Thermal Post-Buckling and Vibration Analysis of Thermally Buckled Antisymmetrically Laminated Beams Using a 20 DOF Finite Element

Misao Mizuno
Embry-Riddle Aeronautical University - Daytona Beach

Follow this and additional works at: https://commons.erau.edu/db-theses
Part of the Aerospace Engineering Commons

Scholarly Commons Citation
https://commons.erau.edu/db-theses/253
THERMAL POST-BUCKLING AND VIBRATION ANALYSIS OF
THERMALLY BUCKLED ANTISYMMETRICALLY LAMINATED BEAMS
USING A 20 DOF FINITE ELEMENT

by

Misao Mizuno

A Thesis Submitted to the
School of Graduate Studies and Research
in Partial Fulfillment of the Requirements for the Degree of
Master of Science in Aerospace Engineering

Embry-Riddle Aeronautical University
Daytona Beach, Florida
August 1992
INFORMATION TO USERS

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleed-through, substandard margins, and improper alignment can adversely affect reproduction. In the unlikely event that the author did not send a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.
THERMAL POST-BUCKLING AND VIBRATION ANALYSIS OF
THERMALLY BUCKLED ANTISYMMETRICALLY LAMINATED BEAMS
USING A 20 DOF FINITE ELEMENT

by

Misao Mizuno

This thesis was prepared under the direction of the
candidate's thesis committee chairman, Dr. Habib Eslami,
Department of Aerospace Engineering, and has been approved
by the members of his thesis committee. It was submitted to
the School of Graduate Studies and Research and was accepted
in partial fulfillment of the requirements for the degree of
Master of Science in Aerospace Engineering.

THESIS COMMITTEE:

[Signatures]

Dr. Habib Eslami
Chairman

Dr. Howard D. Curtis
Member

Dr. Frank J. Radosta
Member

[Signature]
Department Chair, Aerospace Engineering

[Signature]
Dean, School of Graduate Studies and Research

7 July 1992
ACKNOWLEDGEMENTS

First and most importantly, I would like to thank my parents for their moral as well as financial support throughout my college career. Without their support, it would not have been possible to complete my education. I especially thank Dr. Habib Eslami for enhancing my knowledge of structural dynamics, finite element analysis, and composite materials. Without his patience and constant advise during this thesis work, this work would have been impossible. I also thank my thesis committee members Drs. Howard Curtis and Frank Radosta for their time and their suggestions. My thanks extends to Dr. James Ladesic and Mr. John Novy for providing me the teaching assistantship so that I could afford my graduate education.
ABSTRACT

Author: Misao Mizuno
Title: Thermal Post-Buckling and Vibration Analysis of Thermally Buckled Antisymmetrically Laminated Beams Using a 20 DOF Finite Element
Institution: Embry-Riddle Aeronautical University
Degree: Master of Science in Aerospace Engineering
Year: 1992

The purpose of this study is to use the finite element method to analyze thermal buckling, post-buckling, and vibrations of thermally buckled composite beams including shear deformation. The beam element used has ten degrees of freedom at each node: axial displacement, transverse displacement due to bending and shear, twisting angle, inplane shear rotation and their derivatives with respect to x. Hermitian polynomials and Lagrange's equation were used to derive the equations of motion. The equations of motion were divided into static and dynamic parts. For buckling analysis, the eigenvalue problem was solved for the critical temperature. The scaled first mode shape was used as the trial displacement vector for post-buckling deflection analysis, which employed a Newton-Raphson type iterative procedure. Once the final static deflection was achieved, the dynamic part of the equation of motion was used to solve the eigenvalue problem for the natural frequencies and mode shapes of the thermally buckled beams.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copyright Page</td>
<td>ii</td>
</tr>
<tr>
<td>Signature Page</td>
<td>iii</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>iv</td>
</tr>
<tr>
<td>Abstract</td>
<td>v</td>
</tr>
<tr>
<td>Table of Contents</td>
<td>vi</td>
</tr>
<tr>
<td>List of Tables</td>
<td>viii</td>
</tr>
<tr>
<td>List of Figures</td>
<td>ix</td>
</tr>
<tr>
<td>Nomenclature</td>
<td>xii</td>
</tr>
<tr>
<td>1.0 INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Purpose of Study</td>
<td>1</td>
</tr>
<tr>
<td>1.2 Literature Survey</td>
<td>3</td>
</tr>
<tr>
<td>1.2.1 Shear Deformation</td>
<td>3</td>
</tr>
<tr>
<td>1.2.2 Thermal Buckling</td>
<td>7</td>
</tr>
<tr>
<td>1.2.3 Thermal Post-Buckling</td>
<td>8</td>
</tr>
<tr>
<td>1.2.4 Vibration of Thermally Buckled Beams</td>
<td>9</td>
</tr>
<tr>
<td>1.3 Scope of Current Study</td>
<td>10</td>
</tr>
<tr>
<td>2.0 FINITE ELEMENT FORMULATION</td>
<td>17</td>
</tr>
<tr>
<td>2.1 Shear Deformation</td>
<td>17</td>
</tr>
<tr>
<td>2.2 Beam Element Displacement Functions</td>
<td>20</td>
</tr>
<tr>
<td>2.3 Stress-Strain Relations</td>
<td>21</td>
</tr>
<tr>
<td>2.4 Equations of Motion</td>
<td>28</td>
</tr>
<tr>
<td>2.4.1 Linear Element Stiffness Matrix</td>
<td>28</td>
</tr>
<tr>
<td>2.4.2 Nonlinear Element Stiffness Matrix</td>
<td>29</td>
</tr>
<tr>
<td>2.4.3 Stiffness Matrix Due to Thermal Load</td>
<td>33</td>
</tr>
</tbody>
</table>
LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Effect of Shear Deformation on Thermal Buckling, Simply-Supported [30/90/30] Laminate</td>
<td>105</td>
</tr>
<tr>
<td>4.2</td>
<td>Effect of Shear Deformation on Thermal Buckling, Simply-Supported [45/90/45] Laminate</td>
<td>105</td>
</tr>
<tr>
<td>4.3</td>
<td>Effect of Shear Deformation on Thermal Buckling, Simply-Supported [70/90/70] Laminate</td>
<td>106</td>
</tr>
<tr>
<td>4.4</td>
<td>Effect of Shear Deformation on Thermal Buckling, Simply-Supported [90/45/90] Laminate</td>
<td>106</td>
</tr>
<tr>
<td>4.5</td>
<td>Effect of Shear Deformation on Thermal Buckling, Simply-Supported [90/0/90] Laminate</td>
<td>107</td>
</tr>
<tr>
<td>4.6</td>
<td>Effect of Shear Deformation on Thermal Buckling, Simply-Supported [0/45/-45/90] Laminate</td>
<td>107</td>
</tr>
<tr>
<td>4.7</td>
<td>Effect of Shear Deformation on Thermal Buckling for Different Boundary Conditions [45/90/45] Laminate</td>
<td>108</td>
</tr>
<tr>
<td>4.8</td>
<td>Comparison of Buckling Temperature Between [φ/-φ/φ/-φ] and [φ/-φ/-φ/φ] laminate with h/l = 0.01</td>
<td>108</td>
</tr>
</tbody>
</table>
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Geometry of the Beam</td>
<td>14</td>
</tr>
<tr>
<td>1.2</td>
<td>Side View of the Beam</td>
<td>15</td>
</tr>
<tr>
<td>1.3</td>
<td>20 DOF Beam Element</td>
<td>16</td>
</tr>
<tr>
<td>2.1</td>
<td>Shear Deformation in a Beam Element</td>
<td>19</td>
</tr>
<tr>
<td>3.1</td>
<td>Flowchart for Initial Input and Linear Analysis</td>
<td>47</td>
</tr>
<tr>
<td>3.2</td>
<td>Flowchart for Thermal Buckling Analysis</td>
<td>48</td>
</tr>
<tr>
<td>3.3</td>
<td>Flowchart for Thermal Post-Buckling Analysis</td>
<td>49</td>
</tr>
<tr>
<td>3.4</td>
<td>Flowchart for Thermal Post-Buckling (continued) and Vibration of the Thermally Buckled Beams</td>
<td>50</td>
</tr>
<tr>
<td>4.1</td>
<td>Effect of Shear Deformation on Thermal Buckling, Simply-Supported Aluminum Beam</td>
<td>71</td>
</tr>
<tr>
<td>4.2</td>
<td>Effect of Shear Deformation on Thermal Buckling, Simply-Supported [30/90/30] Laminate</td>
<td>72</td>
</tr>
<tr>
<td>4.3</td>
<td>Effect of Shear Deformation on Thermal Buckling, Simply-Supported [45/90/45] Laminate</td>
<td>73</td>
</tr>
<tr>
<td>4.4</td>
<td>Effect of Shear Deformation on Thermal Buckling, Simply-Supported [70/90/70] Laminate</td>
<td>74</td>
</tr>
<tr>
<td>4.5</td>
<td>Angle Orientation and Critical Temperature Simply-Supported $[\phi/90/\phi]$ Laminate</td>
<td>75</td>
</tr>
<tr>
<td>4.6</td>
<td>Angle Orientation and Critical Temperature Simply-Supported $[90/\phi/90]$ Laminate</td>
<td>76</td>
</tr>
<tr>
<td>4.7</td>
<td>Effect of Shear Deformation on Thermal Buckling, Simply-Supported $[0/45/-45/90]_s$ Laminate</td>
<td>77</td>
</tr>
<tr>
<td>4.8</td>
<td>Effect of Shear Deformation on Thermal Buckling, Comparison Between $[45/-45/-45/45]$ and $[-45/45/-45/45]$ Simply-Supported Laminate</td>
<td>78</td>
</tr>
</tbody>
</table>
4.24 First Three Vibration Mode Shapes
Thermally Loaded Simply-Supported Aluminum Beam
at $\Delta T/\Delta T_{cr}$=2........................................ 94

4.25 Frequency Ratio vs Temperature Ratio
[30/-30/-30/30] Beam for Different Boundary
Conditions.................................................. 95

4.26 Frequency Ratio vs Temperature Ratio
Simply-Supported [30/-30/-30/30], Comparison
Between Inclusion and Exclusion of Shear............. 96

4.27 First Three Vibration Mode Shapes
Unloaded Simply-Supported Symmetric Angle-Ply...

4.28 First Three Vibration Mode Shapes
Thermally Loaded Simply-Supported Symmetric
Angle-Ply at $\Delta T/\Delta T_{cr}$=2............................ 98

4.29 Frequency Ratio vs Temperature Ratio
[-30/30/-30/30] Beam for Different Boundary
Conditions.................................................. 99

4.30 Frequency Ratio vs Temperature Ratio
Simply-Supported [-30/30/-30/30], Comparison
Between Inclusion and Exclusion of Shear............. 100

4.31 First Three Vibration Mode Shapes
Unloaded Simply-Supported Unsymmetric Angle-Ply. 101

4.32 First Three Vibration Mode Shapes
Thermally Loaded Simply-Supported Unsymmetric
Angle-Ply at $\Delta T/\Delta T_{cr}$=2............................ 102

4.33 Non-dimensional Mid-Span Deflection vs
Temperature Ratio, Comparison Between
[90/0/90/0], [90/0/0/90]................................. 103

4.34 Non-Dimensional Mid-Span Deflection vs
Temperature Ratio for [90/0/0/90] Laminate
Comparison between w/ and w/o shear................. 104
## NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>[A]</td>
<td>Extensional Stiffness Matrix</td>
</tr>
<tr>
<td>[B]</td>
<td>Coupling Stiffness Matrix</td>
</tr>
<tr>
<td>[D]</td>
<td>Bending Stiffness Matrix</td>
</tr>
<tr>
<td>l</td>
<td>Length of the Beam</td>
</tr>
<tr>
<td>b</td>
<td>Width of the Beam</td>
</tr>
<tr>
<td>h</td>
<td>Thickness of the Beam</td>
</tr>
<tr>
<td>E</td>
<td>Elastic Modulus</td>
</tr>
<tr>
<td>G</td>
<td>Shear Modulus</td>
</tr>
<tr>
<td>I</td>
<td>Moment of Inertia</td>
</tr>
<tr>
<td>J</td>
<td>Polar Moment of Inertia</td>
</tr>
<tr>
<td>v</td>
<td>Poisson's Ratio</td>
</tr>
</tbody>
</table>
| {ε}   | Strain  
  \( (ε_1, ε_2, γ_{12}: \text{Major and Minor Normal Strain, and Shear Strain Respectively}) \) |
| {κ}   | Curvature |
| {σ}   | Stress |
| [Q]   | Reduced Stiffness Matrix |
| [Q]   | Transformed Reduced Stiffness Matrix |
| \( N_x, N_y, N_{xy} \) | Resultant Forces Per Unit Length |
| \( M_x, M_y, M_{xy} \) | Resultant Moment Per Unit Length |
| \( N_1, N_2, N_3, N_4 \) | Hermitian Polynomials |
| u      | Axial Deflections |
| w_b    | Transverse Deflections due to Bending |
| w_s    | Transverse Deflections due to Shear |
| τ      | Twisting Angle |
\( \beta \) Inplane Shear
\( \alpha_1, \alpha_2 \) Major and Minor Thermal Expansion Coefficient
\( U \) Strain Energy
\( W \) Work Done by Forces
\( [K_L] \) Linear Stiffness Matrix
\( [K_{AT}] \) Stiffness Matrix due to Thermal Load
\( [n_1] \) First Order Nonlinear Stiffness Matrix
\( [n_2] \) Second Order Nonlinear Stiffness Matrix
\( \{q\} \) Displacement Vector
\( \{P\} \) Load Vector
\( \Delta T \) Temperature Imposed on Beam
\( \Delta T_{cr} \) Critical Buckling Temperature
\( \{N_{AT}\} \) Thermal Forces and Moments
\( [B_L] \) Strain-Displacement Matrix
\( [B_{NL}] \) Nonlinear Strain-Displacement Matrix
\( [M] \) Mass Matrix
\( T \) Kinetic Energy
\( \rho \) Density of Beam
\( \lambda \) Eigenvalue for Critical Temperature
\( \{\phi\} \) Mode Shapes for Buckling
\( \omega \) Natural Frequencies of Beam
\( \{v\} \) Mode Shapes of vibrations
\( P_{cr} \) Critical Buckling Load
\( \omega_o \) Unloaded Natural Frequency

Subscripts
\( S, T \) Symmetry, Total

xiii
1.0 INTRODUCTION

1.1 PURPOSE OF STUDY

Due to the recent development of advanced aircraft, research for stronger and lighter materials than traditional alloys is one of the greatest interests to aircraft industries. In the aerospace industry, composite materials have been used for a couple of decades. The effective use of those materials can reduce the weight of an aircraft structure significantly. In recent years, composite materials have become widely used not only in secondary structures but also for primary structures. The most attractive properties of composite materials are higher strength-to-weight and stiffness-to-weight ratios compared to metallic materials. The choice of laminate orientation angles in the lay-up of a composite material contributes greatly to its mechanical properties. By changing the laminate orientation angles, a panel can be tailored to different purposes and special applications.

