easily computed from the stress parameters. Stress contours and shear flow vectors might also be of interest, but a graphics package must be integrated or separately coded.

Warped panels should be made available. The stability of the panel, or an approximate critical shear load, should be output to the analyst.
The determination of the disagreement of the test results for the Curtis panel—especially the hybrid panel—must also be addressed and answered, at least from the theoretical perspective.

More general variational principles than PVW or PCVW (e.g., Hellinger-Reisner [11]) might be brought to bear upon the derivation of the shear panel.

The feasibility of ascribing composite material properties to the panel should be investigated.
Another method in determining Garvey's $[F]$ matrix is to determine of the resultant force along each edge. This can be found by integrating the local shear flow along an edge in terms of the perpendicular distance from the baseline PQ (as seen in Fig A.1).

![Diagram of a typical edge of panel with labeled points and distances.]

Figure A.1 Typical edge of panel.

The localized shear flow is described as inversely proportional to the square of the distance from the baseline PQ. This is mathematically expressed in Eq. 3.11. This can then be integrated for an arbitrary line with respect to the baseline PQ as
so that upon evaluating and simplifying, it becomes

\[ F_s = \frac{C_{S_{ij}}}{(d \sin \theta)(d + L_i \sin \theta)} \frac{s}{p} \frac{C_{S_{ij}}}{p} \quad (A.1b) \]

This general solution can then be applied to the four edges of the panel. The sign convention used is the same as Garvey's, i.e., the shear force is positive from the starting node to the ending node. Therefore, the resultant shear forces along the edges of the panel can be expressed in terms of shear flow along edge AB (side 1) using the relationships from Eq. 3.10 as

\[ F_1 = - \frac{s_{1} p_{A} p_{B} L_{AB}}{p_{A} p_{B}} = - s_{1} L_{AB} \]
\[ F_2 = \frac{s_{2} p_{B} p_{C} L_{BC}}{p_{B} p_{C}} = \frac{s_{1} p_{A} p_{B} L_{BC}}{p_{B} p_{C}} = s_{1} \frac{p_{A} L_{BC}}{p_{C}} \quad (A.2) \]
\[ F_3 = - \frac{s_{3} p_{C} p_{D} L_{CD}}{p_{C} p_{D}} = - \frac{s_{1} p_{A} p_{B} L_{CD}}{p_{C} p_{D}} = - s_{1} \frac{p_{A} p_{B} L_{CD}}{p_{C} p_{D}} \]
\[ F_4 = \frac{s_{4} p_{D} p_{A} L_{DA}}{p_{D} p_{A}} = \frac{s_{1} p_{A} p_{B} L_{DA}}{p_{D} p_{A}} = s_{1} \frac{p_{B} L_{DA}}{p_{D}} \]

The matrix \([F]\) can then be formed and is the same as in Eq. 3.33.
APPENDIX B

Computer listing of Cuthill-McKee Optimization Method

optim:
'**********************************************************************
'Optimize the node numbering using the bars
'based on the Cuthill-McKee method.
'Written in Microsoft BASIC for the MacIntosh.
'**********************************************************************
'$INCLUDE "Disc#0:ShearPanel:IncludeFiles:COMMON"
PRINT
i$ = "?"
WHILE i$<"Y" AND i$<"N"
   INPUT"Optimize structure";i$
   i$ = UCASE$(LEFT$(i$,1))
WEND
IF i$ = "N" THEN CHAIN main$
nnodes = a(1)
nbars = a(2)
ndim = a(10)
npanels = a(16)
nlines = a(30)
DIM xyztemp(nnodes,3)
DIM nubars(nbars)
DIM nsnodes(nnodes,4)
DIM nslines(nlines,2)

FOR nb = 1 TO nbars
    nubars(nb) = 0
NEXT nb
nsn = 1
nsnodes(nsn,1) = 1
nsnodes(nsn,2) = 1
nsnodes(nsn,3) = 2
ngdof = 2
IF ndim = 3 THEN
    nsnodes(nsn,4) = 3
    ngdof = 3
END IF

nsl = 0
FOR nn = 1 TO nnodes
    nun = nsnodes(nn,1)
    FOR nbn = 1 TO 2
        nbi = 1
        nbj = 2
        IF nbn = 2 THEN
            nbi = 2
            nbj = 1
        END IF
        FOR nb = 1 TO nbars
            IF bars%(nb,nbi) = nun AND nubars(nb) = 0 THEN
                nsl = nsl + 1
                nslines(nsl,1) = ABS(bars%(nb,5))
                nslines(nsl,2) = ngdof + 1
                ngdof = ngdof + 1
                flag% = 0
                FOR ns = 1 TO nsn
                    IF bars%(nb,nbj) = nsnodes(ns,1) THEN flag% = 1
                NEXT ns
                IF flag% = 0 THEN
                    nsn = nsn + 1
                    nsnodes(nsn,1) = bars%(nb,nbj)
                    nsnodes(nsn,2) = ngdof + 1
                    nsnodes(nsn,3) = ngdof + 2
                END IF
            END IF
        NEXT nb
    NEXT nbn
END FOR

