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Nonlinear Forced Vibration of an Unsymmetrically Laminated Composite Beam

Marie Helene Pompei

Embry-Riddle Aeronautical University - Daytona Beach

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NONLINEAR FORCED VIBRATION OF AN UNSYMMETRICALLY LAMINATED COMPOSITE BEAM

by

Marie-Hélène Pompei

A Thesis Submitted to the Office of Graduate Programs in Partial Fulfillment of the Requirements of the Degree of Master of Science in Aerospace Engineering

Embry-Riddle Aeronautical University
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Marie-Hélène Pompei

This thesis was prepared under the direction of the candidate's thesis committee chairman, Dr. Habib Eslami, Department of Aerospace Engineering, and has been approved by the members of his thesis committee. It was submitted to the Office of the Graduates Programs and was accepted in partial fulfillment of the requirements for the degree of Master of Science in Aerospace Engineering.

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May 17, 1994
The purpose of this thesis is to study the nonlinear forced vibration of an unsymmetrically laminated composite beam. Two cases are considered: unsymmetrical angle-ply and unsymmetrical cross-ply composite beams. The nonlinear governing partial differential equations of motion are obtained by taking into account the von-Karman geometrical nonlinearity. These equations are transformed into nonlinear ODE (Ordinary Differential Equations) by means of the Galerkin method. For both cases the forced Duffing equation was obtained. It is to be noted that only for the unsymmetrical cross-ply laminate simply-supported case are both quadratic and cubic nonlinearities present. The resulting equations for both cases are then solved using the Method of Multiple Scales (MMS). The boundary conditions considered here are simply-supported with immovable edges and clamped at both edges. For both cases the frequency response of the beam associated with the primary resonance was studied. In addition, the subharmonic and superharmonic resonances were analyzed for a simply-supported unsymmetrically laminated angle-ply composite beam. In order to compare the effects of different modes of vibrations, a one-mode solution and a two-mode solution were investigated.
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Upper Case

\( A \)
- Complex forcing amplitude
\( A_j \)
- Coefficient of the extensional stiffness matrix
\( B_j \)
- Coefficient of the coupling stiffness matrix
\( C \)
- Damping coefficient
\( D_t \)
- Time scale derivative
\( D_y \)
- Coefficient of the bending stiffness matrix
\( E \)
- Young's modulus
\( F(t), F_0 \)
- Modified forcing function, Modified forcing amplitude
\( K_1 \)
- Constant defined in equation (2-21)
\( K_2 \)
- Constant defined in equation (2-22)
\( L \)
- Length of the beam
\( L(w,t) \)
- Differential operator
\( M_j \)
- Resultant moment matrix
\( N_j \)
- Resultant force matrix
\( P(t), P_0 \)
- Forcing function, Forcing amplitude
\( T_t \)
- Time scale
\( U \)
- Time function applied in MMS
\( U_0, U_1, U_2 \)
- Terms of the function \( U \)

Lower Case

\( a \)
- Amplitude of the coefficient \( A \)
\( b \)
- Width of the beam
\( f \)
- Amplitude coefficient
\( h \)  
Total thickness of the beam

\( k \)  
Constant coefficient

\( q(t) \)  
Time amplitude function

\( t \)  
Time variable

\( u \)  
Displacement in the \( x \) direction

\( u_0 \)  
Mid-plane displacement in the \( x \) direction

\( v \)  
Displacement in the \( y \) direction

\( v_0 \)  
Mid-plane displacement in the \( y \) direction

\( w \)  
Lateral displacement (in the \( z \) direction)

\( x, y, z \)  
Cartesian coordinates

**Greek Symbols**

\( \alpha_2 \)  
Coefficient of quadratic nonlinearity

\( \alpha_3 \)  
Coefficient of cubic nonlinearity

\( \beta, \gamma \)  
Phase angle

\( \varepsilon_y^0 \)  
Mid-plane strain matrix

\( \varepsilon \)  
Small nondimensional quantity (perturbation scale)

\( \xi \)  
Damping ratio

\( \kappa_y \)  
Mid-plane curvature

\( \kappa \)  
Constant of calculation in MMS

\( \mu \)  
Constant of calculation in MMS

\( \nu \)  
Poisson's ratio

\( \rho \)  
Mass density of the beam

\( \sigma_x, \sigma_y, \sigma_z \)  
Normal stress components

\( \tau \)  
Detuning factor

\( \tau_{xy}, \tau_{xz}, \tau_{yz} \)  
Shear stress components
\( \omega \), Natural frequency of the beam

\( \Omega \), Forcing frequency
Chapter I

Introduction

1.1. Background.

One of the main reasons for the success of composite materials resides in the fact that it corresponds to the exact need of the industrial world, a light yet strong and stiff material. The high strength to weight ratio and high stiffness to weight ratio of composites has been especially enticing for the aircraft industry. Composites can now be found in parts of the airframe structure, in parts of the jet engines, etc ...

There are three main categories of composite materials: fibrous, laminated, and particulate composites. The first one, fibrous composites, consists of an association of fibers and a matrix. The fibers have a high Young's modulus to withstand most of the stress. The matrix, on the other hand, is brittle with a low Young's modulus and its main role is to bond the fibers together, distributing the stress from fiber to fiber, and protect the fibers. The second category, the laminated composites, consists of a stack of layers bonded together. The orientation of the layers can be different in order to strengthen the structure in a multitude of directions. The last category, the particulate composites, consists of particles of one material injected in a matrix of another one. Some composite materials can belong to two different categories at a time. For example, reinforced concrete is both particulate and fibrous and laminated fiber-reinforced materials are both laminated and fibrous. Most commonly, composite structures used in the aircraft industry are laminated structures. It is to be noted that those composites can either be
symmetrically laminated (symmetry of the laminate with respect to the middle surface) or unsymmetrically laminated. Although symmetrically laminated composites are more desirable because of the fact that there is no coupling between bending and extension (i.e. simplification of the calculations), in some cases the non symmetry is necessary to achieve design requirements (e.g., jet turbine fan blades with a pretwist).

The stress-strain relationships are much more complicated for anisotropic composites than they are for conventional isotropic material, and this may result in unexpected behavior. For example, in an isotropic structure, normal stresses cause normal strains only and shear stresses cause shear strains only while in an anisotropic material, normal stresses and/or shear stresses may induce both normal and shear strains. Thus, the study of such materials is important and more general than the conventional isotropic or orthotropic materials. The main characteristic of composite materials is their constitutive equations which will be defined in Chapter II. These equations incorporate an extensional stiffness matrix, $A$, a coupling stiffness matrix, $B$, and a bending stiffness matrix, $D$. Reference [19] presents a detailed analysis of those constitutive equations. For symmetrically laminated composites the coupling matrix is equal to zero, implying that there is no coupling between bending and extension.

Harmonic excitation exists in a variety of mechanical systems, an example among others is the vibration of the airframe of an aircraft at steady level-flight. With the upcoming of composite materials in many industries, it is becoming important to fully understand the behavior of those materials. This can be achieved by first studying in depth simple models of structures, such as beams, and then expanding the theories to more complex shapes. The present thesis will explore the effects of nonlinear forced vibrations of an unsymmetrically laminated composite beam. An application of this subject could be a flap made out of composite materials in an aircraft.

### 1.2. Literature survey.

As the subject of this thesis focuses on composite beams, the literature survey was narrowed down primarily on work and research published on beams. In the first part of this survey, the works
achieved on nonlinear free vibrations of isotropic and composite beams are presented. In the second part, the emphasis is on the published research of nonlinear forced vibrations of beams.

A large number of references exist on nonlinear free vibration of isotropic beams, e.g. references 1 through 18. More recent works have been published for nonlinear free vibration of composite beams. Kapania and Raciti [20] developed a simple one-dimensional finite element for the nonlinear analysis of symmetrically and unsymmetrically laminated composite beams including shear deformation. The formulation, the solution procedure, and the computer program were evaluated by solving a variety of examples in the static response, free vibration, and nonlinear vibrations of isotropic and laminated beams. For unsymmetrically laminated beams, Raciti found that the nonlinear vibrations had a soft spring behavior for certain boundary conditions as opposed to a hard spring behavior observed in isotropic and symmetrically laminated beams. The in-plane boundary conditions were found to have some significant effect on the nonlinear responses. Singh, Rao and Iyengar [21] investigated large-amplitude free vibrations of unsymmetrically laminated composite beams using von Karman large deflection theory. They applied a one dimensional finite element based on classical lamination theory, first-order shear-deformation theory and higher-order shear-deformation theory having 8, 10, and 12 degrees of freedom per node, respectively. This was done to bring out the effects of transverse shear on the large-amplitude vibrations. Because of the presence of bending-extension coupling, the bending stiffness of an unsymmetric laminate becomes direction dependent yielding different amplitudes and spatial deformations for the positive and negative deflection half-cycle. Bangera and Chandrashekhara [22] developed a finite-element model to study the large-amplitude free vibrations of generally-layered laminated composite beams. They considered the effects of the Poisson ratio by including it in the constitutive equation. The direct iteration method was used to solve the nonlinear equation at the point of reversal of motion. The influence of boundary conditions, beam geometry, Poisson effect, and ply orientation on the nonlinear frequencies and mode shapes were demonstrated.

Since the present research focuses on nonlinear forced vibration of beams, special attention and details are given for the work on harmonic forced vibrations of isotropic and composite beams.
Tseng and Dugundji [23] applied the harmonic balance method to solve the problem of a straight isotropic beam with fixed ends subjected to a harmonic excitation at its supporting ends. Alturi [24] applied the method of multiple scales [25] to investigate the response of nonlinear forced vibration of a hinged isotropic beam considering nonlinear inertia terms. He showed that, for some cases, the nonlinearity was of the softening type. Srinivasan [6] solved for free and forced responses of isotropic beams subjected to moderately large-amplitude steady-state oscillations by the averaging method of Ritz. The application of this method transforms the governing partial differential equation into a system of nonlinear algebraic equations. He then applied Newton's method to solve those equations. Nayfeh, Mook and Lobitz [26] presented a numerical-perturbation method for the nonlinear analysis of forced vibration of isotropic beams. A multiple-mode expansion in terms of the linear mode shapes was considered. The problem was then solved using the method of multiple scales, considering internal resonance.

Some research on nonlinear forced vibrations of composites beams has also been carried out. Pai and Nayfeh [27] investigated the forced nonlinear vibration of a symmetrically laminated graphite-epoxy composite beam. The analysis focused on the case of primary resonance of the first flexural-torsional mode. A combination of the fundamental-matrix method, a Galerkin procedure and the method of multiple scales was used to derive four first-order ordinary-differential equations describing the modulations of the amplitudes and phases of the interacting modes with damping, nonlinearity and resonance. The results showed that the motion was non planar although the input was planar. It was further concluded that non planar responses may be periodic motions, amplitude- and phase-modulated motions, or chaotically modulated motions. Chandrashekhara [28] considered the flexural analysis of fiber-reinforced composite beams under static loading based on higher-order shear deformation theory. A von Karman type nonlinearity is incorporated in the formulation of the problem. The finite-element method is used to solve the nonlinear governing equations by direct iteration. Unlike the conventional beam models, Chandrashekhara took into account the \( y \) direction strains. The author investigated the differences in the solutions for the cross-ply laminates and the angle-ply laminates. He concluded that the solution obtained from the two approaches differ slightly in the case of the cross-ply laminates, but there exists a considerable difference in the case of angle-ply. Currently, Kenareh [29] is developing a finite
element code to solve for the response of a symmetrically and unsymmetrically laminated composite beam subjected to nonlinear forced vibration.

