An Investigation of Classical Panel Stiffener Buckling Methods for Modern Airframe Applications

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AN INVESTIGATION OF CLASSICAL PANEL STIFFENER BUCKLING
METHODS FOR MODERN AIRFRAME APPLICATIONS

by

Ryan Timothy Holt

A Thesis Submitted to the
Department of Aerospace Engineering
in Partial Fulfillment of the Requirements for the Degree of
Master of Science in Aerospace Engineering

Embry-Riddle Aeronautical University
Daytona Beach, Florida
May 2008
UMI Number: EP32022

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Ryan Timothy Holt

This thesis was prepared under the direction of the candidate’s thesis committee chairman, Dr. James Ladesic, Department of Aerospace Engineering, and has been approved by the members of the thesis committee. It was submitted to the Department of Aerospace Engineering and was accepted in partial fulfillment of the requirements for the degree of Master of Science in Aerospace Engineering.

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ACKNOWLEDGEMENTS

I would like to extend a special thank you to Dr. James Ladesic for providing me with the opportunity to work on a great project, and for serving as my thesis advisor. I would also like to thank him for imparting me with knowledge, how to think about difficult problems, and how to look for trends. Thank you to Dr. Frank Radosta and Dr. Frederique Drullion for serving on my committee and taking the time to support the research with their expertise. Last but not least I would like to thank my family and friends for their support and motivation over the past years.
ABSTRACT

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Title: An Investigation of Classical Panel Stiffener Buckling Methods for Modern Airframe Applications
Institution: Embry-Riddle Aeronautical University
Degree: Master of Science in Aerospace Engineering
Year: 2008

Classical methods for buckling assessment of aircraft panels reinforced by bulb-stiffened flanges differ regarding symmetric versus asymmetric cross-sections. The present research addresses a number of classical derivations of methods with a focus on the work of Dwight Windenburg as published in “The Elastic Stability of Tee Stiffeners” and the expansion of his work to asymmetric sections by E. F. Bruhn in Analysis and Design of Aerospace Vehicle Structures. Vagueness in the relevance of geometric symmetry of the bulb exists between Windenburg’s plate theory approach, and the accepted industry standard applications defined in the methods of Bruhn. The results presented trace the bibliographic history of sizing bulb-stiffeners to achieve the highest critical stress obtainable by the web, and verify the two sizing procedures theoretically and using Finite Element Analysis software. The results suggest that the theoretical approach presented by Windenburg is correct as stated; however the FEA results suggest that the claim made by Bruhn’s and Windenburg’s sizing process is inadequate.
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List of Abbreviations

a = Length of the plate along the x direction

b = Height of the plate along the y direction

h = Thickness of the plate

f = Width of supporting flange

t = Thickness of the flange

\[ D = \frac{E h^3}{12 (1 - \nu^2)} \quad \text{Plate stiffness} \]

w = Deflection of plate in z direction

E = Young’s modulus

\( \sigma_s = \) Compressive stress

\( \nu = \) Poisson’s Ratio

A = Integration constant

B = Integration constant

\( A_f = \) Area of the flange

\[ \alpha = k \sqrt{\mu + 1} \]

\[ \beta = k \sqrt{\mu - 1} \]

\[ k = \frac{m \pi}{a} \]

\[ \mu = \sqrt{\frac{\sigma_s h}{D k^2}} \]

\( m = \frac{a}{b} \quad \text{Number of sinusoidal half waves the plate buckles into} \)
\[ \Psi = b \sqrt{\frac{\sigma_h h}{D}} \text{ Stress Factor} \]

\[ \Phi = m\pi \frac{b}{a} \text{ Aspect Factor} \]

\[ \theta = \frac{EI}{bD} - \frac{A\psi^2}{bh\phi^3} \text{ Flexural Rigidity Factor} \]

C or \( C_s \) = \( GJ \) Torsional rigidity of the flange

G = Shear modulus

J = Torsion constant

\( \sigma_{cr} \) = Bryan’s Critical Stress

\( \sigma_r \) = Critical twisting stress

\( I_p \) = Polar moment of inertia about the web attachment point to the skin

\( a_{eff} \) = Effective length of the stiffener

\( C_{BT} \) = Torsion bending constant

\[ M_y = -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \text{ Bending moment of plate parallel to the x direction} \]

dl = Diameter of bulb

SSSS = Plate with 4 sides simply supported

SSSF = Plate with 3 sides simply supported and the fourth free
Background

Aircraft manufacturers continue to examine methods of reducing part count as a means for reducing weight, failure points, and manufacturing costs – all part of what is now known as Lean Engineering. Skin-bonded longitudinal metal stiffening members with bulb-flanges can be used instead of other harder-to-produce and assemble sections like Hat- or Tee-stiffeners which have served the industry favorably for decades. Figure 1 illustrates the general cross-section of a bulb-flange as opposed to a regular Tee and I cross-section. Asymmetric bulb-flanges are preferred for the following reasons.

1. Additional inertia provided to the up-standing flange by the bulb
2. Absence of sharp corners
3. One-sided flat surface provides manufacturing opportunities for routing wiring, fastening adjoining structures, frame clips, supporting systems and interior components.

A disadvantage of these asymmetric sections occurs when the panel is loaded in compression. Large compression stresses in the stringer-panel section can cause local buckling and torsional instabilities of the stiffening flange, which in turn can lead to structural failure. As with all structural components of an aircraft, reinforcing stringers...
must be properly sized to control weight without compromising safety. Classical methods routinely applied to flange sizing computations are the focus of this research.

Bulb-stringers are not new to industry application, however, the opportunities afforded by new alloys and bonding technologies have sparked renewed interest in the analysis methods previously derived by Windenburg\(^1\) The fundamental question has remained: how large an area should the bulb have in order to provide both flexural and torsional rigidity comparable to that of a simple support for the up-standing flange?

Approaches derived from plate theory developed by Windenburg were adapted by E.F. Bruhn to form what has become the traditionally accepted method for determining the appropriate bulb size to support the up-standing flange. An initial literature review revealed the Windenburg’s derivations formed the basis for Bruhn’s published and frequently referenced work. However, the study undertaken has noted some disagreement between the results presented by Windenburg and those subsequently interpreted by Bruhn. This discrepancy between these classical methods for sizing a bulb to assure the buckling capacity of the up-standing flange is the motivation of the current research effort.
Problem Statement

The sizing and buckling analysis of bulb-stringers is sometimes considered complicated and confusing based solely on the cited documents. Thus, one goal is to verify and present, in a clear form, the classical method presented by Windenburg for symmetric flanges and explain its expansion by Bruhn for application to asymmetric bulbs.

The aforementioned methods are used to establish initial bulb size. In addition several sized cases are assessed for their performance under loading using Finite Element Method (FEM) software. FEM provides a means for comparison of webs that are simply supported on all four edges to that of webs with three simply supported edges and an elastic support created by an attached flange or bulb. After the primary sizing is complete, the torsional rigidity of the flange is included to calculate the buckling load increase due to the added rigidity. The added stiffness due to the torsional rigidity of the flange is considered because as a bulb becomes too large the primary failure mode becomes torsional instability.

There are many opportunities to clarify the process for determining the torsional properties of a bulb-stringer. Most of the documented research considers thin-walled open sections when determining as a result, many of these coefficients are neither readily available nor easily derived for asymmetric bulbs. The work presented is intended to help clarify the current analysis of symmetric bulbs and to elaborate to include asymmetric sections. The results from this research should provide a starting point for further research in the torsional stability of bulb stringers, and allow for improved
understanding of the methods used to properly size a bulb in order to produce lighter airframe components.

Methods

This work is divided into several distinct phases including: verification of the historic documents through analytical reproduction of the results, clarification of the procedures for determining the size of a stiffening flange, and a finite element analysis to determine the validity of the sizing technique.

Symmetric Flanges

Dwight Windenburg's 1939 paper developed a technique for determining the appropriate size flange to obtain the full buckling strength of the web assuming it was simply supported on four sides.

Figure 2 displays a comparison between the web supported by the flange and supported by four simple supports.
The objective of Windenburg’s procedure was to use the definition of the simple support, infinite-flexural-rigidity, along the free standing edge so that the flange can be sized to approximate this support condition. His procedure began by examining the web as a plate with boundary conditions of three simple supports along edges $x = 0$, $x = a$, and $y = 0$ and an elastic support at $y = b$. These conditions are shown in Figure 3.

Following Windenburg’s methods, the governing equation for plate deflection is given by Timoshenko\(^3\)

\[
\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{\sigma_z h}{D} \frac{\partial^2 w}{\partial x^2} \tag{1}
\]

where: $w = Z$-displacement

$h = $ Thickness of the plate

\[D = \frac{Eh^3}{12(1-v^2)} \text{ Plate stiffness}\]
\[ \sigma_x = \text{Compressive stress} \]

When applying the boundary conditions for edges \( x = 0, x = a, \) and \( y = 0 \) the general solution is obtained:

\[ w = (A \sinh \alpha y + B \sin \beta y) \sin kx \quad (2) \]

where: 
\( A = \text{Integration constant} \)
\( B = \text{Integration constant} \)
\[ \alpha = k\sqrt{\mu + 1} \]
\[ \beta = k\sqrt{\mu - 1} \]
\[ k = \frac{m\pi}{a} \]

The full derivation of Windenburg’s solution is included in Appendix A. The following discussion is used to clarify some ambiguities in Windenburg’s research, and to define a straightforward procedure for properly sizing a flange.

