Investigation of Magnetic Field Profile Effects in Hall Thrusters

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Investigation of Magnetic Field Profile Effects in Hall Thrusters

by

Oren Kornberg

A Thesis submitted in partial fulfillment of the requirements
for the degree of Master of Science in Space Science

Physical Science Department

Embry-Riddle Aeronautical University
Daytona Beach, FL

April 2007
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Investigation of Magnetic Field Profile Effects in Hall Thrusters

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Oren Kornberg

This thesis was prepared under the direction of the candidate's thesis committee chair, Dr. M. Anthony Reynolds, Department of Physical Science, and has been approved by the members of his thesis committee. It was submitted to the Department of Physical Science and was accepted in partial fulfillment of the requirements for the Degree of Master of Science in Space Sciences

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ABSTRACT

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Master of Science in Space Science

2007

The purpose of this study was to show the relationship of different magnetic field profiles to the acceleration region length of a Hall thruster. The general transport equations were simplified and solved using a one-dimensional analysis. Some of the model assumptions include quasineutrality, Maxwellian electrons, and negligible thruster wall effects. The solved equation kept magnetic field as an input to the model for the analysis. The magnetic field was altered by changing the shape through the thruster, while keeping the maximum point fixed, and by shifting the profile, while keeping the shape fixed. Results indicate a strong correlation between the average magnetic field and the length of the acceleration chamber of the Hall thruster.
Acknowledgements

As I come to realize one of my life's dreams, I must give proper thanks to those who have helped me through the years. This project could not have been accomplished solely by myself. Yes, I have done the writing and have defended my work, but many people have played a part to offer their help in their own ways to drive me to this ultimate goal.

I would first like to thank my research advisor, Dr. Mark Anthony Reynolds. He has been the essence of strength and the great angel that sits on my shoulder to guide me down the path of least resistance in this obstacle-filled journey. Not only has he guided my path, he also transformed into a propulsion device to propel me from turtle to rabbit when I was just treading along. When I seemed to run out of steam or did not have any more fuel to burn, he was the miracle that brought forth the energy within to keep me on track and never let me lose my general focus. After a long dry spell, that he so kindly called "writer's block," it seemed he transformed again into a giant hammer: with the power to cause earthquakes that shook me violently back into reality. There are few words that could ever describe the great admiration and respect I have for the person who took me from my lowest times, when I had doubts of ever completing this, to a time when I can be proud of the work that I have accomplished. Not only does he have transformation properties, he contains a seemingly endless supply of sincere energy and time devoted to helping me succeed. So, to put it simply, but with the breadth of the universe in its feeling: Thank you.

In addition, I would also like to thank the rest of my thesis committee: Dr. Mehmet Sozen, Dr. Charles Vuille, and Dr. Eric Perrell for their time and assistance in refining this work. Any errors that remain are my own.
I also want to thank my parents for their undying support and love for me as I embraced this journey. They supported me in times of hardship and celebrated with me in times of accomplishment. For my mother, who always gave her unending and unconditional love and encouragement. And for my father, who has given me a wonderful example in which to follow and showed that, with the right mindset and the right heart, I can accomplish my dreams in life.

Most importantly, I want to thank my wife, Sharon. I want you to know that you have been my rock and redeemer from all of life’s hardships. When I am utterly frustrated or a perturbation has happened to me, you are always there with your psychological wisdom and love to carry me through the toughest times. You will stop everything, as if to stop time itself, and work with me on resolutions to lifes most difficult circumstances. You are also there when I am shining brightly and give me the wind under my wings to sore. You are my life’s truest meaning of hope and love and I could never have accomplished this without you. Thank you for being with me on this amazing passage.
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Nomenclature

\( \varepsilon_0 \)  
permittivity of free space, 8.8542 \( \times 10^{-12} \) 
\([F/m]\)

\( e \)  
electron charge, 1.6022 \( \times 10^{-19} \) 
\([C]\)

\( g \)  
gravitational acceleration, 9.80665 
\([m/s^2]\)

\( m_e \)  
electron mass, 9.1094 \( \times 10^{-31} \) 
\([kg]\)

\( \dot{\mathbf{A}}_\alpha \)  
momentum collision term 
\([kg/m^2 \ s^2]\)

\( \ddot{a} \)  
acceleration vector 
\([m/s^2]\)

\( \mathbf{B} \)  
magnetic flux density 
\([T]\)

\( B \)  
magnitude of magnetic field 
\([T]\)

\( B_0 \)  
maximum magnetic flux 
\([T]\)

\( c \)  
exhaust velocity of the propellant 
\([m/s]\)

\( d \)  
thickness of the plasma sheath 
\([m]\)

\( \mathbf{E}_\perp \)  
electric field perpendicular to magnetic field 
\([V/m]\)

\( \mathbf{E} \)  
electric field 
\([V/m]\)

\( f_\alpha \)  
velocity distribution function of species \( \alpha \) 
[–]
\( \vec{F} \)  force vector of a charged particle  \([N]\)

\( \vec{f} \)  force created by crossing electric and magnetic fields  \([N]\)

\( f(\vec{x}) \)  scaled strength of the magnetic field  \([-]\)

\( I_t \)  total impulse of thruster  \([N \cdot s]\)

\( I_{sp} \)  specific impulse of thruster  \([s]\)

\( \vec{j} \)  current density  \([A/m^2]\)

\( k_a \)  attachment coefficient  \([\text{particles} \ m^3/s]\)

\( k_i \)  ionization coefficient  \([\text{particles} \ m^3/s]\)

\( k_r \)  recombination coefficient  \([\text{particles} \ m^6/s]\)

\( \mathcal{L} \)  length of acceleration region  \([m]\)

\( L_m \)  halfwidth of the Maxwellian profile magnetic field  \([m]\)

\( \dot{m} \)  mass flow rate of the propellant  \([mg/s]\)

\( m_\alpha \)  mass of particle  \([kg]\)

\( M \)  mass of the spacecraft  \([kg]\)

\( M_0 \)  initial mass of the spacecraft with propellant  \([kg]\)

\( \dot{M}_\alpha \)  rate of energy density change due to collisions  \([J/m^3 \cdot s]\)

\( m_p \)  mass of particle  \([kg]\)

\( n_\alpha \)  number density of specie \( \alpha \)  \([m^{-3}]\)

\( n_e \)  electron number density  \([m^{-3}]\)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{P}$</td>
<td>pressure tensor</td>
<td>$[N/m^2]$</td>
</tr>
<tr>
<td>$P$</td>
<td>power in the exhaust of the thruster</td>
<td>$[W]$</td>
</tr>
<tr>
<td>$P_{sys}$</td>
<td>power required by a complete propulsion system</td>
<td>$[W]$</td>
</tr>
<tr>
<td>$q_a$</td>
<td>charge of particle</td>
<td>$[C]$</td>
</tr>
<tr>
<td>$q$</td>
<td>charge of particle</td>
<td>$[C]$</td>
</tr>
<tr>
<td>$\vec{r}$</td>
<td>position vector of a particle</td>
<td>$[m]$</td>
</tr>
<tr>
<td>$r_c$</td>
<td>cyclotron radius of charged particle around the magnetic field</td>
<td>$[m]$</td>
</tr>
<tr>
<td>$S_a$</td>
<td>source term for particles due to collisions</td>
<td>[# particles /s]</td>
</tr>
<tr>
<td>$S_e$</td>
<td>electron source term from collision</td>
<td>[# electrons /s]</td>
</tr>
<tr>
<td>$S_i$</td>
<td>ion source term from collision</td>
<td>[# ions /s]</td>
</tr>
<tr>
<td>$S_n$</td>
<td>neutral source term from recombination</td>
<td>[# neutrals /s]</td>
</tr>
<tr>
<td>$T$</td>
<td>temperature of electrons</td>
<td>$[eV]$</td>
</tr>
<tr>
<td>$T$</td>
<td>thrust</td>
<td>$[N]$</td>
</tr>
<tr>
<td>$U$</td>
<td>velocity of the spacecraft</td>
<td>$[m/s]$</td>
</tr>
<tr>
<td>$u_{e\theta}$</td>
<td>Hall drift velocity in the thruster</td>
<td>$[m/s]$</td>
</tr>
<tr>
<td>$\bar{c}_a$</td>
<td>random or thermal particle velociy</td>
<td>$[m/s]$</td>
</tr>
<tr>
<td>$\bar{u}_a$</td>
<td>average particle velociy</td>
<td>$[m/s]$</td>
</tr>
<tr>
<td>$\bar{v}_a$</td>
<td>particle velocity</td>
<td>$[m/s]$</td>
</tr>
<tr>
<td>$\bar{v}_\perp$</td>
<td>velocity as seen by an observer moving at velocity $\vec{u}_E$</td>
<td>$[m/s]$</td>
</tr>
</tbody>
</table>
\( \vec{v}_\perp \) velocity of particle perpendicular to magnetic field \([m/s]\)

\( \vec{v}_E \) velocity of the plane normal to the magnetic field \([m/s]\)

\( \vec{v} \) velocity of particle \([m/s]\)

\( z \) distance in the axial direction of the thruster \([m]\)

\( z_m \) location where the maximum magnetic flux lies \([m]\)

\( \beta \) scale factor for energy relation \([-\]

\( \chi \) some physical property of a particle in the plasma \([-\]

\( \lambda \) dimensionless parameter inversely proportional to \( \mu_e \) \([-\]

\( \mu_e \) axial electron mobility \([m^2/V \cdot s]\)

\( \eta_{sys} \) efficiency of a complete propulsion system \([-\]