END
ngdof = ngdof + 2
END IF
IF ndim = 3 THEN
  nsnodes(nsn,4) = ngdof + 1
  ngdof = ngdof + 1
END IF
END IF
nubars(nb) = 1
END IF
NEXT nb
NEXT nbn
NEXT nn

i$ = "?"
WHILE i$<>"Y" AND i$<>"N"
  INPUT"Change user defined nodes";i$
  i$ = UCASE$(LEFT$(i$,1))
WEND
IF i$ = "N" THEN CHAIN main$

FOR ns = 1 TO nnodes
  FOR i = 1 TO 3
    xyztemp(ns,i) = xyz(ns,i)
  NEXT i
  NEXT ns

FOR ns = 1 TO nnodes
  nun = nsnodes(ns,1)
  FOR i = 1 TO 3
    xyz(ns,i) = xyztemp(nun,i)
  NEXT i
  NEXT ns

FOR nb = 1 TO nbars
  FOR nbn = 1 TO 2
    flag% = 0
    FOR ns = 1 TO nsn
IF bars%(nb,nbn) = nsnodes(ns,1) AND flag% = 0 THEN
  bars%(nb,nbn) = ns
  IF nbn = 2 AND bars%(nb,1) > bars%(nb,2) THEN
    temp = bars%(nb,1)
    bars%(nb,1) = bars%(nb,2)
    bars%(nb,2) = temp
  END IF
  flag% = 1
END IF
NEXT ns
NEXT nbn
flag% = 0
FOR nl = 1 TO nsl
  IF bars%(nb,5) = nslines(nl,1) AND flag% = 0 THEN
    bars%(nb,5) = nl
    lnode(nl,1) = bars%(nb,1)
    lnode(nl,2) = bars%(nb,2)
    flag% = 1
  END IF
NEXT nl
NEXT nb
FOR np = 1 TO npanels
  FOR npn = 1 TO 4
    flag% = 0
    FOR ns = 1 TO nsn
      IF panels%(np,npn) = nsnodes(ns,1) AND flag% = 0 THEN
        panels%(np,npn) = ns
        flag% = 1
      END IF
    NEXT ns
  NEXT npn
  FOR npl = 7 TO 10
    flag% = 0
    FOR nl = 1 TO nsl
      IF bars%(nb,nbn) = nsnodes(ns,1) AND flag% = 0 THEN
        bars%(nb,nbn) = ns
      END IF
    NEXT nl
  NEXT npl
NEXT np
FOR np = 1 TO npanels
  FOR npn = 1 TO 4
    flag% = 0
    FOR ns = 1 TO nsn
      IF panels%(np,npn) = nsnodes(ns,1) AND flag% = 0 THEN
        panels%(np,npn) = ns
        flag% = 1
      END IF
    NEXT ns
  NEXT npn
  FOR npl = 7 TO 10
    flag% = 0
    FOR nl = 1 TO nsl
      IF bars%(nb,nbn) = nsnodes(ns,1) AND flag% = 0 THEN
        bars%(nb,nbn) = ns
      END IF
    NEXT nl
  NEXT npl
NEXT np
FOR np = 1 TO npanels
  FOR npn = 1 TO 4
    flag% = 0
    FOR ns = 1 TO nsn
      IF panels%(np,npn) = nsnodes(ns,1) AND flag% = 0 THEN
        panels%(np,npn) = ns
        flag% = 1
      END IF
    NEXT ns
  NEXT npn
  FOR npl = 7 TO 10
    flag% = 0
    FOR nl = 1 TO nsl
      IF bars%(nb,nbn) = nsnodes(ns,1) AND flag% = 0 THEN
        bars%(nb,nbn) = ns
      END IF
    NEXT nl
  NEXT npl
NEXT np
IF ABS(panels%(np,npl)) = nslines(nl,1) AND flag% = 0 THEN
  ip = npl - 6
  jp = npl - 5
  IF jp > 4 THEN jp = 1
  panels%(np,npl) = nl
  IF panels%(np,ip) > panels%(np,jp) THEN
  panels%(np,npl) = -nl
  flag% = 1
  END IF
  NEXT nl
  NEXT npl
  NEXT np
CHAIN main$
REFERENCES


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