An irregularity was found in equation-5 of Windenburg’s paper. This equation defines one of the boundary conditions for the elastically supported edge. Windenburg references Timoshenko to obtain the proper boundary condition. The boundary condition equates the bending moment per-unit-length of the web to the twisting moment of the flange. The following equation is a direct replication of the boundary condition found in Timoshenko on page 365.

\[ -D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) = C \frac{\partial^3 w}{\partial x \partial y^2} \quad (3) \]

where:
\( D = \frac{Eh^3}{12(1-\nu^2)} \) Plate stiffness
\[ v = \text{Poisson's Ratio} \]
\[ C = GJ \text{Torsional rigidity of the flange} \]

Windenburg's paper used this boundary condition without any explanation of the sign convention that was used to derive this condition. If the derivation process is continued using the negative sign, the final solution obtained becomes incorrect. Before deciding whether this notation is an error or a flaw in his process, a detailed understanding of the sign convention is required. Figure 4 is drawn using Timoshenko's notation. Positive moments are assumed to be in the direction of the positive axis direction. Since the edge \( y = b \) is of concern, it can be seen from Figure 4 that \( M_y \) is negative. The twisting moment (reaction) of the flange is in the opposite direction of the plate moment, thus making it positive.

![Figure 4 Sign Convention for Elastic Support](image)

Using the definition of \( M_y \) and the signs described above the correct boundary condition is seen to be.
-M_y = + Twisting Moment

\[-D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) = + C \frac{\partial^3 w}{\partial x^2 \partial y} \]

(4)

\[D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) = C \frac{\partial^3 w}{\partial x^2 \partial y} \]

With the sign convention understood, it is evident that the boundary condition used by Windenburg should not have the negative sign for the plate stiffness term. Once this is acknowledged, the derivation follows his procedure correctly.

The final solution for Windenburg’s derivation was written in terms of non-dimensional quantities that allowed for simplified plotting of the solution. His equation 10 was the solution to the plate problem supported on three sides by a simple support and an elastic support on the free standing edge. There was one square root missing in the solution printed by Windenburg. However, it was only a transcription error common of the printing methods for that era, and is corrected in the presentations of Appendix A.

The following equation verifies Windenburg’s results:

\[\sqrt{\psi - \phi} \left[ \psi + (1 - \nu) \phi \right] \cot \sqrt{\phi \psi - \phi^2} = \sqrt{\psi + \phi} \left[ \psi - (1 - \nu) \phi \right] \coth \sqrt{\phi \psi + \phi^2} \]

\[+ 2^\frac{5}{3} \psi \phi \theta + 2^\frac{3}{3} \psi \frac{C}{Db} \sqrt{\psi^2 - \phi^2} \cot \sqrt{\phi \psi + \phi^2} \coth \sqrt{\phi \psi - \phi^2} \]

\[+ \theta \phi^4 \frac{C}{Db} \left( \sqrt{\psi + \phi} \coth \sqrt{\phi \psi + \phi^2} - \sqrt{\psi - \phi} \coth \sqrt{\phi \psi - \phi^2} \right) = 0 \]

(5)

where: \( b = \) Height of the plate along the y direction

\[\psi = b \sqrt{\frac{\sigma h}{D}} \] Stress Factor

\[\phi = \frac{h}{a} \] Aspect Factor
\[ C = GJ \text{ Torsional rigidity of the flange} \]

\[ \vartheta = \frac{EI}{bd} - \frac{A\psi^2}{bh\phi^2} \text{ Flexural Rigidity Factor} \]

\[ v = \text{Poisson’s Ratio} \]

\[ D = \frac{Eh^3}{12(1-v^2)} \text{ Plate stiffness} \]

With the above expression validated, it is helpful to verify Windenburg’s Figure 2a and 2b. Plotting the equation accurately is difficult due to many complications with the function itself. Initially a MATLAB pre-programmed function `ezplot()` was implemented. `Ezplot` is an easy to use built-in plot function. This means that it tries values on a set range of x and y coordinates to plot the function. However, the above function changes shapes so abruptly that singularities occur and the code breaks down. To bypass this problem, the above expression was examined to determine where the shape changes occurred and a three stage dichotomy solver was written. Figure 5 and Figure 6 show a side by side comparison of Windenburg’s plots to those generated by MATLAB.
Figure 5 Windenburg’s Plot (above) vs MATLAB Plot (below) with a Torsional Rigidity Factor of 2
Figure 6 Windenburg’s Plot (above) vs MATLAB Plot (below) with a Torsional Rigidity Factor of 0
Inspection of the plots for values of the torsional rigidity \( \frac{C}{Db} \) equal to zero and two shows both are nearly identical. The peak values for \( \psi \) and the trough location \( \phi \approx 3 \) are identical for each respective plot. On the figures above the heavy line was added to allow for easy comparison of values for \( \psi \) and \( \phi \) when \( \theta \) equals twenty. It can be observed in Figure 6 that as \( \phi \) approaches zero the MATLAB plot differs form Windenburg’s plot. This discrepancy was caused by the changing shape of the function. The important portions of the figures were the peaks and troughs; therefore the shape change was not investigated past the three original locations in the dichotomy solver. With these curves in hand the critical stress for the stiffener is computed using equation(5). This technique will be explained and examined once Windenburg’s method is employed to define the size of the required flange.

It is possible to now discuss the procedure used by Windenburg to properly size the outstanding flange:

1. Size the flange to prevent web buckling
2. Determine the increase in buckling stress due to torsional rigidity of the flange
3. Ensure twisting stability of the stiffener

Each of these three critical steps to the properly sizing the flange are discussed in detail.

**Flange Sizing**

To size the flange to prevent web buckling, Windenburg first used only the flexural rigidity factor, \( \theta \). The assumptions and formulation of this procedure are discussed.
Windenburg observed that for the full buckling load to be obtained \( \theta \) must be infinite.

Thus, simple support effects are produced, but by inspecting Figure 5 and Figure 6 it is noted that as the flexural rigidity factor gets larger, it approaches a constant value of \( \psi \) and \( \phi \). Therefore, it was stated that the stress factor was almost constant for all values of \( \theta \geq 20 \), a value he arbitrarily selected value. Windenburg also used Figure 6 to determine the ratio of \( \frac{\psi}{\phi} \) at the trough as approximately equal to 2. By inserting the appropriate variables into equation (5) an analytical method for sizing the flange is produced. The following equations developed Windenburg’s equation-18.

\[
D = \frac{E h^3}{12(1-\nu^2)}
\]

\[
\theta = 20
\]

\[
\nu = .3
\]

\[
\frac{\psi}{\phi} = 2
\]

\[
\theta = \frac{EI}{bD} - \frac{A\psi^2}{bh\phi^2}
\]

\[
20 \leq \frac{12(1-\nu^2)I}{bh^3} - \frac{4A}{bh}
\]

so

\[
5 \leq \frac{3(1-\nu^2)I}{bh^3} - \frac{A}{bh}
\]

then

\[
5 \leq \frac{3(1-.3^2)I}{bh^3} - \frac{A}{bh}
\]

finally

\[
5 \leq \frac{2.73I}{bh^3} - \frac{A}{bh}
\]
Upon completing this derivation, Windenburg makes two very important statements:

- This equation is only valid for symmetric cross sections.
- The stiffener does not experience any twisting instability.

The twisting stability problem is addressed in section three of the sizing process.

Windenburg solved equation (7) for a Tee cross section in terms of geometric parameters.

The following are the equations for sizing the rectangular flange:

\[ I = \frac{1}{12} f^3 t \]
\[ A = ft \]

\[ 5 \leq \frac{2.73 I}{bh^3} - \frac{A}{bh} \]

\[ 5 \leq \frac{2.73 \frac{1}{12} f^3 t}{bh^3} - \frac{ft}{bh} \]

\[ 5b \leq \frac{.2275 f^3 t}{h^3} - \frac{ft}{h} \]

\[ \frac{22b}{h} \leq \frac{f^3 t}{h^4} - 4.4 \frac{ft}{h^2} \]

\[ 0 \leq \frac{f^3 t}{h^3} - 4.4 \frac{ft}{h^2} - \frac{22b}{h} \]

Following Windenburg’s procedures, the equations to develop the full web buckling strength using a symmetric bulb are:
These previous equations are derived to allow the flange to be sized such that the web can develop a full buckling load as though it had the fourth simple support added to what was previously a free edge. To calculate the buckling stress for the web as though it had a simple support along all four edges is referenced by Windenburg and Bruhn as Bryan’s critical stress case. The critical buckling stress can be calculated as follows:

$$\sigma_{cr} = \left( m \frac{b}{a} + \frac{1}{m} \frac{a}{b} \right)^2 \frac{\pi^2 Eh^2}{12(1-v^2)b^2}$$

Torsional Rigidity Effects

The second step in properly determining the buckling strength of the entire stiffener is to account for the torsional rigidity of the flange. Now that the initial sizing is complete, and again using the simplification that for $\theta \geq 20$ the value of $\psi$ is a constant, the flexural rigidity factor can be taken as infinite ($\infty$), and used to simplify equation (5). This simplification yields the following equation: 

$$I = \frac{\pi R^4}{4}$$

$$A = \pi R^2$$

$$5 \leq \frac{2.73I}{bh^3} - \frac{A}{bh}$$

$$5 \leq \frac{2.73\pi R^4}{4bh^3} - \frac{\pi R^2}{bh}$$

$$5b \leq \frac{2.14R^4}{h^3} - \frac{3.14R^2}{h}$$

$$\frac{2.34b}{h} \leq \frac{R^4}{h^4} - 1.47 \frac{R^2}{h^2}$$

$$0 \leq \frac{R^4}{h^4} - 1.47 \frac{R^2}{h^2} - \frac{2.34b}{h}$$

(9)
\[
2\psi + \phi \frac{3}{2} \frac{C}{Db} \left( \sqrt{\psi + \phi} \coth \sqrt{\psi + \phi} - \sqrt{\psi - \phi} \cot \sqrt{\psi - \phi} \right) = 0 \tag{11}
\]

This equation can then be solved for \( \psi \) after the calculation of the torsional rigidity factor for the specified flange. After the new value for \( \psi \) has been calculated, Windenburg related the increase in the stress factor to a percentage increase of Bryan’s critical buckling stress, using the ratio \( \frac{\psi^2}{4\pi^2} \). This ratio is easily derived based on the definition of \( \psi^2 \). Equation (12) steps through the derivation process to obtain this ratio.