Some work has also been carried out on nonlinear forced vibrations of isotropic and composite plates. Reddy [30] investigated forced motion of laminated plates using a finite element method that accounts for transverse shear deformation, rotary inertia and large rotation (in the von Karman sense). In his paper, he presents numerical results for nonlinear analysis of composite plates, and points out the effects of the plate's thickness, boundary conditions and loading on the deflection and stresses. Mei and Decha-Umphai [31] extended the finite element method to determine the response of large-amplitude forced vibrations of thin isotropic plates. A force matrix under uniform harmonic excitations is developed for nonlinear forced vibration analysis. The results obtained were compared with simple elliptic response, perturbation, and other approximation solutions. The method of multiple scales in conjunction with Galerkin's method was used by Eslami and Kandil [32] to analyze the nonlinear forced and damped response of a rectangular orthotropic plate subjected to a uniformly distributed transverse loading. The analysis considered simply-supported as well as clamped panels. By using the method of multiple scales, all the possible resonances were investigated such as the primary resonance, subharmonic and superharmonic resonances. Hua [33] studied the geometric nonlinear forced flexural vibration of anisotropic symmetrically laminated composite plates under a harmonic force. He presented the effects of angle of orientation of the symmetrically laminated plates on the amplitude-frequency response. Chiang, Xue and Mei [34] presents a finite element formulation for determining the large-amplitude free and steady-state forced vibration response of arbitrarily laminated anisotropic composite thin plates using the Discrete Kirchhoff Theory (DKT) triangular elements. This work focuses on primary resonance only. The nonlinear stiffness and harmonic force matrices of an arbitrarily laminated composite thin plate element are developed for nonlinear free and forced vibration analyses. The effects of damping were not included. Huang [35] investigates the forced nonlinear axisymmetric vibrations of an orthotropic composite plate with fixed boundary conditions. The governing nonlinear partial differential equations are converted into the corresponding nonlinear ordinary differential equations by elimination of the time variable with a Kantorovitch time-averaging method. The solutions of the eigenvalue problem are obtained by using a
Newton iteration technique. The results reveal the effects of finite amplitude and anisotropy of materials upon the fundamental responses.

1.3. Scope of this thesis.

The purpose of this thesis is to study the nonlinear forced vibrations of an unsymmetrically laminated composite beam. This subject, to the author's best knowledge, has never been studied as such. In previous research the effects of unsymmetric laminates was not considered. In this thesis, two cases of unsymmetric laminate are investigated: unsymmetric angle-ply and unsymmetric cross-ply. For both cases, the beams are either simply-supported with immovable edges or clamped-clamped. The governing equations for a general composite beam (symmetric or unsymmetric laminated) are derived in Chapter II. The governing equations are expressed in terms of the lateral displacements and take into account the von Karman geometrical nonlinearities. To solve the fourth order nonlinear partial differential equation obtained, the Galerkin method is applied first. This method provides a mean of transforming the fourth order nonlinear partial differential equation into a nonlinear ordinary differential equation (ODE). The Galerkin method is not the only method that can perform this transformation, yet it is the simplest to apply. The ODE obtained is an expression of the forced Duffing equation which is solved analytically with the Method of Multiple Scales (MMS). By using this method, the primary resonances as well as the subharmonic and superharmonic resonances are studied. This is the first time that subharmonic and superharmonic resonances are studied for composite beams. All the equations leading to the expression of the frequency-response curve are derived in Chapter III. In Chapter IV, the Galerkin method and the MMS are employed for a two-mode solution. Chapter IV will prove that the addition of another term in the solution has no significant effect, and thus, that the derivation of the solution for a one-mode solution in chapter III is sufficient. In Chapter V, numerical examples and the discussion of the results are presented.
Chapter II

Formulation of the Governing Equations of Motion

2.1. Formulation of the problem.

The study of harmonic excitations of a laminated composite beam can be symbolized as follows.

The beam is of length $L$, of thickness $h$ and of width $b$. The load is applied along the $x$ axis of the beam. In Fig 2.1., the beam is simply-supported with immovable edges, yet the governing equations derived in this chapter will be valid for any boundary conditions.
2.2. Governing equations of motion for a composite beam.

The governing equation of motion for a beam can be obtained by using the laminate constitutive equations for a plate,

\[
\begin{bmatrix}
N \\
M
\end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \epsilon^s \\ \kappa \end{bmatrix}
\]

where

\( N \) = resultant force matrix (force per unit length)
\( M \) = resultant moment matrix (moment per unit length)
\( A \) = extensional stiffness matrix
\( B \) = coupling stiffness matrix
\( D \) = bending stiffness matrix
\( \epsilon^0 \) = mid-plane strain matrix
\( \kappa \) = curvature matrix

The strain-displacement and the curvature-displacement relations are given by:

\[
\begin{bmatrix} \epsilon^s \\ \gamma^s \end{bmatrix} = \begin{bmatrix} \epsilon_x^s \\ \epsilon_y^s \\ \gamma_x^s \\ \gamma_y^s \end{bmatrix} = \begin{bmatrix} \frac{\partial \nu_x}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\ \frac{\partial \nu_y}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \\ \partial \nu_x + \partial \nu_y + \frac{1}{2} \frac{\partial \gamma_x}{\partial x} - \frac{1}{2} \frac{\partial \gamma_y}{\partial y} \\ \partial \gamma_x - \frac{1}{2} \frac{\partial \gamma_x}{\partial x} - \frac{1}{2} \frac{\partial \gamma_y}{\partial y} \end{bmatrix}
\]

\[
\begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -\frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}
\]

where

\( \nu \) = inplane displacement in the \( x \) direction
\( v_0 \) = inplane displacement in the \( y \) direction

\( w \) = displacement in the \( z \) direction

The inplane displacements \( u_0 \) and \( v_0 \) are functions of \( x, y \) and \( t \) in general. The transverse displacement \( w \) for a dynamic problem is also a function of \( x, y \) and \( t \). The nonlinearity of the problem appears under the form of the von Karman geometrical nonlinearities. This can be clearly seen in the expression of the mid-plane strain, equation (2-2).

For a beam one of the dimensions \( (x) \) is larger than the two other dimensions \( (y) \) and \( (z) \). Thus, the displacements \( u_0 \), \( v_0 \) are functions of \( x \) only and \( w \) of \( x \) and \( t \). Equations (2-2) and (2-3) are simplified as:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} = \begin{bmatrix}
\frac{\partial u_x}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
0 \\
\frac{\partial v_x}{\partial x}
\end{bmatrix}
\]

(2-4)

\[
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} = \begin{bmatrix}
-\frac{\partial^2 w}{\partial x^2} \\
0 \\
0
\end{bmatrix}
\]

(2-5)

This leads to a new expression of the constitutive equation (2-1) for a composite beam problem:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66}
\end{bmatrix}\begin{bmatrix}
\frac{\partial u_x}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \\
0 \\
-\frac{\partial^2 w}{\partial x^2}
\end{bmatrix}
\]

(2-5a)

The expression of the resultant forces and moments are obtained by expanding (2-5a) as follows:

\[
N_x = A_{11} \left[ \frac{\partial u_x}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{16} \frac{\partial v_x}{\partial x} - B_{11} \frac{\partial^2 w}{\partial x^2}
\]

(2-6)

\[
N_y = A_{12} \left[ \frac{\partial u_x}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{26} \frac{\partial v_x}{\partial x} - B_{12} \frac{\partial^2 w}{\partial x^2}
\]

(2-7)
\[ N_y = A_{m} \left[ \frac{\partial u_y}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + A_{m} \frac{\partial u_y}{\partial y} - B_{n} \frac{\partial^2 w}{\partial x^2} \]  \hspace{1cm} (2-8)

\[ M_x = B_{m} \left[ \frac{\partial u_x}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + B_{m} \frac{\partial u_x}{\partial y} - D_{m} \frac{\partial^2 w}{\partial x^2} \]  \hspace{1cm} (2-9)

\[ M_y = B_{m} \left[ \frac{\partial u_y}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + B_{m} \frac{\partial u_y}{\partial y} - D_{m} \frac{\partial^2 w}{\partial x^2} \]  \hspace{1cm} (2-10)

\[ M_{xy} = B_{m} \left[ \frac{\partial u_{xy}}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right] + B_{m} \frac{\partial u_{xy}}{\partial y} - D_{m} \frac{\partial^2 w}{\partial x^2} \]  \hspace{1cm} (2-11)

The equations of motion for a plate in terms of the resultant forces and moments are given by:

\[ \frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \]  \hspace{1cm} (2-12)

\[ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \]  \hspace{1cm} (2-13)

\[ \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + P = \rho h \frac{\partial^2 w}{\partial t^2} \]  \hspace{1cm} (2-14)

The detailed derivation of these equations is shown in Appendix A. For problems involving beams, equations (2-12) through (2-14) are reduced to:

\[ \frac{\partial N_x}{\partial x} = 0 \]  \hspace{1cm} (2-15)

\[ \frac{\partial N_{xy}}{\partial y} = 0 \]  \hspace{1cm} (2-16)

\[ \frac{\partial^2 M_x}{\partial x^2} + N_x \frac{\partial^2 w}{\partial x^2} + P = \rho h \frac{\partial^2 w}{\partial t^2} \]  \hspace{1cm} (2-17)

Substituting expressions (2-6) and (2-8) into equations (2-15) and (2-16) yields:

\[ A_{11} \frac{\partial^2 u_0}{\partial x^2} + A_{16} \frac{\partial^2 v_0}{\partial x^2} - B_{11} \frac{\partial^2 w}{\partial x^2} + A_{11} \frac{\partial^2 w}{\partial x^2} - B_{16} \frac{\partial^2 w}{\partial x^2} = 0 \]  \hspace{1cm} (2-18)

\[ A_{16} \frac{\partial^2 u_0}{\partial x^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} - B_{16} \frac{\partial^2 w}{\partial x^2} + A_{16} \frac{\partial^2 w}{\partial x^2} - B_{16} \frac{\partial^2 w}{\partial x^2} = 0 \]  \hspace{1cm} (2-19)
By virtue of equation (2-15), it is concluded that \( N_x \) is not a function of \( x \) and that \( N_x = N_x^0 = \) constant. Thus equation (2-17) can be written as:

\[
-D_{41} \frac{\partial^4 w}{\partial x^4} + B_{11} \left[ \frac{\partial^3 u_0}{\partial x^3} + \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \right] + B_{16} \frac{\partial^3 v_0}{\partial x^3} + N_x^0 \frac{\partial^2 w}{\partial x^2} + P = \rho h \frac{\partial^2 w}{\partial t^2}
\]

(2-20)

Equations (2-18) through (2-20) can be reduced to a single equation in terms of \( w \) only. This is shown in the following section.