\( \nu_c \) collision frequency \([s^{-1}]\)

\( \nu_{\alpha\beta} \) collision frequency for momentum transfer \([s^{-1}]\)

\( \Omega_H \) Hall parameter \([-\]

\( \omega_c \) gyrofrequency of a particle moving around the magnetic field \([s^{-1}]\)

\( \phi \) potential in the thruster \([V]\)

\( \phi_a \) potential at the anode of the thruster \([V]\)

\( \phi_w \) potential variation from the plasma to the electrode wall \([V]\)

\( \sigma \) conductivity tensor \([A/V \cdot m]\)

\( \sigma_0 \) longitudinal electrical conductivity \([A/V \cdot m]\)
\( \sigma_H \)  Hall conductivity  
\( \sigma_P \)  Pedersen conductivity  
\( \sigma_z \)  longitudinal conductivity  
\( \alpha \)  specie of particle
Chapter 1

Introduction

1.1 Motivation for Electric Propulsion

The motivation for using electric propulsion rather than the traditional chemical propulsion is that chemical propulsion is limited by the energy release of its propellant. A chemical rocket utilizes the energy released from chemical bonds to produce thermal energy, which is used to expand the propellant through a nozzle, creating thrust. The energy is limited by the actual propellant that is being used in the combustion of the rocket motor. In electric propulsion, there is no limit to the amount of energy that can be deposited into the propellant [3]. Therefore, exhaust velocities for an electric propulsion system can be many times greater than chemical systems. In order to put this into perspective, we look toward gaining information into the performance of a propulsion system.

Both chemical and electric propulsion systems can be modeled using the conservation of momentum:

\[ M(t) \frac{d}{dt} U(t) = T = \dot{m}c, \tag{1.1} \]

where \( M(t) \) and \( U(t) \) are the mass and velocity of the spacecraft, respectively, \( T \) is
the thrust (relative to the rocket), \( \dot{m} \) is the mass flow rate of the propellant, and \( c \) is the exhaust velocity of the propellant. The time-varying mass is due to the expulsion of propellant for thrust production, which, if \( \dot{m} \) is constant, is

\[
M(t) = M_0 - \dot{m}t,
\]

where \( M_0 \) is the initial mass of the spacecraft, with propellant. If we differentiate Eqn. 1.2 we get

\[
\frac{dM}{dt} = -\dot{m},
\]

which seems intuitive since that is the only way the system is losing mass. Substituting Eqn. 1.3 into 1.1, and canceling the time derivatives, we arrive at

\[
dU = -\frac{1}{M} dM.
\]

This can be integrated over the total velocity and mass change to show how the ratio of final to initial spacecraft mass is a function of the velocity increment \( \Delta U \) and the exhaust velocity

\[
\frac{M_f}{M_0} = e^{-\frac{\Delta U}{c}}
\]

Equation 1.5 is known as the rocket equation and shows the relationship of the exhaust velocity to the mass ratio of the spacecraft. This shows that a larger mass fraction is achieved by increasing the exhaust velocity of the propellant for a desired change in spacecraft velocity.

Although the rocket equation is important, it does not give the whole picture of thruster performance. The next specification that we look at is the total impulse of the spacecraft, which is the spacecraft thrust integrated over time. Total impulse has units of Newton-seconds (\( N \cdot s \)) and tells how long a thruster can operate at a
constant thrust:

\[ I_t = \int_0^t T \, dt. \quad (1.6) \]

The total impulse per unit weight of a propellant (under sea-level acceleration of the Earth’s gravity) is called “specific impulse” and is useful to show how effectively a propellant is converted into usable thrust:

\[ I_{sp} = \frac{\int_0^t T \, dt}{g \int_0^t \dot{m} \, dt} = \frac{T}{g \dot{m}} = \frac{c}{g} \quad (1.7) \]

Notice how specific impulse is about one-tenth of the effective exhaust velocity (in SI units). These two quantities are commonly interchanged in the rocket equation. Nominal chemical rocket specific impulse usually ranges between 170-450 s, whereas electric propulsion can reach \( I_{sp} \) of about 10,000 s. A nominal Hall thruster specific impulse ranges between 1000-3000 s [2].

The final specification that is derived and the one that shows the key advantage in electric propulsion is the system efficiency. We start with the idea that a rocket converts energy into thrust. This can be shown with the thrust power equation:

\[ P = \frac{1}{2} \dot{m} c^2 = \frac{1}{2} T c, \quad (1.8) \]

where \( P \) is the kinetic power of the thrust in the thrust direction. Due to thermodynamics, we know that there must be some penalty in producing thrust power from available system power. We define the total system efficiency as the ratio of the thrust power to the input power from the system:

\[ \eta_{sys} = \frac{P}{P_{sys}} = \frac{1}{2} \frac{T c}{P_{sys}} = \frac{1}{2} g I_{sp} \frac{T}{P_{sys}}. \quad (1.9) \]
Equation 1.9 shows how the specific impulse, thrust, and system efficiency and input power can be specified to describe the performance of the space propulsion system.

Although not having the high thrust of its chemical counterparts, the high specific impulse and low fuel usage of some forms of electric propulsion make it possible to have months of constant thrust as opposed to minutes with chemical engines. Allowing thrusters to be on for longer periods of time, with less thrust than chemical propulsion, also proves to be useful in making more precise changes in direction as compared to the chemical thrusters [3].

1.2 Electric Propulsion Concepts

Electric propulsion systems use an external power source to create thrust from a given propellant, whereas chemical propulsion systems use the internally stored molecular bond energy of the propellant. In each form of electric propulsion, energy is transferred to the propellant differently. To differentiate the ways that electric propulsion systems transfer energy to produce thrust, these systems can be categorized into three groups: electrothermal propulsion, electrostatic propulsion, and electromagnetic propulsion.

Electrothermal propulsion uses the same notion as a standard rocket engine of expanding propellant for thrust. The basic principle is using electrical energy to heat a propellant gas and expel the gas through a supersonic nozzle to create thrust. Generally, the energy to heat the gas comes in the form of radiation (a heater coil or heated walls), direct contact with an electrical arc, or by microwave energy. Each of these heating methods have their own advantages and disadvantages.

When a stationary electric field is used to accelerate charged particles, this is categorized as electrostatic propulsion. The basic idea of this system can be seen in
Figure 1.1: **Thruster Using Electrostatic Principles.** An electric field is generated from the high-potential anode (in this case, also the ion source) to the low-potential accelerating electrode. The details of the how the ions are created is left out of this figure, but they enter in from the left of the thruster and accelerate towards the accelerating electrode. As they leave the thruster, a neutralizer (electron emitter of some form) allows for recombination of the ions into neutrals. [1]

Fig. 1.1. Although one accelerating electrode, or grid, is seen on this schematic, the mission thrusters usually have a screen grid and an acceleration grid to pull the ions from the ion source and to accelerate them to the desired velocity, respectively, as explained by Choueri[1].

If both electric and magnetic properties are used for propulsion, mainly to impede and accelerate the velocities of charged particles, we have electromagnetic propulsion. Magnetic fields, although used in other types of propulsion systems to contain plasmas, are used here in combination with the electric field to cause acceleration in the direction perpendicular to both the electric and magnetic fields. As we will see, because there are both electric and magnetic fields involved, there are more possibilities for propulsion as compared to electrothermal and electrostatic thrusters. The best way to describe the principle behind electromagnetic thrusters is with a schematic shown in Fig. 1.2. The crossing of the perpendicular electric and magnetic fields acts
Figure 1.2: Crossed Fields Acting on Ions. This shows a plasma, moving at a velocity \( \vec{u} \), subjected to perpendicular electric and magnetic fields. The electric field drives the current density \( \vec{j} \) (Eqn. 2.29) while the magnetic field interacts with the current density to accelerate it by the force stated in Eqn. 1.10. [1]

on a conducting fluid with a force \( \vec{f} \) given by:

\[
\vec{f} = \vec{j} \times \vec{B},
\]

(1.10)

where \( j \) is the current density and \( B \) is the magnetic field.

Three of the most common electromagnetic propulsion devices are the magnetoplasmadynamic thruster (MPDT), the pulsed-plasma thruster (PPT), and the Hall thruster. These thrusters, although all in the electromagnetic category, use different variations of electric and magnetic fields. A major difference between them is whether they use an applied or self-induced magnetic field to create the force shown by Eqn. 1.10. The MPDT and the PPT both contain self-induced magnetic fields and the Hall thruster uses an applied magnetic field. The concept of a self induced magnetic field is shown in Figure 1.3.

Although there is much to say about the different electromagnetic propulsion devices, only the Hall thruster will be discussed. Furthermore, there are many details which have not been covered for the different electric propulsion concepts and devices.
Current-induced magnetic field (out of page)

Figure 1.3: **Current-Induced Magnetic Field.** The flow of electrons and ions in opposite directions defines a current. A straight current is shown, such as in a wire, although this refers to gaseous plasma. A magnetic field is induced that loops around the center line of the current, just like in a wire. [2]

The interested reader is urged to read works by Choueiri [1] and Humble, Henry, and Larson [2].

### 1.3 Hall Effect

Hall thrusters are sometimes called Hall-effect thrusters because these engines utilize the so-called “Hall-effect” as a fundamental aspect in providing thrust. The Hall effect is the most important phenomenon that occurs in the Hall thruster and is the main physical effect that creates the basis for this thesis. The Hall effect is a basic result of elementary electrodynamics, and was discovered in 1879 by Edwin H. Hall [4]. Figure 1.4 shows a graphical interpretation of this phenomenon. Hall ran electric current through a thin gold conducting sheet and immersed the gold in a magnetic field that was perpendicular to the current that flowed through the sheet. He found that a potential difference $V_H$ (called the Hall voltage) was induced in a direction that was orthogonal to both the current and magnetic fields.