Once the value of \( \psi^2 \) is calculated and plugged into the above ratio, a number slightly larger than one is obtained. The decimal part of this number is the percentage increase in Bryan’s critical stress due to torsional rigidity.

\[
\psi^2 \propto \sigma_{cr}
\]
\[
\psi^2 = b^2 \frac{\sigma_{cr} h}{D}
\]
\[
\sigma_{cr} = 4 \frac{\pi^2 Eh^2}{12(1-\nu^2)b^3}
\]
\[
\frac{\psi^2 D}{b^2 h} \propto \sigma_{cr}
\]
\[
\frac{\psi^2 D}{b^2 h} \propto 4 \frac{\pi^2 Eh^2}{12(1-\nu^2)b^3}
\]
\[
\frac{\psi^2}{12(1-\nu^2)} \propto 4 \frac{\pi^2 Eh^2}{12(1-\nu^2)b^3}
\]
\[
\frac{\psi^2}{4\pi^2} \propto 1 \tag{12}
\]

To verify the results, MATLAB’s Ezplot() function was used to plot the curves for Windenburg’s given values of the torsional rigidity factor. The values of percentage increase in buckling stress for different torsional rigidity factors were calculated and
presented in Windenburg’s Table II. Once the percentage increase over Bryan’s simple support case had been calculated to obtain the new buckling stress, it was only necessary to multiply the calculated ratio by Bryan’s critical stress.

**Twisting Instability**

The final phase in ensuring that the stiffener will not fail due to buckling is to ensure that the stress that causes twisting instability is higher than the critical stress of the stiffener as calculated above. The equation for the critical twisting stress is as follows:

\[
\sigma_t = \frac{1}{I_p} \left( C_s + \frac{\pi^2 E}{a_{eff}^2} C_{BT} \right)
\]

where: 
\( \sigma_t \) = Critical twisting stress 
\( I_p \) = Polar moment of inertia about the web attachment point to the skin 
\( a_{eff} \) = Effective length of the stiffener 
\( C_{BT} \) = Torsion bending constant

Several of the constants in the above equation are defined for only open cross sections. The torsion bending constant is defined as follows:

\[
C_{BT} = C_b + C_T = \int w^2 ds + \frac{\pi^2}{12} \int s^2 ds
\]

Where: 
\( w \) = the normal displacement of the end cross section per unit twist 
\( s \) = distance taken along the cross section

Considerable effort has been applied to developing methods for determining the torsion bending constant \( C_{BT} \). However, most of the available information has been developed for open cross-section shapes in the form of I’s, Tees, Channels, and Zees. Thus a challenge exists for determining the definition for \( C_{BT} \) for a non-uniform bulk
cross-section; such as a bulb as being applied in the present cases. To properly calculate
the twisting stress of a symmetric bulb additional research is needed. Therefore, no
further determination is addressed in the current research.

Windenburg claimed that the three steps previously discussed insured proper
sizing of the symmetric flange to act as a simple support for the web. To ascertain if
indeed this is so and the above steps are clear, the following example problem was
constructed.

The proper size was determined for the outstanding flange for a symmetric Tee
cross-section made of Aluminum 2024-T3 with a Young’s modulus of 10.8×10^6 psi and a
shear modulus of 4.1×10^6 with known plate dimensions of a = 12 in., b = 1 in, and h = t =
0.0625 in. The only quantity to find is the width of the flange, f.

Insert the known dimensional parameters into equation(8) and solve for f:

\[
0 = \frac{f^3}{h^4} - 4.4 \frac{ft}{h^2} - 22b
\]

\[
0 = \frac{f^3}{0.0625^4} - 4.4 \frac{f}{0.0625} - 22 \frac{1}{0.0625}
\]

\[
0 = 4096f^3 - 70.4f - 352
\]

MATLAB was used to quickly solve the above cubic function. The solutions are as
follows.

\[
f = \begin{bmatrix}
0.4543 \\
-0.2271 + 0.3709i \\
-0.2271 - 0.3709i
\end{bmatrix}
\]

Of the three roots found it only makes sense to use the positive real root. Therefore, the
width of the flange is 0.4452 in.
Now that the stiffener is sized, Bryan’s critical stress can be calculated by using equation (10).

\[ \sigma_{cr} = \left( \frac{b}{m} + \frac{1}{m\,b} \right)^2 \frac{\pi^2 Eh^2}{12(1-\nu^2)b^2} \]

\[ \sigma_{cr} = (2)^2 \frac{\pi^2 (10.8 \times 10^6)(0.0625)^2}{12(1-.33^2)1.0^2} \]

\[ \sigma_{cr} = 155,753 \text{ psi} \]

To account for the torsional rigidity of the flange the torsional rigidity factor must be calculated.

\[ C = GJ \]

\[ C = 4.1 \times 10^6 \left( \frac{1}{3} \text{m}^3 \right) \]

\[ C = 4.1 \times 10^6 \left( \frac{1}{3} (0.4425)(0.0625)^3 \right) \]

\[ C = 147.64 \text{ lb} \cdot \text{in}^2 \]

Dividing the torsional rigidity factor by \( D \) and \( b \), a value of 0.6387 is obtained. This value can be inserted into equation (11), and then plotted using MATLAB to determine the minimum value for \( \psi \).
To determine the percentage increase in the critical buckling stress added by the torsional rigidity the ratio $\frac{\psi^2}{4\pi^2}$ will be used. The percentage increase of buckling stress over Bryan's case for this particular stiffener is 18.1%. The modified critical stress is found as follows:

$$\sigma_{mcy} = \sigma_c \cdot \frac{\psi^2}{4\pi^2}$$

$$\sigma_{mcy} = 183,988 \text{ psi}$$

The third and final check is to insure that the critical twisting stress is larger than the modified critical stress for buckling. For the simple geometry of the symmetric Tee-section, the equation for critical twisting stress has been derived.
After calculating the critical twisting stress, it is evident that the flange is not proportioned well enough to refrain from twisting instabilities. Additional iterations of the parameters used in the three sizing steps suggest that the thickness of the flange needs to become much larger to increase the resistance to twisting. The final dimensions that meet all the above criteria can be seen in Figure 9. The dimensions of the web are unchanged. The final sizing meets all of Windenburg’s criteria, and therefore by his theory should develop the full buckling capacity of the web.

$$\sigma_t = 18,228 \text{ psi}$$

$$\sigma_t = 18,228 \text{ psi}$$

(20)

After calculating the critical twisting stress, it is evident that the flange is not proportioned well enough to refrain from twisting instabilities. Additional iterations of the parameters used in the three sizing steps suggest that the thickness of the flange needs to become much larger to increase the resistance to twisting. The final dimensions that meet all the above criteria can be seen in Figure 9. The dimensions of the web are unchanged. The final sizing meets all of Windenburg’s criteria, and therefore by his theory should develop the full buckling capacity of the web.

$$\sigma_t \geq \sigma_{mcr}$$

(21)

$$240,030 \text{ psi} \geq 183,988 \text{ psi}$$

Figure 9 Flange Sized to Satisfy Windenburg’s Conditions
Asymmetric Flanges

E.F. Bruhn\textsuperscript{2} derived a technique for determining the size of an asymmetric flange such as an angle or bulb cross-section. The process for sizing these asymmetric sections was a simple extrapolation on the work of Windenburg. The main difference between the sizing of asymmetric and symmetric flanges was the difference in the moment of inertia terms. The moment of inertia was increased because of the parallel axis theorem. Figure 10 depicts the geometry of the asymmetric bulb with the distance between the bulb’s and web’s centroidal y-axes labeled.

Figure 10 Asymmetric Bulb Dimensions
The following two equations develop the sizing function for a lip, and then for an asymmetric bulb. Equation (22) is the development of the sizing function for a lip. It should be noted that the thickness of the flange and web are required to be equivalent \( t = h \), based on the following derivation.

\[
5 \leq \frac{2.73I}{bh^3} - \frac{A}{bh}
\]

\[
I = \frac{hf^3}{3}
\]

\[
A = hf
\]

\[
0.910 \left( \frac{f}{h} \right)^3 - \frac{f}{h} = 5 \frac{b}{h}
\]

Equation (23) is the sizing function for an asymmetric bulb cross-section:

\[
5 \leq \frac{2.73I}{bh^3} - \frac{A}{bh}
\]

\[
I = \frac{\pi d^4}{64} + \frac{\pi d^2}{4} \left( \frac{d - h}{2} \right)
\]

\[
A = \frac{\pi d^2}{4}
\]

\[
\left( \frac{d}{h} \right)^4 - 1.6 \left( \frac{d}{h} \right)^3 - 0.374 \left( \frac{d}{h} \right)^2 = 7.44 \frac{b}{h}
\]

With these adaptations made to Windenburg’s theory, it is important to recall he specifically stated that equation (7) is only valid for symmetric cross-sections. There is no supporting evidence or attempt made to validate Bruhn’s claim. This raises concerns and leads to a more in depth study of the effects asymmetry may have on the sizing of the bulbs. A comparison was conducted between Windenburg’s theory for a symmetric bulb
and Bruhn’s method. The only difference in the technique used to initially size the bulb is the use of equation(9) for symmetric bulb and equation(23) for Bruhn’s asymmetric bulb. A short MATLAB code was written to solve each of the sizing functions. The comparison in bulb sizes can be seen in Table 1.