Because composite materials have been used for primary structures in high speed airplanes in recent years, the research into their behavior under high thermal loads is in demand. The fibers must be bonded together with the proper matrix material to form a viable composite structure. Those
materials must withstand high thermal loads and maintain their properties at that condition. The most commonly used polymer matrices, epoxy resins, deteriorate at high temperature and high humidity conditions. Therefore, new thermoplastic resins that can operate in a wide range of temperatures are being considered.

Increasing interest in composite materials has created a considerable amount of research. However, the analysis of composite beams and plates, particularly antisymmetric laminates, is more complicated than for isotropic materials due to the existence of bending-stretching and bending-twisting coupling effects. In such cases the effect of in-plane shear deformation cannot be ignored.

The purpose of this study is to analyze thermal buckling, post-buckling deflection, and vibration of thermally buckled arbitrary laminated composite beams including shear deformation. The finite element method using a 20 degrees of freedom (DOF) beam element is employed to study the behavior of such beams.
1.2 LITERATURE SURVEY

1.2.1 SHEAR DEFORMATION

Much research on composite beams and plates has shown the importance of shear deformation. As far as isotropic beams are concerned, Shames and Dym [1] studied in detail the effects of shear deformation and rotatory inertia on vibration. They compared the dimensionless frequencies of the classical theory involving only rotatory inertia, only shear effects, and combined effects. According to the results of their study, there is little contribution from the shear effect for large slenderness ratios and low frequency modes. However, the shear deformation effect is very significant for short beams and high frequency modes. Analyses of the vibration of plates including shear deformation are also given by others, such as Whitney and Pagano [2]; Reddy and Kuppusamy [3]; and Bhashyam and Gallagher [4]. The literature on composite beams includes Chen and Yang [5]; Teh and Haung [6]; Suzuki [7]; Teoh and Haung [8]; Murty and Shimpi [9]; Miller and Adams [10]; and Hu, Kolsky and Pepkin [11]. All of these studies are restricted to symmetrically laminated cases only. Symmetrically laminated composite materials are relatively easier to analyze compared to unsymmetrically laminated ones because there is no bending-stretching coupling mode. Some
of the studies for unsymmetrically laminated cases were carried out by Lin and King [12]; Jones [13]; Raciti [14], [15]; and Singh, Rao and Iyenger [16]. Kapania and Raciti [14] studied non-linear vibrations of unsymmetrically laminated beams including shear deformation. They developed a composite beam element based on first order shear deformation theory in order to investigate large amplitude vibrations. Their beam element has total of twenty degrees of freedom, with ten degrees of freedom at each node. They assumed that the spatial variation of transverse displacement in the non-linear range to be the same as the linear mode. By doing so, they reduced dynamic finite element equation of motion to a single second order non-linear ordinary differential equation, the so-called "Duffing equation." Singh, Rao, and Iyenger [16] presented a study similar to Kapania and Raciti. However, in their work, the beam element used was based on higher order shear deformation theory. They also investigated the beam element with different degrees of freedom, such as eight and twelve degrees of freedom per node.

In laminated composite beams, the transverse shear deformation significantly affects the lateral displacement, the natural frequency of the vibration, and the buckling loads. The classical beam theory, which is based on the Bernoulli-Euler assumption, can result in high percentage
error when analyzing anisotropic beams. According to Reddy [17], the classical theory predicts natural frequencies that are 25% higher than those predicted by a shear deformation theory for plates with side-to-side thickness ratio of 10. For buckling, Raciti [14] showed that as the slenderness ratio decreases the buckling load $P_{cr}$ for the element with no shear deformation becomes much higher than for the element with shear deformation. For a laminated beam with $[0/90/0]^T$ lay-up, when the thickness to length ratio is 0.1, $P_{cr}$ for the element with shear deformation is nearly 30% lower than $P_{cr}$ for the element without shear deformation.

The first analytical method including deformations due to transverse shear appears in the book "Applied Mechanics" by W.J.M. Rankine. The theory was revised by S.P. Timoshenko [18]. It is now widely used in finite element formulations such as that given in [19]. Although many researchers experienced difficulty incorporating all the boundary conditions of the Timoshenko beam, Thomas and Abbas [40] developed a model for the dynamic analysis of Timoshenko beam which can satisfy all the boundary conditions by using strain and kinetic energy expressions and assigning cubic polynomial expressions for slope and deflections. They used a finite element with total deflection, bending slope, and their first derivatives. The equation of motion was derived by the use of Lagrange's equation. However, their analysis
was limited to isotropic materials.

For the free vibrations of composite beams, Abarear and Cunniff [20] approached the problem by dividing the slope of the deflection curve into two terms. One term contains direct bending and induced bending due to twisting and another term contains the shearing angle of distortion due to application of shear force. Murty and Shimpi [9] took care of shear deformation in different way. They divided the total displacement into bending displacement and the transverse shear deflection.

One important assumption for the Timoshenko beam is that shear strain is constant through the depth. This assumption reduces the difficulties of the analysis. One usually compensates for the error induced by using a shear correction factor $k$. Although quite a few studies exist for the determination for the shear correction factor $k$, for example [41],[42], there is no general agreement. One of the most accepted way is to take ratio of the average shear strain on a section to the shear strain at the centroid. Bert and Gordaninejad [21] showed a closed form solution for the shear factor for laminated materials.
1.2.2 THERMAL BUCKLING

For isotropic beams, the critical temperature can be found by setting the well known-expression for the Euler buckling load, $\pi^2EI/L^2$, equal to the thermally induced axial load $EA\alpha\Delta T$, where $E, A, \alpha$, and $\Delta T$ are Young's modulus, cross-sectional area, thermal expansion coefficient, and critical temperature, respectively. However, for composite materials, the analysis is not so easy because of the existence of two different thermal expansion coefficients, and variety of fiber angle orientations. So far, for composite materials, the literature is limited. Problems of this type have been approached both analytically and by using finite element techniques. Meyers and Heyer [22] used the Rayleigh-Ritz method in conjunction with the first and second order variations of the total potential energy to study the buckling of a quasi-isotropic rectangular plate and an orthotopic plate. Chen and Chen [23] used the principle of minimum potential energy in a finite element approach to the problem of the thermal buckling of composite plates. In their analysis, thermal stresses were calculated first, and those stresses were used to calculate the critical buckling temperature. Most recently, Gray [24] and Gray and Mei [25] used the finite element approach to study thermal buckling, post-buckling, and vibration of thermally buckled plates. Their plate element was arbitrarily
laminated and had a total of twenty four degrees of freedom. They solved the problem for uniform as well as nonuniform thermal load. To solve the eigenvalue problem for the buckling temperature, they also included the first order non-linear stiffness matrix due to membrane forces. However, if the contribution from the membrane stress is small enough, just the linear term is sufficient. In their formulations, Gray and Mei did not include shear deformation which could make a significant contribution to the critical temperature and subsequently affect the post-buckling analysis.

1.2.3 THERMAL POST-BUCKLING

For isotropic plates, Paul [27] employed a classical approach and a 25 term series approximation to determine the post-buckling deflection of thin, clamped aluminum plates with aspect ratios of one and two. Rao and Raju [28] developed a simple finite element for the post-buckling behavior of circular isotropic plates. Yang and Hang [29] developed a higher order triangular element model for a more precise finite element formulation of isotropic plates under the influence of mechanical loading. They used a 54 degree of freedom triangular element. Meyers and Heyer [22] studied post-buckling behavior of symmetrically laminated composite plates under uniform temperature loads. For this
analysis they used the Rayleigh-Ritz approach. Gray [24],[26] and Gray and Mei [26] also recently studied post-buckling behavior of plates with initial imperfection using the finite element method. They used a Newton-Raphson type iterative procedure to determine the post-buckling deflection.

1.2.4. VIBRATIONS OF THERMALLY BUCKLED PLATES

There are few reported analyses of the vibrations of thermally buckled plates. So far literature for beams on such problems is not available. In the 1950's, Bisplinghoff and Pian [30] worked on the problem of vibrations of thermally buckled isotropic plates. In their analysis, they used an extension of Marguerre's theory [40] in order to obtain solutions for rectangular plates in longitudinal compression. They have verified their results for an aspect ratio of three with experimental data.

Gray [24], [25] and Gray and Mei [26] used the finite element method to approach this problem for composite plates. Gray's composite plates are arbitrarily laminated, and the element has a total of 24 degrees of freedom, as described in previous two sections. To derive the equations of motion for this vibration analysis, Gray [24] considered the von Karman type geometrical non-linearity in his
formulation and the effect of initial imperfection. The first- and second-order non-linear stiffness matrices due to the large deflection caused by the thermal load were determined at the end of the thermal post-buckling iterations.

Currently there is no literature available on the analysis of thermal buckling, thermal post-buckling, and vibrations of thermally buckled composite beams with lamination and including shear deformations.

1.3 SCOPE OF CURRENT STUDY

Advanced high speed aircraft and spacecraft structures are expected to operate well at elevated temperature and high levels of vibrations. Analytical and experimental studies have shown that aircraft panels tend to buckle when subjected to relatively small variation in temperature. Thus post-buckling analysis using large deflection theory must be employed to determine the deflections and stresses. A buckled panel subjected to free oscillation experiences additional large deflections and large stresses. Therefore it is of paramount importance to study thermal buckling and the vibrations of thermally buckled modern aircraft structural components made of composite materials.

To account for the treatment of arbitrarily boundary
conditions, arbitrary laminated panels and the shear deformation effect, the finite element method will be used to derive the governing non-linear equation of motion. This analysis of the thermal buckling, thermal post-buckling, and vibration of thermally buckled composite beams including shear deformation is new. In some cases it yields significant differences from the existing results.

The geometry of the beam element is shown in Fig. 1.1. The 1 direction is aligned along the ply of the lamina and 2 is perpendicular to it. The side view of the element is shown in Fig. 1.2. The beam has no initial imperfection. First a thermal load is imposed on the beam producing some static deflection \( q \). When the temperature is beyond the critical temperature, the beam's thermal post-buckling deflection is obtained. Finally the vibration behavioral of the thermally post-buckled beam is analyzed.

As shown in Fig. 1.3, the beam element has a total of twenty degrees of freedom, with ten degrees of freedom at each node. They are the axial displacement \( u \), the deflection due to bending \( w_b \), the deflection due to shear \( w_s \), the twisting angle \( \tau \), the inplane shear \( \beta \) (\( du/dy \)), and all of their derivatives with respect to \( x \). Both \( \tau \) and \( \beta \) are assumed to be constant throughout the depth. Two coordinate systems are used. An x-y cartesian coordinate system is used as the
global system, and $\xi-y$ local coordinate system is used in order to perform Gaussian integration.

In Chapter 2, the governing equations of motion are obtained. The nodal displacement functions are interpolated in terms of displacements using Hermitian polynomials. In deriving the equation of motion using the Lagrange's equation, the linear stiffness matrix due to thermal load as well as the first- and second-order non-linear stiffness matrices due to the large deflection are obtained. The equations of motion are derived by using the Lagrange's equation.

In Chapter 3, the computational procedures are explained. To obtain the critical buckling temperature for the beam element, a linear eigenvalue problem is solved. To obtain the post-buckling deflection for a given thermal load, a Newton-Raphson type iteration is used to solve the non-linear equation relating the deflection of the beam and the thermal load. At the end of the iteration, the first- and second-order non-linear stiffness matrices are formed. They are used together with the linear stiffness matrices. Then the eigenvalue problem is solved for the natural frequencies.

In Chapter 4, the discussion of results is presented. The
finite element results are compared with existing results using the classical method for isotropic beams. Thermal buckling for the beam element with and without shear deformation is also compared. Also, vibrations for several different ply angle orientations are compared.

In this study, orientation of the lay-up of the composite beam is important. Isotropic material can be considered as the special case of the laminate. Both symmetric and unsymmetric cases are studied for several different angle orientations. In this study the thermal load applied to the beam is uniform throughout the beam. Also no initial deflections are imposed on the beam.

Chapter 5 draws conclusions for this study and it will also suggest the possible future work.
Figure 1.1 Geometry of the Beam
Figure 1.2 – Side View of the Beam
Figure 1.3 – 20 DOF Beam Element
2.0 FINITE ELEMENT FORMULATION

In this chapter, the non-linear equations of motion for a beam element are obtained using Hermitian polynomials for interpolation of displacement functions and Lagrange's equation. The non-linearity is due to a non-linear term in the strain-displacement relationship, the so-called "von Karman type geometrical non-linearity." The temperature that is imposed on the beams is steady-state and uniform throughout the beam. The beam is considered to be flat (no initial imperfections). The beam element is made of layers of orthotopic material. As mentioned before, the beam element has a total of 20 degrees of freedom or 10 degrees of freedom at each node. It is used to evaluate thermal buckling load, compute the thermal post-buckling deflections, and analyze the vibrations of thermally post-buckled beams. A computer program written in FORTRAN is used for the calculations.

2.1 SHEAR DEFORMATION

The widely-used Bernoulli-Euler beam theory does not include the effect of shear deformation. However, as explained in Chapter 1, the shear deformation may have significant effects on the vibration and buckling of laminated beams.
Timoshenko presented a study of the correction factors for shear of prismatic bars [18] and rectangular sections [30]. In Bernoulli-Euler beam theory, it is assumed that cross sections originally perpendicular to the neutral axis of the beam remain plane and perpendicular to the neutral axis. So, in that case the slope is simply equal to the derivative of the deflection, \( \frac{dw}{dx} \). In Timoshenko beam theory the plane sections remain plane, but not necessarily perpendicular to the axis of the beam. In this case any initially rectangular element tends to deform into a parallelogram, so the slope is diminished by the angle due to shear \( \psi - \frac{dw}{dx} \). This is shown in Fig. 2.1. As can be seen in this figure \( \psi(x) \) is not equal to the slope \( \frac{dw}{dx} \) of the deflection curve. Thus in Timoshenko beam theory, two displacement function, \( w \) and \( \psi \) must be determined and there are two elastic equations, which are

\[
\psi - \frac{dw}{dx} = \frac{V}{kAG} \quad (2.1a)
\]

\[
\frac{d\psi}{dx} = \frac{M}{EI} \quad (2.1b)
\]

where \( A \) is the cross-sectional area, \( G \) is the shear modulus, \( k \) is the shear correction factor, which depends on the shape of the cross-section. In this case, the dynamical equation for vibration problems are as follows:

\[
\frac{d}{dx} \left( EI \frac{d\psi}{dx} \right) + kAG \left( \frac{dw}{dx} - \psi \right) - J\psi = 0 \quad (2.1c)
\]
perpendicular to the face

tangent to the deflection curve

Figure 2.1 Shear Deformation in a Beam Element
\[
\rho A \ddot{w} - \frac{d}{dx} \left[ kAG \left( \frac{dw}{dx} - \Psi \right) \right] = 0 \quad (2.1d)
\]

By eliminating \( \Psi \) between these coupled equations we obtain,

\[
EI \frac{\partial^4 w}{\partial x^4} + pA \frac{\partial^2 w}{\partial t^2} - (J + \rho EI \frac{kG}{} \frac{\partial^2 w}{\partial x^2 \partial t^2} + \rho J \frac{\partial^4 w}{\partial t^4} = 0 \quad (2.2)
\]

The terms which include \( J \) (polar moment of inertia) are referred to as the rotatory inertia terms. Thus, Timoshenko beam theory accounts for both shear deformation and rotatory inertia.