The applied current through the gold sheet implies an electric field as is defined
Figure 1.4: The Electronic Hall Effect. This shows a thin conducting plate with an applied electric current $I$ in the $x$ direction. A constant, uniform magnetic field is applied through the thickness of the plate (in the $z$ direction). The Lorentz magnetic force for a negatively charged particle, such as an electron, as shown in the top left of the diagram (as this one is assuming that $q$ is the magnitude of the charge), defines the force that acts upon a moving charged particle with velocity $\vec{v}$ in the presence of a magnetic field $\vec{B}$. This states that the electrons will rotate around the magnetic field in a circular motion. As the electrons are forced to rotate around $\vec{B}$, they also experience collisions with the other electrons that flow in the $-x$ direction, due to the applied current. This causes the electrons to drift to the $-y$ side of the plate and creates a detectable voltage difference through the plate in the $y$ direction. (Image taken from www.eeel.nist.gov/812/effe.htm)
by the simplified Ohm's law in the $x$ direction of Figure 1.4:

$$j_x = \sigma_0 E_x,$$  \hspace{1cm} (1.11)

where $j_x$ is the current density of charged particles moving through the gold sheet, $E_x$ is the implied electric field also in the $x$ direction, and $\sigma_0$ is the conductivity and is defined as:

$$\sigma_0 = \frac{n_e e^2}{m_e \nu_c},$$  \hspace{1cm} (1.12)

where $n_e$ is the electron particle density, $e$ is the charge, $m_e$ is the mass of an electron, and $\nu_c$ is the collision frequency between electrons and neutrals.

A uniform magnetic field forces the charge carriers (electrons in this case) to circulate about the magnetic field lines with a force relative to their velocity, as stated by the Lorentz force in a uniform magnetostatic field:

$$\vec{F} = q \vec{E} + q \left( \vec{v} \times \vec{B} \right),$$  \hspace{1cm} (1.13)

where $\vec{E}$ is the electric field, $\vec{B}$ is the magnetic field and $q$ and $\vec{v}$ are the electric charge and velocity of the particle, respectively. For the sake of explanation, let’s assume, for now, that there is no electric field $\vec{E} = 0$. Since the electron velocity is perpendicular to the magnetic field anywhere on the gold sheet, and the magnetic field is always pointing in the $z$ direction, Equation 1.13 can be simplified to

$$\vec{F} = q v_{\perp} B_z \hat{y}.$$  \hspace{1cm} (1.14)

This shows that the Lorentz force will provide a centripetal force for the particle in
a uniform magnetic field and thus:

\[ qv_\perp B_z = \frac{m_p v_\perp^2}{r_c}, \]  

(1.15)

where \( m_p \) is the mass of the particle and \( r_c \) is the radius of orbit, also known as the cyclotron radius:

\[ r_c = \frac{m_p v_\perp}{qB}. \]  

(1.16)

When the effects of the magnetic field and the electric field are present, as is the case, Equation 1.13 becomes:

\[ m^\perp = q(E_\perp + v_\perp B), \]  

(1.17)

We have assumed no electric field perpendicular to \( B \). In order to solve this equation, let

\[ \vec{v}_\perp = \vec{v}_\perp' + \vec{v}_E, \]  

(1.18)

where \( \vec{v}_E \) is the constant velocity in the plane normal to \( B \) and \( \vec{v}_\perp' \) is the velocity as seen by an observer in the reference frame moving at velocity \( \vec{v}_E \). Substituting Eqn. 1.18 into Eqn. 1.17 and writing the component of the electric field perpendicular to \( B \) as

\[ \vec{E}_\perp = -\left( \frac{\vec{E}_\perp \times \vec{B}}{B^2} \right) \times \vec{B}, \]  

(1.19)

we attain

\[ m^\perp \frac{d\vec{v}_\perp}{dt} = q \left( \vec{v}_\perp' + \vec{v}_E - \frac{\vec{E}_\perp \times \vec{B}}{B^2} \right) \times \vec{B}, \]  

(1.20)

This shows that on a plane normal to \( B \) and moving with a velocity

\[ \vec{v}_E = \frac{\vec{E}_\perp \times \vec{B}}{B^2}, \]  

(1.21)
the particle is governed by

$$\vec{F} = q \left( \vec{v}_\perp \times \vec{B} \right), \quad (1.22)$$

which is essentially a more general form of equation 1.14 for the gold sheet. Again, this states that in a plane normal to $\vec{B}$ and moving at $\vec{v}_E$, the particle is moving in a circular motion with a radius defined by Eqn. 1.16.

The total resulting motion of a particle immersed in orthogonal magnetic and electric fields is the superposition of the velocity from equations 1.21 and 1.15:

$$\vec{v} = \frac{qBr_c}{m_p} + \frac{\vec{E}_\perp \times \vec{B}}{B^2}, \quad (1.23)$$

The first term on the right hand side represents the cyclotron circular motion, while the second term represents the guiding center drift of the particle. The first term can be simplified to

$$\vec{v}_\perp = \hat{\omega}_c \times \vec{r}_c, \quad (1.24)$$

where $\hat{\omega}_c = \frac{qB}{m}$ which is the gyrofrequency of the particle around the magnetic field at a radius $r_c$.

The resultant motion of Eqn. 1.23 describes a cycloid as shown in Figure 1.5. The physical explanation that follows is supported if we look at the motion of the electrons in Figure 1.5. As the electrons gyrate around the magnetic field in a counter-clockwise motion, they also experience the force from the electric field, which for the electrons, is in a direction opposite of $E$. The electric field accelerates the electrons when they are on their downward leg of the motion and decelerates them when they are on their upward leg of the motion. Equation 1.16 states that the cyclotron radius increases as the velocity increases, therefore varying the curved path that the electron follows in the presence of both a magnetic and electric field.
Figure 1.5: **Cycloid Motion of Charged Particles.** Crossed electric and magnetic fields are shown with the particle motion in the perpendicular plane to the magnetic field. The charged particles have a circular motion around the magnetic field superimposed with a constant velocity in the $\vec{E} \times \vec{B}$ direction. This results in a cycloid motion. The particles are accelerated by the electric field so as to either increase or decrease their velocity as they rotate about the magnetic field. The electron speed, for example, increases when its motion is in the $-\vec{E}$ direction and decreases in the $\vec{E}$ direction, causing its gyroradius to increase and decrease, respectively. (Part of the image taken from www.phy6.org/Education/wdrift.html)
A positively charged particle will revolve in the opposite direction, but still have the same overall drift. An ion is much heavier than an electron and will have a much larger gyroradius, but its cyclotron frequency will be smaller, making the $\vec{E} \times \vec{B}$ drift velocity the same for both types of particles.

Since both species drift at the same speed, one could assume that there would be no resulting current, but this is only the case in a collisionless plasma, where ions and electrons are free to move together. In a normal circumstance, such as in the gold sheet used by Hall, the ion-neutral collision frequency is much greater than electron-neutral collision frequency, and this generates an electric current. This current is in the $- (E \times B)$ direction and is considered the Hall current.

### 1.4 Hall Thrusters

The main ingredient of the Hall thruster is the plasma, which it uses to produce thrust. Plasma, as coined by Langmuir in 1928 [5], is an ionized gas containing an approximately equal number of positive ions and electrons, and therefore is macroscopically neutral in charge. Since the atoms are ionized, the ions and electrons are no longer bound to each other, and the result is that plasmas exhibit a behavior richer than that of neutral gases. Plasma can be produced by energizing a neutral gas (via light, heat, or electric fields), which compels the outermost orbital electrons to overcome their binding energies. Xenon (as well as other noble gases) is a common element used as fuel for Hall thrusters, since it is non-reactive in its natural state and it has a large molecular weight.

In order to use this plasma for propulsion, an external, fixed radial magnetic field and an axial electric field are generated in the cylindrical Hall thruster. Figure 1.6 illustrates the Hall thruster to aid in understanding the concepts. The electric field
is produced between an anode, at the upstream\(^1\) end of the plasma, and a cathode near the exhaust. It is self-consistently generated by the plasma. This electric field interacts with the radial magnetic field, which would normally generate an azimuthal Hall electric field, as stated in section 1.3. Although, in the Hall thruster, no electric field is created because the electrons are free to move in the azimuthal direction around the axis of symmetry in the Hall thruster and generate an azimuthal Hall current near the thruster exit. The electrons in the Hall current are used to ionize the neutrals. The magnetic field strength and the electron collisions alter the Hall current and the electron mobility toward the anode, which is discussed in section 2.3.2. This azimuthal Hall current also crosses the magnetic field perpendicularly, and creates a current in the \((\vec{E} \times \vec{B}) \times \vec{B}\) direction, which accelerates the plasma ions axially outward. The electric field in the thruster is sustained by the current flow within the thruster, as stated by Ohm's law, not by the electrostatic potential from two electrodes.