2.3. Governing equation of motion in terms of \( w(x,t) \) only.

As one of the main objectives of this thesis is to find the maximum transverse deflection of the beam for different values of the forcing frequency, it is important to obtain a single equation in terms of \( w \) only. In order to do so, \( u_0 \) and \( v_0 \) need to be eliminated from equations (2-18), (2-19) and (2-20) and that can be done by solving equation (2-18) and (2-19) for \( \frac{\partial^2 u_0}{\partial x^2} \) and \( \frac{\partial^2 v_0}{\partial x^2} \):

\[
\frac{\partial^2 u_0}{\partial x^2} = K_1 \frac{\partial^3 w}{\partial x^3} - \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \tag{2-21}
\]

\[
\frac{\partial^2 v_0}{\partial x^2} = K_2 \frac{\partial^3 w}{\partial x^3} \tag{2-22}
\]

where

\[
K_1 = \left( \frac{A_{16} B_{16} - A_{66} B_{11}}{A_{16} - A_{11} A_{66}} \right)
\]

\[
K_2 = \left( \frac{A_{16} B_{11} - A_{11} B_{16}}{A_{16} - A_{11} A_{66}} \right)
\]

Differentiating equations (2-21) and (2-22) with respect to \( x \) yields:

\[
\frac{\partial^3 u_0}{\partial x^3} = K_1 \frac{\partial^4 w}{\partial x^4} - \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \tag{2-23}
\]

\[
\frac{\partial^3 v_0}{\partial x^3} = K_2 \frac{\partial^4 w}{\partial x^4} \tag{2-24}
\]
Now, multiplying equation (2-6) by $dx$ and integrating the resulting equation over the length of the beam yields:

$$
\int_0^L N_x \, dx = A_{11} \int_0^L \frac{\partial u_0}{\partial x} \, dx + \frac{A_{16}}{2L} \int_0^L \left( \frac{\partial w}{\partial x} \right)^2 \, dx + A_{14} \int_0^L \frac{\partial v_0}{\partial x} \, dx - B_{11} \int_0^L \frac{\partial^2 w}{\partial x^2} \, dx
$$  \hspace{1cm} (2-25)

This can also be written as:

$$
N_x^0 = \frac{A_{11}}{L} [u_0(L) - u_0(0)] + \frac{A_{16}}{2L} \left[ v_0(L) - v_0(0) \right] + \frac{A_{14}}{L} \int_0^L \frac{\partial w}{\partial x} \, dx - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} \, dx \hspace{1cm} (2-26)
$$

Expressions $[u_0(L) - u_0(0)]$ and $[v_0(L) - v_0(0)]$ represent the in-plane boundary conditions of the beam. Here, only the immovable edge conditions are considered, thus:

$$
[u_t(L) - u_t(0)] = 0 \\
v_o(L) - v_o(0)] = 0
$$

Therefore, equation (2-26) simplifies to:

$$
N_x^0 = \frac{A_{11}}{2L} \int_0^L \left( \frac{\partial w}{\partial x} \right)^2 \, dx - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} \, dx \hspace{1cm} (2-27)
$$

The governing equation of motion in the z-direction is then obtained by plugging equations (2-23), (2-24), and (2-27) into equation (2-20) which yields:

$$
\left( B_{11} K_1 + B_{16} K_2 - D_{11} \right) \frac{\partial^4 w}{\partial x^4} + \frac{A_{11}}{2L} \int_0^L \left( \frac{\partial w}{\partial x} \right)^2 \, dx - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} \, dx \right) \frac{\partial^2 w}{\partial x^2} + P - \frac{\partial^2 w}{\partial t^2} - C \frac{\partial w}{\partial t} = 0
$$  \hspace{1cm} (2-28)

Equation (2-28) represents the general equation of motion of an antisymmetric laminated composite beam when subjected to a forced vibration $P$. It is to be noted that a damping term $C \frac{\partial w}{\partial t}$ has been added to the initial equation of motion, as usually damping exists in a real life problem. The expression of the nonlinearity of the problem is contained in the terms in parenthesis. The first term
involving the coefficient $A_{11}$ will create a cubic nonlinearity whilst the term multiplying $B_{11}$ will create a quadratic nonlinearity.
Chapter III

Method of Solution

3.1. The Galerkin method.

The governing partial differential equation of motion given by equation (2-28) is solved here by means of the Galerkin method. This method transforms a partial differential equation into algebraic equations for a static problem and into ordinary differential equations for a dynamic problem. In the case exposed here a nonlinear ordinary differential equation of the Duffing type is obtained. This method is successful in dealing with both linear and nonlinear problems, as well as stability and buckling problems.

The method can be described as follows:

Assuming that the motion of a beam structure is represented by:

\[ L(w,t) - P = 0 \]  \hspace{1cm} (3-1)

where:

- \( L \) = a differential operator
- \( P \) = forcing function acting on the structure
- \( w \) = displacement function

and that \( \delta w \) is a small arbitrary variation of the displacement of the beam, then the external and internal work done by the system are:
\[ \delta W_{\text{ext}} = \int_{\text{Area}} (P\delta w) \cdot dA \] (3-2)

\[ \delta W_{\text{int}} = \int_{\text{Area}} [L(w,t)\delta w] \cdot dA \] (3-3)

If \( w \) happens to be the exact solution of (3-1), then (3-2) and (3-3) would be identical. Now assuming that \( w \) is expressed in the form

\[ w = a_1 \psi_1 + a_2 \psi_2 + a_3 \psi_3 + \cdots + a_n \psi_n \] (3-4)

where

- \( a_i \), unknown variable to be determined
- \( \psi_i \), function of the variable of \( w \), and constituting a complete set

then equality \( \delta W_{\text{ext}} = \delta W_{\text{int}} \) yields the system of equations

\[ \int_{\text{Area}} [L(w,t) - P] \cdot \psi_1 \cdot dA = 0 \]

\[ \int_{\text{Area}} [L(w,t) - P] \cdot \psi_2 \cdot dA = 0 \]

\[ \vdots \]

\[ \int_{\text{Area}} [L(w,t) - P] \cdot \psi_n \cdot dA = 0 \] (3-5)

For a uniform beam \( dA = (\text{width})dx \), so equations (3-5) simplify to a system where

\[ \int_{L} [L(w,t) - P] \cdot \psi_i \cdot dx = 0 \] (3-5a)

By substituting the expression

\[ w = \sum_{i=1}^{n} a_i \cdot \psi_i(x) \]

into equations (3-5a), the resulting equation will be a set of algebraic equations (for a static problem) or a set of differential equations (for a dynamic problem). The resulting equations yield the solution for \( a_i \).
3.1.1. The Galerkin method applied to a simply-supported beam.

In this section, a simply-supported beam with immovable edges subjected to a harmonic excitation $P(x, t) = P_0 \cos \Omega t$ is analyzed.

The boundary conditions for a simply-supported beam are as follows

\[ \begin{align*}
\text{at } x = 0 & \quad w = 0 \\
& \quad M_x = 0 \\
\text{at } x = L & \quad w = 0 \\
& \quad M_x = 0
\end{align*} \tag{3-6} \]

The immovable edge conditions are by definition

\[ \begin{align*}
\nu_0(0) &= 0 \\
\nu_0(L) &= 0 \\
\nu_0(0) &= 0 \\
\nu_0(L) &= 0
\end{align*} \]

Equation (2-28) needs to be solved with the above boundary conditions.

For a one-mode solution, the displacement function is assumed as follows

\[ w(x, t) = h q(t) \sin \left( \frac{\pi}{L} x \right) \tag{3-7} \]

where

- $h = \text{total thickness of the composite beam}$
- $q(t) = \text{time amplitude function}$
It is to be noted that this solution satisfies the main geometrical boundary conditions, that being the
displacement at both ends of the beam is equal to zero. For a generally antisymmetric laminated beam
and for an unsymmetrical cross-ply beam the boundary condition \( M_x = 0 \) is not satisfied. However
for an angle-ply composite beam all the boundary conditions are satisfied.

Now, applying the Galerkin’s method to equation (2-28) i.e. setting:

\[
\int_{0}^{L} \left[ L(w, t) - P \right] \sin \left( \frac{\pi}{L} x \right) dx = 0
\]  

(3-8)

where

\[
L = (B_{11} K_1 + B_{16} K_2 - D_{11}) \frac{\partial^4}{\partial x^4} + \left( \frac{A_{11}}{2 L} \int_{0}^{L} \left( \frac{\partial}{\partial x} \right)^2 dx - \frac{B_{11}}{L} \int_{0}^{L} \frac{\partial^2}{\partial x^2} dx \right) \frac{\partial^2}{\partial x^2} - p \frac{\partial^2}{\partial t} - C \frac{\partial}{\partial t}
\]

with

\[
K_1 = \left( \frac{A_{66} B_{11} - A_{11} B_{16}}{A_{16}^2 - A_{11} A_{66}} \right)
\]

\[
K_2 = \left( \frac{A_{11} B_{16} - A_{16} B_{11}}{A_{16}^2 - A_{11} A_{66}} \right)
\]

and performing the integration (3-8) yields,

\[
\ddot{q} + 2 \xi \omega_0 \dot{q} + \omega_0^2 q + \alpha_2 q^2 + \alpha_3 q^3 = F_0 \cos \Omega t
\]  

(3-9)

where

\[
\omega_0^2 = (B_{11} K_1 + B_{16} K_2 + D_{11}) \frac{\pi^4}{p h L^4}
\]

\[
\xi = \frac{C}{2 p h \omega_0}
\]

\[
\alpha_2 = 2 B_{11} \frac{\pi^3}{p L^4}
\]

\[
\alpha_3 = A_{11} \frac{h \pi^4}{4 p L^4}
\]

\[
F_0 = \frac{2}{\rho L h^2} \int_{0}^{L} P_0 \cos \Omega t \sin \left( \frac{\pi}{L} x \right) dx
\]

It is to be noted that \( \omega_0 \) represents the natural frequency of the beam, \( \xi \) the damping ratio and \( \alpha_2, \alpha_3 \)
spring forces.
Equation (3-9) is known as the **forced Duffing equation**. It can be noticed that the nonlinearities are of the quadratic and cubic form. This equation will be solved using a Perturbation Method, namely the **Method of Multiple Scales**. This method will be presented later in this chapter.

### 3.1.2. The Galerkin method applied to a clamped-clamped beam.

In this section a clamped-clamped beam with immovable edges subjected to a forcing function $P(x,t)$ is considered.

![Figure 3-2. Clamped-Clamped Beam.](image)

For a clamped-clamped beam, the boundary conditions are as follows:

- $@ x = 0 \quad w = 0$
- $\frac{\partial w}{\partial x} = 0$
- $@ x = L \quad w = 0$
- $\frac{\partial w}{\partial x} = 0$

A one-mode solution of the following form is assumed:

$$w(x,t) = hq(t) \left[ 1 - \cos \left( \frac{2\pi x}{L} \right) \right] \quad (3-10)$$

Once again it can be easily shown that this form of solution satisfies all the boundary conditions.

Applying the same procedures as for the simply-supported case, the following differential equation is obtained:

$$\ddot{q} + 2\xi\omega_0 \dot{q} + \omega_0^2 q + \alpha_3 q^3 = F_0 \cos \Omega t \quad (3-11)$$
with

\[ \omega_0^2 = \left( B_{11} K_1 + B_{16} K_2 + D_{11} \right) \frac{(2\pi)^4}{3\rho h^4} \]

\[ K_1 = \left( \frac{A_{16} B_{11} - A_{11} B_{16}}{A_{16} - A_{11}} \right) \]

\[ K_2 = \left( \frac{A_{11} B_{16} - A_{16} B_{11}}{A_{16} - A_{11}} \right) \]

\[ \xi = \frac{C}{2\rho h \omega_0} \]

\[ \alpha_3 = \frac{h(2\pi)^4}{12\rho h^4} \]

\[ F_0 = \frac{2}{3L^4} \int_0^L P_0 \cos \Omega \left[ 1 - \cos \frac{2\pi x}{L} \right] dx \]

It can be noted that only a cubic nonlinearity exists for this case.