2.2 BEAM ELEMENT DISPLACEMENT FUNCTIONS

The displacement functions are interpolated by Hermitian polynomials as follows:

\[
u(x) = N_1 u_1 + N_2 u_1' + N_3 u_2 + N_4 u_2' \quad (2.3)
\]

\[
\beta(x) = N_1 \beta_1 + N_2 \beta_1' + N_3 \beta_2 + N_4 \beta_2' \quad (2.4)
\]

\[
w_b(x) = N_1 w_{b_1} + N_2 w_{b_1}' + N_3 w_{b_2} + N_4 w_{b_2}' \quad (2.5)
\]

\[
w_s(x) = N_1 w_{s_1} + N_2 w_{s_1}' + N_3 w_{s_2} + N_4 w_{s_2}' \quad (2.6)
\]

\[
\tau(x) = N_1 \tau_1 + N_2 \tau_1' + N_3 \tau_2 + N_4 \tau_2' \quad (2.7)
\]

The prime stands for \( d/dx \) and \( N's \) are Hermitian polynomials:
\[ N_1 = 1 - 3 \left( \frac{X}{L} \right)^2 + 2 \left( \frac{X}{L} \right)^3 \] (2.8)

\[ N_2 = x - 2 \left( \frac{X^2}{L^2} \right) + \left( \frac{X^3}{L^2} \right) \] (2.9)

\[ N_3 = 3 \left( \frac{X}{L} \right)^2 - 2 \left( \frac{X}{L} \right)^3 \] (2.10)

\[ N_4 = -\left( \frac{X^2}{L^2} \right) - 2 \left( \frac{X^3}{L^2} \right) \] (2.11)

To perform Gaussian integration, which requires the limits from -1 to 1, it is necessary to change the functions above as follows:

\[ N_1 = \frac{(2 - 3\xi + \xi^4)}{4} \] (2.12)

\[ N_2 = \frac{(\xi^3 - \xi^2 - \xi + 1)}{4} \] (2.13)

\[ N_3 = \frac{(2 + 3\xi - \xi^3)}{4} \] (2.14)

\[ N_4 = \frac{(\xi^3 + \xi^2 - \xi - 1)}{4} \] (2.15)

where

\[ \xi = 2 \left( \frac{X}{L} \right) - 1 \] (2.16)

and \(-1 \leq \xi \leq 1\)

2.3 STRESS-STRAIN RELATIONSHIPS

For initially stress free orthotopic materials without thermal load, the plane stress-strain relationships relative
to the ply direction can be written as follows [35]:

\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{pmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{21} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{pmatrix}
e_1 \\
e_2 \\
\gamma_{12}
\end{pmatrix}
\]  

(2.18)

Including thermal effect, the equation can be rewritten as

\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{pmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{21} & Q_{22} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{pmatrix}
e_1 - \alpha_1 \Delta T \\
e_2 - \alpha_2 \Delta T \\
\gamma_{12}
\end{pmatrix}
\]  

(2.19)

where \( \alpha_1 \) and \( \alpha_2 \) are the major and minor coefficients of thermal expansion, respectively, and \( \Delta T \) is the temperature change. For the \( i \)-th layer with angle of orientation \( \phi \), the stresses in the \( x,y \) direction can be determined as follows [36]:

\[
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{pmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{12} & Q_{22} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66}
\end{bmatrix}
\begin{pmatrix}
e_x - \alpha_x \Delta T \\
e_y - \alpha_y \Delta T \\
\gamma_{xy} - \alpha_{xy} \Delta T
\end{pmatrix}
\]  

(2.20)

where \([\tilde{Q}]\) is the transformed reduced stiffness matrix, which depends on lamination angle. The relationship between the reduced stiffness matrix and transformed reduced stiffness matrix can be shown as [35]

\[
[\tilde{Q}] = [T_\phi]^{-1} [Q] [T_\phi]^{-T}
\]  

(2.21)

where
\[
[T] = \begin{bmatrix}
\cos^2\phi & \sin^2\phi & 2\sin\phi\cos\phi \\
\sin^2\phi & \cos^2\phi & -2\sin\phi\cos\phi \\
-\sin\phi\cos\phi & \sin\phi\cos\phi & \cos^2\phi - \sin^2\phi
\end{bmatrix}
\] (2.22)

The stress resultants per unit length \(\{N\}\) and \(\{M\}\) in the \(i\)-th laminate are defined as follows:

\[
(\{N\}, \{M\})_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \{\sigma\}_i(1, z) \, dz 
\] (2.23)

The response of laminated beams under external loads or excitations is predicted by classical laminate theory [35]. According to the theory, the constitutive relationships without thermal effect are

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\] (2.24)

When considering the thermal loads, this becomes

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} =
\begin{bmatrix}
A & B \\
B & D
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\begin{bmatrix}
N_{x\Delta T} \\
N_{y\Delta T} \\
N_{xy\Delta T}
\end{bmatrix}
\] (2.25)

The laminate stiffness matrices \([A],[B]\), and \([D]\) are as follows:
\[ A_{ij} = \sum_{k=1}^{n} (h_k - h_{k-1}) \overline{Q}_{ij}^{(k)} \]  

\[ B_{ij} = \sum_{k=1}^{n} \frac{1}{2} (h_k^2 - h_{k-1}^2) \overline{Q}_{ij}^{(k)} \]  

\[ D_{ij} = \sum_{k=1}^{n} \frac{1}{3} (h_k^3 - h_{k-1}^3) \overline{Q}_{ij}^{(k)} \]

where \( i, j = 1, 2, 6 \) and \( n \) and \( h_k \) are the total number of laminates and the thickness of the \( k \)-th laminate, respectively.

Since this study deals only with beams

\[ N_y = M_y = N_y \Delta_T = M_y \Delta_T = 0 \]  

However, the inplane strain \( \varepsilon_y \) and the bending curvature \( \kappa_y \) are assumed to be non-zero. Expressing \( \varepsilon_y \) and \( \kappa_y \) in terms of \( \varepsilon_x \), \( \gamma_{xy} \), \( \kappa_x \), and \( \kappa_{xy} \), they become

\[ \varepsilon_y = a_1 \varepsilon_x + a_2 \gamma_{xy} + a_3 \kappa_x + a_4 \kappa_{xy} \]  

\[ \kappa_y = b_1 \varepsilon_x + b_2 \gamma_{xy} + b_3 \kappa_x + b_4 \kappa_{xy} \]

where

\[ a_1 = \frac{(A_{12} - \frac{B_{22}B_{12}}{D_{22}})}{\frac{B_{22}^2}{D_{22}} - A_{22}} \]  

(2.32)
\[ a_4 = \frac{(A_{26} - \frac{B_{26}B_{22}}{D_{22}})}{(B_{22}^2 - A_{22})} \]  
\[ (2.33) \]

\[ a_3 = \frac{(B_{12} - \frac{D_{12}B_{22}}{D_{22}})}{(B_{22}^2 - A_{22})} \]  
\[ (2.34) \]

\[ a_4 = \frac{(B_{26} - \frac{D_{26}B_{22}}{D_{22}})}{(B_{22}^2 - A_{22})} \]  
\[ (2.35) \]

\[ b_1 = \frac{(B_{26} - \frac{B_{22}A_{12}}{A_{12}})}{(B_{22}^2 - A_{22})} \]  
\[ (2.36) \]

\[ b_2 = \frac{(B_{26} - \frac{A_{26}B_{22}}{A_{22}})}{(B_{22}^2 - A_{22})} \]  
\[ (2.37) \]

\[ b_3 = \frac{(D_{12} - \frac{B_{12}B_{22}}{A_{22}})}{(B_{22}^2 - A_{22})} \]  
\[ (2.38) \]

\[ b_4 = \frac{(D_{26} - \frac{B_{26}B_{22}}{A_{22}})}{(B_{22}^2 - A_{22})} \]  
\[ (2.39) \]
Finally, the constitutive relations become

\[
\begin{pmatrix}
N_x \\
N_{xy} \\
M_x \\
M_{xy}
\end{pmatrix} =
\begin{pmatrix}
\overline{D}_{11} & \overline{D}_{12} & \overline{D}_{13} & \overline{D}_{14} \\
\overline{D}_{21} & \overline{D}_{22} & \overline{D}_{23} & \overline{D}_{24} \\
\overline{D}_{31} & \overline{D}_{32} & \overline{D}_{33} & \overline{D}_{34} \\
\overline{D}_{41} & \overline{D}_{42} & \overline{D}_{43} & \overline{D}_{44}
\end{pmatrix}
\begin{pmatrix}
e_x \\
y_{xy}
\end{pmatrix} -
\begin{pmatrix}
N_{xAT} \\
N_{xyAT} \\
M_{xAT} \\
M_{xyAT}
\end{pmatrix}
\]

(2.40)

The detailed procedure to obtain Eqs. (2.30) through (2.40) is described in Appendix A. The shear deformation term is included in this formulation by dividing the transverse displacement into two parts:

\[
w = w_s + w_b
\]

(2.41)

Defining \( Q_x \) as the transverse shear force, the force-strain relationship becomes [36]

\[
Q_x = kD_{44}y_{xx}
\]

(2.42)

where \( k \) is the shear correction factor.

Including the shear deformation term, the constitutive relations become

\[
\{N\} = [\overline{D}]\{\epsilon\} - \{N_{AT}\}
\]

(2.43)

where

\[
[\overline{D}] =
\begin{pmatrix}
\overline{D}_{11} & \overline{D}_{12} & \overline{D}_{13} & \overline{D}_{14} & 0 \\
\overline{D}_{21} & \overline{D}_{22} & \overline{D}_{23} & \overline{D}_{24} & 0 \\
\overline{D}_{31} & \overline{D}_{32} & \overline{D}_{33} & \overline{D}_{34} & 0 \\
\overline{D}_{41} & \overline{D}_{42} & \overline{D}_{43} & \overline{D}_{44} & 0 \\
0 & 0 & 0 & 0 & \overline{D}_{55}
\end{pmatrix}
\]

(2.44)
\( \{N\} = \begin{bmatrix} N_x \\ N_{xy} \\ M_x \\ M_{xy} \\ Q_x \end{bmatrix} \) \hspace{1cm} (2.45)

\[
\{e\} = \begin{bmatrix} e_x \\ \gamma_{xy} \\ \kappa_x \\ \kappa_{xy} \\ \gamma_{xz} \end{bmatrix}
\] \hspace{1cm} (2.46)

\[
\{N_{i}^{\Delta T}\} = \begin{bmatrix} N_{x_{i}^{\Delta T}} \\ N_{xy_{i}^{\Delta T}} \\ M_{x_{i}^{\Delta T}} \\ M_{xy_{i}^{\Delta T}} \\ Q_{x_{i}^{\Delta T}} \end{bmatrix}
\] \hspace{1cm} (2.47)

\[ D_{55} = k \sum_{i=1}^{n} \bar{Q}_{i4}^i (h_{i+1} - h_i) \] \hspace{1cm} (2.48)

The thermal load and moment resultants are defined as follows [36]:

\[ N_{x_{i}^{\Delta T}} = \Delta T \int_{-\frac{h}{2}}^{\frac{h}{2}} (\bar{Q}_{11} \alpha_x + \bar{Q}_{12} \alpha_y + \bar{Q}_{16} \alpha_{xy}) dz \] \hspace{1cm} (2.49)

\[ N_{xy_{i}^{\Delta T}} = \Delta T \int_{-\frac{h}{2}}^{\frac{h}{2}} (\bar{Q}_{16} \alpha_x + \bar{Q}_{26} \alpha_y + \bar{Q}_{66} \alpha_{xy}) dx \] \hspace{1cm} (2.50)

\[ M_{x_{i}^{\Delta T}} = \Delta T \int_{-\frac{h}{2}}^{\frac{h}{2}} (\bar{Q}_{11} \alpha_x + \bar{Q}_{12} \alpha_y + \bar{Q}_{16} \alpha_{xy}) zdz \] \hspace{1cm} (2.51)
The equations of motion are derived by means of Lagrange's equation. It is necessary to form the strain energy and kinetic energy expressions for dynamic analysis as well as form the thermal load vector for static analysis.

2.4 EQUATIONS OF MOTION

The strain energy for the beam element is given as follows:

\[ U = \frac{D}{2} \int_0^L \{e\}^T \{N\} \, dx \]  \hspace{1cm} (2.54)

Substitution of \{N\} from Eq.(2.43) into (2.54) yields

\[ U = \frac{b}{2} \int_0^L \{e\}^T \{D\} \{e\} \, dx \]  \hspace{1cm} (2.55)

The right side of this equation can be divided into two parts,

\[ U_1 = \frac{b}{2} \int_0^L \{e\}^T \{D\} \{e\} \, dx \]  \hspace{1cm} (2.56)

\[ U_2 = -\frac{b}{2} \int_0^L \{e\}^T \{N_{AT}\} \, dx \]  \hspace{1cm} (2.57)
The strain vector \( \{\epsilon\} \) can be written in terms of the displacement vector as

\[
\{\epsilon\} = [B]\{q\} \quad (2.58)
\]

where \( \{q\} \) and \([B]\) are shown in Appendix B. Substitution of Eq. (2.58) into Eqs. (2.56) and (2.57) yields

\[
U_1 = \frac{B}{2} \{q\}^T \int_0^L [B]^T [D] [B] \, dx \{q\} \quad (2.59)
\]

\[
U_2 = -\frac{B}{2} \{q\}^T \int_0^L [B]^T \{N\} \, dx \quad (2.60)
\]

For the static case, the load vector and the stiffness matrix can be extracted from Eqs. (2.59) and (2.60). The load-displacement equation for this case is

\[
\{F\} = [k]\{q\} \quad (2.61)
\]

where \([k]\) is the 20 by 20 element stiffness matrix

\[
[k] = b \int_0^L [B]^T [D] [B] \, dx \quad (2.62)
\]

and \(\{F\}\) is the 20 component applied load vector corresponding to each element

\[
\{F\} = b \int_0^L [B]^T \{N\} \, dx \quad (2.63)
\]

2.4.2 NONLINEAR ELEMENT STIFFNESS MATRIX

For the linear case, the strains and curvatures are [14]
By including the nonlinear term, this equation becomes

\[
\{e\} = \left( \begin{array}{c}
\frac{\partial u}{\partial x} \\
\frac{\partial \beta}{\partial x} \\
\frac{\partial^2 w_b}{\partial x^2} \\
2 \frac{\partial \tau}{\partial x} \\
\frac{\partial w_s}{\partial x}
\end{array} \right) + \left( \begin{array}{c}
0 \\
0 \\
1 \left( \frac{\partial w_b}{\partial x} \right)^2 \\
0 \\
0
\end{array} \right) = \{e_L\} + \{e_{NL}\}
\]  