<table>
<thead>
<tr>
<th>Sizing Method</th>
<th>h (in)</th>
<th>b (in)</th>
<th>radius (in)</th>
<th>% of Radius</th>
<th>Area (in²)</th>
<th>% of Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>Windenburg</td>
<td>0.0625</td>
<td>0.6960</td>
<td>0.1516</td>
<td>100%</td>
<td>0.0722</td>
<td>100%</td>
</tr>
<tr>
<td>Bruhn</td>
<td>0.0625</td>
<td>0.6960</td>
<td>0.1110</td>
<td>73%</td>
<td>0.0387</td>
<td>54%</td>
</tr>
</tbody>
</table>

It is clear from the above data that Bruhn under sizes the bulb based on Windenburg’s criteria. The bulb is sized such that Bruhn’s bulb has 46% less area than Windenburg’s. This draws some question to the methodology used by Bruhn to adapt the process for symmetric cross-sections to asymmetric sections.

After discovering the large deviation in the theoretical sizing techniques more research is needed to be able to draw conclusions about the sizing methods. It is for this reason the finite element analysis was conducted.
Finite Element Analysis

Upon analytically sizing the bulbs and other flanges using the aforementioned techniques, the stringers are modeled in two finite element software packages: NEiNastran and ANSYS Workbench. These models are intended to explore and determine if the assumption that a bulb sized according to either Bruhn's or Windenburg's methods actually develop the same buckling stress as if they are simply supported on all four edges.

Modeling

The first step in this analysis is to determine which test cases are important to the problem statement. To accurately compare the historic sizing techniques using FEA a number of cases that are directly applicable to the theory are needed. Table 2 displays the cases that are selected to be examined for this purpose. Each case is sized according to the methods of Windenburg or Bruhn, respectively.

<table>
<thead>
<tr>
<th>$\theta=20$</th>
<th>$a$ (in)</th>
<th>$b$ (in)</th>
<th>$h$ (in)</th>
<th>$t$ or $R$ (in)</th>
<th>$f/offset$ (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Web SSSS</td>
<td>12</td>
<td>0.9375</td>
<td>0.0625</td>
<td>0.0625</td>
<td>0.0625</td>
</tr>
<tr>
<td>Web SSSF</td>
<td>12</td>
<td>0.9375</td>
<td>0.0625</td>
<td>0.0625</td>
<td>0.0625</td>
</tr>
<tr>
<td>Tee Flange</td>
<td>12</td>
<td>1</td>
<td>0.0625</td>
<td>0.0625</td>
<td>0.4543</td>
</tr>
<tr>
<td>Square Tee Flange</td>
<td>12</td>
<td>1</td>
<td>0.0625</td>
<td>0.287</td>
<td>0.287</td>
</tr>
<tr>
<td>Symmetric Bulb</td>
<td>12</td>
<td>1</td>
<td>0.0625</td>
<td>0.164</td>
<td>N/A</td>
</tr>
<tr>
<td>Asymmetric Bulb 1</td>
<td>12</td>
<td>1</td>
<td>0.0625</td>
<td>0.164</td>
<td>0.0625</td>
</tr>
<tr>
<td>Asymmetric Bulb 2</td>
<td>12</td>
<td>1</td>
<td>0.0625</td>
<td>0.164</td>
<td>0.1</td>
</tr>
<tr>
<td>Asymmetric Bulb 3</td>
<td>12</td>
<td>1</td>
<td>0.0625</td>
<td>0.164</td>
<td>tangent</td>
</tr>
<tr>
<td>Asymmetric Bulb Bruhn</td>
<td>12</td>
<td>1</td>
<td>0.0625</td>
<td>0.120</td>
<td>tangent</td>
</tr>
</tbody>
</table>

All of these cross-sections can be found in Appendix B.
Once the geometry is defined it is necessary to decide how to model the stringers in the FEA programs. Initially the use of plate elements for the web and beam elements for the bulb led to difficulties with the asymmetric cases, primarily on how to achieve tangency between one side of the plate. Plate elements are only a surface with the thickness accounted for by the element type, and beams are defined by their centerline. For this reason, the parametric feature-based properties of CATIA V5R16 prove useful in creating solid models of each stringer that can subsequently be manipulated in FEM. Figure 11 shows the 3-D model of a stand-alone web, a Tee cross-section, and an asymmetric bulb.

![Figure 11 Catia 3-D Models](image)

The solid models are constructed and then converted to .stp files and imported into ANSYS Workbench and into NEiNastran. At this point the boundary conditions to properly constrain the models are applied. The initial thought was to constrain the faces of the model in the proper directions in both programs. Figure 12 depicts the boundary conditions on the web that was supported by three simple supports and a free edge. The plate had the same boundary conditions as the other models.
A 1.0 psi pressure load is applied to the face of the stringer that was only constrained in the y-direction. The pressure load is set to 1.0 psi because the linear buckling analysis conducted by the FEA solvers calculates a load multiplier. Therefore, if the load is applied as $\frac{lb}{in^2}$, the eigenvalue calculated by the solvers is directly related to the critical buckling stress.

Upon review of the results, a problem was discovered with the boundary conditions; the face constraints were adding stiffness to the problem which was damping out the expected mode-1 buckling shape, a half-sine wave. This phenomenon is illustrated in Figure 13 which was a case run in ANSYS.
This odd damping was believed to occur because constraining the faces created an inadvertent clamped condition. The clamped condition was able to resist the moment and required the slope of the plate to be zero. After confirming this result with the NEiNastran model, a new approach was taken in applying the simple support conditions to avoid adding stiffness to the problem.

The new approach took the x- and z-displacement constraints and moved them from face constraints to edge conditions. In Figure 14 the orange dashed line is the z-displacement constraint and the green solid line is the x-constraint.
The results from these new boundary conditions were compared with the previous face constraints and showed that the critical buckling stress was lower. This lowering of the critical stress revealed that some of the stiffness added by the boundary conditions had been removed by the new constraints. However, damping of the buckling was still occurring. Another iteration of boundary condition implementation was required.

The new boundary condition required a re-design of the solid models being imported into the FEA solvers because the boundary conditions need to be applied along the centerline of the cross-section. ANSYS needed a line or a surface to apply constraints and (to remain consistent) the Nastran models were constrained identically. CATIA was used to split the solid models along the model’s x-z plane. Figure 15 displays the two halves of the Tee cross-section which were then imported into the FEA solvers as two individual bodies.
Both of these sections were imported and ANSYS automatically created a connection between the two bodies; however, the line in the middle was still active and able to be selected for constraints. Figure 16 displays the locations of the boundary conditions and pressure load in ANSYS.
NEiNastran imports the bodies very similarly to ANSYS. Except for setting up a connection, the user has to merge coincident nodes and entities to allow the model to act as one solid stringer. Nastran uses the numbers 1, 2, and 3 to represent the x, y, and z directions respectively. To allow a clear easier to manipulate graphical representation ANSYS was used to Femap model. Figure 17 is the ANSYS interpretation of the Nastran /Femap constraints. This software was used because of the ability to view the nodal locations of the constraints.

![Figure 17 Tee Cross-Section Boundary Conditions form Nastran](image)

With the boundary conditions now applied along the centerline of the cross-section the results for the critical stress should again be smaller than the previous two cases. Table 3 shows that the new boundary conditions along the centerline do indeed relieve some of the stiffness created by the other attempts.
Table 3 Comparison of Edge and Centerline Boundary Conditions

<table>
<thead>
<tr>
<th>Boundary Conditions</th>
<th>Mode 1 $\sigma_{cr}$ ANSYS (psi)</th>
<th>MODE 1 $\sigma_{cr}$ Nastran (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edge support</td>
<td>17,861</td>
<td>18,004</td>
</tr>
<tr>
<td>Centerline support</td>
<td>16,917</td>
<td>16,942</td>
</tr>
</tbody>
</table>

It was not sufficient to only determine the critical stress, but necessary to evaluate the shape of the mode-1 buckle as well. Figure 18 shows the mode shape for the Tee cross-section as computed by both ANSYS and Nastran. It was clear that the expected mode shape of the half-sine wave was developed.

![ANSYS Mode 1](image)

![Nastran Mode 1](image)

Figure 18 Buckling Mode 1 for Tee Cross-Section

Now that the best method for applying the boundary conditions is determined, it is necessary to set up the requirements for test cases. The requirements include the selection of a material for the stringer and mesh sizing. The material chosen for all the test cases was Aluminum 2024-T3 using the properties from Metallic Material Properties.
Development and Standardization-01 (MMPDS-01) for extrusions. Table 4 summarizes the properties for the test cases.

Table 4 Aluminum 2024-T3 Extrusion Properties

<table>
<thead>
<tr>
<th>AL 2024-T3</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>E (ksi)</td>
<td>v</td>
<td>G (ksi)</td>
</tr>
<tr>
<td>10,800</td>
<td>0.33</td>
<td>4,100</td>
</tr>
</tbody>
</table>

The overall dimensions of the models examined are 12 inch x 1 inch x 0.0625 inch. The mesh sizing was the same for ANSYS and Nastran. The cross-sectional surfaces are meshed with a surface sizing of 0.03125 inch; the longitudinal lines are meshed using a line sizing of 0.125 inch. The lines defining the thickness of the stringer are meshed using a line sizing of two elements over the length. When using finite element software it is necessary to have enough elements in the model to ensure the accuracy of the results. With the sizing used above for flanged webs the average number of elements in the Nastran models are 60,000 elements and 20,000 elements in ANSYS. For the cases of only the web the Nastran model has 12,288 elements and the ANSYS model has 900 elements.