3.2. Method of Multiple Scales (MMS).

3.2.1. Introduction.

The main difficulty in solving the Duffing equation resides in the presence of a cubic nonlinearity and for some cases, also a quadratic nonlinearity. Many methods, analytical or numerical, may be applied to solve the Duffing equation. The Method of Multiple Scales is employed herein to obtain an approximate solution to equation (3-9) and (3-11). This method is a powerful technique in dealing with the following resonances cases:

- \( \Omega \equiv \omega_0 \) primary resonance,
- \( \Omega \equiv \frac{1}{3} \omega_0 \) superharmonic resonance,
- \( \Omega \equiv 3\omega_0 \) subharmonic resonance.

The solutions obtained are free of secular terms and small divisor terms. Those terms will be defined in the following subsections.

One of the advantages of the Method of Multiple Scales over other methods, as for example the method of Harmonic Balance, is that the superharmonic and subharmonic resonances are simpler
to calculate. Furthermore, the MMS can deal with multi-mode solution very easily. This will be shown in Chapter IV.

In the following paragraphs, the forcing function will be uniform along the x axis. Thus the forcing excitation $P$ will only be function of time. This can be represented as follows:

![Uniform Excitation on a Simply-Supported Beam](image)

Figure 3-3. Uniform Excitation on a Simply-Supported Beam.

For the clamped-clamped beam, the loading will also be uniform.

3.2.2. Primary resonance $\Omega \equiv \omega_0$.

Case 1. Unsymmetrically laminated angle-ply composite beam.

By definition, for an unsymmetrically laminated angle-ply composite structure, the following coefficients are set to zero (Reference 19):

$$
A_{16} = A_{26} = 0 \\
B_{11} = B_{12} = B_{21} = B_{22} = 0 \\
D_{16} = D_{26} = 0
$$

(3-12)

Thus equation (3-9) can be simplified and becomes:

$$
\ddot{q} + 2\xi_0 \omega_0 q + \omega_0^2 q + \alpha_3 q^3 = F_0 \cos \Omega t
$$

(3-13)

where the coefficients for a constant $P_0$ are presented in Table 3-1.

Equation (3-11) for a clamped-clamped beam has the same expression as equation (3-13).
Table 3-1. Coefficients of Equation (3-13)

<table>
<thead>
<tr>
<th></th>
<th>Simply-supported</th>
<th>Clamped-clamped</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_0^2$</td>
<td>$\left(D_{11} - \frac{B_{12}^2}{A_{11}}\right)\pi^2$</td>
<td>$\left(D_{11} - \frac{B_{12}^2}{A_{11}}\right)(2\pi)^4$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$A_i h \left(\frac{\pi}{L}\right)$</td>
<td>$A_i h \left(\frac{2\pi}{L}\right)$</td>
</tr>
<tr>
<td>$F_0$</td>
<td>$\frac{4P_0}{\pi \rho h^2}$</td>
<td>$\frac{2P_0}{3 \rho h^2}$</td>
</tr>
</tbody>
</table>

Making the following change of variables:

\[ q = e^{0.5}U \]
\[ \xi = 0.0 = \mu e \]

into equation (3-13), and dividing the resulting equation by $e^{0.5}$ yields:

\[ \ddot{U} + \omega_0^2U = f \cos \Omega t - 2\mu \dot{U} - \alpha_3 \varepsilon U^3 \]  

\[ (3-14) \]

The vicinity of the forcing frequency around the natural frequency of the beam can be expressed as follows:

\[ \Omega = \omega_0 + \varepsilon \sigma \]  

\[ (3-15) \]

where

\[ \sigma = \text{detuning parameter} \]
\[ \varepsilon << 1 \text{ with } \varepsilon \neq 0 \]

By examining equation (3-14) it can be noticed that the cubic nonlinearity and the damping term are of the same order. In order to have a uniformly valid approximation of the solution, the forcing excitation needs to appear when the damping and the nonlinearity occur. Thus, the forcing function is expressed as $f = e^\varepsilon k$.

To determine an approximate solution to equation (3-14), a solution of the following form is assumed:

\[ U(t, \varepsilon) = U_0(T_0, T_1, \ldots) + \varepsilon U_1(T_0, T_1, \ldots) + \ldots \]

where the time scale is defined as $T_n = e^{\varepsilon t}$

\[ (3-16) \]

Thus, the derivatives become:
\[ \frac{d}{dt} = D_0 + e D_1 + \cdots \]  
\[ \frac{d^2}{dt^2} = D_0^2 + 2e D_0 D_1 + \cdots \]  

(3-17)

where:

\[ D_n = \frac{\partial}{\partial T_n} \]  

(3-18)

Now, substituting equations (3-16) and (3-17) into (3-14) and separating with the order of \( e \) yields:

\[ O(e^0) : \]

\[ D_0^2 U_0 + \omega_0^2 U_0 = 0 \]  

(3-19)

and

\[ O(e^1) : \]

\[ D_0^2 U_1 + \omega_0^2 U_1 = k \cos (\omega_0 T_0 + \sigma T_1) - 2 \mu D_0 U_0 - \alpha_3 U_0^3 - 2 D_0 D_1 U_0 \]  

(3-20)

Equation (3-19) represents a linear homogeneous ordinary differential equation with constant coefficients. A general solution to this equation is:

\[ U_0 = A(T_1) e^{i \omega_0 T_0} + A(T_1) e^{-i \omega_0 T_0} \]  

(3-21)

where \( A(T_1) \) is the complex conjugate of \( A(T_1) \).

When inserting (3-21) into equation (3-20), this yields

\[ D_0^2 U_1 + \omega_0^2 U_1 = -\left[ 2i \omega_0 (A' + \mu A) + 3\alpha_3 A^2 \bar{A} - \frac{1}{2} ke^{i(\sigma T_1)} \right] e^{i \omega_0 T_0} - \alpha_3 A^3 e^{i 3 \omega_0 T_0} + C.C. \]  

(3-22)

where \( C.C. \) stands for the complex conjugate of the preceding terms.

Due to the fact that the homogeneous solution of equation (3-22) contains terms such as \( e^{i \omega_0 T_0} \), all the non-homogeneous terms containing \( e^{i \omega_0 T_0} \) produce solutions such as \( T_0 \sin \omega_0 T_0 \) or \( T_0 \cos \omega_0 T_0 \). These terms go to infinity as \( T_0 \to \infty \) and they are called secular terms. In order to have a uniformly valid expansion i.e. equation (3-16), the solution must be free of the so-called "secular terms". Thus:

\[ 2i \omega_0 (A' + \mu A) + 3\alpha_3 A^2 \bar{A} - \frac{1}{2} ke^{i \sigma T_1} = 0 \]  

(3-23)

By assuming

\[ A = \frac{1}{2} ae^{i \beta} \]

\[ \sigma T_1 - \beta = \gamma \]  

(3-24)
and separating the imaginary and real part of (3-23), the following two equations must be verified:

\[
a' = -\mu a + \frac{1}{k} \frac{k}{\omega_0} \sin \gamma
\]
\[
\alpha' = \sigma a - \frac{3}{8} \frac{\alpha_3}{\omega_0} a^2 - \frac{1}{2} \frac{k}{\omega_0} \cos \gamma
\]

(3-25)

For the steady state solution \(a'\) and \(\gamma'\) are set to zero. By squaring the above equations and summing the resultant ones, the following expression is obtained:

\[
\left[ \mu^2 + \left( \sigma - \frac{3}{8} \frac{\alpha_3}{\omega_0} a^2 \right)^2 \right] a^2 = \frac{k^2}{4\omega_0^2}
\]

(3-26)

Equation (3-26) is the steady-state amplitude-frequency response corresponding to the primary resonance.

**Case 2. Unsymmetrically laminated cross-ply composite beam.**

For an unsymmetrically laminated cross-ply composite beams the following coefficients are equal to zero:

\[
A_{16} = A_{26} = 0
\]
\[
B_{12} = B_{16} = B_{26} = B_{66} = 0
\]
\[
D_{16} = D_{26} = 0
\]

(3-27)

Thus equation (3-9) can be written as follows:

\[
\ddot{q} + 2\xi \omega_0 \dot{q} + \omega_0^2 q + \alpha_2 q^2 + \alpha_3 q^3 = F_0 \cos \Omega t
\]

(3-28)

where the coefficients for a constant \(P_0\) are presented in Table 3-2.
Table 3-2. Coefficients of Equation (3-28)

<table>
<thead>
<tr>
<th></th>
<th>Simply-supported</th>
<th>Clamped-clamped</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_0^2$</td>
<td>$D_{11} - \frac{B_{11}^2}{A_{11}} \frac{\pi^4}{\rho h L^4}$</td>
<td>$D_{11} - \frac{B_{11}^2}{A_{11}} \frac{(2\pi)^4}{3\rho h L^4}$</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>$B_{11} \frac{2\pi^3}{\rho L^4}$</td>
<td>0</td>
</tr>
<tr>
<td>$\alpha_3$</td>
<td>$A_{11} \frac{h\pi^4}{4\rho L^4}$</td>
<td>$A_{11} \frac{h(2\pi)^4}{12\rho L^4}$</td>
</tr>
<tr>
<td>$F_0$</td>
<td>$\frac{4P_0}{\pi \rho h^2}$</td>
<td>$\frac{2P_0}{3\rho h^2}$</td>
</tr>
</tbody>
</table>

It is to be noted that for a simply-supported beam, equation (3-28) contains a quadratic nonlinearity in addition to the cubic nonlinearity. Once again, the method of multiple scales will be used in order to solve for the response frequency curve. The method is a little more complex because of the addition of the quadratic nonlinearity.

Due to this quadratic nonlinear term the following changes of variable are required

$q = \varepsilon U$

and $\xi \omega_0 = \mu \varepsilon^2$

Substituting into equation (3-28) and dividing the resultant equation by $\varepsilon$ yields:

$$\ddot{U} + \omega_0^2 U = \frac{F_0}{\varepsilon} \cos \Omega t - 2\mu \varepsilon^2 \dot{U} - \alpha_2 \varepsilon U^2 - \alpha_3 \varepsilon^2 U^3 \quad (3-29)$$

Assuming the forcing frequency to be in the vicinity of the natural frequency:

$$\Omega = \omega_0 + \varepsilon^2 \sigma \quad (3-30)$$

and the forcing amplitude term to be of the following form:

$$\frac{F_0}{\varepsilon} \cos \Omega t = \varepsilon^2 k \cos \Omega t \quad (3-31)$$

Thus, equation (3-29) can be expressed as

$$\ddot{U} + \omega_0^2 U = \varepsilon^2 k \cos \Omega t - 2\mu \varepsilon^2 \dot{U} - \alpha_2 \varepsilon U^2 - \alpha_3 \varepsilon^2 U^3 \quad (3-32)$$

A solution to equation (3-32) is assumed to be of the following form:
\[ U(t, \varepsilon) = U_0(T_0, T_1, T_2) + \varepsilon U_1(T_0, T_1, T_2) + \varepsilon^2 U_2(T_0, T_1, T_2) \] (3-33)

The solution needs to be of the second order as the cubic nonlinearity is multiplied by \( \varepsilon^2 \).