Correspondingly, since the thrust force in the Hall thruster acts both on electrons and ions, the space charge limitation that is experienced by an electrostatic thruster is not applicable with this type of electromagnetic thruster design. This is explained from the principles of the Child-Langmuir Law:

\[
j = \frac{4}{9} \left(\frac{2q}{m}\right) \frac{1}{d^2} \epsilon_0 \phi_w \frac{\phi_w}{\phi_w + d}\frac{1}{\epsilon_0} \frac{1}{d^2},
\]

where \(\epsilon_0\) is the permittivity of free space, \(\phi_w\) is the potential variation from the plasma to the electrode wall, \(d\) is the thickness of the electron-free region near the wall, and \(\frac{q}{m}\) is the charge-mass ratio of the charged particle used for thrust. With a given particle charge-mass ratio, and a given applied potential, there is a direct

\(^1\)Figure 1.6 can assist in showing that the anode is located at the opposite end of the exhaust and therefore is upstream of the plasma flow.
relation between how far apart the electrodes must be to achieve a certain current density, which in turn provides thrust. This is a problem in thrusters that use a grid to accelerate the ion, such as in Figure 1.1.

Along with supplying the high potential of the internal electric field, the anode serves as the injector for the Xenon gas into the Hall thruster. A typical flow rate is on the order of 1-10 mg/s of Xenon for a 5kW thruster. Providing the low potential of the electric field is the cathode. It is used to feed electrons into the Hall thruster and to neutralize the ions in the exhaust. Usually, hollow cathodes are used in Hall thrusters. A hollow cathode, shown in Figure 1.7 consists of a cylinder with an entrance and exit port to receive neutral gas and expel ions and electrons, respectively. Inside, a cylindrical insert with a low work function gives off electrons when initially heated [2]. These electrons ionize the incoming neutral gas as well as neutrals formed by recombination in the chamber. A cathode keeper, surrounding the cathode is set at a positive voltage with respect to ground, but more negative than the anode. This ensures that only electrons will be emitted by the cathode system. Other variants which are not hollow cathodes, may use a heated filament to allow for thermionic release of electrons. This method degrades the filament quickly and is less reliable than the hollow cathode, but uses no propellant [3].

Although the electrons from the cathode are intended for ionization and neutralization, they also contribute to losses in efficiency by colliding with the insulator channel walls. When primary electrons (directly from the cathode) hit the dielectric material, they embed themselves deep in the material and, because of the loss of kinetic energy, only secondary electrons are released back into the chamber. These secondary electrons lack the energy needed to ionize the Xenon, and their lower energy means they remain tied to the magnetic field lines until they diffuse toward the anode [3]. The insulator channel, as seen in Figure 1.6, has a width that is usually
Figure 1.6: Cutaway View of Hall a Thruster. Perpendicular electric and magnetic fields are applied by using an anode and cathode for the electric field and electromagnets for the magnetic field. The radial magnetic field interacts with the axial electric field producing a Hall current in the azimuthal $(\vec{E} \times \vec{B})$ direction. This traps electrons from the cathode near the exit of the thruster and forms a zone where the neutrals from the anode are ionized by the electrons. The ions, due to their larger mass, are not significantly affected by the magnetic field, allowing them to be accelerated outward by the electric field to produce thrust. Some of the electrons from the cathode then combine with the expelled ions and neutralize them so that the thruster does not build a negative charge. [3]
Figure 1.7: Hollow Cathode Schematic. The hollow cathode is initially heated to temperatures of about 1000°C to allow thermionic emissions for electrons from the insert. As propellant is fed into the cathode, it is ionized by the interior discharge and is emitted, along with the electrons, through a hole on the order of 1 mm in diameter. In order to keep the emissions to just the electrons, a cathode keeper is placed just outside of the cathode exit and is set to a potential higher than ground, but lower than the anode, to allow for emitted electrons to flow toward the anode. Since the particle density is low within the cathode, the electrons are able to attain a high kinetic energy (several eV), which makes a more efficient system for ionizing additional propellant [2].

10-15 percent of its outer diameter. It is one of the features unique to a Hall thruster called the stationary plasma thruster (SPT). It has a great deal to do with efficiency of the thruster. There is a greater probability, due to area and confinement, that both ions and electrons will hit the insulator and not each other. The main loss of efficiency comes from atoms that have been ionized and hit the dielectric material. Although they may be reemitted to the Hall chamber, the process of striking the dielectric material will have caused the ions to recombine to their atomic ground state.

There is a type of Hall thruster that does not have an insulator. This thesis deals only with the SPT, and the reader is urged to read Choueiri [6] for information on the difference between the thruster variants.
1.5 Research on Hall Thrusters

Development on Hall thrusters began experimentally in the United States in the early 1960's by Seikel, Reshotko, Brown and Pinsley [7, 8]. In the mid-1960's, Russia also began developing Hall thrusters with the main design from A.I. Morozov [9, 10] and work by Zharimov and Popov [11].

Although the first working devices were reported by Americans, the Hall thruster was further developed by the Russians. This was partly due to the abandonment of U.S. Hall thruster efforts in 1970, because Americans were unable to make these thrusters as efficient as gridded ion thrusters. The other part is that, at the same time, the Russians abandoned the ion thruster efforts because they were unsuccessful in developing the grids needed for those thrusters [12]. Hall thrusters were first flown in space in December 1971, on a Meteor satellite [13], and development continued through the 1970's with papers from Morozov [14] and Bishaev [15], just to name two. Since then, over one hundred Hall thrusters have flown in space for orbit maintenance and station keeping [16, 17].

Experimental research on Hall thrusters has involved increasing the efficiency and thrust of existing designs, and looking toward higher power thrusters [18, 19, 20, 21]. While computational research has dealt with the same topics as experimental research, there is a large area of research devoted to making better numerical models to more accurately describe the true physical processes in experiments. An example of difficulty in modeling experiments is that there is some form of anomalous electron transport that has not been investigated in detail. Research has been conducted to understand this effect more closely, namely by Keidar, Boyd, and Beilis [22]. This has since been blamed for electron current along the plasma sheath connected to the cylindrical walls, toward the anode. Now, the computational models are trying to
appropriately incorporate that issue into existing models of the Hall thruster.

The work being done in experimentation costs time and money, which is why a large part of Hall thruster research involves numerical analysis. Computers allow for designing and testing of many variations of a thruster in one day. Life-time tests, which take thousands of hours to run, take a fraction of the required time on computers, with time-scaling. Usually, computational research includes the modeling of Hall thrusters in one and two dimensions. One-dimensional models can provide a great deal of insight into the physics and workings of the Hall thruster. The one-dimensional models by Choueiri [6], Manzella [23] and many others in references [24, 25, 26, 27, 28] (to name a few) have brought forth many important results to the development of the thruster.

1.6 Overview of the Thesis

This thesis will deal with the effect of the magnetic field topology inside the Hall thruster. Of the many contributions to research, I feel that there are fewer attempts to characterize the effect of the magnetic field profile upon the performance parameters in the Hall thruster. I will look at how the specific shape of the magnetic field profile affects the Hall thruster when its peak is moved from the inside to the outside of the acceleration chamber. Overall, my research will hopefully give a starting point at which another can derive a two dimensional analysis or a more comprehensive one dimensional analysis of the physics in the thruster, while optimizing the magnetic field for different models of the thruster.

The plasma in the hall thruster will be divided into three fluids. The electron, ion, and neutral fluids will each have the moments of the general transport equations adapted by assumptions, to give a working model of the thruster. The three species
are modeled as fluids in thermal equilibrium. Poisson's equation is not solved, and therefore, the explicit time dependence of the charged fluids is neglected. This also leads to the assumption of quasineutrality of the plasma. This states that the electrons are so mobile, compared to the ions, that they react instantaneously to the motion of the ion fluid in order to keep the balance of neutral charge for the plasma as a whole; however, electric current is still allowed.

Plasma sheaths are left for future work, since they are largely non-neutral regions in the plasma. These would be located around the outer and inner wall of the cylindrical chamber, and attached to the anode. However, since these regions are shown to be small compared to the dimensions of the thruster, they are said to be negligible and good results can still be acquired without their consideration.

Collisions are limited to electron-neutral and ion-neutral collisions. The electron-neutral collisions are the only source of ionization, and momentum exchange is governed by both of the collision types.
Chapter 2

Governing Equations

2.1 General Transport Equation

Since this thesis deals with a three-fluid model of the plasma in the Hall thruster, the following equations derive the equations that govern the macroscopic plasma parameters. In this derivation, we follow Bittencourt [29]. Beginning with the generalized Boltzmann equation

\[
\frac{Df_\alpha}{Dt} + \nabla f_\alpha + \overline{a} \cdot \nabla f_\alpha = \left( \frac{\delta f_\alpha}{\delta t} \right)_{\text{coll}},
\]

where \( f_\alpha \) is the velocity distribution function of species \( \alpha \). Note that \( f \) is not directly measurable, but "moments" of \( f \) are, and we now derive these moments.