The edge spacing is able to be visualized using both ANSYS and Nastran to ensure a quality mesh before using the computing time to actually mesh the model. Figure 19 is the graphical representation of the edge sizing given by ANSYS prior to meshing.
The mesh was one area of the test cases where ANSYS and Nastran differ. ANSYS has an automatic mesh method that the user only has to size using the aforementioned sizing. This method was a solid element meshing scheme using quadrilateral elements. In Nastran it was necessary to use tetrahedral elements to get a good mesh quality that would map the geometry around the sharp corners and circular cross-section of the bulbs. Figure 20 displays the ANSYS mesh, and Figure 21 displays the Nastran mesh.
Figure 20 ANSYS Quadrilateral Mesh of a Tee Cross-Section
Figure 21 Nastran Tetrahedral Mesh of a T Cross-Section
For visualization purposes Figure 22 and Figure 23 display a mesh similar to the above figures except they are for an asymmetric bulb.

Figure 22 ANSYS Quadrilateral Mesh on an Asymmetric Bulb
Figure 23 Nastran Tetrahedral Mesh of an Asymmetrical Bulb
After the boundary conditions and meshing techniques were satisfied, the test cases from Table 2 were analyzed. The following section discusses the findings of the FEA cases.

Test Case Results

The first and most crucial test cases are the stand-alone webs with four simply supported edges and with three simple supports with one longitudinal edge free. These cases are the most important because there is a known theoretical solution which will allow the FEA models to be validated.

Test Case 1 Web SSSS

The web with four sides simply supported was important to the current research because the classical sizing techniques claimed the bulb could provide the fourth simple support\(^1,2\). The theoretical value for the critical buckling stress could be easily calculated from theory using the equation\((24)\).

\[
\text{BRYAN'S CASE SSSS}
\]

\[
\sigma_{cr} = \frac{k\pi^2 Eh^2}{12(1-\nu^2)b^2}
\]

\(k = 4\)  

The theoretical value is listed in Table 5 as well as those calculated by ANSYS and Nastran. A percent error calculation is also included to determine how well the model matches the theoretical value.
There is very close agreement between ANSYS and Nastran. The fact that the percent error is within 5% of the theoretical value verifies that the modeling techniques employed for the research were adequate. The ANSYS and Nastran models, seen in Figure 24 and Figure 25 respectively, agree on the total displacement as well as the mode shape. The theory of plate buckling states that when the plate is simply supported on four sides the number of half waves, m, is determined by the height of the web. Therefore, since the web dimensions are b = 1.0 inch and a = 12.0 inches long the plate should buckle into 12 half-waves. Both the ANSYS and Nastran model both buckle into 12 half waves.
Test Case 2 Web SSSF

Analysis of a web with a free longitudinal edge verified the need for the extra support along the freestanding edge. This case had a very straightforward calculation to determine the critical stress$^3$. Equation (25) can be used to directly solve for the critical stress.

**TIMOSHENKO CASE SSSF**

\[
\sigma_{cr} = \frac{k \pi^2 Eh^2}{12(1-v^2)b^3}
\]

\[k = 0.456 + \frac{b^2}{a^2}\]
This is the second case that can be used to verify the quality of the modeling techniques implemented. Table 6 displays the values necessary to conduct a comparison of the FEA models to the theoretical value.

Table 6 Test Case 2 Web SSSF Results

<table>
<thead>
<tr>
<th>Test Case</th>
<th>ANSYS $\sigma_{cr}$ (psi)</th>
<th>Nastran $\sigma_{cr}$ (psi)</th>
<th>Theoretical $\sigma_{cr}$ (psi)</th>
<th>ANSYS %Error</th>
<th>Nastran %Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Web SSSF</td>
<td>15,392</td>
<td>15,503</td>
<td>18,026</td>
<td>14.61</td>
<td>14.00</td>
</tr>
</tbody>
</table>

Even though the values were farther away from the theoretical, the general trend that the FEA models are less than the theoretical holds true. The mode shapes for this very simple case can be seen in Figure 26 and Figure 27.

Figure 26 Web Simply Supported on 3 Sides Mode 1 ANSYS
Test Case 3 Tee Section Sized according to Windenburg

This Tee cross-section had the same geometrical dimensions as the flange sized in the example using the flexural rigidity factor equation. This is the last case that was used to further validate the modeling technique. It has already been shown in the example that though the originally sized flange solves equation (7), it still falls for the simply supported case due to twisting instability. The critical stress for twisting has previously been solved and a straight comparison of the FEA models to this value can be seen below.
Table 7 Test Case 3 Tee Cross-Section Results

<table>
<thead>
<tr>
<th>Test Case</th>
<th>ANSYS $\sigma_{cr}$ (psi)</th>
<th>Nastran $\sigma_{cr}$ (psi)</th>
<th>Theoretical $\sigma_{cr}$ (psi)</th>
<th>ANSYS %Error</th>
<th>Nastran %Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tee Flange</td>
<td>16,917</td>
<td>16,942</td>
<td>18,228</td>
<td>7.19</td>
<td>7.05</td>
</tr>
</tbody>
</table>

When looking at the mode shapes created by the FEA software it is evident that the flange is undergoing a major twist. This can be seen from the total deformation plots in ANSYS, Figure 28, and in Nastran, Figure 29.
With the first three test cases in such close agreement, the other test cases will be presented in a general manner.

Test Case

The reason for multiple test cases is current theories contain large gaps in their derivations; thoroughness is of the utmost importance. The square cross-section flange was implemented to see how sensitive the stringers are to twisting instability and warping.

Intuitively, it was determined that the buckling stress should increase as the unsupported flange width became smaller. The symmetric bulb is chosen for its ability to
be sized directly from Windenburg's equations. The first set of asymmetric bulbs is sized exactly like that of the symmetric bulb, allowing only for a small offset between the centerline of the web to centerline of the bulb. The final case examined is an asymmetrical bulb sized according to Bruhn. Table 8 displays all the FEA results for every test case.

<table>
<thead>
<tr>
<th></th>
<th>ANSYS $\sigma_{cr}$ (psi)</th>
<th>Nastran $\sigma_{cr}$ (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Web SSSS</td>
<td>148,670</td>
<td>148,817</td>
</tr>
<tr>
<td>Web SSSF</td>
<td>15,392</td>
<td>15,503</td>
</tr>
<tr>
<td>Tee Flange</td>
<td>16,917</td>
<td>16,942</td>
</tr>
<tr>
<td>Square Tee Flange</td>
<td>29,270</td>
<td>30,066</td>
</tr>
<tr>
<td>Symmetric Bulb</td>
<td>31,069</td>
<td>30,747</td>
</tr>
<tr>
<td>Asymmetric Bulb 1</td>
<td>31,972</td>
<td>32,268</td>
</tr>
<tr>
<td>Asymmetric Bulb 2</td>
<td>32,928</td>
<td>33,441</td>
</tr>
<tr>
<td>Asymmetric Bulb 3</td>
<td>34,995</td>
<td>35,995</td>
</tr>
<tr>
<td>Asymmetric Bulb Bruhn</td>
<td>23,991</td>
<td>23,814</td>
</tr>
</tbody>
</table>

From the information provided about the values the critical buckling stress never developed to the level of a simple support. To help with this visualization, Table 9 has been constructed to display the percent of simple support the given bulb dimensions supply.
Table 9 Percent of Simple Support Developed

<table>
<thead>
<tr>
<th></th>
<th>ANSYS % of SS</th>
<th>Nastran % of SS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Web SSSS</td>
<td>100.00</td>
<td>100.00</td>
</tr>
<tr>
<td>Web SSSF</td>
<td>10.35</td>
<td>10.42</td>
</tr>
<tr>
<td>Tee Flange</td>
<td>11.38</td>
<td>11.38</td>
</tr>
<tr>
<td>Square Tee Flange</td>
<td>19.69</td>
<td>20.20</td>
</tr>
<tr>
<td>Symmetric Bulb</td>
<td>20.90</td>
<td>20.66</td>
</tr>
<tr>
<td>Asymmetric Bulb 1</td>
<td>21.51</td>
<td>21.68</td>
</tr>
<tr>
<td>Asymmetric Bulb 2</td>
<td>22.15</td>
<td>22.47</td>
</tr>
<tr>
<td>Asymmetric Bulb 3</td>
<td>23.54</td>
<td>24.19</td>
</tr>
<tr>
<td>Asymmetric Bulb Bruhn</td>
<td>16.14</td>
<td>16.00</td>
</tr>
</tbody>
</table>

These values demonstrate that every bulb or flange sized using Windenburg’s and Bruhn’s method was drastically undersized for the claim that it would approximate a simple support. After observing these results, additional studies were deemed necessary. The first case is to determine how large a bulb is needed to approximate a simple support according to Nastran. The next case is required to examine the effect on critical stress that results from a fillet blending the web and the bulb.