Substituting equation (3-33) into (3-32) yields the following system of equations ordered by the power of \( \varepsilon \):

\[ O(\varepsilon^0) : \]
\[ D_0^2 U_0 + \omega_0^2 U_0 = 0 \] (3-34)

\[ O(\varepsilon^1) : \]
\[ D_0^2 U_1 + \omega_0^2 U_1 = -2D_0 \partial D_1 U_0 - \alpha_2 U_0^2 \] (3-35)

\[ O(\varepsilon^2) : \]
\[ D_0^2 U_2 + \omega_0^2 U_2 = -2D_0 \partial D_1 U_1 - \left(2D_0 \partial D_1 + D_1^2 \right) U_0 - 2\mu D_0 U_0 - 2\alpha_2 U_0 U_1 - \alpha_3 U_0^3 + k \cos(\omega_0 T_0 + \sigma T_2) \] (3-36)

Solving (3-34) gives:
\[ U_0 = A(T_1, T_2)e^{i\omega_0 T_0} + \overline{A(T_1, T_2)}e^{-i\omega_0 T_0} \] (3-37)

Inserting equation (3-37) into equation (3-35) yields the following:
\[ D_0^2 U_1 + \omega_0^2 U_1 = -2i\omega_0 A' e^{i\omega_0 T_0} - \alpha_2 A^2 e^{2i\omega_0 T_0} - \alpha_3 A \overline{A} + C.C. \] (3-38)

By setting the secular terms to zero, \( A' = 0 \), thus the following conclusion can be drawn:
\[ A = A(T_2) \] (3-39)

Solving equation (3-38) yields the following equation:
\[ U_1(T_0, T_1, T_2) = \frac{\alpha_2}{\omega_0^2} \left[ \frac{1}{3} A^2 e^{2i\omega_0 T_0} + \frac{1}{3} \overline{A^2} e^{-2i\omega_0 T_0} - 2A \overline{A} \right] \] (3-40)

Substituting the above equation into equation (3-36) yields:
\[
D_0^2 U_2 + \omega_0^2 U_2 = -2i\omega_0 A e^{i\omega_0 T_0} + 2i\omega_0 A^* e^{-i\omega_0 T_0} - 2i\mu\omega_0 A e^{i\omega_0 T_0} + 2i\mu\omega_0 A^* e^{-i\omega_0 T_0}
- 2\frac{\alpha^2}{\omega_0^3} \left[ \frac{A^3}{3} e^{i3\omega_0 T_0} + \left( \frac{A^2}{3} - 2A^2 A^* \right) e^{i\omega_0 T_0} + C.C. \right] + k \cos(\omega_0 T_0 + \sigma T_2)
- \alpha^3 \left[ e^{3\omega_0 T_0} + 3A^2 A e^{i\omega_0 T_0} + C.C. \right]
\]

(3-41)

The secular terms need to be set to zero, thus the following must be satisfied:

\[
2\mu\omega_0 \left( A' + \mu A \right) + A^2 \left( 3\alpha^2 - \frac{10\alpha^2}{3\omega_0^2} \right) - \frac{1}{2} ke^{i\sigma T_2} = 0
\]

(3-42)

Assuming the following:

\[
A = \frac{1}{2} e^{i\theta}
\]

\[
\sigma T_2 - \beta = \gamma
\]

(3-43)

and by separating real and imaginary parts of equation (3-42) yields:

\[
a' = -\mu a + \frac{k}{2\omega_0} \sin \gamma
\]

\[
\alpha' = a\gamma = a\sigma - \frac{9\alpha_3 \omega_0^2 - 10\alpha^2}{24\omega_0^2} a^3 - \frac{k}{2\omega_0} \cos \gamma
\]

(3-44)

Solving for the steady-state solution yields equation (3-45):

\[
a^6 - \frac{2\sigma}{\beta} a^4 + \left( \frac{\sigma^2 + \mu^2}{\beta^2} \right) a^2 \left( \frac{F_0}{2e^3 \omega_0 \beta} \right)^2 = 0
\]

(3-45)

where

\[
\beta = \frac{9\alpha_3 \omega_0^2 - 10\alpha^2}{24\omega_0^3}
\]

For the clamped-clamped case, the equation becomes:

\[
\left[ \mu^2 + \left( \sigma - \frac{3\alpha^2}{8\omega_0^2} \right)^2 \right] a^2 = \frac{k^2}{4\omega_0^2}
\]

(3-46)

which is the same as equation (3-26) with different coefficients of course.
3.2.3. **Superharmonic resonance** $\Omega = \frac{1}{3} \omega_0$.

In the case of superharmonic resonance, the forcing frequency is expressed in the following manner:

$$3\Omega = \omega_0 + \varepsilon \sigma$$ \hspace{1cm} (3-47)

In addition to the terms that produce secular terms (coefficients of $e^{\omega_0 T_0}$), there is a new term that produces secular terms in $u_1$ that is the coefficients of $e^{3\omega_0 T_0}$.

Equation (3-14) becomes a system of equations ordered by the following power of $\varepsilon$:

**$O(\varepsilon^0)$**:

$$D_0^2 U_0 + \omega_0^2 U_0 = f \cos \Omega T_0$$ \hspace{1cm} (3-48)

**$O(\varepsilon^1)$**:

$$D_0^2 U_1 + \omega_0^2 U_1 = -2\mu D_0 U_0 - \alpha_3 U_0^3 - 2D_0 D_1 U_0$$ \hspace{1cm} (3-49)

The solution to equation (3-48) is:

$$U_0 = A(T_1) e^{\omega_0 T_0} + \Lambda e^{\Omega T_0} + \overline{A}(T_1) e^{-\omega_0 T_0} + \Lambda e^{-\Omega T_0}$$ \hspace{1cm} (3-50)

with

$$\Lambda = \frac{1}{2} \left( \frac{\kappa}{\omega_0^2 - \Omega^2} \right)$$ \hspace{1cm} (3-51)

Inserting the equation (3-50) into (3-49), using $3\Omega T_0 = \omega_0 T_0 + \sigma T_1$, and setting the secular terms to zero yields the following equation:

$$2i\omega_0 \left[ A' + \mu A \right] + 6\alpha_3 A \Lambda^2 + 3A^2 \overline{A} \alpha_3 + \alpha_3 A^3 e^{\sigma T_1} = 0$$ \hspace{1cm} (3-52)

Assuming

$$A = \frac{1}{2} \alpha e^{\beta}$$ \hspace{1cm} (3-53)

and separating the real and imaginary part of equation (3-52) yields the system:

$$a' = -\mu a - \frac{\alpha_3 A^3}{\omega_0} \sin(\sigma T_1 - \beta)$$

$$a \beta' = \frac{3}{8} \omega_0 \left[ \Lambda^2 + \beta^2 \right] a + \frac{\alpha_3 A^3}{\omega_0} \cos(\sigma T_1 - \beta)$$ \hspace{1cm} (3-54)
Assuming $\gamma = \sigma T_1 - \beta$ and solving for the steady-state solution yields:

$$\mu^2 + \left( \sigma - \frac{3\alpha_3 \Lambda^2}{\omega_0^3} - \frac{3\alpha_3 \Lambda^2}{8\omega_0} a^2 \right)^2 a^2 = \frac{\alpha_2^2 \Lambda^6}{\omega_0^2}$$  \hspace{1cm} (3-55)

3.2.4. Subharmonic resonance $\Omega \equiv 3\omega_0$

For this case, the forcing frequency is expressed as follows:

$$\Omega = 3\omega_0 + e\sigma$$  \hspace{1cm} (3-56)

As for the superharmonic case, in addition to the secular terms produced by the coefficients of $e^{i\omega_0 T_0}$, the term proportional to $e^{i(\Omega-2\omega_0)T_0}$ also produces a secular term in $u_1$ as:

$$(\Omega - 2\omega_0)T_0 = \omega_0 T_0 + e\sigma T_0 = \omega_0 T_0 + \sigma T_1$$  \hspace{1cm} (3-57)

By setting the secular producing terms to zero, the following equation is obtained:

$$2i\omega_0 (A' + \mu A) + 3\alpha_3 A^2 \overline{A} + 6\alpha_3 \Lambda^2 A + 3\alpha_3 \Lambda A^2 e^{\sigma T_1} = 0$$  \hspace{1cm} (3-58)

As the procedure is identical to the subharmonic case, only the final equation is presented here:

$$\left[ 9\mu^2 + \left( \sigma - \frac{9\alpha_3 \Lambda^2}{\omega_0^3} - \frac{9\alpha_3 \Lambda^2}{8\omega_0} a^2 \right)^2 \right] a^2 = \frac{81\alpha_3 \Lambda^2}{16\omega_0^4}$$  \hspace{1cm} (3-59)

with

$$\Lambda = \frac{1}{2} \frac{\kappa}{(\omega_0^2 - \Omega^2)}$$

$$\alpha_3 = \frac{A_1 h \pi^4}{4\rho L^3}$$

It is to be noted that in this case, the final equation is a quadratic equation in terms of $a^2$. 

Chapter IV

Two-Mode Analysis

4.1. Introduction.

In the previous chapter, the analysis was performed with a one-mode solution. The purpose of this chapter is to consider a two-mode solution. As it will be seen in the result's chapter (Chapter V), the predominant mode is the first one. The addition of a second-mode solution proves to have no significant effect on the behavior of the beam. Only primary resonances are considered in this chapter.

Rewriting the governing equation for an unsymmetrically laminated angle-ply

\[
L(w,t) - P = \left( D_{11} - \frac{B_{16}^2}{A_{66}} \right) \frac{\partial^4 w}{\partial x^4} - \left( \frac{A_{11}}{2L} \int_0^L \left( \frac{\partial w}{\partial x} \right)^2 \, dx \right) \frac{\partial^2 w}{\partial x^2} + \rho_h \frac{\partial^2 w}{\partial t^2} + C \frac{\partial w}{\partial t} - P \tag{4-1}
\]

Once again, here it is assumed that the forcing excitation is only function of time.

The assumed two-mode solution is of the following form:

\[
w(x,t) = q_1 h \sin \left( \frac{\pi}{L} x \right) + q_2 h \sin \left( \frac{3\pi}{L} x \right) \tag{4-2}
\]

This solution satisfies all the boundary conditions of a simply-supported unsymmetrically laminated angle-ply laminated composite beam. It is to be noted that the absence of the term \( q_2 h \sin \left( \frac{2\pi}{L} x \right) \) in equation (4-2) is due to the fact that only symmetric modes are excited for the forced vibration of simply-supported beams for a uniform loading.
4.2. The Galerkin method applied to equation (4-1).

To apply the Galerkin method for a two-mode solution, the following must be set:

\[
\int_0^L \left[L(w, t) - P\right] \sin \left(\frac{\pi x}{L}\right) \, dx = 0 \tag{4-3}
\]

and

\[
\int_0^L \left[L(w, t) - P\right] \sin \left(\frac{3\pi x}{L}\right) \, dx = 0 \tag{4-4}
\]

Inserting equation (4-2) into equations (4-3) and (4-4) and simplifying yields:

\[
\ddot{q}_1 + 2\xi_1 \omega_1 \dot{q}_1 + \omega_1^2 q_1 + \alpha_1 q_1^3 + \gamma_1 q_1 q_2^2 = \frac{4P}{\rho h^2 \pi} \tag{4-5}
\]

\[
\ddot{q}_3 + 2\xi_3 \omega_3 \dot{q}_3 + \omega_3^2 q_3 + \alpha_3 q_3^3 + \gamma_3 q_3 q_2^2 = \frac{4P}{3\rho h^2 \pi} \tag{4-6}
\]

where the coefficients are given in Table 4-1.