(2.65)

where \(\{e_L\}\) and \(\{e_{NL}\}\) are the linear and nonlinear strain vectors, respectively. The nonlinear strain vector can be written as [14]

\[
\{e_{NL}\} = \frac{1}{2} \left\{ A \frac{\partial w_b}{\partial x} \right\} = \frac{1}{2} [A] \theta
\]

(2.66)

where
The derivative of \( w_b \) is related to nodal displacements by

\[
\frac{\partial w_b}{\partial x} = \frac{2}{L} \left( N'_1 w_{b_2} + N'_2 \theta_{b_1} + N'_3 w_{b_2} + N'_4 \theta_{b_2} \right) \tag{2.69}
\]

where prime represents the partial derivative with respect to \( \xi \). In matrix form

\[
\theta = [G] \{q^b\} \tag{2.70}
\]

where

\[
[G] = \frac{2}{L} \begin{bmatrix} N'_1 & N'_2 & N'_3 & N'_4 \end{bmatrix} \tag{2.71}
\]

\[
\{q^b\} = \begin{bmatrix} w_{b_1} \\ \theta_{b_1} \\ w_{b_2} \\ \theta_{b_2} \end{bmatrix} \tag{2.72}
\]

Therefore, the strain displacement matrix can be written as

\[
[B] = [B_L] + \frac{1}{2} [B_{NL}(q^b)] \tag{2.73}
\]
$[B_{NL}]$ is found by taking a variation of $\{ \epsilon_{NL} \}$ with respect to $\{q^b\}$ as follows:

$$\delta\{\epsilon_{NL}\} = \frac{1}{2} \delta\{A\} \theta + \frac{1}{2} \{A\} \delta \theta = \{A\} \{G\} \delta\{q^b\} \tag{2.74}$$

Therefore

$$[B_{NL}] = [A][G] \tag{2.75}$$

The total stiffness matrix in the non-linear case is obtained by substituting $[B]$ from Eq.(2.73) into Eq.(2.62)

$$[K] = \int_0^L (b[B_L]^T[D][B_L]) dx$$

$$+ \frac{1}{2} \int_0^L b(2[B_{NL}(q^b_I)]^T[D][B_L])$$

$$+ [B_L]^T[D][B_{NL}(q^b_I)]) dx$$

$$+ \frac{1}{3} \int_0^L \frac{3}{2} b([B_{NL}(q^b_I)]^T[D][B_{NL}(q^b_I)]) dx \tag{2.76}$$

Therefore, the stiffness matrix including nonlinear terms become

$$[K] = [K_L] + 1/2[ n_1 ] + 1/3[ n_2 ] \tag{2.77}$$

where

$$[K_L] = b \int_0^L [B_L]^T[D][B_L] dx \tag{2.78}$$
\[ [n_1] = b \int_0^L (2 [B_{NL}(q_i^b)]^T [D] [B_L]) \]
\[ + [B_L]^T [D] [B_{NL}(q_i^b)]) \, dx \] (2.79)

\[ [n_2] = \frac{3}{2} b \int_0^L [B_{NL}(q_i^b)]^T [D] [B_{NL}(q_i^b)] \, dx \] (2.80)

2.4.3 STIFFNESS MATRIX DUE TO THERMAL LOAD

When a beam is subjected to a set of external forces, \( F_1, F_2, \ldots, F_i \), but regarded as fixed against all displacements \( q \) except \( q_i \), which occurs at point \( i \), however, no presence of axial force work done by the force is equal to the strain energy \( U \), following relationship is given [37]:

\[ F_i = \frac{\partial W}{\partial q_i} = \frac{\partial U}{\partial q_i} \] (2.81)

In the presence of axial force, following relationship is given [37]:

\[ W = W_L + W_a = U \] (2.82)

where \( W_L, W_a, U \) are the work done by lateral force, work done by axial force, and the strain energy stored during the bending respectively. If the restrained condition employed in the Eq.(2.81) is considered, it follows that [37]

\[ \frac{\partial W}{\partial q_i} = \frac{\partial W_L}{\partial q_i} \] (2.83)
Then from Eqs. (2.81) to (2.82), following expression is given [37]:

\[ F_i = \frac{\partial}{\partial d_i} (U - W) \]  
\[ (2.84) \]

The work done by the axial force is

\[ W_a = \frac{1}{2} \int_0^L P \frac{\partial w}{\partial x}^2 dx \]  
\[ (2.85) \]

where \( w = w_b + w_s \).

In our case \( P \) is the thermally induced load. The work done by the thermal load can be shown to be

\[ W_A = [-\alpha_x \overline{Q}_{11} \frac{\alpha_{x\overline{Q}_{12}}}{\overline{Q}_{22}} + \frac{\alpha_{x\overline{Q}_{26}}}{\overline{Q}_{22}} \overline{Q}_{12} - \alpha_{xy} \overline{Q}_{16}] \left( \frac{A}{2} \right) \int_0^L \Delta T \left( \frac{\partial w}{\partial x} \right)^2 dx \]  
\[ (2.86) \]

The details of this derivation are given in Appendix C. From Eq. (2.86), it can be shown that the stiffness matrix due to thermally induced load may be written in the form for Gaussian quadrature,

\[ [K_{\Delta T}] = \frac{AL}{2} [-\alpha_x \overline{Q}_{11} \frac{\alpha_{x\overline{Q}_{12}}}{\overline{Q}_{22}} + \frac{\alpha_{x\overline{Q}_{26}}}{\overline{Q}_{22}} \overline{Q}_{12} - \alpha_{xy} \overline{Q}_{16}] \Delta T \sum N_i'(\xi) N_j'(\xi) H_k \]  
\[ (2.87) \]

where \( H_k \) is the weighting factor for the Gaussian quadrature and \( i \) and \( j \) correspond to the transverse displacements. Abscissas and weights are listed in Appendix D. Observe that the \( x-y \) coordinate has been transformed to the local \( \xi \) coordinate for the Gaussian quadrature. After the
integration the matrix must be transformed back to the
global x-y coordinates by following manner.

\[ [k_G] = [T]^T[k_L][T] \]  \hspace{1cm} (2.88)

where \([k_G], [k_L],\) and \([T]\) are the stiffness matrices in
global coordinate, local coordinate, and transformation
matrix respectively.

The transformation matrix \([T]\) is obtained by noting that
derivatives of the displacements in the local and global
coordinates are related by

\[ \frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}/\left(\frac{\partial \xi}{\partial x}\right) \]  \hspace{1cm} (2.89)

while the displacements are the same in both coordinates.

From Eq.(2.16), following expression is obtained

\[ \frac{\partial x}{\partial \xi} = \frac{L}{2} \]  \hspace{1cm} (2.90)

Therefore the transformation matrix in Eq.(2.88) contains
elements on the main diagonal equal to 1 corresponding to
the displacements, and equal to \(L/2\) corresponding to the
derivatives of the displacements. All other components of
the matrix will be zero.

2.4.4 THERMAL LOAD VECTOR

The integral in equation (2.57) can be written in terms of
the non-dimensional coordinate \(\xi\) (Eq.(2.16)) as
\[ U_2 = -\frac{d}{2} \int_{-1}^{1} [N_{A_2}] (\frac{L}{2}) d\xi \]  

(2.91)

Using Eq. (2.58) it can be written as:

\[ U_2 = -\frac{d}{2} \int_{-1}^{1} [B_L] [N_{A_2}] (\frac{L}{2}) d\xi \]  

(2.92)

But \( U_2 \) can also be shown to be

\[ U_2 = -(\vec{q}^T P_{A_2}) \]  

(2.93)

Equating (2.92) and (2.93),

\[ \frac{L}{2} \frac{d}{2} \int_{-1}^{1} [B_L] [N_{A_2}] d\xi = (\vec{q}^T P_{A_2}) \]  

(2.94)

Therefore the thermal load vector becomes

\[ \{P_{A_2}\} = \frac{d}{2} \frac{L}{2} \int_{-1}^{1} [B_L] [N_{A_2}] d\xi \]  

(2.95)

The integration will be done using Gaussian quadrature.

2.4.5 MASS MATRIX

For the dynamic analysis, it is also necessary to form a mass matrix. The kinetic energy to be substituted into the Lagrange's equations of motion is [14]

\[ T = \frac{1}{2} \int_{0}^{L} \rho A \left( \frac{\partial w}{\partial t} \right)^2 dx + \frac{1}{2} \int_{0}^{L} \rho I \left( \frac{\partial \psi}{\partial t} \right)^2 dx \]  

(2.96)

The first term is due to the translatory motion and the second term is due to the rotatory motion. Previous studies such as [1] have shown that the rotatory inertia can be neglected for thin beams. In that case, the kinetic energy
becomes

\[ T = \frac{1}{2} \int_0^L \rho A (\frac{\partial w}{\partial t})^2 dx \]  \hspace{1cm} (2.97)

The deflection function of the beam is

\[ w = w - y t \]  \hspace{1cm} (2.98)

where \( w = w_b + w_s \). Substituting equation (2.97) into (2.98), the kinetic energy becomes [14]

\[ T = \frac{1}{2} \int_0^L \rho A w^2(x) dx + \frac{J}{2} \int_0^L t^2(x) dx \]  \hspace{1cm} (2.99)

where \( A \) is the cross sectional area and \( J \) is the polar mass moment of inertia. The dot is the derivative with respect to time. From this kinetic energy equation, the element mass matrix can be calculated as follows [14]

\[ m_{ij} = \int_0^L \rho A N_i(x) N_j(x) dx \]  \hspace{1cm} (2.100)

where \( i, j = 3, 4, 13, \) and \( 14 \) and

\[ m_{ij} = \int_0^L J N_i(x) N_j(x) dx \]  \hspace{1cm} (2.101)

where \( i, j = 7, 8, 17, \) and \( 18 \). Eqs. (2.100) and (2.101) correspond to the transverse displacement and torsional displacement, respectively. Converting the integration domain into the local coordinate system for Gaussian quadrature, these integrals become
where $i,j = 3, 4, 13, \text{ and } 14$

and

\[ m_{ij} = \int_{-1}^{1} \rho AN_i(\xi) N_j(\xi) \frac{L}{2} d\xi \quad (2.102) \]

where $i,j = 7, 8, 17, \text{ and } 18$

\[ m_{ij} = \int_{-1}^{1} JN_i(\xi) N_j(\xi) \frac{L}{2} d\xi \quad (2.103) \]

2.4.6 EQUATIONS OF MOTION

Substituting the strain energy, Eqs. (2.59) and (2.60), and the kinetic energy Eq. (2.96), into the well known Lagrange's equation

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) + \frac{\partial U}{\partial q_i} = F_i = 0 \quad (2.104)
\]

we can obtain the system equations of motion

\[
[K_L] - [K_{\Delta T}] + \frac{1}{2} [n_1] + \frac{1}{3} [n_2] \{\dot{q}\} + [M] \{\ddot{q}\} - \{P_{\Delta T}\} = 0 \quad (2.105)
\]

The displacement vector $\{q\}$ can be written as:

\[
\{q\} = \{q(t)\}_d + \{q\}_{\Delta T} \quad (2.106)
\]

The subscripts $d$ and $\Delta T$ denote the deflections due to the dynamic vibrations and static deflection due to thermal load respectively. Substituting Eq. (2.106) into (2.105) yields the following equation of motion:
Recalling Figure 1.2, the following assumptions can be made:

\[ \{q\}_{d} < \{q\}_{\Delta T} \] (2.108)

\[ \{q\}_{d}^2 = 0 \] (2.109)

\[ \{q\}_{\Delta T} = 0 \] (2.110)

Eq. (2.107) can be divided into static and dynamic terms.

The static part of the equation is

\[ \left[ [K_L] - [K_{\Delta T}] + \frac{1}{2} [n_1]_{\Delta T} + \frac{1}{2} [n_1]_d + \frac{1}{3} [n_2]_{\Delta T} \right] \{q\}_{\Delta T} = \{P\}_{\Delta T} \] (2.111)

and the dynamic part is

\[ \left[ [K_L] - [K_{\Delta T}] + \frac{1}{2} [n_1]_{\Delta T} + \frac{1}{2} [n_1]_d + \frac{1}{3} [n_2]_{\Delta T} + \frac{2}{3} [n_2]_{\Delta T, d} \right] \{q\}_{d} \]

\[ + \left[ \frac{1}{2} [n_1]_d + \frac{2}{3} [n_2]_{\Delta T, d} \right] \{q\}_{\Delta T} + [M] \{\ddot{q}\}_{d} = 0 \] (2.112)

Because of the assumptions Eq. (2.109), the fourth and sixth terms of equation (2.112) vanish. Also because of the linear dependence of the first order stiffness matrix on the displacement, the subscripts on the seventh term can be interchanged as follows

\[ \frac{1}{2} [n_1]_d \{q\}_{\Delta T} = \frac{1}{2} [n_1]_{\Delta T} \{q\}_{d} \] (2.113)

The second order stiffness matrix is quadratically dependent
on the displacement, and a similar approach shows that the subscripts can be interchanged in eighth term, so that
\[
\frac{2}{3} [N_{2\Delta T,d}] \{q_{\Delta T}\} = \frac{2}{3} [N_{2\Delta T}] \{q_d\} \tag{2.114}
\]
Appendix E explains the subscript manipulations that are performed above. Finally, the dynamic equation can therefore be rewritten as
\[
[[K_L] - [K_{\Delta T}] + [n_{1\Delta T}] + [n_{2\Delta T}] \{q_d\} + [M]\{q_d\} = 0 \tag{2.115}
\]

2.5 THERMAL BUCKLING

For a beam subjected to a uniform temperature distribution \(\Delta T\), using linear stiffness matrix and stiffness matrix due to thermally induced ,the familiar eigenvalue problem for buckling analysis can be shown [37]
\[
[[K_L] - \lambda [K_{\Delta T}]] \{q\} = 0 \tag{2.116}
\]
or
\[
[K]\{\phi\} = \lambda [K_{\Delta T}]\{\phi\} \tag{2.117}
\]
where \(\lambda\) is the eigenvalue and \(\phi\) is the displacement eigenvector. The smallest eigenvalue is used for the critical buckling temperature of the beam,
\[
\Delta T_{cr} = \lambda_1 \tag{2.118}
\]
2.6 THERMAL POST-BUCKLING

The static system equation in (2.111) incremental form becomes [24]

\[
\begin{bmatrix} [K] - [K_{AT}] \end{bmatrix} + \begin{bmatrix} n_{1AT} \end{bmatrix} + \begin{bmatrix} n_{2AT} \end{bmatrix} \Delta q_{AT} = \Delta P
\]

(2.119)

where

\[
\Delta P = \begin{bmatrix} P \end{bmatrix} - \begin{bmatrix} [K] - [K_{AT}] \end{bmatrix} + \frac{1}{2} \begin{bmatrix} n_{1AT} \end{bmatrix} + \frac{1}{3} \begin{bmatrix} n_{2AT} \end{bmatrix} \{q_{AT}\}
\]

(2.120)

where \([N_1AT]\) and \([N_2AT]\) are the first order and second order nonlinear stiffness matrices from Eqs. (2.79) and (2.80). Equation (2.119) is solved for the critical \(\{q_{AT}\}\) through the Newton-Raphson type iterative procedure, wherein

\[
(q_{AT})_{i+1} = (q_{AT})_i + \Delta q_{AT}
\]

(2.121)

This equation is used for the convergence towards the static thermal deflections for the given temperature step \(\Delta T\). It is necessary to guess some initial value to start this iteration. For this study, the initial trial values were obtained from the solution for the critical buckling temperature. The first eigenvector \(\{\phi_1\}\) is scaled to a small amplitude. Because of the nonlinear relationship between the temperature and deflections and also the use of the tangential stiffness matrix (Eq. (2.119)), it is necessary to use small temperature steps. If the temperature step is too large the deflection \(\{q_{AT}\}_i\) sequence might not converge.
2.7 FREE VIBRATION OF THE BUCKLED BEAM

Once the final static thermal deflections are found the final nonlinear stiffness matrices \([N_{1AT}]\) and \([N_{2AT}]\) are determined using the final deflections. Then they are substituted into the dynamic system equation (2.115) which can be written

\[
[K_{total}] \{q_d\} + [M] \{\ddot{q}_d\} = 0 \tag{2.122}
\]

where \([K_{total}] = [K_L] - [K_{AT}] + [n_{1AT}] + [n_{2AT}]\)

This equation leads to an eigenvalue problem for the linear vibration analysis

\[
[[K_{total}] - \omega^2 [M]] \{q\} = 0 \tag{2.123}
\]

or

\[
[K_{total}] \{v\} = \omega^2 [M] \{v\} \tag{2.124}
\]

where \(\omega\) is a natural frequency and \(\{v\}\) is the corresponding mode shape.
3.0 COMPUTATIONAL PROCEDURE

Although some of the computational procedures have been outlined in Chapter 2, in this chapter they will be explained in more detail. For this study, a FORTRAN program was written. The program is capable of doing the linear vibration analysis including the mode shapes, the thermal buckling analysis, and the vibration analysis of the thermally buckled beams. Furthermore, when the boundary conditions are imposed, it is possible to analyze the beams with or without shear deformation so that comparisons can be made.