The first study was conducted by picking a range of values for percentage increase in area and calculating a new diameter of the bulb. Then the models were analyzed in Nastran to determine the critical buckling stress. The percentage of simple support provided by the area was then calculated. The new diameter was divided by b to obtain a non-dimensional quantity for plotting the trends. Table 10 presents the data that was calculated and used to plot the trends in Figure 30.
<table>
<thead>
<tr>
<th>Technique</th>
<th>d/b</th>
<th>Asymmetric (%)</th>
<th>Symmetric (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30% of Windenburg Bruhn</td>
<td>0.180</td>
<td>11.36</td>
<td>10.50</td>
</tr>
<tr>
<td>Tangent Windenburg</td>
<td>0.239</td>
<td>16.00</td>
<td>14.51</td>
</tr>
<tr>
<td>25% increase in area</td>
<td>0.328</td>
<td>24.19</td>
<td>20.66</td>
</tr>
<tr>
<td>50% increase in area</td>
<td>0.367</td>
<td>26.54</td>
<td>23.35</td>
</tr>
<tr>
<td>75% increase in area</td>
<td>0.402</td>
<td>28.26</td>
<td>24.86</td>
</tr>
<tr>
<td>100% increase in area</td>
<td>0.434</td>
<td>29.51</td>
<td>25.94</td>
</tr>
<tr>
<td>200% increase in area</td>
<td>0.464</td>
<td>30.47</td>
<td>26.62</td>
</tr>
<tr>
<td>500% increase in area</td>
<td>0.568</td>
<td>32.70</td>
<td>27.89</td>
</tr>
<tr>
<td>1000% increase in area</td>
<td>0.733</td>
<td>32.31</td>
<td>30.10</td>
</tr>
<tr>
<td></td>
<td>1.037</td>
<td>22.89</td>
<td>41.15</td>
</tr>
</tbody>
</table>

The trends were plotted for both symmetric and asymmetric bulbs. Both curves were suited best by cubic functions. However, there was a significant difference between the shape of the symmetric and asymmetric trend. First, looking at the symmetric bulb’s trend: as the area increased the percent of simple support achieved increases. Following simple physics, a simple support acts as an infinite flexural support. To achieve infinite flexural support the moment of inertia of the bulb must become larger, and since the bulb is symmetric about the web the shear center was located on the z axis of the cross-section. This is an advantage because the load was acting through the shear center, which caused no twisting. On the asymmetric bulb, the shear center was not in plane with the load, and therefore twisting is especially a problem when the bulb becomes much larger than web thickness. This can be observed in Figure 30 where the maximum value for percentage of simple support is around 34%- any larger diameter starts to decrease the effectiveness of the bulb.
Table 11 depicts the values used in this analysis, to determine if the fillet between the web and the bulb have any significant contributions to the buckling stress.

<table>
<thead>
<tr>
<th>Rf/R</th>
<th>Nastran Buckling Stress (psi)</th>
<th>% increase over no fillet</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>35,995</td>
<td>0.00</td>
</tr>
<tr>
<td>0.125</td>
<td>37,240</td>
<td>3.46</td>
</tr>
<tr>
<td>0.25</td>
<td>38,289</td>
<td>6.37</td>
</tr>
<tr>
<td>0.5</td>
<td>39,079</td>
<td>8.57</td>
</tr>
<tr>
<td>0.75</td>
<td>40,044</td>
<td>11.25</td>
</tr>
<tr>
<td>1</td>
<td>40,883</td>
<td>13.58</td>
</tr>
</tbody>
</table>
Plotting the above data reveals a power law relationship. This means that as the radius of the fillet enlarged, the higher the buckling capacity became. The gain in buckling capacity is related directly to the added area, and thus the moment of inertia of the fillet.

**Percent Increase in Buckling Capacity vs Rf/R**

![Graph showing percent increase in buckling capacity vs Rf/R](image)

\[ y = 0.136x^{0.630} \]

\[ R^2 = 0.984 \]

Figure 31 Percent Increase in Buckling Capacity versus Rf/R

With such varying results between the FEA and theoretical values for critical buckling stress it is required to run one more test to determine why the historical methods to size a bulb have worked for many years in industry. This case will be the asymmetric bulb as sized by Bruhn with a fixed boundary condition along the centerline along which a web would join with the panel. This was done to examine if the added stiffness from the surrounding skin section and mating flange is enough to achieve a critical stress.
approaching that of the four-sided simple support case. The results from this analysis can be seen in Table 12 and Figure 32.

Table 12 Percent of Simple Support Case with Attachment Line Fixed

<table>
<thead>
<tr>
<th>Test Case</th>
<th>Nastran $\sigma_{cr}$ (psi)</th>
<th>% of SS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asymmetric bruhn fixed</td>
<td>27,199</td>
<td>18.28</td>
</tr>
</tbody>
</table>

Figure 32 Nastran Mode 1 for Fixed Attachment Line
Conclusions

The methods for sizing symmetric and non-symmetric flanges and bulbs based on classical methods according to the findings of the present work appears insufficient at best. However, the theory behind Dwight Windenburg’s work is correct and can be clearly explained. An example sizing case was run to help illustrate this procedure. The first step was to neglect the torsional rigidity and only size the flange based on the flexural rigidity factor. After successfully sizing the flange or bulb, it was necessary to include the stiffness induced by the torsional rigidity of the flange. Including this value with Bryan’s Critical stress for a plate with four sides simply supported will add an increase to the buckling capacity. Step 3 was used to determine if there would be any instability due to twisting. As seen in the example case, this step is not negligible since several iterations were required to obtain acceptable values for the critical twisting stress. With this step directly affecting the sizing of the flange, it is important to calculate the twisting stress for symmetric and asymmetric bulbs. However, the Torsional-Bending Constant ($C_{BT}$) is complicated to calculate for bulb cross-sections. The lack of understanding on how $C_{BT}$ will be affected by an obscure cross-section like the asymmetric bulb makes step 3 a point for further research.

After sizing is complete, the FEA results were analyzed to reveal that the flanges and the bulbs sized by both the theoretical techniques do not provide a simple support for the web.
Table 9 displays the percentage of the simply supported buckling stress developed by each of the flanges. With the highest percentage being 24%, questions arise as to why the theory claims a simple support with the sizing technique. The theory claims the flange to act like a simple support because all of the assumptions used to derive the sizing function are based on the simple support condition.

A last FEA case was run (treating the bottom boundary condition as a fixed constraint) to check if the critical stress would reach the level of the simple support. Had the test verified that the web being fixed on the bottom was sufficient to increase the buckling stress, then the theory would need to be augmented to account for the fixed boundary condition. However, the fixed boundary condition did not add enough stiffness to account for the gap in the theoretical sizing techniques. Lastly, the incorrect sizing of these bulb-stiffened stringers has not been a problem in industry because the compressive stress distributed throughout the entire panel structure does not allow the load in the stringer to reach even a large percent of the critical buckling stress.

For more conclusions to be drawn further research must be conducted in the areas of twisting instability and load transfer from the aircraft skin to the stringer. Also, compression testing should be completed to further examine the classical sizing techniques.
References


Appendix A – Derivation of Windenburg’s Results

General Solution

\[ w = (A \sinh \alpha y + B \sin \beta y) \sin kx \]  
\[ \text{(1)} \]

First Boundary Condition

\[ D \left[ \frac{\partial^3 w}{\partial y^3} + (2 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} \right] - EI \frac{\partial^4 w}{\partial x^4} - A \sigma_y \frac{\partial^3 w}{\partial x^2} = 0 \]
\[ \text{(2)} \]

Second Boundary Condition

\[ D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) = C \frac{\partial^3 w}{\partial x^2 \partial y} \]
\[ \text{(3)} \]

Substituting (1) into (2)

\[ \frac{\partial^3 w}{\partial y^3} = \left[ A \alpha^2 \cosh \alpha y - B \beta^2 \cos \beta y \right] \sin kx \]
\[ \frac{\partial^3 w}{\partial x \partial y^2} = -k^2 \left[ A \alpha \cosh \alpha y + B \beta \cos \beta y \right] \sin kx \]
\[ \frac{\partial^4 w}{\partial x^4} = k^4 \left[ A \sinh \alpha y + B \sin \beta y \right] \sin kx \]
\[ \frac{\partial^2 w}{\partial x^2} = -k^2 \left[ A \sinh \alpha y + B \sin \beta y \right] \sin kx \]
\[ \text{(4)} \]
\[
D\left[ A\alpha^2 \cosh \alpha y - B\beta^2 \cos \beta y \right] + (2-v) \left( -k^2 \left[ A\alpha \cosh \alpha y + B\beta \cos \beta y \right] \right) \\
-Elk^4 \left[ A \sinh \alpha y + B \sin \beta y \right] - A, \sigma \left( -k^2 \left[ A \sinh \alpha y + B \sin \beta y \right] \right) = 0
\]

\[
DA\alpha^3 \cosh \alpha y - DB\beta^3 \cos \beta y - D(2-v)k^2 \left( A\alpha \cosh \alpha y + B\beta \cos \beta y \right) \\
-Elk^4 A \sinh \alpha y - Elk^4 B \sin \beta y + A, \sigma, k^2 A \sinh \alpha y + A, \sigma, k^2 B \sin \beta y = 0
\]

(5)

\[
A \left[ D\alpha^3 \cosh \alpha y - D(2-v)k^2 \alpha \cosh \alpha y - Elk^4 \sinh \alpha y + A, \sigma, k^2 \sinh \alpha y \right] \\
+ B \left[ -D\beta^3 \cos \beta y - D(2-v)k^2 \beta \cos \beta y - Elk^4 \sin \beta y + A, \sigma, k^2 \sin \beta y \right] = 0
\]

\[
A \left[ (D\alpha^3 - D(2-v)k^2 \alpha) \cosh \alpha y + (-Elk^4 + A, \sigma, k^2) \sinh \alpha y \right] \\
+ B \left[ -(D\beta^3 + D(2-v)k^2 \beta) \cos \beta y + (-Elk^4 + A, \sigma, k^2) \sin \beta y \right] = 0
\]
Simplifying change of variables and divide through by $D$

$$s = \alpha^2 - \nu k^2$$
$$t = \beta^2 + \nu k^2$$
$$\varepsilon = \frac{EI}{D} - \frac{A_r \sigma_s}{Dk^2}$$

$$A \left[ (\alpha^3 - (2 - \nu) k^2 \alpha) \cosh \alpha y + \left( \frac{Elk^4}{D} + \frac{A_r \sigma_s k^2}{D} \right) \sinh \alpha y \right]$$
$$+ B \left[ -(\beta^3 + (2 - \nu) k^2 \beta) \cos \beta y + \left( \frac{Elk^4}{D} + \frac{A_r \sigma_s k^2}{D} \right) \sin \beta y \right] = 0$$