<table>
<thead>
<tr>
<th>Table 4-1. Coefficients of equations (4-5) and (4-6)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Equation (4-5)</strong></td>
</tr>
<tr>
<td>$\omega_1^2 = \left(D_{11} - \frac{B_{16}^2}{A_{66}}\right) \frac{1}{\rho h \left(\frac{L}{\pi}\right)^4}$</td>
</tr>
<tr>
<td>$\alpha_1 = \frac{A_{11}}{2L} \frac{h}{\rho \left(\frac{L}{2}\right) \left(\frac{\pi}{L}\right)^4}$</td>
</tr>
<tr>
<td>$\gamma_1 = \frac{A_{11}}{2L} \frac{h}{\rho \left(\frac{L}{2}\right)^2 \left(\frac{\pi}{L}\right)^2}$</td>
</tr>
</tbody>
</table>

It can be noted that equation (4-5) and (4-6) are coupled and will need to be solved as a system.

4.3. The Method of Multiple Scales.

To obtain the primary resonance responses of the coupled equations (4-5) and (4-6) using the method of multiple scales, two cases must be considered. But first, the following transformations are made:
\[ q_1 = e^{0.5U_1} \]
\[ q_3 = e^{0.5U_3} \]
\[ \xi_{\omega_1} = \mu_1 \varepsilon \]
\[ \xi_{\omega_3} = \mu_3 \varepsilon \]  \hspace{1cm} (4-7)

\[ \dot{U}_1 + 2\mu_1 \varepsilon U_1 + \omega_1^2 U_1 + \alpha_1 \varepsilon U_1^3 + \gamma_1 \varepsilon_1^2 U_1 = \frac{f_1(t)}{e^{0.5}} = \frac{F_0}{e^{0.5}} \cos \Omega \]  \hspace{1cm} (4-8)

\[ \dot{U}_3 + 2\mu_3 \varepsilon U_3 + \omega_3^2 U_3 + \alpha_3 \varepsilon U_3^3 + \gamma_3 \varepsilon_3^2 U_3 = \frac{f_3(t)}{e^{0.5}} = \frac{F_0}{e^{0.5}} \cos \Omega \]  \hspace{1cm} (4-9)

**Case 1.** \( \Omega \equiv \omega_1 \)

The closeness of \( \Omega \) to \( \omega \) is represented by the following equation:

\[ \Omega = \omega_1 + \varepsilon \sigma \]

Equations (4-8) and (4-9) can be expressed as a system of equations ordered by the power of \( \varepsilon \).

**Order of \( O(\varepsilon^0) \)**

\[ D_0^2 U_{10} + \omega_1^2 U_{10} = 0 \]  \hspace{1cm} (4-10)
\[ D_0^2 U_{30} + \omega_3^2 U_{30} = 0 \]  \hspace{1cm} (4-11)

**Order of \( O(\varepsilon^1) \)**

\[ D_0^2 U_{11} + \omega_1^2 U_{11} = k_1 \cos(\omega_1 T_0 + \sigma T_1) - 2D_0 D_1 U_{10} - 2\mu_1 D_0 U_{10} - \alpha_1 U_1^3 - \gamma_1 U_3^2 U_{10} \]  \hspace{1cm} (4-12)
\[ D_0^2 U_{31} + \omega_3^2 U_{31} = k_3 \cos(\omega_3 T_0 + \sigma T_1) - 2D_0 D_3 U_{30} - 2\mu_3 D_0 U_{30} - \alpha_3 U_3^3 - \gamma_3 U_1^2 U_{30} \]  \hspace{1cm} (4-13)

A solution of equation (4-10) and (4-11) is given by:

\[ U_{10} = A_1(T_1)e^{\omega_1 T_0} + A_1^\dagger(T_1)e^{-\omega_1 T_0} \]  \hspace{1cm} (4-14)
\[ U_{30} = A_3(T_1)e^{\omega_3 T_0} + A_3^\dagger(T_1)e^{-\omega_3 T_0} \]  \hspace{1cm} (4-15)

Inserting equations (4-14) and (4-15) into (4-12) and (4-13) yields the following system:

\[ D_0^2 U_{11} + \omega_1^2 U_{11} = \left[2i\omega_1 \left(A_1^\dagger + \mu_1 A_1\right) + 3\alpha_1 A_1^2 A_1 + 2\gamma_1 A_3 A_1 A_3 + \frac{1}{2} k_1 e^{i(\sigma T_1)} \right] e^{\omega_1 T_0} - \alpha_1 A_1^3 e^{3\omega_1 T_0} - \gamma_1 A_1^2 A_1 e^{(\omega_1 + 2\omega_3) T_0} + CC. \]  \hspace{1cm} (4-16)

\[ D_0^2 U_{31} + \omega_3^2 U_{31} = \left[2i\omega_3 \left(A_3^\dagger + \mu_3 A_3\right) + 3\alpha_3 A_3^2 A_3 + 2\gamma_3 A_1 A_3 A_1 \right] e^{\omega_3 T_0} - \alpha_3 A_3^3 e^{3\omega_3 T_0} + \frac{1}{2} k_3 e^{i\left(\omega_1 T_0 + \sigma T_1\right)} - \gamma_3 A_3^2 A_3 e^{(\omega_3 + 2\omega_1) T_0} + CC. \]  \hspace{1cm} (4-17)
The secular terms need to be set to zero. For (4-16) the secular terms are the coefficients of $e^{i\omega_1 t_0}$, and for (4-17) the coefficients of $e^{i\omega_2 t_0}$. It is important to note that $\gamma_1 A_3^2 A_1 e^{i(\omega_1 + 2\omega_3) t_0}$ does not create a secular term as $\omega_3$ is much larger than $\omega_1$.

Thus the following equations must be verified:

\[
2i\omega_1 (A'_1 + \mu_1 A_1) + 3\alpha_1 A_1^2 A_1 + 2\gamma_1 A_2 A_3 A_1 = \frac{1}{2} k_1 e^{i\omega_1 t_0} \quad (4-18)
\]
\[
2i\omega_3 (A'_1 + \mu_3 A_3) + 3\alpha_3 A_3^2 A_3 + 2\gamma_3 A_4 A_5 A_3 = 0 \quad (4-19)
\]

Assuming:
\[
A_1 = \frac{1}{2} a_1 e^{i\theta_1}
\]
\[
A_3 = \frac{1}{2} a_3 e^{i\theta_3}
\]

and separating real and imaginary parts of equations (4-18) and (4-19) yields:

The imaginary part of equations (4-18) and (4-19):
\[
a'_1 + \mu_1 a_1 = \frac{1}{2} \frac{k_1}{\omega_1} \sin(\sigma T_1 - \beta_1) \quad (4-20)
\]
\[
a'_3 + \mu_3 a_3 = 0 \quad (4-21)
\]

The real part of equation (4-18):
\[
a_1 \beta'_1 = \frac{3 \alpha_1}{8 \omega_1} a_1^3 + \frac{1}{4} \frac{\gamma_1}{\omega_1} a_1^2 a_3 - \frac{1}{2} \frac{k_1}{\omega_1} \cos(\sigma T_1 - \beta_1) \quad (4-22)
\]

By integrating equation (4-21) an expression of $a_3$ is found:
\[
a_3 = e^{-i\omega_3 t_0} \quad (4-23)
\]

Solving for the steady state solution ($a'_1 = \beta'_1 = 0$) and letting $\delta_1 = \sigma T_1 - \beta_1$ yields:
\[
\mu_1 a_1 = \frac{1}{2} \frac{k_1}{\omega_1} \sin \delta_1 \quad (4-24)
\]
\[
a_1 \sigma - \frac{3 \alpha_1}{8 \omega_1} a_1^3 = \frac{1}{4} \frac{\gamma_1}{\omega_1} a_1^2 a_3 + \frac{1}{2} \frac{k_1}{\omega_1} \cos \delta_1 \quad (4-25)
\]

For the steady solution $a_3 \to 0$ as $T_1 \to 0$. Thus, by simplifying equation (4-25), and squaring the above equations yields:
\[
\left[ \mu_1^2 + \left( \sigma - \frac{3 \alpha_1}{8 \omega_1} a_1^2 \right)^2 \right] a_1^2 = \frac{k_1^2}{4 \omega_1^2} \quad (4-30)
\]
It can be noted that this is exactly what was found with a one-mode solution for the primary resonance equation (3-26).

**Case 2.** $\Omega \equiv \omega_3$

Considering the case where $\Omega = \omega_3 + \delta \sigma$ and applying the same method as previously, yields the following equation:

$$\left[ \mu_3^2 + \left( \sigma - \frac{3}{8} \frac{a_3}{a_5} \right)^2 \right] a_3^2 = \frac{k_3^2}{4\omega_3^2}$$

(4-31)
Chapter V

Numerical Examples and Discussions

The Method of Multiple Scales was used in order to determine the response of a nonlinear forced vibration of an unsymmetrically laminated angle-ply composite beam and an unsymmetrically laminated cross-ply composite beam. The results that will be presented include the study of a simply-supported and a clamped-clamped beam with immovable edges. The beams considered are made out of graphite-epoxy, and have the following material properties:

Table 5-1. Material Properties of the Graphite-Epoxy Beams.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>Length</td>
<td>12 in</td>
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<td>1 in</td>
<td></td>
</tr>
<tr>
<td>$E_1$</td>
<td>18.566 psi</td>
<td></td>
</tr>
<tr>
<td>$E_2$</td>
<td>1.6e6 psi</td>
<td></td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>0.6566 psi</td>
<td></td>
</tr>
<tr>
<td>$v_{12}$</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>$P$</td>
<td>1.4234e-4 lb.sec^2/in^4</td>
<td></td>
</tr>
</tbody>
</table>

The thickness and the angle of orientation of the layers depend on the case considered, and will be given individually for each beam.
5.1. Validity of the results.

In order to verify the validity of the analysis in chapter IV, it was necessary to compare the results with previously published results. The first comparison was held with free vibration theories. Fig. 5-1 presents the superposition of different methods of calculation. In order to obtain free vibration with the Method of Multiple Scales (MMS), the forcing function and the damping coefficient were set to zero. Although the MMS gives a better approximation for the forced vibration than the free, yet, as it seen in Fig. 5-1, the comparison with the other well-known theories is good.

The regular perturbation method approximates the response with the following equation:

\[
\frac{\Omega}{\omega_0} = \sqrt{1 + \frac{3}{16}A^2} \tag{5-1}
\]

where

\(\Omega\) = Forcing frequency

\(\omega_0\) = Natural frequency of the beam

\(A\) = Amplitude / Radius of Gyration

For the MMS, the equation for an isotropic simply-supported beam is found by using equation (3-26) with the damping factor and the coupling stiffness matrix \(Bij\) being zero. The equation obtained is as follows:

\[
\frac{\Omega}{\omega_0} = 1 + \frac{3}{32}A^2 \tag{5-2}
\]

Furthermore, comparison was made with forced vibrations for an isotropic beam. Presented in Figure 5-2 is the superposition of the results from the Method of Harmonic Balance and the Method of Multiple Scales. It can be seen that there is a good agreement between the two methods.
5.2. Angle-ply composite beam.

All the figures presented in this section will apply to an unsymmetrically laminated angle-ply beam made out of 4 layers (thickness per layer is 0.012", thus \( h = 0.048 \) inches) of graphite-epoxy with the following angle of orientation \([30/-30/30/-30]\) or other when specified.

5.2.1. Primary resonance (single-mode solution).

Fig. 5-3 represents a 'typical' nonlinear response of a beam to harmonic excitation. The response is called the frequency response curve. It presents the variation of the maximum amplitude over thickness function of the frequency ratio. The main characteristic of this curve is its multivaluedness at certain frequency ratios. To justify this, an experiment could be set up starting the forcing frequency at a value lower than the natural frequency of the beam. As the forcing frequency is increased the amplitude is slowly increased until the peak amplitude is reached, then any small increase in the frequency will make the response curve drop to a much lower amplitude. From this point any increase in the frequency causes slowly decreasing amplitude. But the experiment could also be performed from a point beyond the natural frequency, for example point A (Figure 5-3). As the forcing frequency is decreased the amplitude is slowly increased until point B is reached. At this point the amplitude jumps to point C. As the frequency ratio is decreased further the amplitude will decrease slowly. The portion of the response curve from point B to the peak amplitude is unstable.