3.1 LINEAR VIBRATION ANALYSIS

The linear vibration analysis will be performed first. The beam is straight and no thermal load is applied. The frequencies and the mode shapes are obtained from solving the ordinary eigenvalue problem using the linear stiffness matrix \([K]\) and the mass matrix \([M]\),

\[
\begin{bmatrix} [K] - \omega^2[M] \end{bmatrix} = 0 \tag{3.1}
\]

The results will be compared with the vibration analysis of the thermally buckled beams.
3.2 THERMAL BUCKLING

To find the critical temperature for thermal buckling, the eigenvalue problem for the buckling will be solved. The beam is initially straight. The thermal load applied to the beam can be considered as equivalent to a mechanical load as explained in section 2.4.3. Once the stiffness matrix $[K_{AT}]$ due to the thermal load for the given temperature is formed the eigenvalue problem

$$[K_L] \{\phi\} = \lambda [K_{AT}] \{\phi\}$$

(3.2)

is solved. The smallest eigenvalue gives the critical buckling temperature. The solution procedures are shown in figure 3.2.

3.3 THERMAL POST-BUCKLING

Once the buckling analysis is done, the program proceeds to the post-buckling analysis. In this procedure, the program computes the deflection and slope of the beam due to the high thermal loading. For this analysis, the Newton-Raphson iterative procedure was used to obtain a solution for $q_{AT}$ of the non-linear equation (2.111). For this procedure, it is necessary to guess some initial value for the deflection vector of the beam. In this study the first eigenvector $\{\phi_1\}$ is scaled to about one half thickness of the beam, and used as the initial trial displacement. Once the desired
temperature step and the trial displacement vector are
given, the program starts the iterative procedure. First it
calculates the thermal force \( N_{\Delta T} \) and the thermal moment
\( M_{\Delta T} \) from Eqs.(2.49)-(2.53). Then using the trial
displacements the nonlinear stiffness matrices \( N_{1\Delta T} \) and
\( N_{2\Delta T} \) are calculated from Eqs.(2.79) and (2.80). Then
using the nonlinear stiffness matrices, the thermal load,
and the initial trial deflections, the incremental thermal
load \( \Delta P \) is found from Eq.(2.120). Using the incremental
thermal load, the incremental deflections \( \Delta q_{\Delta T} \) are
calculated by using the tangential stiffness matrix \([K_T]\)
\[
\{\Delta q_{\Delta T}\} = [K_T]^{-1}\{\Delta P\} \tag{3.4}
\]
where
\[
[K_T] = [K] - [K_{\Delta T}] + [N_{1\Delta T}] + [N_{2\Delta T}] \tag{3.5}
\]
Once the incremental deflection is obtained, it is then
checked against the convergence criterion which is \(10^{-4}\)
for this analysis:
\[
\max \{\Delta q_{\Delta T}\} < (10^{-4}) \tag{3.6}
\]
If the convergence criterion is not met, then the
incremental deflection \( \Delta q_{\Delta T} \) is added to the initial trial
displacement and the new deflection become the next trial
displacement.
\[
(q_{\Delta T})_{i+1} = (q_{\Delta T})_i + \Delta q_{\Delta T} \tag{3.7}
\]
The new trial deflection is used to compute the new
nonlinear stiffness matrices, and another iterative procedure begins. Once the incremental deflection meets the convergence criterion, the final deflection is used to compute the final nonlinear stiffness matrices for the vibration analysis of the buckled beam. During this procedure, it is necessary to use small temperature steps even if the final $\Delta T$ is large. As mentioned previously, if the temperature step is too large, the iteration might not converge \cite{24}. The program used for this analysis allows the user to choose the temperature step to adjust the computation time.

3.4 VIBRATION OF THE BUCKLED BEAMS

Because of the assumptions (2.108) through (2.110), the problem to be solved here is an ordinary linear vibration problem. Once the nonlinear stiffness matrices are calculated at the end of the post-buckling procedure, they will be added to the linear stiffness matrix as indicated in Eqs. (2.115). Then the regular eigenvalue problem Eq. (2.124) is solved for the frequencies and the mode shapes.
START

User Input:
Geometry, Material Property
Boundary Conditions

Forming Matrices:
Stiffness
Stiffness due to thermal load
Mass

Linear Vibration Analysis
Eigenvalue Problem

Solutions
Natural Frequencies
Mode Shapes

END

Buckling, Post-Buckling
Vibration of buckled beam

Figure 3.1 – Flowchart for Initial Input and Linear Analysis
Figure 3.2 – Flowchart for Thermal Buckling Analysis
Given $\Delta T_{cr}$, $\{\phi_1\}$

Compute Initial Trial Displacement $\{q\}_{\Delta T} = \{0.5 \ h \{\phi_1\}\}$

Increment $\Delta T$

Calculate $\{N_{\Delta T}\}$, $\{M_{\Delta T}\}$, then $\{P\}$

Using $\{q\}_{\Delta T}$, calculate $[N1]$ and $[N2]$

Calculate $\Delta P$

Calculate $[K_T]$

Calculate $\{\Delta q\}_{\Delta T}$ by $\{\Delta q\}_{\Delta T} = [K_T]^{-1} \{\Delta P\}$

No Converge ?

Yes

C D

Figure 3.3 – Flowchart for Thermal Post-Buckling Analysis
Figure 3.4 Thermal Post-Buckling (Continued) and Vibration of the Thermally Buckled Beam
4.0 DISCUSSION OF RESULTS

In this chapter, several examples are presented to evaluate the solution procedures and the finite element computer program. The results were obtained including shear deformation and not including shear deformation, and comparisons were made. The models that were used for this computation include both symmetrically and unsymmetrically laminated ones. Shear correction factor $k$ used that is used throughout this research is $5/6$. Although the author made the best effort to compare the results with other work, it was not possible in many cases because the analysis of thermal post-buckling behavior and vibration of thermally buckled beams including shear deformation have not been done previously.

4.1 THERMAL BUCKLING

For thermal buckling of beams, isotropic beams, symmetrically laminated beams, and unsymmetrically laminated beams were analyzed. The boundary conditions used were both ends simply-supported and both ends clamped. Inclusion or exclusion of shear deformation terms ($w_s$, $\theta_s$) was specified in the boundary conditions.
The isotropic beams were considered as the special or the most simplified case of the composite beams. There is no bending stretching term and the beam can be considered as a single layer. The beam element used was aluminum and had following properties:

- Thickness of the Beam: 0.064 in.
- Width of the Beam: 1.0 in.
- Total Length of the Beam: 10.0 in.
- Elastic Modulus: \( E = 10.0 \times 10^6 \) psi
- Shear Modulus: \( G = 3.85 \times 10^6 \) psi
- Poisson's Ratio: \( v = 0.3 \)
- Thermal Expansion Coefficient: \( \alpha = 12.5 \times 10^{-6} \) in/in/F
- Number of Elements Used: 6

Figure 4.1 shows the effect of shear deformation on buckling temperature. When shear is neglected the critical temperature is always higher than the when shear is included. According to Timoshenko [32], the critical buckling load under the effect of shear deformation is

\[
P_{cr} = \frac{P_e}{\frac{nP_e}{AG} + 1}
\]

(4.1)

where \( P_e \) is the critical buckling load when neglecting shear deformation, \( A \) is the cross sectional area, \( G \) is the shear modulus, and \( n \) is a constant which depends on the shape of
the cross section. \( n = 1.2 \) is used for the rectangular cross section. The axial load due to the temperature rise in an isotropic beam is

\[
P = \Delta T \alpha E A
\]

(4.2)

Therefore the relationship between the critical temperature with shear and without shear becomes

\[
\Delta T_{cr} = \frac{\Delta T_e}{n \Delta T_e} \frac{1}{1 + \frac{n \Delta T_e}{AG}}
\]

(4.3)

Fig. 4.1 is consistent with Eq.(4.3). In general the effect of shear deformation on buckling is negligible for isotropic beams. Even when \( h/l \) is 0.15 the difference in critical temperature is only about 5%. However, recall that \( h \) of the beam used here is 0.064 in. In the case of \( h/l = 0.15 \), the length of the beam .4267 in. while width of the beam is 1 in. Furthermore, the critical temperature is over 10,000 F which is absolutely impractical.

4.1.2 LAMINATED BEAMS

In the case of laminated beams, the thermal buckling behavior was different from mechanical buckling in general for the symmetrically laminated beams and unsymmetrically laminated beams. Symmetrically laminated beams are used far more commonly than unsymmetrically laminated beams, and they were analyzed for many different ply angle orientations. The
materials used were graphite/epoxy and the properties of the material are as follows:

Major Elastic Modulus: \( E_1 = 22.2 \times 10^6 \) psi

Minor Elastic Modulus: \( E_2 = 1.85 \times 10^6 \) psi

Shear Moduli:
\[
G_{12} = 0.81 \times 10^5 \text{ psi} \\
G_{13} = 0.5 \times 10^6 \text{ psi}
\]

Poisson's Ratio: \( v = 0.3 \)

Thermal Expansion Coefficients:
\[
\alpha_1 = -0.43 \times 10^{-6} \text{ in/in/F} \\
\alpha_2 = 13.6 \times 10^{-6} \text{ in/in/F}
\]

Ply Thickness: 0.012 in./layer

Width of the Beam: 1 in.

First, the effect of shear deformation on buckling was analyzed for several angle orientations. Figs. 4.2 through 4.5 show the effect of shear deformation on both mechanical buckling and thermal buckling. For the case of \( [\phi/90/\phi]_T \) orientation, as \( \phi \) increases the effect of the shear deformation decreases for thermal and mechanical buckling. Figure 4.2 shows the significance of the shear deformation particularly for mechanical buckling. When \( h/l \) is equal to 0.1, the difference between the critical load with and without shear deformation is already close to 30% for mechanical buckling, while the difference between critical temperature is 20% for thermal buckling. However the \( \Delta T_{cr} \) with shear is 9131.09 F (Table 4.1) which is far beyond the operational range of any organic composite material.
Obviously, as the length of the beam increased, the critical temperature decreases, as does the effect of shear deformation. For $[30/90/30]_T$ orientation, although the effect of shear deformation is on buckling becomes significant at high $h/l$ ratios, in actuality the beam is nearly immune from buckling because of correspondingly high critical temperatures.

For the angle orientation of $([\phi/90/\phi]_T)$, both $\Delta T_{cr}$ and the effect of shear deformation become progressively smaller as the angle $\phi$ increases (Tables 4.1 through 4.3). At the same time difference between the influence of shear deformation on thermal buckling and mechanical buckling decreases, and at $[70/90/70]$, thermal buckling is more affected by shear deformation although it is hard to see from the graph. In all cases, the critical temperature is reasonable when the effect of shear deformation is very small, while the effect of shear deformation is significant at critical temperatures which are impractically high. Therefore, the effect of shear deformation can be ignored for the buckling analysis of such angle orientations.

Fig. 4.6 shows the effect of the angle orientation on the critical temperature for the $[\phi/90/\phi]_T$ layup at $h/l=0.01$. While $\phi$ is small, the critical temperature is extremely high and as $\phi$ increases, the critical temperature decreases and
lowest at $\phi$ is equal to 90 degrees. Note that the critical temperature is not zero even when $\phi$ is equal to 90 degrees. The reason the beam has such high critical temperature when $\phi$ is small is that $\alpha_1$, which is the major thermal expansion coefficient, is negative. So if $\phi$ is 0 or close to zero, the top and bottom layers shrink with increasing temperature. So even though the second layer tries to expand rather rapidly along the x-axis, the beam as a whole expands very slowly. However, in the case of plates the results are quite different from the beam as shown by Gray [24], unlike a beam, a plate can also buckle in the y direction.

Again using only three layers but the orientation was changed to $[90/\phi/90]_T$. For $[90/45/90]_T$ (Table 4.4) the effect of shear deformation is slightly less than and otherwise appears similar to those for $[45/90/45]_T$, except the critical temperature is lower. Buckling temperatures are much lower at small $\phi$ than in the $[\phi/90/\phi]_T$ orientations, as can be seen in Fig. 4.6. and stays low. The $\Delta T_{cr}$ trend in $[90/\phi/90]_T$ is similar to that in $[\phi/90/\phi]$: higher at low $\phi$ and decreasing as $\phi$ increases.

One of the most widely used orientation $[0/45/-45/90]_s$ beam was also analyzed. Since the beam has 8 layers and is thicker than the previous two cases, the length of the
modelled beam was adjusted to maintain same h/l ratio.
Although Fig.4.7 shows that shear deformation is significant for h/l greater than about h/l=0.04, as can be seen from Table 4.6, the corresponding critical temperature exceeds 2000 F. Therefore, for all practical cases, the effect of the shear deformation on buckling can be neglected.