(6)

$$\beta^2 = \alpha^2 - 2k^2$$
$$\alpha^2 = \beta^2 + 2k^2$$

$$A \left[ \alpha \left( \beta^2 + \nu k^2 \right) \cosh \alpha y - k^4 \varepsilon \sinh \alpha y \right]$$
$$+ B \left[ -\beta \left( \alpha^2 - \nu k^2 \right) \cos \beta y - k^4 \varepsilon \sin \beta y \right] = 0$$

$$A \left[ \alpha t \cosh \alpha y - k^4 \varepsilon \sinh \alpha y \right] - B \left[ \beta s \cos \beta y + k^4 \varepsilon \sin \beta y \right] = 0$$
Substituting (1) into (3)

\[
\frac{\partial^2 w}{\partial y^2} = \left[ A\alpha^2 \sinh \alpha y - B\beta^2 \sin \beta y \right] \sin kx
\]

\[
\frac{\partial^2 w}{\partial x^2} = -k^2 \left[ A\sinh \alpha y + B\sin \beta y \right] \sin kx
\]

\[
\frac{\partial^3 w}{\partial x^2 \partial y} = -k^2 \left[ A\alpha \cosh \alpha y + B\beta \cos \beta y \right] \sin kx
\]

\[
D \left[ A\alpha^2 \sinh \alpha y - B\beta^2 \sin \beta y \right] + v \left( -k^2 \left[ A\sinh \alpha y + B\sin \beta y \right] \right)
\]

\[
= C \left( -k^2 \left[ A\alpha \cosh \alpha y + B\beta \cos \beta y \right] \right)
\]

\[
DA\alpha^2 \sinh \alpha y - DB\beta^2 \sin \beta y - Dv k^2 A \sinh \alpha y + Dv k^2 B \sin \beta y
\]

\[
= -Ck^2 A\alpha \cosh \alpha y - Ck^2 B\beta \cos \beta y
\]

\[
A \left[ D\alpha^2 - Dv k^2 \right] \sinh \alpha y + B \left[ -D\beta^2 - Dv k^2 \right] \sin \beta y
\]

\[
= -Ck^2 A\alpha \cosh \alpha y - Ck^2 B\beta \cos \beta y
\]

Simplifying change of variables and divide through by D
\[ s = \alpha^2 - v k^2 \]
\[ t = \beta^2 + v k^2 \]
\[ r = \frac{C k^2}{D} \]

\[
A \left[ \alpha^2 + v k^2 \right] \sinh \alpha y - B \left[ \beta^2 + v k^2 \right] \sin \beta y = - \frac{C k^2}{D} A \alpha \cosh \alpha y - \frac{C k^2}{D} B \beta \cos \beta y
\]

(9)

\[
A \left[ \alpha^2 + v k^2 \right] \sinh \alpha y - B \left[ \beta^2 + v k^2 \right] \sin \beta y = - r \left( A \alpha \cosh \alpha y + B \beta \cos \beta y \right)
\]

\[ A \left[ s \sinh \alpha y + r \alpha \cosh \alpha y \right] + B \left[ - t \sin \beta y + r \beta \cos \beta y \right] = 0 \]

The two equations found are

\[
A \left[ \alpha \cosh \alpha y - k \beta \sinh \alpha y \right] - B \left[ \beta s \cos \beta y - k \beta \sin \beta y \right] = 0
\]

(10)

\[
A \left[ r \alpha \cosh \alpha y + s \sinh \alpha y \right] + B \left[ r \beta \cos \beta y - t \sin \beta y \right] = 0
\]

(11)

Since these equations are simultaneous, homogenous, linear equations the constants A and B can be determined as follows.

\[
\begin{vmatrix}
\alpha \cosh \alpha y - k \beta \sinh \alpha y & - \left( \beta s \cos \beta y + k \beta \sin \beta y \right) \\
r \alpha \cosh \alpha y + s \sinh \alpha y & r \beta \cos \beta y - t \sin \beta y
\end{vmatrix} = 0
\]

(12)
\[
(\alpha \cosh \alpha y - k^4 \varepsilon \sinh \alpha y) (r \beta \cos \beta y - t \sin \beta y)
\]
\[
-(r \alpha \cosh \alpha y + s \sinh \alpha y) (-\beta s \cos \beta y - k^4 \varepsilon \sin \beta y)
\] = 0

\[
\alpha \beta r \cosh \alpha y \cos \beta y - \alpha r^2 \cosh \alpha y \sin \beta y - k^4 \varepsilon \beta \sinh \alpha y \cos \beta y + k^4 \varepsilon \sin \beta y
\]
\[
+ \alpha \beta s \cosh \alpha y \cos \beta y + k^4 \varepsilon \alpha \cosh \alpha y \sin \beta y + \beta s^2 \sinh \alpha y \cos \beta y + k^4 \varepsilon s \sinh \alpha y \sin \beta y = 0
\]

\[
[\sinh \alpha y \sin \beta y]
\left[
(\alpha \beta r + \alpha \beta s) \coth \alpha y \cot \beta y + (k^4 \varepsilon r - \alpha r^2) \coth \alpha y
\right]
\]
\[
+ (\beta s^2 - k^4 \varepsilon \beta) \cot \beta y + k^4 \varepsilon (t + s)
\] = 0

\[
[\sinh \alpha y \sin \beta y]
\left[
\alpha \beta r (t + s) \coth \alpha y \cot \beta y + \alpha (k^4 \varepsilon r - t^2) \coth \alpha y
\right]
\]
\[
+ \beta (s^2 - k^4 \varepsilon r) \cot \beta y + k^4 \varepsilon (t + s)
\] = 0 \quad (13)

Variable substitution to obtain Windenburg’s equation 9

\[
s = \alpha^2 - vk^2
\]
\[
t = \beta^2 + vk^2
\]
\[
\alpha = k \sqrt{\mu + 1}
\]
\[
\beta = k \sqrt{\mu - 1}
\]

\[
[\sinh \alpha y \sin \beta y]
\left[
\alpha \beta r (t + s) \coth \alpha y \cot \beta y + \alpha (k^4 \varepsilon r - t^2) \coth \alpha y
\right]
\]
\[
+ \beta (s^2 - k^4 \varepsilon r) \cot \beta y + k^4 \varepsilon (t + s)
\] = 0 \quad (14)

\[
[\sinh \alpha y \sin \beta y]
\left[
2k^4 r \mu \sqrt{\mu^2 - 1} \coth \alpha y \cot \beta y + \alpha k^4 (\varepsilon r - (1 - \mu - \nu)^2) \coth \alpha y
\right]
\]
\[
+ \beta k^4 ((1 + \mu - \nu)^2 - \varepsilon r) \cot \beta y + 2k^6 \mu \epsilon
\] = 0

Divide by \(k^4\) and rearrange
\[
[\sinh \alpha y \sin \beta y] \left[ \beta (1 + \mu - \nu)^2 \cot \beta y - \alpha (1 - \mu - \nu)^2 \coth \alpha y + 2k^2 \mu e \right. \\
\left. + 2r \mu \mu^2 - 1 \cot \alpha y \cot \beta y + \varepsilon r (\alpha \coth \alpha y - \beta \cot \beta y) \right] = 0 \tag{15}
\]

Change of variables to achieve non-dimensional variables

\[
\alpha = \frac{\sqrt{\phi}}{b} \sqrt{\psi + \phi} \\
\beta = \frac{\sqrt{\phi}}{b} \sqrt{\psi - \phi} \\
r = \frac{C \phi^2}{Db^2} \\
\theta = \frac{EI}{bD} - \frac{Ay^2}{bh\phi^2} \tag{16}
\]

Substituting 16 into the second term of 15 yields the final solution

\[
\sqrt{\psi - \phi} \left[ \psi + (1 - \nu) \phi \right]^2 \cot \phi - \phi^2 - \sqrt{\psi + \phi} \left[ \psi - (1 - \nu) \phi \right]^2 \coth \phi + \phi^2 \\
+ 2\phi^2 \psi \psi + 2\phi^2 \psi - \frac{C}{Db} \sqrt{\psi^2 - \phi^2} \coth \phi \psi + \phi^2 \cot \phi \psi - \phi^2 \\
+ \phi^4 \theta \frac{C}{Db} \left( \sqrt{\psi + \phi} \coth \phi \psi + \phi^2 - \sqrt{\psi - \phi} \cot \phi \psi - \phi^2 \right) = 0 \tag{17}
\]
Appendix B – Cross-Sections used for FEA
Figure 39 Asymmetric Bulb Bruhn Cross-Section

Figure 40 Enlarged Area Symmetric Bulb Cross-Section
Appendix C FEA Mode Shape Figures

Figure 42 Square Tee-Mode-Shape Nastran

Total Deformation
Type: Total Deformation
Load Multiplier: 29270
Unit: in
5/1/2008 1:28 PM

Figure 43 Square Tee-Mode-Shape ANSYS

0.000 3.500 7.000 (in)
1.750 5.250
Figure 44 Symmetric Bulb Mode-Shape Nastran

Total Deformation
Type: Total Deformation
Load Multiplier: 31069
Unit: in
5/1/2008 1:30 PM

Figure 45 Symmetric Bulb Mode-Shape ANSYS
Figure 46: Asymmetric Bulb 1 Mode-Shape Nastran

Total Deformation
Type: Total Deformation
Load Multiplier: 31927
Unit: in
5/1/2008 1:31 PM

Figure 47: Asymmetric Bulb 1 Mode-Shape ANSYS
Figure 48 Asymmetric Bulb 2 Mode-Shape Nastran

Total Deformation
Type: Total Deformation
Load Multiplier: 32928
Unit: in
5/1/2008 1:32 PM

Figure 49 Asymmetric Bulb 2 Mode-Shape Ansys
Figure 50 Asymmetric Bulb 3 Mode-Shape Nastran