5.2.1.a. Simply-supported case.

Equation (3-26) includes three coefficients \( k, \mu, \) and \( \alpha \), values of which are functions of three parameters respectively \( F_0 \), the damping ratio \( \xi \) and the laminate angle of orientation \( \theta \). The variation of the three parameters are considered below:
The effect of three different values of $F_0 = 250, 350 \text{ and } 500 \text{ 1/sec}^2$, which corresponds to $P_0 = 6.44 \times 10^{-5}, 9.02 \times 10^{-5}, 1.29 \times 10^{-4} \text{ psi}$ respectively, is presented in Fig. 5-4. As expected, the larger the forcing amplitude gets, the larger the amplitude of the response will be. It is also noticeable that the peak amplitude occurs at higher frequency ratios as the amplitude of $F_0$ is increased.

The variation of the damping ratio is presented in Fig. 5-5. As the damping is increased the peak amplitude is decreased.

The increase of the angle of ply orientation of the composite beam increased the amplitude response as well as the frequency ratio where the maximum occurred. This is shown in Fig. 5-6, 5-7 and 5-8. For angles of 60 degrees or more, the peak amplitude of the response occurs at a very high frequency ratio. For 80 degrees the amplitude of oscillations jumps down to lower values only when the forcing frequency is nearly 8 times the natural frequency of the beam. This shows that the smaller the angle of orientation, the stiffer the composite beam. Another way of visualizing the effects of the change in angles can be viewed with Fig. 5-9 and 5-10. As the angle increases the value of the amplitude at resonance increases in a nonlinear way as well as the maximum amplitude.

The effects of a symmetrically versus an unsymmetrically laminated angle-ply composite beam was considered. The curves obtained are shown in Fig. 5-11 and it can be seen that for the symmetrical case the beam reaches a lower maximum amplitude at an earlier stage. This is due to the fact that there is no coupling between bending and extension.

5.2.1.1. Clamped-clamped case.

The response of a nonlinear forced vibration of a clamped-clamped beam is similar to the one of a simply-supported beam (Fig. 5-12). The value of the damping ratio had to be lowered in order to visualize correctly the Jump phenomena. It can be seen that the amplitude of the response is much smaller, and that the peak value is reached at lower frequency ratios when compared to the previous case. It can be therefore concluded that the nonlinearities have a more visible effect on a simply-supported beam then on a clamped-clamped one. The same conclusions can be drawn with the variation of the damping factor and the change in the angles of orientation of the layers.
5.2.2. Superharmonic case.

As mentioned earlier, the superharmonic case was only considered for the unsymmetrically laminated simply-supported angle-ply beam. In order to initiate the phenomena the forcing function applied to the beam was amplified compared to the primary resonance case. A lower value of the forcing function would create the division of the equation by small numbers. Thus, mathematically the problem would give a solution, but it would have no real physical meaning. Fig. 5-13 presents the results for $F_0 = 5,000 \text{ 1/sec}^2$ which corresponds to $P_0 = 2.46 \times 10^3 \text{ psi}$. The response is still nonlinear, as expected, but it can be noticed that although the forcing function is 10 times greater than that at primary resonance, the amplitudes reached are rather small. As the amplitude of the forcing function is increased, the curve starts bending a little more to higher frequency ratios and reaches a greater amplitude. The increase in the damping ratio tends to reduce the amplitude of the response as seen in Fig. 5-14. The comparison between a symmetrically laminated angle-ply beam and an unsymmetrical (Fig. 5-15) shows that the latter case undergoes more vibrations. The amplitude for the same loading is nearly 1.7 times greater when the orientation of the layers is unsymmetrical. This would tend to show that one way of reducing the risk of failure at the superharmonic resonance is to have a symmetrically laminated composite structure.

5.2.3. Subharmonic case.

The subharmonic resonance has only been recently studied in depth. Before, scientists never explored what could occur when the forcing frequency would reach values three times the one of the natural frequency. The study of this behavior became indispensable when they realized that this phenomena could cause serious damages to structures. Lefshetz (1956) described the destruction of a commercial airplane. The propellers had induced a subharmonic vibration of order 0.5 in the wings which in turn had induced a subharmonic of order 0.25 in the rudder, resulting in the breaking of the aircraft.
Equation (3-46), describing the response for the subharmonic case, will always give a mathematical solution to a problem, but in order for it to describe a real physical behavior, some extra conditions need to be satisfied. They are as follows (Reference [26]):

\[ A^2 < \frac{4\omega_0\sigma}{27\alpha} \]

and

\[ \frac{\alpha A^2}{\omega_0} \left( \sigma - \frac{63\alpha A^2}{8\omega_0} \right) - 2\mu^2 \geq 0 \]

Given these conditions, the response will actually describe the real behavior of the beam. For the figures presented in this section, the conditions implied:

- a value of \( F_0 = 2,500 \) 1/sec\(^2\) corresponding to \( P_0 = 6.44 \times 10^{-4} \) psi
- a damping coefficient of 0.001
- a value of epsilon of 0.1

Unlike the primary and the superharmonic resonances, the response at the subharmonic resonance gives two stable solutions. The amplitude of the response is always increasing as can be seen in Fig. 5-16. Although the amplitude will never be greater than the one reached by the primary resonance, it will never disappear unless failure occurs. In order to present a clear graph, the value of \( F_0 \) was set to 50,000 (or \( P_0 = 0.0128 \) psi), although the phenomena initiates itself at lower values. It appears on Fig. 5-17, that the response of a symmetrically laminated beam starts at a lower frequency ratio but reaches higher values of amplitude compared to the unsymmetrical case. The influence of the variation of the angle of orientation of the layers is not constant for all values of the forcing function. It was noted that for a high value of \( F_0 \) (around 50,000) the increase in the angle shifted the phenomena to a higher frequency ratio. Around 2,500 the effect of the increase in the angle was reversed, higher angles shifted the phenomena to lower frequency ratios.
5.2.4. Resonances.

Fig. 5-18 puts into perspective the different influences of the three resonances studied. It is obvious that the primary resonance is predominant, although the subharmonic case might cause failure. Once the primary resonance has occurred, the beam drops its amplitude and reestablishes its path of vibration along the subharmonic case. The input values were primarily chosen to initiate the subharmonic case. The response for the superharmonic case is not really detectable because of the very small value of the damping coefficient.

5.2.5. Two-mode solution for the primary resonance.

Adding a second term to the initial approximated solution did not have much effect on the response. Fig. 5-19 shows the response of the beam when the forcing frequency approaches the second-mode natural frequency. It can be clearly seen by referring to Fig. 5-3, that the amplitude of the response is of one order of magnitude less than for the first-mode response. It can be concluded that the second-mode has no real effects, and can be neglected in the analysis of the behavior.

5.3. Cross-ply composite beam (primary resonance).

All the cross-ply beams presented in this section are made out of 6 layers (thickness per layer is 0.01", thus the total thickness $h = 0.06$ in) of graphite-epoxy with the following angle of orientation $[0/90/0/90/0/90]$ or other when specified.

5.3.1. Simply-supported beam.

Fig. 5-20 presents the response of an unsymmetrically laminated cross-ply composite beam. The behavior is not much different from the angle-ply case. The major difference between the
governing equation of the angle-ply composite beam and the cross-ply beam when simply-supported is the addition for the later case of a quadratic nonlinear term. In many analyses, the quadratic term is left out. Fig 5-21 shows the impact of setting the quadratic term to zero. The difference is detectable when looking at the frequency ratio where the jump phenomena occurs.

Comparing Fig 5-20 and 5-22, the data foresee that the curves are identical, yet the angle of orientation of the layers is different. The difference between the two orientations resides only in the sign of the coefficient $B_{11}$ and not its value. Since $B_{11}$ is squared in the governing equation, the difference disappears.

As the forcing amplitude is amplified (Fig 5-23), the response curve reaches a higher amplitude and bends more to the higher frequency ratios. The effects of varying the damping coefficient (Fig 5-24) is identical to the angle-ply composite beam.

5.3.2. Clamped-clamped beam.

The response to a nonlinear forced vibration applied on an unsymmetrically laminated cross-ply composite beam is presented in Fig 5-25. The governing equation has in this case only a cubic nonlinearity. In fact, the general expression is identical to the one developed for a simply-supported angle-ply composite beam with of course different values of the coefficients. Therefore the results will be similar to the ones presented in the primary resonance paragraph for angle-ply.
Chapter VI

Conclusion


This thesis analytically investigates an unsymmetrically laminated composite beam subjected to harmonic excitation. The governing equation was derived based on the von Karman geometrical nonlinearity. The Galerkin method was then applied by means of transforming the nonlinear partial differential equation into an ordinary differential equation of the Duffing type. This equation was then solved with the Method of Multiple Scales. By applying this method, the response of the beam for all ranges of frequencies including primary resonance, subharmonic and superharmonic was clearly studied. The effects of a two-mode solution were also investigated.

The following conclusions can be drawn:

• the increase of the forcing amplitude and the decrease of the damping ratio increases the peak amplitude for both angle-ply and cross-ply unsymmetrical laminates.

• for an unsymmetrical angle-ply laminate, the increase of the angle of orientation increases the peak amplitude and the frequency ratio where it occurs. Thus, it can be concluded that the smaller the angle is, the stiffer the beam is.

• for an unsymmetrical cross-ply laminate, the Duffing equation includes a quadratic nonlinear term, absent in the angle-ply case. This extra term has a definite effect on the response of the beam. Neglecting that term simplifies the mathematics but also approximates the response a little too much.
• the addition of a mode in the approximated solution proved to have practically no effects on the response.

• the method of multiple scales was applied to study the subharmonic and superharmonic responses which are difficult to obtain with other methods, the harmonic balance method for example. Results for the subharmonic and superharmonic responses proved their importance and their significant effects at low damping ratios or high forcing amplitudes.

6.2. Future work.

This thesis cannot be considered, by any means, as completely covering the analysis of nonlinear vibrations of laminated composite beams. Further studies are definitely warranted in the areas of induced shear deformation, and experimental validation of the results to quote a few.
REFERENCES


Appendix A

Formulation of the Governing Equations of Motion

This appendix shows the derivation of the governing equations in terms of the resultant forces and moments presented in Chapter II, equations (2-12) through (2-14). A 3-D stressed element of a structure can be represented as follows:

Figure A-1. Stressed 3-D Element
In the x-direction

The equation of motion for the kth layer of the laminate in the x-direction is given by

$$\sum F_x = ma_x$$

The forces in the x-direction can be determined on Figure A-1. Thus, the above equation yields

$$\sum F_x = \sum \sigma_x dy dz \quad (A-1)$$

where $\rho_0^{(k)}$ is the mass density of the kth layer lamina. After simplification the equation reduces to:

$$\frac{\partial \sigma_x^{(k)}}{\partial x} + \frac{\partial \tau_{xy}^{(k)}}{\partial y} + \frac{\partial \tau_{xz}^{(k)}}{\partial z} = \rho_0^{(k)} \frac{\partial^2 u}{\partial t^2} \quad (A-2)$$

In the y-direction

Similarly, the equation of motion in terms of stresses in the y-direction for the kth layer is

$$\frac{\partial \tau_{xy}^{(k)}}{\partial x} + \frac{\partial \sigma_y^{(k)}}{\partial y} + \frac{\partial \tau_{yz}^{(k)}}{\partial z} = \rho_0^{(k)} \frac{\partial^2 v}{\partial t^2} \quad (A-3)$$

In the z-direction

The equation of motion in the z-direction is more complicated than the other two directions. It will be necessary to determine the resultant force due to all the stress components.