Unsymmetrically laminated beams were also analyzed. The relationship between the critical temperature and angle orientation for \([\phi/-\phi/\phi/-\phi]\) and \([\phi/-\phi/-\phi/\phi]\) layers is given in Table 4.8. For unsymmetric \([\phi/-\phi/\phi/-\phi]\) and symmetric \([\phi/-\phi/-\phi/\phi]\) angle plies, the critical temperatures are exactly same. Fig. 4.8 shows the effect of shear deformation when angle orientations are \([45/-45/45/-45]\) and \([45/-45/-45/45]\). Again the behavior is exactly same for both. The reason is the \(B_{11}\) component of \([B]\) matrix of the beams. For the case of the symmetric laminate, obviously all the terms in the \([B]\) matrix are zero. Therefore there is no bending-stretching coupling. For the unsymmetric laminate, most of the components in \([B]\) matrix are non zero. In the case of \([\phi/-\phi/\phi/-\phi]\), \(B_{16}\) and \(B_{26}\) are not zero. However, the \(B_{11}\), which is the most dominant term in the \([B]\) matrix, is still zero. Because \(B_{11}\) is zero there was no difference in buckling temperature between symmetric laminate \([\phi/-\phi/-\phi/\phi]\) and unsymmetric laminate \([\phi/-\phi/\phi/-\phi]\).
To prove this, another example was analyzed. This time, the symmetric laminate and unsymmetric laminate were [90/0/0/90] and [90/0/90/0], respectively. The buckling temperature for [90/0/0/90] is 46 F while buckling temperature for [90/0/90/0] is 123 F. $B_{11}$ for [90/0/90/0] is non-zero while $B_{11}$ for [90/0/0/90] is zero. From the examples above, unsymmetricity itself does not necessarily affect the critical temperature and shear deformation effect, but the $B_{11}$ term does. Although almost all unsymmetric angle plies have non-zero $B_{11}$, it is not the case for unsymmetric angle plies. The explanation suggests that both [30/-30/30/-30] and [30/-30/-30/30] should have the same buckling temperature. However Gray [24] reported a difference in buckling temperature between those two for plates. He explained the difference as due to $D_{16}$ and $D_{26}$ components of the matrices. For $[\phi/-\phi/-\phi/\phi]$, $D_{16}$ and $D_{26}$ will be zero while $[\phi/-\phi/-\phi/\phi]$ orientation has positive $D_{16}$ and $D_{26}$ values equal in magnitude to the rest of the $[D]$ matrix components. Therefore the $[\phi/-\phi/-\phi/\phi]$ orientation has a higher buckling stiffness. However, the present results show no difference in those two cases. The reason is that the twisting degrees of freedom $\tau$ and $\tau'$ were neglected from the buckling analysis in this study. $D_{16}$ and $D_{26}$ represent the bending-twisting coupling. By neglecting the twisting degrees of freedom, the terms affected by $D_{16}$ and $D_{26}$ are eliminated from the stiffness matrix while the computer
program forms it. Therefore in the comparison between the symmetric and unsymmetric angle ply, there was no difference in buckling temperature.

The relationship between critical temperature and the boundary conditions was also investigated. Fig. 4.9 compares the shear deformation effect on buckling of the beam ([45/90/45]) for the simply-supported case and the clamped-clamped case as well as one end simply-supported and the other end clamped. From the graph, it is obvious that the beam with the clamped-clamped boundary condition is affected most by shear deformation, followed by simply-supported-clamped combination. The beam with the simply-supported boundary condition was least affected. The possible reason for this is the effective length. Let the length of the beam be L. The effective length of the simply-supported beam is L while the effective length of the simply-supported-clamped and clamped-clamped beams are 0.7L and 0.5L respectively. The shear deformation affect the effective length rather than actual length of the beam. If this is correct, $\Delta T_{cr}/\Delta T_{cr, no shear}$ for simply-supported (S-S) with h/l of 0.1 and clamped-clamped (C-C) with h/l of 0.05 should be equal. According to Table 4.7 those two values are very close. The values for S-S with h/l of 0.15 and C-C of 0.075 are also close. The comparisons were also done between S-S case and simply-supported-clamped (S-C) case.
In this case $\Delta T_{cr}/\Delta T_{cr}$, no shear for S-S with 0.075 and S-C with 0.05 should be close. According Table 4.7 the values are not very close to each other. However 70% of 0.075 is actually 0.0525 and the difference from 0.05 was big enough to make the value inaccurate. Overall Fig.4.9 and Table 4.7 support the claim that shear deformation effects correlate on the effective length of the beam rather than actual length of the beam.

4.2 POST-BUCKLING DEFLECTION

For both isotropic and composite beams, if the temperature increase beyond $\Delta T_{cr}$ is very small, post-buckling deflections can be considered as linear. However as the deflection increases, the linear theory is no longer accurate and it is necessary to use non-linear theory. The post-buckled deflection with and without shear deformation terms were compared.

4.2.1 VERIFICATION OF NONLINEAR MATRICES

Since there is no previous published work for post-buckling and vibration analysis of composite beams, direct comparison with other work cannot be made. Therefore indirect comparisons were made by using non-linear free vibration results such as previously published work by Raciti [14] and
Mei [38]. To do this a few modifications were made in the computer program. For the analysis of post-buckling deflections, the scaled buckling mode shapes were fed into the subroutine that creates non-linear stiffness matrices. For the non-linear vibration analysis, the mode shapes for linear vibration analysis were fed into the same subroutine. Therefore by having good agreement with the previous non-linear vibration results, the non-linear matrices can be considered correct. Recalling the equations (2.77)-(2.79), (2.100), and (2.101) and using the Lagrange's equation, the equation of motion becomes

$$[[K] + [N_1]} + [N_2]}] \{q\} + [M] \{\ddot{q}\} = 0$$

(4.4)

The matrices can be subdivided as follows [14]

$$\begin{bmatrix} K_{uu} & K_{uw} \\ K_{wu} & K_{ww} \end{bmatrix} \{q_u\} + \begin{bmatrix} 0 & N_{uw1} \\ N_{uw1} & 0 \end{bmatrix} \{q_w\} + \begin{bmatrix} 0 & 0 \\ 0 & N_{ww2} \end{bmatrix} \{q_w\} + \begin{bmatrix} 0 & 0 \\ 0 & M_{ww} \end{bmatrix} \{\ddot{q}_w\} = \{0\}$$

(4.5)

where the subscripts u and w refer to the inplane and transverse displacements respectively. For cases other than unsymmetric laminate beams, [K_{uw}], [K_{wu}], and [N_{ww1}] will be zero since there is no bending-stretching coupling. Raciti [14] made an assumption that the spatial variation of the transverse displacement in the non-linear range is the same as the linear mode, and this assumption is valid for moderately large amplitude vibrations. For the case of very high amplitudes, the effect of other modes may be considered by using multi mode analysis.
Now let [14]

\[ \{q_w\} = \{\overline{q}_w\} \tau(t) \]  \hspace{1cm} (4.6)

where

\[ \tau = A \cos \Omega t \]  \hspace{1cm} (4.7)

and \{q_w\}, A, and \Omega are the normalized linear mode, the amplitude of vibration, and the nonlinear frequency, respectively. By substituting Eq. (4.6) into Eq. (4.5), the inplane displacement vector can be written as [14],

\[ \{q_d\} = -[K_{uu}]^{-1}[K_{uw}][\overline{q}_w] \tau - [K_{uu}]^{-1}[N_{uw1}][\overline{q}_w] \tau^2 \]  \hspace{1cm} (4.8)

By substituting Eq. (4.7) into Eq. (4.5), the equation of motion becomes [14],

\[ [K_{ww} - K_{wu} K_{uu}^{-1} K_{uw}][\overline{q}_w] \tau + [N_{wu1} - K_{wu} K_{uu}^{-1} N_{uw1} - N_{wu1} K_{uu}^{-1} K_{uw}][\overline{q}_w] \tau^2 \]

\[ + [N_{wu2} - N_{wu1} K_{uu}^{-1} N_{uw1}][\overline{q}_w] \tau^3 + [M_{ww}][\overline{q}_w] \dot{\tau} = 0 \]  \hspace{1cm} (4.9)

By multiplying all terms by \{q_w\}^T, the equation (4.9) becomes [15],

\[ \ddot{\tau} + \mu_1 \tau + \mu_2 \tau^2 + \mu_3 \tau^3 = 0 \]  \hspace{1cm} (4.10)

Then the solution to Eq. (4.9) is given by [14],[15], and [40] as

\[ \Omega = \omega \left[ 1 + A^2 \left( \frac{3}{4} \mu_2 - \frac{5}{6} \left( \frac{\mu_2}{\mu_1} \right)^2 \right) \right]^{-\frac{1}{2}} \]  \hspace{1cm} (4.11)

where \mu_1, \mu_2, \mu_3, and \omega are the coefficients of \tau^1, \tau^2, \tau^3, and linear frequency, respectively.
Fig. 4.10 shows the comparison of the present results with those of Mei [41] for an isotropic beam. The boundary conditions were both ends simply-supported. The agreement is good. Results for composite beam were also compared. The laminate orientation is [0/90/90/0] and the material used was graphite-epoxy, whose properties are $E_1=18.5 \times 10^6$ psi, $E_2=1.6 \times 10^6$ psi, $v_{12}=0.25$, $G_{12}=0.65 \times 10^6$ psi, and $\rho=0.055$ lb/in$^3$. Fig. 4.11 shows the comparison between present results and those of Raciti [14] for simply-supported beam. Again, the agreement is good. Therefore the present computer program is producing the correct non-linear stiffness matrices.

4.2.2 ISOTROPIC BEAMS

Although there are some detailed work for thermal post-buckling of isotropic plates, such as those of Paul [27] and Gray [24], there is no published research for beams. Bisplinghoff and Pian [30] worked on isotropic beams. However their results were limited to vibration of the buckled beams and contain no information on thermally buckled deflection.

The isotropic beam used for this analysis has the exactly same properties and geometry as the one employed for the thermal buckling analysis, and the number of elements used
was six. Boundary conditions were both ends simply-supported. Fig. 4.12 shows the lengthwise thermal post-buckling deflections at various $\Delta T/\Delta T_{cr}$. As can be seen from the figure, when the thermal load is increased the beam does not jump to a higher deflection mode. It simply deflects further in the first mode. Fig. 4.13 compares the number of elements in the finite element analysis. Ideally, as the number of elements increases, the results will converge towards an "exact" solution. Since in this case no exact solution was available, such a comparison could not be made. However, the difference between the two can be considered as the error produced by using too few elements. When the number of elements is small, the beam is too stiff and the nondimensional deflection is smaller than actual.

4.2.3 SYMMETRICALLY LAMINATED BEAMS

The material used for this analysis was graphite-epoxy whose properties were given in Section 4.1.2. The angle orientation used was $[30/-30/-30/30]$ and the width and length of the beam were 1.0 in and 10.0 in respectively. Three kinds of boundary conditions were considered: both ends simply-supported, clamped-simply-supported, and both ends clamped. Figs. 4.14, 4.15 and 4.16 show the deflection for simply-supported, clamped-simply-supported, and clamped-
clamped respectively. Again, in all cases, increasing the thermal load did not make the deflection jump into a higher mode. It simply increased the deflection in the first mode as found in isotropic case. For the case of the uniform thermal load, the only way to get higher deflection mode shapes is to constrain an interior point which obviously increases the critical temperature. Note that for the clamped-simply-supported case, the maximum deflection occurs off center closer to simply-supported side rather than center of the beam.

Fig. 4.17 shows the relationships between temperature ratio and the dimensionless mid-span deflections for different boundary conditions. Changing the boundary conditions affects the non-dimensional deflection.

Fig. 4.18 shows the comparison between the deflection with and without shear deformation. From the graph, it is clear that shear deformation hardly affects the post-buckling deflection. However that was rather predictable, because the formulations for the post-buckling deflection do not include anything related shear deformation directly. Therefore if shear deformation affects the post-buckling deflection, the difference occurred at buckling. For the present case, the thickness-to-length ratio is 0.0064, and according to the buckling analysis, the influence of shear
deformation is negligible. The thickness of the beam can be increased if one wants to see the effect of shear deformation. However, this might increase the critical temperature beyond the operational temperature of epoxy. Therefore in almost all the post buckling analysis, the effect of shear deformation can be neglected.

4.2.4 UNSYMMETRICALLY LAMINATED BEAMS

Post-buckling deflections of unsymmetrically laminated beams were also analyzed. The material properties used were the same as for the symmetric case. The angle orientation is [-30/30/-30/30], and the thickness and length were 0.012 in/layer and 10 in respectively. Fig. 4.19 shows the deflection of a simply-supported beam. As in the isotropic and symmetric laminate cases, the deflections were increased in the first mode only. Fig. 4.20 shows the effect of boundary conditions on the maximum deflection. The trend appears to be same as the symmetric case. The effect of shear deformation was also investigated. Again the effect of shear on thermal post-buckling appears to be negligible according to Fig. 4.21.

For the case of cross-ply, some difference between unsymmetric-ply and symmetric-ply were expected because of presence of a non-zero B_{11} component. Fig. 4.33 shows the
difference between the symmetric and unsymmetric angle-ply.
Fig. 4.34 shows the insignificance of shear deformation on cross-ply, too.

4.3 VIBRATION ANALYSIS OF THERMALLY BUCKLED BEAMS

The natural frequencies of beams change considerably during the post-buckling process because of the major change in the stiffness matrix. During the pre-buckling stage, natural frequencies decrease as the temperature increases and become zero at the critical temperature, if there is no initial imperfection. At the critical temperature the stiffness of beam goes to zero. In the post-buckling stage, the natural frequencies increase as thermal load increases. This section shows how the fundamental frequencies change as thermal load increases for different materials and boundary conditions. The effect of shear deformation was also investigated.

As previously mentioned, Bisplinghoff and Pian [30] worked on the vibration of thermally buckled isotropic beams. Fig. 4.22 shows the comparison between their 1-D analysis and the present result. \( \omega_0 \) in Fig.4.22 is the natural frequency of the unloaded beam. The agreement is good. As Fig. 4.22 shows, the fundamental frequency goes to zero as the thermal load increases in the pre-buckling stage, and reaches zero.
when $\Delta T$ is equal to $\Delta T_{cr}$. The fundamental frequency increases as the thermal load increases during the post-buckling phase. Figs. 4.23 and 4.24 are the mode shapes of vibration for unloaded case and for $\Delta T/\Delta T_{cr}=4$, respectively. It appears that thermal-postbuckling deflection hardly affects any of the mode shapes.