In order to determine the average value of a certain plasma parameter \( \chi (\vec{r}, \vec{v}, t) \), we multiply the distribution function by that parameter and integrate over all velocities (see equation 2.3). The most important parameters are powers of the velocity, and such integrals are called moments of the distribution function. Hence, multiplying
Eq 2.1b by $\chi(\vec{r}, \vec{v}, t)$ and integrating over velocity space results in a "transport" equation for that particular plasma parameter, see eq 2.4

$$\int_v \chi \frac{\partial f_\alpha}{\partial t} d^3v + \int_v \chi \vec{v} \nabla f_\alpha d^3v + \int_v \chi \alpha \nabla_v f_\alpha d^3v = \int_v \chi \left( \frac{\delta f_\alpha}{\delta t} \right)_{coll} d^3v \tag{2.2}$$

Evaluating each term separately and using the definition of the average value of the property $\chi(\vec{r}, v, t)$

$$\langle \chi(\vec{r}, \vec{v}, t) \rangle_\alpha = \frac{1}{n_\alpha(\vec{r}, t)} \int_v \chi(\vec{r}, \vec{v}, t) f_\alpha(\vec{r}, \vec{v}, t) d^3v, \tag{2.3}$$

where $n$ is the number density (particles per unit volume), which is taken from Bittencourt [29], results in the general transport equation

$$\frac{\partial}{\partial t} (n_\alpha \langle \chi \rangle_\alpha) - n_\alpha \langle \partial \chi \rangle_\alpha + \nabla \cdot (n_\alpha \langle \chi \vec{v} \rangle_\alpha) - n_\alpha \langle \vec{v} \cdot \nabla \chi \rangle_\alpha = \left[ \frac{\delta}{\delta t} (n_\alpha \langle \chi \rangle_\alpha) \right]_{coll} \tag{2.4}$$

If the plasma property $\chi$ is only a function of $\vec{v}$ ($\chi = \chi(\vec{v})$), and not explicitly a function of time or space, then the general transport equation becomes

$$\frac{\partial}{\partial t} (n_\alpha \langle \chi \rangle_\alpha) + \nabla \cdot (n_\alpha \langle \chi \vec{v} \rangle_\alpha) - n_\alpha \langle \vec{v} \nabla \chi \rangle_\alpha = \left[ \frac{\delta}{\delta t} (n_\alpha \langle \chi \rangle_\alpha) \right]_{coll} \tag{2.5}$$

The latter version of the general transport equation is the form which will be used to derive the three lowest moments continuity, momentum, and energy.

One definition that must be made before proceeding is

$$\vec{v}_\alpha = \vec{u}_\alpha + \vec{c}_\alpha, \tag{2.6}$$

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where \( \bar{u}_\alpha = \langle \bar{v}_\alpha \rangle \) as defined in Eq 2.3 and the random, or “thermal” particle velocity \( \bar{v}_\alpha \). It is also imperative to state that \( \langle \bar{v}_\alpha \rangle = 0 \), since the average of a random property is zero. Therefore, \( \langle \bar{v}_\alpha \rangle = \langle \bar{u}_\alpha \rangle = \bar{u}_\alpha \), since an average of an average property does not change its value.

### 2.1.1 Continuity Equation

The number continuity equation is derived by setting the plasma factor to 1. It is the zeroth-order moment of the transport equation, since \( \chi^0 = 1 \)

\[
\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \bar{u}_\alpha) = S_\alpha \tag{2.7}
\]

If Eq 2.7 is multiplied by \( m_\alpha \) or \( q_\alpha \) it results in the mass continuity and electric charge continuity, respectively shown as

\[
\frac{\partial n_\alpha m_\alpha}{\partial t} + \nabla \cdot (n_\alpha m_\alpha \bar{u}_\alpha) = m_\alpha S_\alpha \tag{2.8}
\]

\[
\frac{\partial n_\alpha q_\alpha}{\partial t} + \nabla \cdot (n_\alpha \bar{u}_\alpha) = q_\alpha S_\alpha \tag{2.9}
\]

where \( \bar{j}_\alpha = n_\alpha q_\alpha \bar{u}_\alpha \) is the charge current density. The term \( S_\alpha \) is the source term for particles due to collisions, which will be defined later.

### 2.1.2 Momentum Equation: Equation of Motion

The plasma factor used in deriving the momentum equation is \( \chi(v) = m_\alpha \bar{v} \). Substituting this into Eq 2.5 gives

\[
\frac{\partial}{\partial t} (n_\alpha m_\alpha (\bar{u})_\alpha) + \nabla \cdot (n_\alpha m_\alpha (\bar{v})_\alpha) - n_\alpha (\bar{F} - \nabla \cdot (\bar{u})_\alpha) = \bar{A}_\alpha, \tag{2.10}
\]
where $\vec{F}$ is the force term from the multiplication of $m_\alpha$ and the third term on the left hand side of Eq. 2.5. The term on the right hand side of the equation, $\vec{A}_\alpha$, is the collision term, which defines the rate of change of the average momentum due to particle collisions. The definition of $\vec{A}_\alpha$ is shown in section 2.1.4.

After substituting Eq. 2.6 into Eq. 2.10, noting that $\langle \vec{c}_\alpha \rangle = 0$, and simplifying terms, the following equation results:

$$n_\alpha m_\alpha \frac{\partial \vec{u}_\alpha}{\partial t} + \vec{u}_\alpha \frac{\partial n_\alpha m_\alpha}{\partial t} + \nabla \cdot (n_\alpha m_\alpha \vec{u}_\alpha \vec{u}_\alpha) + \nabla \cdot (n_\alpha m_\alpha \langle \vec{c}_\alpha \vec{c}_\alpha \rangle) - n_\alpha \langle \vec{F} \rangle_\alpha = \vec{A}_\alpha. \quad (2.11)$$

The only other simplification, for now, is to recognize that the fourth term on the left hand side of Eq. 2.11 is the divergence of the pressure tensor, $\nabla \cdot \vec{P}_\alpha$, where

$$\nabla \cdot \vec{P}_\alpha = \nabla \cdot (n_\alpha m_\alpha \langle \vec{c}_\alpha \vec{c}_\alpha \rangle). \quad (2.12)$$

Eq. 2.11 is therefore written

$$n_\alpha m_\alpha \frac{\partial \vec{u}_\alpha}{\partial t} + \vec{u}_\alpha \frac{\partial n_\alpha m_\alpha}{\partial t} + \nabla \cdot (n_\alpha m_\alpha \vec{u}_\alpha \vec{u}_\alpha) + \nabla \cdot \vec{P}_\alpha - n_\alpha \langle \vec{F} \rangle_\alpha = \vec{A}_\alpha. \quad (2.13)$$

Later, it will be shown how the continuity equation and the momentum equation are used to further simplify the previous equation.
2.1.3 Energy Equation

In order to derive the energy equation, the kinetic energy of a particle, \( \frac{1}{2} m_a v_a^2 \) is substituted for \( \chi(v) \) in Eq. 2.5. The resulting equation is

\[
\frac{D}{Dt} \left( \frac{3p_a}{2} \right) + \left( \frac{3p_a}{2} \right) \nabla \cdot \vec{u}_a + \frac{\partial}{\partial t} \left( \frac{1}{2} n_a m_a u_a^2 \right) + \nabla \cdot \left( \frac{1}{2} n_a m_a u_a^2 \vec{u}_a \right) + \nabla \cdot \left( \vec{P} \cdot \vec{u}_a \right) + \nabla \cdot \vec{q}_a = n_a u_\alpha \langle \vec{F} \cdot \vec{u} \rangle = M_a. \quad (2.14)
\]

2.1.4 Collision Terms

The collision terms affect all of the moments of the transport equation by serving as the source or sink of particles. Collision processes involve ionization, recombination, attachment, momentum transfer and others, for higher moments of the transport equation, not dealt with in this thesis.

The collision term for the continuity equation involves processes that render particle production or loss. In this thesis, these processes will be due to inelastic collisions such as ionization, and recombination. For charged species, ionization indicates a source of particles, while recombination is a loss of particles. For the neutral species, ionization indicates the loss of particle, and recombination indicates a source of particles.

In the case of the number continuity, shown in Eq. 2.7, the source term \( S_\alpha \) for electrons, ions, and neutrals is given by

\[
S_e = k_i n_e - k_r n_e^2 - k_a n_e, \quad (2.15)
\]
\[
S_i = k_i n_e - k_r n_e^2 - k_a n_i, \quad (2.16)
\]
\[
S_n = k_r n_e^2 - k_i n_e. \quad (2.17)
\]
where $k_t$, $k_r$, and $k_a$ are the ionization, recombination, and thruster wall attachment coefficients, respectively. The units of the terms on the right hand side of Eqs. 2.15, 2.16, and 2.17 are number of particles created, recombined, or attached per unit time. Attachment is recombination of ions or electrons at the wall of the thruster.

The collision term for the momentum equation states the change in the average momentum of the particular fluid species of concern. Collisions of two particles from the same species does not change the momentum of the system, or of that particular fluid. The momentum transfer is given by the following equation:

$$A_\alpha = n_\alpha m_\alpha \sum_{\beta \neq \alpha} \nu_{\alpha\beta} (\bar{u}_\alpha - \bar{u}_\beta),$$ (2.18)

where $\alpha$ and $\beta$ are different species.

In the energy equation, shown as Eq. 2.14, the collision term is signified by $M_\alpha$. This represents the rate of energy density change due to collisions. It can be written as:

$$M_\alpha = \frac{1}{2} m_\alpha \int_v v^2 \left( \frac{\delta f_\alpha}{\delta t} \right)_{coll} dv = \left[ \frac{\delta \left( \frac{1}{2} n_\alpha m_\alpha \langle v^2 \rangle_\alpha \right)}{\delta t} \right]_{coll}$$ (2.19)

### 2.2 Magnetic Field

The magnetic field $\vec{B}$ is an externally controlled function of position. For simplicity, I have chosen a Maxwellian profile in the axial ($z$) direction. Figure 2.1 shows a reproduced example of a magnetic field profile in a 1-D model of a Hall thruster, taken from Cohen-Zur et al. [26], which is given by

$$B = B_0 \exp \left[ -\frac{(z - z_m)^2}{L_m^2} \right],$$ (2.20)
where $B_0$ is the maximum magnetic flux, $z$ and $z_m$ are the distance from the anode, and the distance from where the peak of $B$ lies, respectively, and $L_m$ is the halfwidth of the Maxwellian profile. Each of these parameters can be varied.