Total Deformation
Type: Total Deformation
Load Multiplier: 34995
Unit: in
5/1/2008 1:50 PM

Figure 51 Asymmetric Bulb 3 Mode-Shape ANSYS
Figure 52 Asymmetric Bulb Bruhn Mode-Shape Nastran

Total Deformation
Type: Total Deformation
Load Multiplier: 23991
Unit: in
5/1/2008 1:49 PM

Figure 53 Asymmetric Bulb Bruhn Mode-Shape ANSYS
Appendix D MATLAB Codes

MATLAB CODE 1 Windenberg_plots_dichotomy_solver_rl.m

```matlab
clear;
clc;
close all;
warning off all

% User modified values
x_lim_l = .010; % x lower limit for plotting (zero seems to break the function)
x_lim_r = 10.00; % x upper limit for plotting (7 is a default value, but the plots
                % converge by 5)
step_size = .10; % the step size for the solver. smaller steps should be more accurate at
                 % the cost of CPU time
Rf = 0;
v = 0.3; % Poisson's ratio
error = .100; % error to solve within
max_iter = 30000; % an escape check if the solver doesn't converge

%Plotting
color=[r',g',b',c',m',y',k',r',g',b',c',m',y',k'];% Changes the line color for visualization
                                   % purposes ->Red, green, blue, cyan,
                                   % magenta, yellow, black

j=1;
hold on
%end plotting

tic %start the timer

for th = 0:5:40 % This allows for a family of curves for various values of theta to be
               % generated form: lowervalue:stepsize:uppervalue
    i=1;
    for ph = x_lim_l:step_size:x_lim_r

        x(i) = ph; % Track the values of phi in a vector for later plotting.
        iter = 0; % Track the number of iterations until the desired error-level is reached.
                   % This is also used against the max_iter value to provide an escape if the
                   % convergence isn't met.
        erf = 999999; % Initializes the error at an unacceptably high value.
    
        % check what side of the asymptote we are one
        k=1;
```
for k=1:2
    ps = pi^2/ph+ph/k/100;
    a = (sqrt(ps-ph).*(ps+(1-v).*ph).^2).*cot(sqrt((ph.*ps)-ph.^2));
    b = (sqrt(ps+ph).*((ps-(1-v).*ph).^2).*coth(sqrt((ph.*ps)+ph.^2)));
    c = (2.*ph.^((5/2).*ps.*th));
    d = (2.*ph.^((3/2).*ps.*Rf.*sqrt(ps.^2-ph.^2).*coth(sqrt((ph.*ps+ph.^2)))).*cot(sqrt((ph.*ps-ph.^2))));
    e = (ph.^4.*th.*Rf.*(sqrt(ps+ph).*coth(sqrt((ph.*ps+ph.^2)))-(sqrt(ps-ph).*cot(sqrt((ph.*ps-ph.^2)))))
    cs(k) = a + b + c + d + e;
end

%%%%%%%%%%%
%% CASE 1 %%%
%%%%%%%%%%%
if cs(2)<cs(1)
    ps_l = pi^2/ph+ph + 0.01;
    if (ps_l>0)
        ps_l = pi^2/ph+ph - 0.005;
    end
    ps_r = ps_l + 2;
    if (ps_r <=0)
        ps_r = ps_l + 10
    end
while (erf>error) && (iter<=max_iter)
    ps = (ps_l + ps_r)/2;
    % This section is a breakdown of Windenburg's work as is in the paper.
    a = (sqrt(ps-ph).*((ps+(1-v).*ph).^2).*cot(sqrt((ph.*ps)-ph.^2)));
    b = (sqrt(ps+ph).*((ps-(1-v).*ph).^2).*coth(sqrt((ph.*ps)+ph.^2)));
    c = (2.*ph.^((5/2).*ps.*th));
    d = (2.*ph.^((3/2).*ps.*Rf.*sqrt(ps.^2-ph.^2).*coth(sqrt((ph.*ps+ph.^2)))).*cot(sqrt((ph.*ps-ph.^2))));
    e = (ph.^4.*th.*Rf.*(sqrt(ps+ph).*coth(sqrt((ph.*ps+ph.^2)))-(sqrt(ps-ph).*cot(sqrt((ph.*ps-ph.^2)))))
    eql = a + b + c + d + e;
    %Dichotomy for CASE 1
    if (eql<0)
        ps_l = ps;
    elseif (eql>0)
        ps_r = ps;
    else
        fprintf('ERROR... CASE 1 DICHOT!');
    end
erf = abs(eq1);
iter = iter + 1;
end

fprintf('Iteration %d yields eq1 = %d \n',iter,eq1);

y(i) = ps;
plot(x(i),ps_g,'o','markersize',1.5);
i = i+1;

%%%%%% CASE 2 %%%
elseif(cs(2)>cs(1)) && (ph > 4)
    ps_r = pi^2/ph+ph - 0.0005;
    if(ps_r>0)
        fprintf('here i am\n');
        ps_r = pi^2/ph+ph + 0.00005;
    end
    ps_l=0.01;
    if(ps_l<=0)
        ps_l = ps_r - 3
    end
while (erf>error) && (iter<=max_iter)
    ps = (ps_l + ps_r)/2;
    a = (sqrt(ps-ph)*((ps+(1-v)*ph)^2)*cot(sqrt((ph*ps)-ph^2)));
    b = (sqrt(ps+ph)*((ps-(1-v)*ph)^2)*coth(sqrt((ph*ps)+ph^2)));
    c = (2*ph^(3/2)*ps*th);
    d = (2*ph^4*th*Rf*(sqrt(ps+ph)*coth(sqrt(ph*ps+ph^2))-(sqrt(ps-ph)*coth(sqrt(ph*ps-ph^2)))));
    eq1 = a + c + d + e;

    Dichotomy for CASE 2
    if(eq1>0)
        ps_l = ps;
    elseif(eq1<0)
        ps_r = ps;
    else
        fprintf('ERROR... CASE 2 DICHOT!');
    end

D3
erf = abs(eq1);
iter = iter + 1;
end

%fprintf('Iteration %d yields eq1 = %d \n',iter,eq1);

y(i) = ps;
%plot(x(i),ps_g,'o','markersize',1.5);
i = i+1;


%%%%% CASE 3 %%%

elseif(cs(2)>cs(l))&&(ph<= 4)

ps = 3.0;
erf=999999;
while (erf>0) && (iter<=max_iter)

ps = ps + .0001;
% This section is a breakdown of Windenburg's work as is in the paper.
a=(sqrt(ps-ph)*((ps+(l-v)*ph^2)*cot(sqrt((ph*ps)-ph^2))));
b = (sqrt(ps+ph)*((ps-(l-v)*ph^2)*coth(sqrt((ph*ps)+ph^2))));
c = (2*ph^((5/2)*ps*th));
d = (2*ph^((3/2)*ps*RF*sqrt(ps^2-

ph^2)*coth(sqrt((ph*ps+ph^2)))*cot(sqrt((ph*ps-ph^2))));
e = (ph^4*th*RF*(sqrt((ps+ph)*cot(sqrt((ph*ps+ph^2)))-(sqrt(ps-

ph)*cot(sqrt((ph*ps-ph^2))))));

eql = a - b + c + d + e;

% Case 3 doesn't use a dichotomy. Rather it will increment
% the value slowly until the sign goes negative.
erf = (eq1);
iter = iter + 1;
%fprintf('%d %d %d\n',iter,ps,erf)
end

fprintf('Iteration %d yields eq1 = %d \n',iter,eq1);

y(i) = ps;
i = i+1;

else
fprintf('
C(2) was equal to C(1) or else something happened... So,
the case-check broke down here!\n');
end
end
plot(x,y,color(j)) \% will have problems if more than 14 plots at a time due to color
indexing, this can be fixed by adding more colors to color[]

hold on
j=j+1;
end
toc \%CPU run time was

figure(1)
xlim([0,6]);
ylim([0,10]);
title('\psi versus \phi')
xlabel('\phi')
ylabel('\psi')
MATLAB CODE 2 bulbsizing.m

```matlab
% MATLAB code for bulb sizing

clear;
clc;

syms f r d

t = 0.0625;
h = 0.0625;
b = 1;

%Sizing using Theta=20
xf = double(solve((f^3*t)/h^4-4.4*(f*t)/h^2-22*b/h,f))
x = double(solve(r^4/h^4-1.47*r^2/h^2-2.33*b/h,r))
xb = double(solve(d^4/h^4-1.6*d^3/h^3-.374*d^2/h^2-7.44*b/h,d))

%Sizing using Theta=10
%x = double(solve((f^3*t)/h^4-4.4*(f*t)/h^2-11*b/h,f))
%x = double(solve(r^4/h^4-1.47*r^2/h^2-1.17*b/h,r))
%x = double(solve(d^4/t^4-1.6*d^3/t^3-.374*d^2/t^2-7 44*b/t,d))

%Sizing Theta = 40
xf40 = double(solve((f^3*t)/(12*h^4)-(6.4615*f*t)/h^2-3.6615*(b/h),f))
x_40 = double(solve(r^4/h^4-25.846*r^2/h^2-4.6620*b/h,r))
x_40_b = double(solve(d^4/h^4-1.6001*d^3/h^3-19.8835*d^2/h^2-14.9232*b/h,d))
```

D6
clear
clc

syms phi psi
% C = 5.7950e+005
% h = .0625
% D = (10.8e6*h)/(12*(1-.3^2))
% b = 1-.0625
x = 7.3

ezplot(2*psi+phi^(3/2)*(x)*sqrt(psi+phi)*coth(sqrt(phi*psi+phi^2))-sqrt(psi-phi)*cot(sqrt(phi*psi-phi^2)),[0.8,0.10])