Having verified the analysis for isotropic beams, the analysis was extended to symmetric and unsymmetric laminates. Fig. 4.25 shows the frequency ratio vs $\Delta T/\Delta T_{cr}$ for $[30/-30/-30/30]$ symmetric ply for three different boundary conditions. Although the difference among frequency ratios for the three boundary conditions is small when $\Delta T/\Delta T_{cr}$ is relatively small, as $\Delta T/\Delta T_{cr}$ increases it appears that difference in the frequency ratios get larger. The frequency ratio is directly related to how much the beam deflects. The larger the deflection is, the larger the frequency ratio is. It appears that simply-supported beams deflect more easily than the clamped-clamped and clamped-simply-supported beams even at high $\Delta T$. That could be the reason why the differences among the frequency ratios for the three boundary conditions become larger as temperature increases. Fig. 4.26 shows the difference between frequency ratios with and without shear deformation for both ends simply-supported. There is very little affect of shear deformation for this boundary condition. Again, this is an
expected result. From the post-buckling deflection analysis, it is known that shear deformation has very little effect on the deflection. Since deflections are nearly the same, the non-linear stiffness matrices will also be nearly identical. Therefore there should be little affect on vibration from shear deformation. Figs. 4.27 and 4.28 show the first three mode shapes of vibration for the unloaded case and at the $\Delta T/\Delta T_{cr}=4$, respectively. Again there is very little change in the shape for any of the modes. Gray [24] reported that for plates there is little change in mode shapes before and after the thermal loading for most cases. However, he showed that there was a big change in the shape of the first mode after the thermal loading for a $[-30/30]_s$ laminate. However, from the figure he presented, it appears that there is some kind of coupling between the vibrations along x-axis and y-axis. The present beam analysis cannot yield such results, since it deals only with one dimension.

The unsymmetrically laminated case was also analyzed. Fig. 4.29 shows the frequency ratio vs. temperature ratio for three boundary conditions. For this case, too, the differences in frequency ratios among the three boundary conditions at low $\Delta T/\Delta T_{cr}$ are small. As $\Delta T/\Delta T_{cr}$ increases, the differences become larger. Fig. 4.30 shows the difference between frequency ratios with and without shear deformation for a simply-supported beam with $[-30/30/-30/30]$
layup. For this case, too, the effect of shear deformation is negligible. Figs. 4.31 and 4.32 show the first three mode shapes of vibration before and after the thermal loading. The differences between the two are nearly imperceptible. Gray [24] reported that for plate as well, there is very little change in the mode shapes for the angle orientation of [-30/30/-30/30].
Effect of Shear Deformation on Buckling

Simply-Supported Aluminum Beam

Figure 4.1
Effect of Shear Deformation on Buckling

Simply-Supported [30/90/30]

![Graph showing the effect of shear deformation on buckling for a simply-supported [30/90/30] structure. The graph compares the critical load ratio $P_{cr}/P_{cr, no shear}$ with the ratio of critical load differences $\Delta P_{cr}/\Delta P_{cr, no shear}$ as a function of $h/l$. The graph includes three cases: neglecting shear, including shear (thermal buckling), and including shear (mechanical buckling). The legend indicates the symbols used for each case. The graph is labeled as Figure 4.2.]
Effect of Shear Deformation on Buckling

Simply-Supported [45/90/45]

Figure 4.3
Effect of Shear Deformation on Buckling

Simply-Supported [70/90/70]

Figure 4.4

- ○ Neglecting Shear
- □ Including Shear (Thermal Buckling)
- ◆ Including Shear (Mechanical Buckling)
Angle Orientation and Critical Temperature

Simply-Supported, $h/l=0.01$  $[\phi/90/\phi]$

Figure 4.5
Angle Orientation and Critical Temperature

Simply-Supported, h/l=0.01  [90/ϕ/90]

Figure 4.6
Effect of Shear Deformation on Thermal Buckling

Simply-Supported [0/45/-45/90/90/-45/45/0]

$\frac{\Delta T_{cr}}{\Delta T_{cr, no shear}}$ against $h/l$

Figure 4.7
Effect of Shear Deformation on Thermal Buckling

Simply-Supported Symmetric and Unsymmetric Angle-Ply

Figure 4.8
Effect of Shear Deformation on Buckling

Figure 4.9

![Graph showing the effect of shear deformation on buckling](image)
Amplitude-Frequency Curve
Simply-Supported Isotropic Beam

Figure 4.10
Amplitude-Frequency Curve

Simply-Supported Symmetric Cross-Ply Beam

- Present Result
- Raciti

Figure 4.11
Deflection of Simply-Supported Isotropic Beam

Figure 4.12
Non-dimensional Mid-Span Deflection vs Temperature Ratio

Simply-Supported Aluminum Beam

- 6 elements
- 4 elements
- 2 elements

Figure 4.13
Deflection of Simply-Supported Beam

[30/-30/-30/30]

\[ \Delta T/\Delta T = 2 \]

\[ \Delta T/\Delta T = 3 \]

\[ \Delta T/\Delta T = 4 \]

Figure 4.14
Deflection of Clamped-Simply-Supported Beam

Figure 4.15
Deflection of Clamped-Clamped Beam

[30/-30/-30/30]

\[ \Delta T/\Delta T = 2 \]
\[ \Delta T/\Delta T = 3 \]
\[ \Delta T/\Delta T = 4 \]

Figure 4.16
Non-dimensional Mid-Span Deflection vs Temperature Ratio

\[ [30/-30/-30/30], h/l=0.0048 \]

- Simply-Supported
- Clamped-Simply-Supported
- Clamped-Clamped

\[ \frac{\Delta T}{\Delta T_{cr}} \]

Figure 4.17
Non-dimensional Mid-Span Deflection vs Temperature Ratio

\[ [30/-30/-30/30], \ h/l=0.0048 \]

- Including Shear Deformation
- Neglecting Shear Deformation

Figure 4.18
Deflection of Simply-Supported Beam

\[-30/30/-30/30\]

\[\Delta T/\Delta T = 2\]
\[\Delta T/\Delta T = 3\]
\[\Delta T/\Delta T = 4\]

Location (inch)

Deflection (inch)

Figure 4.19
Non-dimensional Mid-Span Deflection vs Temperature Ratio

[-30/30/-30/30], h/l=0.0048

- Simply-Supported
- Clamped-Simply-Supported
- Clamped-Clamped

Figure 4.20
Non-Dimensional Deflection vs Temperature Ratio

[-30/30/-30/30], h/l=0.0048

Figure 4.21
Frequency Ratio vs Temperature Ratio

Simply-Supported Aluminum Beam

\[ \frac{\omega}{\omega_0} \] vs \( \frac{\Delta T}{\Delta T_{cr}} \)

- Present Finite Element Result
- Pian

Figure 4.22
First Three Vibration Mode Shapes

Unloaded Simply-Supported Aluminum Beam

Figure 4.23
First Three Vibration Mode Shapes

Thermally Loaded Simply-Supported Aluminum Beam

Figure 4.24
Frequency Ratio vs Temperature Ratio

[30/-30/-30/30]

- Simply-Supported
- Clamped-Simply-Supported
- Clamped-Clamped

Figure 4.25
Frequency Ratio vs Temperature Ratio

Simply-Supported Symmetric Angle-Ply [30/-30/-30/30]

G—© Including Shear

Neglecting Shear

Figure 4.26
First Three Vibration Mode Shapes

Unloaded Simply-Supported Symmetric Angle-Ply

Figure 4.27
First Three Vibration Mode Shapes

Thermally loaded Simply-Supported Symmetric Angle-Ply Beam

Figure 4.28
Frequency Ratio vs Temperature Ratio

\([-30/30/-30/30]\)

- Simply-Supported
- Clamped-Simply-Supported
- Clamped-Clamped

\[\frac{\omega}{\omega_0} \text{ vs } \frac{\Delta T}{\Delta T_{cr}}\]

Figure 4.29
Frequency Ratio vs Temperature Ratio
Simply-Supported Unsymmetric Angle-Ply [-30/30/-30/30]

Figure 4.30
First Three Vibration Mode Shapes

Unloaded Simply-Supported Unsymmetric Angle-Ply

Figure 4.31
First Three Vibration Mode Shapes

Thermally Loaded Simply-Supported Unsymmetric Angle-Ply

Figure 4.32
Non-dimensional Mid-Span Deflection vs Temperature Ratio

\[
\eta = 0.0048
\]

\[
\begin{align*}
[90/0/0/90] \\
[90/0/90/0]
\end{align*}
\]

Figure 4.33
Non-dimensional Mid-Span Deflection vs Temperature Ratio

h/l = 0.0048

- Including Shear
- Neglecting Shear

Figure 4.34
Table 4.1 Effect of Shear Deformation on Thermal Buckling
Simply-Supported [30/90/30] Laminate

<table>
<thead>
<tr>
<th>h/l</th>
<th>(\Delta T_{cr _shear} F)</th>
<th>(\Delta T_{cr _no _shear} F)</th>
<th>(\Delta T_{cr.s}/\Delta T_{cr.ns})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>114.090</td>
<td>114.256</td>
<td>0.99855</td>
</tr>
<tr>
<td>0.025</td>
<td>702.999</td>
<td>713.710</td>
<td>0.98499</td>
</tr>
<tr>
<td>0.05</td>
<td>2686.20</td>
<td>2856.28</td>
<td>0.94045</td>
</tr>
<tr>
<td>0.075</td>
<td>5625.47</td>
<td>6423.67</td>
<td>0.87574</td>
</tr>
<tr>
<td>0.10</td>
<td>9131.09</td>
<td>11429.9</td>
<td>0.79888</td>
</tr>
</tbody>
</table>

Table 4.2 Effect of Shear Deformation on Thermal Buckling
Simply-Supported [45/90/45] Laminate

<table>
<thead>
<tr>
<th>h/l</th>
<th>(\Delta T_{cr _shear} F)</th>
<th>(\Delta T_{cr _no _shear} F)</th>
<th>(\Delta T_{cr.s}/\Delta T_{cr.ns})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>18.9699</td>
<td>18.9748</td>
<td>0.99974</td>
</tr>
<tr>
<td>0.025</td>
<td>118.213</td>
<td>118.738</td>
<td>0.99558</td>
</tr>
<tr>
<td>0.05</td>
<td>466.655</td>
<td>475.095</td>
<td>0.98224</td>
</tr>
<tr>
<td>0.075</td>
<td>1027.67</td>
<td>1068.45</td>
<td>0.96183</td>
</tr>
<tr>
<td>0.10</td>
<td>1774.92</td>
<td>1899.36</td>
<td>0.93448</td>
</tr>
<tr>
<td>0.125</td>
<td>2675.52</td>
<td>2969.55</td>
<td>0.90098</td>
</tr>
<tr>
<td>0.15</td>
<td>3689.68</td>
<td>4273.91</td>
<td>0.86330</td>
</tr>
</tbody>
</table>
### Table 4.3 Effect of Shear Deformation on Thermal Buckling
Simply-Supported [70/90/70] Laminate

<table>
<thead>
<tr>
<th>h/l</th>
<th>( \Delta T_{cr, shear} F )</th>
<th>( \Delta T_{cr, no shear} F )</th>
<th>( \Delta T_{cr,s}/\Delta T_{cr,ns} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>6.59788</td>
<td>6.59938</td>
<td>0.99977</td>
</tr>
<tr>
<td>0.025</td>
<td>41.1616</td>
<td>41.2425</td>
<td>0.99804</td>
</tr>
<tr>
<td>0.05</td>
<td>56.2932</td>
<td>56.8128</td>
<td>0.99107</td>
</tr>
<tr>
<td>0.075</td>
<td>363.810</td>
<td>371.291</td>
<td>0.98004</td>
</tr>
<tr>
<td>0.10</td>
<td>636.831</td>
<td>660.124</td>
<td>0.96471</td>
</tr>
</tbody>
</table>

### Table 4.4 Effect of Shear Deformation on Thermal Buckling
Simply-Supported [90/45/90] Laminate

<table>
<thead>
<tr>
<th>h/l</th>
<th>( \Delta T_{cr, shear} F )</th>
<th>( \Delta T_{cr, no shear} F )</th>
<th>( \Delta T_{cr,s}/\Delta T_{cr,ns} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>8.24683</td>
<td>8.25553</td>
<td>0.99894</td>
</tr>
<tr>
<td>0.025</td>
<td>51.5049</td>
<td>51.5697</td>
<td>0.99874</td>
</tr>
<tr>
<td>0.05</td>
<td>204.148</td>
<td>206.283</td>
<td>0.98965</td>
</tr>
<tr>
<td>0.075</td>
<td>453.957</td>
<td>464.254</td>
<td>0.97762</td>
</tr>
<tr>
<td>0.10</td>
<td>793.39</td>
<td>825.162</td>
<td>0.96170</td>
</tr>
</tbody>
</table>
Table 4.5 Effect of Shear Deformation on Thermal Buckling
Simply-Supported [90/0/90] Laminate

<table>
<thead>
<tr>
<th>h/l</th>
<th>$\Delta T_{cr, shear F}$</th>
<th>$\Delta T_{cr, no shear F}$</th>
<th>$\Delta T_{cr,s}/\Delta T_{cr,ns}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>15.8080</td>
<td>15.8126</td>
<td>0.9995</td>
</tr>
<tr>
<td>0.025</td>
<td>98.4500</td>
<td>98.8913</td>
<td>0.9955</td>
</tr>
<tr>
<td>0.05</td>
<td>390.177</td>
<td>395.271</td>
<td>0.9871</td>
</tr>
<tr>
<td>0.075</td>
<td>864.544</td>
<td>889.357</td>
<td>0.9720</td>
</tr>
<tr>
<td>0.10</td>
<td>1503.78</td>
<td>1581.68</td>
<td>0.9507</td>
</tr>
</tbody>
</table>

Table 4.6 Effect of Shear Deformation on Thermal Buckling
Simply-Supported [0/45/-45/90]s Laminate

<table>
<thead>
<tr>
<th>h/l</th>
<th>$\Delta T_{cr, shear F}$</th>
<th>$\Delta T_{cr, no shear F}$</th>
<th>$\Delta T_{cr,s}/\Delta T_{cr,ns}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>126.511</td>
<td>126.938</td>
<td>0.99664</td>
</tr>
<tr>
<td>0.025</td>
<td>779.393</td>
<td>793.298</td>
<td>0.98247</td>
</tr>
<tr>
<td>0.05</td>
<td>2963.23</td>
<td>3172.68</td>
<td>0.93398</td>
</tr>
<tr>
<td>0.075</td>
<td>6160.59</td>
<td>7141.36</td>
<td>0.86266</td>
</tr>
<tr>
<td>0.10</td>
<td>9892.51</td>
<td>12690.9</td>
<td>0.77950</td>
</tr>
</tbody>
</table>
Table 4.7 Effect of Shear Deformation on Thermal Buckling
(\( \Delta T_{cr,s}/\Delta T_{cr,ns} \)) for Different Boundary Conditions
[45/90/45] Laminate

<table>
<thead>
<tr>
<th>h/l</th>
<th>Simply-Supported</th>
<th>Simply-Supported-Clamped</th>
<th>Clamped-Clamped</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.99974</td>
<td>0.99831</td>
<td>0.99676</td>
</tr>
<tr>
<td>0.025</td>
<td>0.99558</td>
<td>0.99120</td>
<td>0.98239</td>
</tr>
<tr>
<td>0.05</td>
<td>0.98224</td>
<td>0.96819</td>
<td>0.93429</td>
</tr>
<tr>
<td>0.075</td>
<td>0.96183</td>
<td>0.92630</td>
<td>0.86325</td>
</tr>
<tr>
<td>0.10</td>
<td>0.93448</td>
<td>0.87616</td>
<td>0.78021</td>
</tr>
</tbody>
</table>

Table 4.8 Comparison of Buckling Temperature
Between \([\phi/-\phi/\phi/-\phi]\) and \([\phi/-\phi/\phi/\phi]\)
with \(h/l = 0.01\)

<table>
<thead>
<tr>
<th>Angle, (\phi)</th>
<th>(\Delta T_{cr}[, [\phi/-\phi/\phi/-\phi]] F)</th>
<th>(\Delta T_{cr}[, [\phi/-\phi/-\phi/\phi]] F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.0</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>20.0</td>
<td>----</td>
<td>----</td>
</tr>
<tr>
<td>27.0</td>
<td>1127.98</td>
<td>1127.98</td>
</tr>
<tr>
<td>30.0</td>
<td>260.290</td>
<td>260.290</td>
</tr>
<tr>
<td>40.0</td>
<td>38.8564</td>
<td>38.8564</td>
</tr>
<tr>
<td>50.0</td>
<td>14.3822</td>
<td>14.3822</td>
</tr>
<tr>
<td>60.0</td>
<td>8.69118</td>
<td>8.69118</td>
</tr>
<tr>
<td>70.0</td>
<td>6.89055</td>
<td>6.89055</td>
</tr>
<tr>
<td>80.0</td>
<td>6.22609</td>
<td>6.22609</td>
</tr>
<tr>
<td>90.0</td>
<td>6.04650</td>
<td>6.04650</td>
</tr>
</tbody>
</table>

with shear deformation
5.0 CONCLUSIONS AND FUTURE WORK

For the symmetric laminate of \([\phi/90/\phi]\), as the angle \(\phi\) increases effect of shear becomes smaller (Fig.4.2-4.4).