### 2.3 Assumptions and Modifications

Although all of the species of the different moments described in the previous sections can be used in modeling the Hall thruster, the more efficient and reasonable method is to choose the equations that have the greatest contribution to the plasma model for the particular problem at hand. This section is devoted to making simplifying assumptions to the moments of the transport equation and to state the required equations necessary to effectively model the Hall thruster and keep the magnetic field...
as an input to the system of equations. It is in this way that the magnetic field profile will be imported into the system of equations and changed in order to find the optimum efficiency of the thruster.

The simplified physical model that we choose to use is that developed by Choueiri [6]. The required equations to model the plasma sufficiently in this thesis are ion and electron continuities, the electron momentum, and an assumption about the energy that relates the temperature to the electric potential.

In the explicit evaluation of the moment equations, a coordinate system must be chosen. I use a standard cylindrical coordinate system, since that is the natural geometry of the Hall thruster. As usual, the radial direction is denoted by $\rho$, the azimuthal direction by $\theta$, and the axial direction by $z$. Azimuthal symmetry is assumed to hold so that the derivatives $\frac{\partial}{\partial \theta}$ are zero. The derivatives $\frac{\partial}{\partial \rho}$ are also zero since the plasma is assumed to not change significantly in the radial direction. In reality, this is not correct, but the greatest variation takes place only in a narrow sheath near the walls, which make up a small fraction of the acceleration cross-sectional area. Including the sheath region would complicate the analysis immensely since the plasma is not quasineutral in this region.

### 2.3.1 Simplified Continuity Equations

The form of the continuity equation used for both the electrons and ions is the charge continuity as follows:
Electrons

By removing the temporal term and removing the unnecessary terms from the divergence in Eq. 2.9, the following equation results:

$$\frac{dj_{ez}}{dz} = k, ne,$$  \hspace{1cm} (2.21)

where $S_e$ has been explicitly written in terms of the rate of ionization.

Ions

Starting with the same formulation as the electrons, Eq. 2.9 is now written explicitly to show the ion-specific terms:

$$\nabla \cdot en\vec{\mu}_i = \nabla \cdot \vec{j}_i = S_i$$  \hspace{1cm} (2.22)

Assuming that the pressure is low and that the ion current is much greater than the electron current, the source term due to ionization is negligible. Therefore the ion current is a constant as follows:

$$\nabla \cdot en\vec{\mu}_i = \nabla \cdot \vec{j}_i = 0$$  \hspace{1cm} (2.23)

$$enu_{iz} = \text{constant} = j_{i0},$$  \hspace{1cm} (2.24)

where $j_{i0}$ is the ion current at the end of the acceleration channel.

Notice how the constant current has been defined as $j_{i0}$, which is taken as the ion current at the end of the acceleration channel. Also, only the $z$ component of the current has been kept due to simplification, since $u_{i\theta} \approx 0$.

If Eq. 2.24 is solved for the density and the velocity is taken from setting the kinetic and electric field energies equal, the following final equation results.
\[ n = \frac{j_{i0}}{e \sqrt{\frac{2e(\phi_x - \phi)}{m_i}}} \]  \hfill (2.25)

This is the form of the ion continuity equation that will be used as one of the closed-set equations.

### 2.3.2 Simplified Momentum Equation

Since the ion momentum is taken into account in the velocity definition for the ion continuity equation, Eq. 2.25, it is only necessary to derive the electron momentum equation here.

Starting from Eq. 2.13, assuming a steady state (so that all time derivatives are zero), and ignoring the third term on the left-hand side, which is second-order in the velocity, I obtain

\[ \nabla \cdot \vec{P}_e - n(\vec{F})_e = \vec{A}_e. \]  \hfill (2.26)

The distribution of the random particle velocities is taken to be isotropic, and therefore the divergence of the pressure tensor reduces to the gradient of the scalar pressure, \( \nabla p_e \). Furthermore, the gradient is removed and replaced by \( \frac{d}{dz} \), since only the change in the \( z \) direction is of importance.

The force term on the LHS of Eq. 2.26 is taken to be the electromagnetic Lorentz force:

\[ n(\vec{F})_e = ne (\vec{E} + \vec{u}_e \times \vec{B}) \]  \hfill (2.27)

This force term is further simplified by letting the electric field be shown as the derivative of the potential, and only taking the axial component of the cross product between the velocity and magnetic field, which is taken to point in the radial direction (\( \rho \)). This simplification is reached with the assumption that forces in the other
directions are zero, since this is a one-dimensional model. Therefore the important velocity is the Hall drift, which only points in the $-\theta$ direction.

$$ne \left( \vec{E} + \vec{u}_e \times \vec{B} \right) = ne \left( -\frac{d\phi}{dz} + (u_e \theta B)_{\parallel} \right)$$  \hspace{1cm} (2.28)

In order to simplify our set of equations further, it is important to model the physics of the electron and ion species correctly for a Hall thruster device. For a partially ionized plasma immersed in a magnetic field, the relationship between an applied electric field and the resulting current is given by a conductivity tensor

$$\vec{j} = \vec{E} \cdot \sigma$$

$$= \begin{pmatrix} \sigma_p & \sigma_H & 0 \\ -\sigma_H & \sigma_p & 0 \\ 0 & 0 & \sigma_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$  \hspace{1cm} (2.29)

where $z$ is taken to be the magnetic field direction, and $\sigma_p$ and $\sigma_H$ are the Pedersen and Hall conductivities, respectively.\footnote{For completeness, the "parallel conductivity," $\sigma_z$, is just the usual Ohm's law parameter, which in the present case takes on the value $nq^2/m\nu$.} Note that a field in the $x$ direction, say, results in currents in both the $x$ and $y$ directions. The ratio of these conductivities is

$$\frac{\sigma_H}{\sigma_p} = \frac{\omega_c}{\nu} = \Omega_H$$  \hspace{1cm} (2.30)

where $\nu$ is the collision frequency with neutrals, and $\Omega_H$ is called the "Hall parameter." For the electrons in a Hall thruster, $\Omega_H$ is large (because the magnetic field is large and they experience relatively few neutral collisions), which implies that the axial electric field imparts an electron current primarily in the azimuthal ($\theta$) direction.
In this way, the electrons spend a lot of time in the acceleration region where they continue to collisionally ionize the incoming neutral gas. The ions, on the other hand, have a much smaller Hall parameter, and this, along with their large cyclotron radius, means that they are effectively unmagnetized. Their velocity (and hence the ion current) is primarily in the axial direction, where they leave the rear of the thruster with substantial momentum, and hence thrust, to the spacecraft.

From the \( \theta \) component of the electron momentum equation, given our present assumptions we obtain

\[
\dot{u} = u_{\theta} \Omega_H, \tag{2.31}
\]

where \( \Omega_H \) is the (large) Hall parameter, as explained above. In addition, the definition of the electron current is just

\[
j = neu_{\theta}, \tag{2.32}
\]

which means that Eq. 2.28 can be written as

\[
j \left( \frac{B_p \Omega_H}{e} + \frac{m_e \nu \omega_c}{e} \right) = ne \frac{d\phi}{dz} - \frac{d(nT)}{dz} \tag{2.33}
\]

Here, I've expressed the pressure as \( p = nT \), making the ideal gas assumption, which holds for hot plasmas. By using the definition of \( \Omega_H \) and the definition of the electron gyrofrequency, \( \omega_c = \frac{eB}{m_e} \), the simplified version of the electron momentum equation can be written as such:

\[
j = \mu_e \left( ne \frac{d\phi}{dz} - \frac{d(nT)}{dz} \right), \tag{2.34}
\]

where \( \mu_e \) is the Pedersen mobility

\[
\mu_e = \frac{e}{m_e \nu \omega_c \Omega_H^2 + 1} \tag{2.35}
\]
The mobility $\mu$ is simply the proportionality constant between the applied electric field and the resulting charged particle velocity $u = \mu E$. Of course, just like the conductivity, $\mu$ is a tensor. Since the current and velocity are related by $\gamma = nqu$, the conductivity and mobility tensors are simply proportional to one another, $\sigma = nq\mu$, and therefore $\mu$ has both Pedersen and Hall elements (as well as a parallel element) just like the conductivity.