The resultant force acting in the z-direction due to the normal stress $\sigma_z$ alone is given by

$$\left(F_z\right)_z = \int \left[ \sigma_x + \frac{\partial \sigma_x^{(k)}}{\partial z} \right] dy dz \left[ \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right] - \left( \sigma_z dy dz \right) \frac{\partial w}{\partial x} \quad (A-4)$$

By neglecting high order terms, equation (A-4) can be rewritten as
\[(F_z)_{\sigma_x} = \frac{\partial}{\partial x} \left( \sigma_x \frac{\partial w}{\partial x} \right) dx dy dz \]  \hspace{1cm} (A-5)

Note: the superscript (k) has been temporarily dropped. Similarly, the resultant force acting in the z-direction due to the shear stress \(\tau_{xz}\) alone is

\[(F_z)_{\tau_{xz}} = \left[ \tau_{xz} + \frac{\partial \tau_{xz}}{\partial x} \right] dx dz \left[ \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right] dx dy dz \]  \hspace{1cm} (A-6)

By eliminating the higher order terms, the above equation yields

\[(F_z)_{\tau_{xz}} = \frac{\partial}{\partial x} \left( \tau_{xz} \frac{\partial w}{\partial x} \right) dx dy dz \]  \hspace{1cm} (A-7)

The resultant force acting in the z-direction due to \(\tau_{xy}\) alone

\[(F_z)_{\tau_{xy}} = \left[ \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} \right] dx dz \left[ \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right] dx dy dz \]  \hspace{1cm} (A-8)

Simplifying the above expression yields

\[(F_z)_{\tau_{xy}} = \frac{\partial}{\partial y} \left( \tau_{xy} \frac{\partial w}{\partial y} \right) dx dy dz \]  \hspace{1cm} (A-9)

Considering only the stress components acting in the y-direction, the resultant force acting in the z-direction due to \(\sigma_y\) alone is

\[(F_z)_{\sigma_y} = \frac{\partial}{\partial y} \left( \sigma_y \frac{\partial w}{\partial y} \right) dx dy dz \]  \hspace{1cm} (A-10)

The resultant force due to \(\tau_{xy}\) alone and \(\tau_{yz}\) alone gives the following two equations:

\[(F_z)_{\tau_{xy}} = \frac{\partial}{\partial x} \left( \tau_{xy} \frac{\partial w}{\partial y} \right) dx dy dz \]  \hspace{1cm} (A-11)
The resultant force acting in the z-direction due to the stress components \( \sigma_z \), \( \tau_{xz} \), and \( \tau_{yz} \) is given by

\[
(F_z)_{\sigma_z, \tau_{xz}, \tau_{yz}} = \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} \right) \, dx \, dy \, dz
\]  

Therefore, the equation of motion in the z-direction for the \( k \)th layer lamina of the laminated thin plate is given by

\[
\sum F_z = ma_z
\]

Equations (A-2), (A-3) and (A-14) are the equations of motions in terms of the stress components in the \( k \)th layer lamina of a laminate.

These equations can be expressed in terms of the resultant forces, transverse shear resultants, and resultant bending moments for a plate. Integrating equation (A-2) with respect to \( z \) yields

\[
\int_{-h/2}^{h/2} \frac{\partial \sigma_z^{(k)}}{\partial x} \, dx + \int_{-h/2}^{h/2} \frac{\partial \tau_{xz}^{(k)}}{\partial y} \, dy + \int_{-h/2}^{h/2} \frac{\partial \tau_{yz}^{(k)}}{\partial z} \, dz = \int_{-h/2}^{h/2} \rho_o \frac{\partial^2 u}{\partial t^2} \, dz
\]

The definition of the resultant force matrix is given by

\[
[N] = \begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix}
\sigma_x^{(k)} \\
\sigma_y^{(k)} \\
\tau_{xy}^{(k)}
\end{bmatrix} \, dz
\]

Interchanging the order of differentiation and integration, neglecting rotatory inertia terms, and using (A-16) yields a new expression of equation (A-15):
where \( \rho \) stands for the average mass density of the laminate, and is defined as

\[
\rho = \int_{-h/2}^{h/2} \rho_0^{(k)} \, dz
\]

(A-18)

Similarly, integrating equation (A-3) yields the following equation:

\[
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \rho \frac{\partial^2 v_0}{\partial t^2}
\]

(A-19)

Finally, integrating equation (A-14) with respect to \( z \) yields

\[
\begin{align*}
N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial w}{\partial x} \left( \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) + P = \rho \frac{\partial^2 w}{\partial t^2}
\end{align*}
\]

(A-20)

where

\[
P = \sigma^{(k)} \bigg|_{-h/2}^{h/2}
\]

is the distributed transverse loading or in general, it is the sum of the external forces, and \( u_0, v_0, w \) are the displacements of the middle surface.

In most engineering applications of thin plates the inplane load inertia effects \( \frac{\partial^2 u_0}{\partial t^2}, \frac{\partial^2 v_0}{\partial t^2} \) can be neglected and the motion of the plates are predominantly transverse. Under this condition, equations (A-17), (A-19) and (A-20) become

\[
\begin{align*}
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0 \\
N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial w}{\partial x} \left( \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) + P &= \rho \frac{\partial^2 w}{\partial t^2}
\end{align*}
\]

(A-21, A-22, A-23)

where

\[
\begin{bmatrix}
Q_x \\
Q_y
\end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix}
\tau^{(k)}_{xz} \\
\tau^{(k)}_{yz}
\end{bmatrix} \, dz
\]
By definition, the resultant moment can be written in a matrix form as follows

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix}
\sigma_x^{(k)} \\
\sigma_y^{(k)} \\
\tau_{xy}^{(k)}
\end{bmatrix} dz
\]  

(A-24)

Multiplying both sides of equation (A-2) by \( z \) and integrating the resulting equation with respect to \( z \) over the thickness of the plate and using equation (A-24) yields

\[
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \int_{-h/2}^{h/2} \frac{\partial \sigma_x^{(k)}}{\partial z} z dz = \int_{-h/2}^{h/2} \rho_0 \frac{\partial^2 u}{\partial t^2} z dz
\]  

(A-25)

If surface forces in the \( x \)-direction are not considered, equation (A-25) becomes:

\[
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0
\]  

(A-26)

Similar operations with equation (A-3) yields

\[
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0
\]  

(A-27)

Differentiating equations (A-26) and (A-27) with respect to \( x \) and \( y \) respectively yields

\[
\frac{\partial Q_x}{\partial x} = \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y}
\]  

(A-28)

\[
\frac{\partial Q_y}{\partial y} = \frac{\partial^2 M_{xy}}{\partial x^2} + \frac{\partial^2 M_y}{\partial x \partial y}
\]  

(A-29)

Finally, substituting these equations into equation (A-23) yields

\[
\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + P = \rho \frac{\partial^2 w}{\partial t^2}
\]  

(A-30)

Equations (A-21), (A-22), and (A-30) constitute the equations of motion for laminated thin plates, and they have the same form of those of classical homogeneous, and isotropic plate theory.
Appendix B

Figures of Chapter V
Figure 5.1 Comparison of Published Results for Free Vibration.
Figure 5-2. Comparison of Method of Multiple Scales and Harmonic Balance for Forced Vibrations.
Figure 5-3. Nonlinear Response of a Beam to Harmonic Excitation.
Figure 5-4. Variation of the Forcing Amplitude for [30/-30/30/-30] and a Damping Ratio of 0.01.
Figure 5-5. Variation of the Damping Ratio for [30/-30/30/-30] and $F_0 = 500$. 

- Damping = 0.01
+ Damping = 0.02

Amplitude $A$ 
Thickness $h$
Figure 5-6. Variation of the Angle of Orientation of the Laminate for a Damping Ratio of 0.01 and $F_0 = 500$. 

\[ \frac{\text{Amplitude}}{\text{Thickness}} = \frac{A}{h} \]
Figure 5-7. Variation of the Angle of Orientation of the Laminate for a Damping Ratio of 0.01 and $F_n = 500$. 

$\frac{Amplitude}{Thickness} = \frac{A}{h}$

$\frac{Frequency Ratio}{\Omega}{\omega_n}$

- [40/-40/40/-40]
- [50/-50/50/-50]
- [60/-60/60/-60]
Figure 5-8. Variation of the Angle of Orientation of the Laminate for a Damping Ratio of 0.01 and $F_0 = 500$.
Amplitude $A$  
Thickness $h$

\[
\frac{\text{Amplitude}}{\text{Thickness}} = \frac{A}{h}
\]

Figure 5-9. Angle of Orientation versus the Peak Amplitude.
Figure 5-10. Angle of Orientation versus the Frequency Ratio where the Peak Amplitude Occurs.
Figure 5-11. Symmetrical versus Unsymmetrical for a Damping Ratio of 0.01 and $F_0 = 500$. 

- $[45/-45/-45/45]$ Symmetrical
- $[45/-45/45/-45]$ Unsymmetrical
Figure 5-12. Nonlinear Response of a Clamped-Clamped Beam for a Damping Ratio of 0.005 and $F_0 = 500$. 

\[
\frac{\text{Amplitude}}{\text{Thickness}} = \frac{A}{h}
\]
Figure 5-13. Superharmonic Resonance for [30/-30/30/-30] with $F_0 = 5,000$ and a Damping Ratio of 0.01.
Figure 5-14. Variation of the Damping Ratio for [30/-30/30/-30] with $F_0 = 10,000$. 

Equation: \[
\frac{\text{Amplitude}}{\text{Thickness}} = \frac{A}{h}
\] 

- Damping = 0.01
- Damping = 0.02
Figure 5-15. Symmetrical versus Unsymmetrical with $F_0 = 10,000$ and a Damping Ratio of 0.01.
Figure 5-16. Subharmonic Resonance for [30/-30/30/-30J with $f_0 = 50,000$ and a Damping Ratio of 0.01.
Figure 5-17. Subharmonic Resonance: Symmetrical versus Unsymmetrical with $F_0 = 50,000$ and a Damping Ratio of 0.01.
Figure 5-18. Primary, Superharmonic and Subharmonic Resonances.
Figure 5-19. Two-mode Solution Response around $\omega_1$ for $F_n = 500$ and a Damping Factor of 0.01.
Figure 5-20. Nonlinear Response to Harmonic Excitation (Cross-Ply) with $F_0 = 500$ and a Damping Factor of 0.01.
Figure 5-21. Effects of the Quadratic Nonlinearity.

Quadratic term present
- Quadratic term set to zero
Figure 5-22. Nonlinear Response to Harmonic Excitation [90/0/90/0/90/0].
Figure 5-23. Variation of the Forcing Amplitude for [0/90/0/90/0/90] and Damping Ratio of 0.01.
Figure 5-24. Variation of the Damping Ratio for [0/90/0/90/0/90/0] with $F_0 = 500$. 

$Amplitude \over Thickness = \frac{A}{h}$

$Frequency Ratio \frac{\Omega}{\Omega_0}$
Figure 5-25. Nonlinear Response to Harmonic Excitation for a Clamped-Clamped Beam with $F_0 = 500$ and Damping Ratio of 0.01.