For the case of cross ply, there was significant difference in buckling temperature because of the presence of B11 component.

For thermal post-buckling deflections, shear deformation has very little effect for both symmetric and unsymmetric laminate. Boundary conditions affect the magnitude of deflection and therefore affect the post-buckled vibration.

Difference of frequency ratios among the three boundary conditions appeared to be significant at high \(\Delta T/\Delta T_{cr}\) ratio. As the temperature ratio decreases, the difference appears to be smaller. There was no difference in frequency ratios for pre-buckling phase. Again, shear deformation does not have any significance on post-buckled vibrations.

Several conclusions can be made. First of all, in most cases, the effect of shear deformation can be neglected. As the thickness-to-length ratio increases, the effect of shear deformation increases. However it means that the critical temperature also increases. In certain cases the critical
temperature goes beyond the epoxy resins' operational temperature. If fiber glass is used for the fiber, the critical temperature might also go beyond the glass transition temperature. Therefore in practicality, shear deformation can be neglected for thermal buckling. However, depending on layup, slenderness ratio, and boundary conditions, it might be necessary to incorporate shear deformation for accurate mechanical buckling analysis, since the load applied to the beam is external and independent from the temperature.

For future work, initial imperfection should be added for the analysis. In practicality, there are few beams that have no initial imperfection. When there is an imperfection, the beam does not have a bifurcation point and the natural frequency does not go to zero. Therefore, when comparison between theory and experiment is made, inclusion of initial imperfection is critical. Also non-uniform temperature distributions should be analyzed. According to Gray [24], when a non-uniform temperature load is applied (tent like load distribution and cosine curve), as the $\Delta T/\Delta T_{cr}$ increases, instead of staying in the first mode shape of deflection, the plate jumped into third mode shape.

Finally and most importantly, experiments should be performed to verify the results, because currently there is
no other published work with which to compare the results.
References


APPENDIX A

The [D] matrix
Originally the constitutive relationship is

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & DB_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
e_x \\
e_y \\
\gamma_{xy} \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]  

(A.1)

For the case of beams,

\[
N_y = A_{12} e_x + A_{22} e_y + A_{26} \gamma_{xy} + B_{12} \kappa_x + B_{22} \kappa_y + B_{16} \kappa_{xy} = 0
\]  

(A.2)

\[
M_y = B_{12} e_x + B_{22} e_y + B_{26} \gamma_{xy} + D_{12} \kappa_x + D_{22} \kappa_y + D_{26} \kappa_{xy} = 0
\]  

(A.3)

However \(\epsilon_y\) and \(\kappa_y\) are assumed to be non zero. Solving the equations (A.2) and (A.3) simultaneously,

\[
\epsilon_y = \frac{1}{(\frac{B_{22}^2}{D_{22}} - A_{22})} \left[ (A_{12} - \frac{B_{22} B_{12}}{D_{22}}) e_x + (A_{26} - \frac{B_{26} B_{22}}{D_{22}}) \gamma_{xy} \right] + (B_{12} - \frac{D_{12} B_{12}}{D_{22}}) \kappa_x + (B_{26} - \frac{D_{26} B_{22}}{D_{22}}) \kappa_{xy}
\]  

(A.4)

\[
\kappa_y = \frac{1}{(\frac{B_{22}^2}{A_{22}} - D_{22})} \left[ (B_{12} - \frac{B_{22} A_{12}}{A_{22}}) e_x + (B_{26} - \frac{A_{26} B_{22}}{A_{22}}) \gamma_{xy} \right] + (D_{12} - \frac{B_{12} B_{22}}{A_{22}}) \kappa_x + (D_{26} - \frac{B_{26} B_{22}}{A_{22}}) \kappa_{xy}
\]  

(A.5)

or they can be shown as equations (2.30) through (2.39).
Then the constitutive relationship becomes

\[
\begin{bmatrix}
N_x \\
N_{xy} \\
M_x \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{16} & B_{11} & B_{16} \\
A_{16} & A_{66} & B_{16} & B_{66} \\
B_{11} & B_{16} & D_{11} & D_{16} \\
B_{16} & B_{66} & D_{16} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\gamma_{xy} \\
\kappa_x \\
\kappa_{xy}
\end{bmatrix} +
\begin{bmatrix}
A_{12} & B_{12} \\
A_{26} & B_{26} \\
B_{12} & D_{12} \\
B_{26} & D_{26}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_y \\
\kappa_y
\end{bmatrix}
\]  
(A.6)

Using the equations (2.31) and (2.32), the second term of the equation (A.6) becomes

\[
\begin{bmatrix}
A_{12} & B_{12} \\
A_{26} & B_{26} \\
B_{12} & D_{12} \\
B_{26} & D_{26}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_y \\
\kappa_y
\end{bmatrix} =
\begin{bmatrix}
A_{12} & B_{12} \\
A_{26} & B_{26} \\
B_{12} & D_{12} \\
B_{26} & D_{26}
\end{bmatrix}
\begin{bmatrix}
a_1 & a_2 & a_3 & a_4 \\
a_1 & a_2 & a_3 & a_4 \\
b_1 & b_2 & b_3 & b_4 \\
b_1 & b_2 & b_3 & b_4
\end{bmatrix}
\begin{bmatrix}
\gamma_{xy} \\
\kappa_x \\
\kappa_y
\end{bmatrix}
\]  
(A.7)

So the constitutive relationship becomes

\[
\begin{bmatrix}
N_x \\
N_{xy} \\
M_x \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{16} & B_{11} & B_{16} \\
A_{16} & A_{66} & B_{16} & B_{66} \\
B_{11} & B_{16} & D_{11} & D_{16} \\
B_{16} & B_{66} & D_{16} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\gamma_{xy} \\
\kappa_x \\
\kappa_{xy}
\end{bmatrix} +
\begin{bmatrix}
A_{12} & B_{12} \\
A_{26} & B_{26} \\
B_{12} & D_{12} \\
B_{26} & D_{26}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_y \\
\kappa_y
\end{bmatrix}
\]  
(A.7)

Finally the constitutive relationship becomes

\[
\begin{bmatrix}
N_x \\
N_{xy} \\
M_x \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
\overline{D}_{11} & \overline{D}_{12} & \overline{D}_{13} & \overline{D}_{14} \\
\overline{D}_{21} & \overline{D}_{22} & \overline{D}_{23} & \overline{D}_{24} \\
\overline{D}_{31} & \overline{D}_{32} & \overline{D}_{33} & \overline{D}_{34} \\
\overline{D}_{41} & \overline{D}_{42} & \overline{D}_{43} & \overline{D}_{44}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\gamma_{xy} \\
\kappa_x \\
\kappa_{xy}
\end{bmatrix}
\]  
(A.8)
APPENDIX B

Strain-Displacement Matrix and Displacement Vector
Strain-Displacement Matrix

\[
[B] = \frac{2}{L} \begin{bmatrix}
N'_1 & N'_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & \frac{L}{2}N'_1 & \frac{L}{2}N'_2 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 2N'_1 & 2N'_2 & 0 & 0
0 & 0 & 0 & 0 & N'_1 & N'_2 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
N'_3 & N'_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & \frac{L}{2}N'_3 & \frac{L}{2}N'_4 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 2N'_3 & 2N'_4 & 0 & 0
0 & 0 & 0 & 0 & N'_3 & N'_4 & 0 & 0 & 0 & 0
\end{bmatrix}
\] (B.1)

Displacement Vector

\[
\{q\} = [u_1 \ u'_1 \ w_{b_1} \ \theta_{b_1} \ w_{s_1} \ \theta_{s_1} \ \tau_1 \ \tau'_1 \ \beta_1 \ \beta'_1]
\]

\[
u_2 \ u'_2 \ w_{b_2} \ \theta_{b_2} \ w_{s_2} \ \theta_{s_2} \ \tau_2 \ \tau'_2 \ \beta_2 \ \beta'_2]^T
\] (B.2)
APPENDIX C

Thermally Induced Load
For the case of isotropic beam, the thermal equivalent load is simply

\[ P = E\alpha \Delta T \quad \text{(C.1)} \]

or

\[ P = \sigma_x A \quad \text{(C.2)} \]

In general, the stress-strain relationship is

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x - \alpha_x \Delta T \\
\epsilon_y - \alpha_y \Delta T \\
\gamma_{xy} - \alpha_{xy} \Delta T
\end{bmatrix}
\quad \text{(C.3)}
\]

For beams, assuming that \( \epsilon_x = \gamma_{xy} = 0 \)

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
-\alpha_x \Delta T \\
-\alpha_y \Delta T \\
-\alpha_{xy} \Delta T
\end{bmatrix}
\quad \text{(C.4)}
\]

Therefore \( \sigma_x \) becomes

\[ \sigma_x = -\bar{Q}_{11}\alpha_x \Delta T + \bar{Q}_{12}(\epsilon_y - \alpha_y \Delta T) - \bar{Q}_{16}\alpha_{xy} \Delta T \quad \text{(C.5)} \]

at the same time

\[ \sigma_y = 0 = -\bar{Q}_{12}\alpha_x \Delta T + \bar{Q}_{22}(\epsilon_y - \alpha_y \Delta T) - \bar{Q}_{26}\alpha_{xy} \Delta T \quad \text{(C.6)} \]

or

\[ \epsilon_y - \alpha_y \Delta T = \frac{(\bar{Q}_{12}\alpha_x + \bar{Q}_{26}\alpha_{xy}) \Delta T}{\bar{Q}_{22}} \quad \text{(C.6)} \]
Using Eqs. (C.2), (C.5), and (C.7), the thermally induced load becomes

\[ P = \left[ -\alpha_x \overline{\varphi}_{11} + \left( \frac{\alpha_x \overline{\varphi}_{12} + \alpha_{xy} \overline{\varphi}_{26}}{\overline{\varphi}_{22}} \right) \overline{\varphi}_{12} - \alpha_{xy} \overline{\varphi}_{16} \right] \Delta T \]  

(C.8)
APPENDIX D

Abscissas and Weights in Gaussian Quadrature, Ref.[33]
<table>
<thead>
<tr>
<th>n</th>
<th>Abscissas, $x_i$</th>
<th>Weight, $H_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$\pm .577350269189$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$8/9$</td>
</tr>
<tr>
<td></td>
<td>$\pm .77459666241$</td>
<td>$5/9$</td>
</tr>
<tr>
<td>4</td>
<td>$\pm .339981043584856$</td>
<td>.652145154862546</td>
</tr>
<tr>
<td></td>
<td>$\pm .861136311594053$</td>
<td>.347854845137454</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>.568888888888888888888889</td>
</tr>
<tr>
<td></td>
<td>$\pm .538469310105683$</td>
<td>.478628670499366</td>
</tr>
<tr>
<td></td>
<td>$\pm .906179845938664$</td>
<td>.236926885056189</td>
</tr>
</tbody>
</table>
APPENDIX E

Derivation of Equations (2.113) and (2.114)
In Eq. (2.113) is defined as follows, according to Eq. (2.79),

\[ [n_1 d] = b \int_0^L ([B_{NL}(q^b)_d]^T[\overline{D}] [B_L] + [B_L]^T[\overline{D}] [B_{NL}(q^b)_d]) \, dx \quad (E.1) \]

Likewise, for \([n_{1\Delta T}]\) we have

\[ [n_{1\Delta T}] = b \int_0^L ([B_{NL}(q^b)_{\Delta T}]^T[\overline{D}] [B_L] + [B_L]^T[\overline{D}] [B_{NL}(q^b)_{\Delta T}]) \, dx \quad (E.2) \]

\([n_{2d}]\) in equation (2.114) is, according to Eq. (2.80),

\[ [n_2 d] = \frac{3}{2} b \int_0^L [B_{NL}(q^b)_d]^T[\overline{D}] [B_{NL}(q^b)_d] \, dx \quad (E.3) \]

Likewise, \([n_{2\Delta T}, d]\) and \([n_{2\Delta T}]\) are

\[ [n_{2\Delta T}, d] = \frac{3}{2} b \int_0^L [B_{NL}(q^b)_{\Delta T}]^T[\overline{D}] [B_{NL}(q^b)_d] \, dx \quad (E.4) \]

\[ [n_{2\Delta T}] = \frac{3}{2} b \int_0^L [B_{NL}(q^b)_{\Delta T}]^T[\overline{D}] [B_{NL}(q^b)_{\Delta T}] \, dx \quad (E.5) \]

In the above equations

\[ [B_{NL}]_{\Delta T} = \{A_{\Delta T}\} [G] = \begin{bmatrix} \frac{\partial w_p}{\partial x} \\ \frac{\partial w_p}{\partial x} \end{bmatrix} \]

\[ [G] \]

(E.6)
Examining Eqs. (C.1) through (C.9), the element matrices show following properties

$[B_{NL}]_d = [A_d] [G] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(E.7)

and

$$\left( \frac{\partial w_b}{\partial x} \right)_{\Delta T} = \frac{2}{L} (N'_1 w_{b1_d} + N'_2 \theta_{b1_d} + N'_3 w_{b2_d} + N'_4 \theta_{b2_d})$$

(E.8)

$$\left( \frac{\partial w_b}{\partial x} \right)_d = \frac{2}{L} (N'_1 w_{b1_d} + N'_2 \theta_{b1_d} + N'_3 w_{b2_d} + N'_4 \theta_{b2_d})$$

(E.9)

as claimed in Eqs. (2.113) and (2.114).