2.3.3 Energy Specification

A simple energy relationship proposed by Choueiri [6] is

$$T_e = \beta e\phi$$  \hspace{1cm} (2.36)

While this proportionality is not strictly correct, the logic for this assumption comes from the fact that the electrons have some directed velocity, but as they are accelerated in the axial direction, they collide with the neutrals, and some of that bulk kinetic energy is transformed to random, thermal kinetic energy. That is, some fraction ($\beta < 1$) of the potential energy gained by the electrons goes into heating the electron fluid. This relationship allows the electron temperature to increase with increasing potential. Since the solution iterations will start at the cathode, and $\phi_{\text{cathode}} = 0$, the electrons are assumed to enter the acceleration zone with no thermal velocity.
2.4 Scaling

The equations 2.21 and 2.34 represent a set of two, first-order, coupled ordinary differential equations. Equations 2.25 and 2.36 are relationships that allow the differential equations to be expressed as one second-order equation, with $\phi$ as the dependent variable. One approach to solving such an equation is to render them dimensionless by scaling the variables (position, time, density, etc.) by their characteristic values. A scaled set of equations is derived in order to remove the dimensional terms from the solution set. In order to non-dimensionalize equations 2.21, 2.25, 2.34, and 2.36, the relevant variables are non-dimensionalized using characteristic or known* values. The characteristic values are length, number density, current density, and magnetic field strength, while the known values are anode potential and electron charge. The non-dimensional variables are:

$$
\tilde{\phi} \equiv \frac{\phi}{\phi_0}, \quad \tilde{z} \equiv \frac{z}{l}, \quad \tilde{\bar{n}} \equiv \frac{n}{\bar{n}}, \quad \tilde{\bar{j}_e} \equiv \frac{\bar{j}_e}{\bar{j}'},
$$

$$
\bar{T} \equiv \frac{T}{e\phi_0}, \quad f(\tilde{z}) \equiv \frac{B(\tilde{z})}{B'}, \quad \tilde{\Omega}_e(\tilde{z}) \equiv \frac{eB'}{m_e\nu_e} f(\tilde{z}).
$$

(2.37)

$\tilde{\Omega}$ is dimensionless, and it is the usual definition of the Hall parameter, with the understanding that it is a function of position. $B'$ is chosen such that:

$$
B' = \frac{m_e\nu_e}{e},
$$

(2.38)

$$
\tilde{\Omega}_e(\tilde{z}) = f(\tilde{z}),
$$

(2.39)

so that when $B = B'$, $\mu = \mu_0/2$. 

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2.5 Solution Set

By replacing the terms from Eq. 2.37 in Eqs. 2.21, 2.25, 2.34, and 2.36, and then simplifying, we arrive at a dimensionless, closed solution set:

\[
\frac{\partial \vec{j}_{ez}}{\partial z} = Q \bar{n}, \tag{2.40}
\]

\[
\bar{n} = A (1 - \bar{\rho})^{-\frac{1}{2}}, \tag{2.41}
\]

\[
\vec{j}_{ez} = C(\bar{z}) \left[ \bar{n} \frac{d\phi}{d\bar{z}} - \frac{d (\bar{n} \bar{T})}{d\bar{z}} \right], \tag{2.42}
\]

\[
\bar{T} = \beta \bar{\phi}, \tag{2.43}
\]

where

\[
Q = \frac{\nu \epsilon l' n'}{j'}, \quad A = \frac{j_{10}}{n' \sqrt{\frac{2 \sigma \phi_a}{m_e}}}, \quad C(\bar{z}) = \mu_e(\bar{z}) \frac{e \phi_a n'}{v' j'}. \tag{2.44}
\]

and

\[
\mu_e(\bar{z}) = \mu_0 \frac{1}{1 + f^2(\bar{z})}, \tag{2.45}
\]

where

\[
\mu_0 = \frac{e}{m_e v_c}. \tag{2.46}
\]

\(\mu_e\) is the usual definition of Pederson mobility, with the caveat that it, too, is a function of position. \(\mu_e = \mu_0\) when \(f = 0\).

The magnetic field, although a constant value, will be manipulated by use of the non-dimensional \(f(\bar{z})\) term. This will directly affect the value of the Pederson mobility and will change the value of \(C(\bar{z})\).

In order find a solution to this set, equations 2.40 to 2.43 are combined. To combine these, Eq. 2.42 is differentiated and set equal to Eq. 2.40, with Eqs. 2.41 and 2.43 used to eliminate \(\bar{n}\) and \(\bar{T}\) from the equation. The resulting equation is as
follows:

$$\frac{d^2}{dz^2} \left[ 2 \left(1 - \bar{\phi}\right)^{\frac{1}{2}} + (1 - \bar{\phi})^{-\frac{1}{2}} \beta \bar{\phi} \right] + \frac{Q}{C'(z)} (1 - \bar{\phi})^{-\frac{1}{2}} = 0. \quad (2.47)$$

Eq. 2.47 is a nonlinear, second-order ordinary differential equation for \( \bar{\phi} \), the electric potential, as a function of axial position in the Hall thruster. Because of its complexity, this equation (2.47) cannot be solved analytically, hence we resort to a numerical scheme. The solution is well-behaved, and the standard fourth-order Runge-Kutta method is adequate to obtain an accurate solution.

Solutions were calculated with various values of \( \beta \) between 0 and 1. Although \( \beta \) comes from the energy relation, it is found to have a small effect on the results of the equation, in comparison to changing the magnetic field parameters. Therefore, I arbitrarily use \( \beta = .001 \).
Chapter 3

Results

3.1 Constant Hall Parameter

To investigate the general characteristics of the solution to Eq. 2.47, I first solve it when the external magnetic field is constant. That is, when

\[ f(\bar{z}) = C, \]  

(3.1)
is not a function of \( \bar{z} \). The only "free parameter" in Eq. 2.47 is the coefficient of the last term, which I define as \( \lambda \)

\[ \lambda(\bar{z}) = \frac{Q}{C(\bar{z})}, \]  

(3.2)
a dimensionless parameter that is inversely proportional to the electron mobility (see Eq. 2.44)

\[ \lambda(\bar{z}) = 1 + f^2 \propto \frac{1}{\mu_e}. \]  

(3.3)

When the magnetic field strength is constant, the parameter \( \lambda \) can be absorbed
into the scaling of the axial position, \( z \), to obtain

\[
\hat{z} = \sqrt{\lambda z}, \tag{3.4}
\]

and Eq. 2.47 becomes

\[
\frac{d^2}{d\hat{z}^2} \left[ 2 \left( 1 - \phi \right)^{\frac{1}{2}} + \left( 1 - \phi \right)^{-\frac{1}{2}} \beta \phi \right] + \left( 1 - \phi \right)^{-\frac{1}{2}} = 0. \quad \tag{3.5}
\]

Note that the electric potential is a function of \( \hat{z} \). Therefore, the entire family of solutions is, in fact, only one solution—one simply has to rescale the axial dimension. Indeed, for magnetic field strengths large compared with \( B' = m_e \nu_e/c \approx 1/\mu T \), which is typically the case in Hall thrusters, we have

\[
\lambda \approx f^2 \propto B^2, \tag{3.6}
\]

so that \( \hat{z} \approx B\hat{z} \), which means that as the magnetic field strength increases, the length of the acceleration region decreases. This can also be shown by solving for the characteristic length from the equation \( Q = C = 1 \), which incorporates two of the non-dimensionalization terms. After simplifying, the characteristic length is shown as

\[
l' = l'_0 \frac{1}{\sqrt{1 + f^2(\hat{z})}}, \tag{3.7}
\]

where

\[
l'_0 = \sqrt{\frac{\mu_0 \phi_a}{\nu_{1z}}}. \tag{3.8}
\]

Equation 3.7 shows that when \( f = f_0 = 0 \), then \( l' = l'_0 \). If you recall, a value of 0 for \( f \) is the same as stating that the Hall parameter is equal to 0 (\( \Omega = 0 \)). Therefore, if our thruster were not a Hall thruster at all, then the length required by this model,
to give the ions a potential drop of $\phi_a$ would be $l'_0$.

This canonical solution is shown in Figure 3.1. Note in Fig. 3.1 that in properly scaled units ($\tilde{z}$), the acceleration region is approximately one unit in length. The reason that we see an increase in potential in the plots is because we start our numerical analysis from the cathode, where $\phi = 0$ and work our way to the anode, where $\phi = \phi_a$.

When the magnetic field is large and this length of the acceleration region is inversely proportional to the magnetic field strength, I can express it as

$$L \approx l' \frac{B'}{B},$$

(3.9)

The restricted case considered by Choueiri is when the magnetic field is not taken into account for the solution. When there is no magnetic field $f = 0$, we have $\lambda = 1$. Therefore, the solution equation is:

$$\frac{d^2}{dz^2} \left[ 2 \left(1 - \bar{\phi}\right)^{\frac{1}{4}} + \left(1 - \bar{\phi}\right)^{-\frac{3}{4}} \beta \bar{\phi} \right] + \left(1 - \bar{\phi}\right)^{-\frac{1}{4}} = 0.$$  (3.10)

Eq. 3.10 is the same as Eq. 2.47, where $\frac{Q}{c(t_4)} = \lambda = 1$, and the ODE used by Choueiri [6]. I have taken this case further as I have added the variable magnetic field affects.

### 3.2 Variations in the Hall Parameter

The main thrust of this thesis, however, is to investigate the shape of $\bar{\phi}$ (and the resulting length of the acceleration region) when the external magnetic field is not constant. As discussed in Section 3.2, a useful profile is a Gaussian, because I have
control over the location of the maximum field strength as well as its gradient. ¹

As opposed to constant magnetic field strength, which fixes one value throughout the acceleration region of the Hall thruster, a varying magnetic field can be accomplished in the following ways:

1. Changing the location where the magnetic field reaches its maximum ($z_{B_{\text{max}}}$),

2. Changing the shape of the magnetic field,

3. Changing the maximum value of the field.

¹Unfortunately, this generalization introduces a subtle physical issue that we must ignore for the problem to remain one-dimensional, as well as tractable. In a cylindrical Hall device, the radial magnetic field component ($B_\rho$) can vary with axial position, $B_\rho(z)$. Strictly speaking, any externally generated magnetic field must satisfy Maxwell's equations: $\nabla \cdot B = 0$ as well as Ampere's Law with no current, $\nabla \times B = 0$. Once $B_\rho$ is a function of $z$, Maxwell's equations require that the axial component ($B_z$) is non zero, and moreover, that it varies with radial position, $B_z(\rho)$. Unfortunately, the field I have proposed to use does not satisfy these conditions. However, the inclusion of $B_z(\rho)$ renders the problem inherently two-dimensional, requiring a much greater computational burden (without the simply physical insight that we gain here). In addition, I assume that the variation in $B_\rho$ is small enough that the resulting axial component is negligible and does not affect the essential 'Hall effect' physics.
These are the variations that are used to view the effect of a changing magnetic field on our model of the Hall thruster.

The only way to show such a wide array of changes in an analytical setting is to plot the different solutions with the representative $\vec{B} - \text{field}$ as a reference to what is happening. Had we plotted $\vec{B}$ for the previous section, we would have simply seen a constant line showing the same value of the magnetic field from the beginning to the end of the acceleration chamber.

### 3.2.1 Maximum B-Field in Varying Locations

In this section, I chose one profile for the magnetic field, by fixing the maximum magnetic field and the value of $L$ in the magnetic field equation 2.20. I have chosen the maximum magnetic field strength to be 10 and $L = 1$. One of the plots in Figure 3.4 shows this scenario when the maximum magnetic field is located at the thruster exit. Figure 3.2 shows the potential and magnetic field profiles where the peak magnetic field shifts between the cathode and the anode of the thruster. Figure 3.2 suggests that a locally strong magnetic field results in a locally strong gradient in $\phi$. Therefore, it turns out that the location of the strong magnetic field only weakly affects the length of the acceleration region. All that matters is that the field is strong somewhere. Where there are no large magnetic gradients, the potential follows the same potential profile as if there were no magnetic field, until the ions reach the cathode at full velocity and $\phi = 0$.

In order to find a relationship between the magnetic profile shift and the acceleration chamber length, I average the magnetic field for each unique magnetic field profile at the different positions along the thruster. The results are shown in Figure 3.3 which proposes that the length of the acceleration region is dependent upon the

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2 Only a subset of magnetic field positions are shown in Fig 3.2 (b)
Figure 3.2: Magnetic Field and Potential Profiles for Changing $\bar{z}_{B_{\text{max}}}$. a) The potential profiles associated with changing the location of the maximum magnetic field b) The magnetic field profiles that shift from the cathode (left side) to the anode. $\bar{z}_{B_{\text{max}}}$ dictates the location of maximum $\bar{B}$. 

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average magnetic field strength within the thruster with the following relation:

\[ \mathcal{L} \propto \frac{1}{B_{\text{avg}}} \quad (3.11) \]

when \( B > 1 \), which is in the applicable regime that a Hall thruster would operate, as reinforced by Eqn. 2.30. It also confirms that the relationship is not directly\(^3\) dependent upon the location of the maximum magnetic field.

### 3.2.2 Maximum B-Field at Cathode

In this set of solutions, we will set the location of the maximum magnetic field at the end of the acceleration chamber, which is usually the case in realistic Hall thrusters.

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\(^3\)I state *directly* because the length of the acceleration region is, in fact, dependent upon the location of the magnetic field, but only in the sense that the average magnetic field changes with respect to the location of maximum \( B \).
We will then change the shape of the magnetic field by working with the “flatness”, or halfwidth, of the graph, ranging from a constant line at constant magnetic field, to a graph where the magnetic field strength starts at the same point, but drops rapidly to zero.

The plots in Figure 3.4 show the different magnetic field profiles and the corresponding potential profile for different halfwidth values. Each corresponding potential profile indicates the needed length of the acceleration chamber to allow the ions to utilize all of the potential drop for acceleration. Each of the magnetic field profiles starts at the same maximum value and is altered by changing \( L \) in Equation 2.20. As \( L \) increases, the magnetic field profile flattens, until \( L \) is large enough to cause the magnetic field to appear constant. The main idea of Figure 3.4 is to show that for the same maximum magnetic field strength, the acceleration region will be shorter for a flatter magnetic field profile, since the magnetic field has a stronger overall profile.

In order to generalize the findings from Figure 3.4, one could plot the potential profiles for different magnetic field maximums. This would essentially show similar data as Fig. 3.4, in a different range of magnetic field maximums.

When the magnetic field was constant, the acceleration length was only dependent on the strength of the magnetic field. Now that there are two variables, the magnetic field strength and the magnetic field profile (changed by \( L \)), there needs to be a way to see how these variables factor into the change of the acceleration length. It is seen that for each value of a maximum magnetic field and changing \( L \), we will get a self-similar plot to Fig. 3.4, with a different acceleration region length. For this reason, Fig. 3.4 is taken as the basis for discussion and separate plots are not shown (but data is calculated) for different maximum magnetic fields.

If we choose one value for the flatness of the profile \((L = 10\) in this case) and we change the maximum value of the magnetic field \((f_{\text{max}})\), we arrive at Fig. 3.5. It
Figure 3.4: Magnetic Field and Potential Profiles for Changing $L$. a) The potential profiles associated with changing the shape of the maximum magnetic field. b) The different magnetic field profiles created by changing $L$ in the magnetic field equation.
Figure 3.5: **Effect of Changing Maximum Magnetic Field.** The basic profile of the magnetic field is held constant, by fixing $L$. As the maximum magnetic field changes for any value of fixed $L$, the average magnetic field throughout the acceleration chamber is affected. The ratio of the length of the acceleration region to the average magnetic field set up by changing the maximum magnetic field is -1:1.

shows the relationship between the average magnetic field value and the acceleration length to be what we have already discovered in the previous section when $B$ is large:

$$L \propto \frac{1}{B_{\text{avg}}}.$$

(3.12)

If we choose one value for the maximum magnetic field ($f_{\text{max}} = 2000$ for this case) and we vary $L$, we get Figure 3.6. This shows how changing the flatness of the magnetic field profile changes the average magnetic field and affects the length of the acceleration region. We see that as we flatten the magnetic profile ($L$ is larger), we change the average magnetic field for a certain maximum value for the magnetic field. Therefore, a flatter profile for the same maximum magnetic field strength, will require a shorter length for acceleration chamber. The actual relation that sets up a slope of $-\frac{1}{2}$ in Fig. 3.6 is buried within the relationship between $L$ and $B$ which is
Figure 3.6: **Effect of changing** $L$ **for a chosen** $f_{\text{max}}$. For a fixed maximum magnetic field ($f_{\text{max}} = 2000$ in this case) there is a change in the required acceleration length for a given $L$, or flatness in the magnetic field profile. As different values for $L$ are chosen, the average magnetic field changes with a ratio of -2:1 to the length of the acceleration region.

found in equation 2.20. This relationship is not investigated further, as I just observe the results.

To better understand the concepts of this section, all of the self-similar plots that could have been created of Figures 3.5 and 3.6 are plotted together in Fig. 3.7. This confirms and broadens our view that both the maximum magnetic field and the flatness of the magnetic field profile have an impact on the length of the acceleration region in the Hall thruster. By following the lines of $\text{Slope} = -1$ and $\text{Slope} = -\frac{1}{2}$, we can see that the same acceleration region length may be acquired by several sets of $f_{\text{max}}$ and $L$. Of the two, the maximum magnetic field strength has more of an impact on the length of the acceleration region.
Figure 3.7: **Overview of Effects in Changing Magnetic Field Parameters.** The effects on the average magnetic field when changing the maximum magnetic field and the magnetic field profile is seen here. As $f_{max}$ changes, it follows a ratio of -1:1 between the average magnetic field and the length of the acceleration region. As $L$ changes, it follows a ratio of approximately -2:1 between the average magnetic field and the length of the acceleration region. It is seen that there are multiple sets of $f_{max}$ and $L$ that can be used to achieve a certain length of the acceleration region.
Chapter 4

Conclusion

The main search in this thesis was to see what effect changing the magnetic field within a Hall thruster would have on the length required by the acceleration region to give the ions a scaled potential of 0 when they exited the thruster (in turn giving them the maximum velocity).

The general transport equations for a plasma were simplified with the assumptions stated in Chapter 2. A non-dimensional, nonlinear, second-order differential equation was solved using a standard fourth-order Runge-Kutta method that incorporated the varying magnetic field.

When we simply defined the magnetic field profile as a constant value, we discovered that the length of the acceleration chamber was inversely proportional to the magnetic field strength. This can be found exactly from the equations, as in Eqn. 3.7. This is shown as

\[ L \propto \frac{1}{B} \]  \hspace{1cm} (4.1)

In order to make the magnetic field variable for further analysis, we changed the location of the actual magnetic field profile, while keeping the shape constant, and then we varied the shape of the magnetic field while keeping the location of
its maximum stationary. Surprisingly, in both of these circumstances, we discovered that the length of the acceleration region was weakly affected by the shape or the location of the magnetic field. When \( \vec{B} \) was large, as is the case in Hall thrusters, it was inversely proportional to the average of the magnetic field strength within the thruster.

\[
\mathcal{L} \propto \frac{1}{B_{\text{Avg}}}.
\]  

This is the significant result of this research and is my contribution to previous Hall thruster research.

In order to incorporate all values of the magnetic field strength (even when \( B \) is small), the complete relation is shown as:

\[
\mathcal{L} \propto \frac{1}{\sqrt{1 + B_{\text{Avg}}^2}}.
\]  

In the future, a better model of the electron temperature is desired. Along with this idea, secondary electron emissions from the thruster walls will need to be modeled or taken into account. While still performing a 1D analysis, one could incorporate more of the collision terms into the equation or make the fluid equations more robust. This comes with a precaution that the model will tend to diverge when the ion velocity approaches the speed of sound for the plasma. If one is inclined to go further, an analysis of the plasma sheath region would be a complex addition to the model. A 2D analysis would be interesting in order to see how the relationships hold with the new dimension.
Bibliography


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