Aeroelastic Analysis of a Flexible Membrane Wing

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Aeroelastic Analysis of a Flexible Membrane Wing

by

Isaac Wanjohi

A Thesis Submitted to the Graduate Studies Office
in Partial Fulfillment of the Requirements for the Degree of
Master of Science in Aerospace Engineering

Embry-Riddle Aeronautical University
Daytona Beach, Florida
Summer 2004
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Isaac Nguri Wanjohi

This thesis was prepared under the direction of the candidate's thesis committee chairman, Dr. Jim Ladesic, Department of Aerospace Engineering, and has been approved by the members of his thesis committee. It was submitted to the Department of Aerospace Engineering and was accepted in partial fulfillment of the requirements for the degree of Master of Science in Aerospace Engineering.

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Abstract

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The focus of this research proposal is the investigation of the aeroelastic effects of a flexible lift augmentation system (LAS) wing. This research involves characterization of the forced vibration response of a wing appendage used to augment short field take off and landing (STOL) operations.

Although flutter theory is now well understood, the LAS presents the complications of a highly deformable airfoil shape as well as larger structural damping values compared to metal wings.

The proposed research will involve derivation of the equations of motion aided by experimental data from nodal excitation of the wing; stiffness and rigidity modeling from static wing loading and collection of flight test data to characterize the potential flow around the membrane wing.

The knowledge gained from this research will be of critical importance in assuring the safety of flight by identifying the critical flutter speeds, as well as establishing a good basis for the structural design of future lift augmentation structures.
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Chapter 1
Introduction

1.1 Problem Statement

Aeroelasticity is often defined as a science which studies the mutual interaction between aerodynamic forces and elastic forces and the influence of this interaction on airplane design. Since the beginning of powered flight, aeroelastic problems have played a huge role in design of aircraft; a classic example being the trend of early designers towards biplane wings after the success of the Wright brother's biplane over Langley's monoplane wing, which was destroyed by torsional divergence.

The lift augmentation system (LAS) wing is a flexible double surface wing commonly used for ultra-light aircraft. The wing is attached to a Cessna 337 and is attached at the vertex of a quad pod structure stemming from the four spar cap fittings on top of the fuselage. The main purpose of this modification to the airplane is to investigate the improvements in take-off and landing performance. The final design of the wing is intended to be retractable into a pod during the cruise mission of the airplane and deployed only during take off and landing. Amongst the growing concerns during initial testing of an initial design LAS wing, are the intense oscillations of the wing tips during fast taxi runs.

The LAS wing is not designed for speeds exceeding 86 knots and therefore the aeroelastic characteristics above these speeds are unknown. Wing interference and prop wash effects of the host aircraft may also play a vital role in changing the normal aeroelastic interaction of aerodynamic and inertial forces of the LAS wing.
A good understanding of the flight characteristics of the LAS wing as well as the aerodynamic stability of the wing will be necessary in ascertaining the safety and design of future systems.

1.2 Objective of the Research

The main focus of this thesis is to determine the critical flutter speed for the LAS wing and to quantify the different aerodynamic as well as structural parameters influencing the wing in flight so as to determine the frequencies and critical speeds during different flight regimes. The analytical method used in the analysis will require the identification of the type of aeroelastic phenomenon to be expected.

The wing is then modeled as a cantilever structure with a mid span spring support as shown in Figure 1.1. The mechanical stiffness and inertial mass data are obtained both experimentally and analytically while the aerodynamic data are defined purely from experimental flight test data.

The equations of motion will then be derived using from the LAS wing mass, stiffness, and aerodynamic data. A Rayleigh type analysis method proposed by Carson Yates Jr. [6] will be used to identify the flutter speeds and frequencies. These parameters will then be compared to the observed wing vibrations.
The aeroelastic model that is generated from this analysis is then going to be used as a platform for future designs. This will be effective in estimating the effects of increasing or decreasing structural stiffness, moving of the elastic axis or mass distribution to various locations of the wing. A direct relation can then be established to clearly identify the onset of flutter and specific recommendations for this wing may help increase the flutter speed to a safer limit.

1.3 LAS wing Geometry and Material

The LAS wing is a typical trike wing as shown in Figure 1.1 and is manufactured by Air Creation*. The wing is composed of a network of cables and aluminum tubes covered by a polyester woven fabric and Mylar film laminate. Figure 1.2 shows the leading edge tubes pinned together at the keel, and constrained in this position by the cross tubes stretching from the middle of the keel. This main frame is kept taut by the fabric wing cover, the cables from the top of the kingpost as well as the sides of the control bar. The shear stress and normal force are carried by battens (tubes) that run chord wise in pockets inside the fabric wing. These battens also add a nonlinear feature to the chord wise stiffness.

*Air Creation - Aerodrome de Lanas 07200 AUBENAS – France
Tel: 33 (0)4 75 93 66 66 - Fax: 33 (0)4 75 35 04 03 - E-mail: info@aircreation.fr
The LAS wing has a plan form area of 161.5 ft\(^2\). This is approximately 80% of the host aircraft wing area. The span is 32.8 ft long, aspect ratio is 6.66 and sweep angle of 30\(^0\). The root chord of the LAS wing is 8.23 ft long and has a taper ratio of 0.13.

Most of the tubing is made from high tensile strength aluminum 7075, the cables and fittings are made from stainless steel while the wing fabric is made from trilam and polyester. The total weight of the structure is 117 lbs. Figure 1.3 shows the interior structure of the LAS wing.
The flexible wing is mounted on top of a Cessna 337 by means of a quad pod structure, see Figure 1.3; the quad pod is firmly attached to the four spar cap fittings of the airplane and rises 5 feet into a vertex for single bolt attachment of the LAS wing. The Cessna 337 is a light utility airplane of a push-pull engine configuration, with a wing area of 202 square feet and 38 foot span.

Figure 1.3 LAS wing on Aircraft during a short take-off run
Chapter 2
Background

2.1 Aeroelastic analysis Applications

When flying in an airplane, many of us with a window seat can see the wing flexing up and down. Many people are disconcerted by this elastic motion of the wing, and some even suspect it may very well break off. A lot of design effort goes into reducing the probability of that happening and this forms the basis of aeroelastic analysis.

Aeroelastic wing problems appeared when designers abandoned bi-plane construction with its relatively high torsional rigidity, in favor of monoplane types. The latter often had insufficient torsional rigidity, resulting in loss of aileron effectiveness and deformation effects on load distribution.

Flutter is seen to be an aeroelastic, self excited vibration, in which the external source of energy is the air stream. The classical type of flutter is associated with clean potential flow, and aerodynamic forces which mostly (though not necessarily) involve coupling of several degrees of freedom of the structure. The non-classical type of flutter, which is more difficult to analyze on a purely theoretical basis, may involve flow separation, stalling conditions, shock, and various types of time lag effects between the flow pattern and the motion; examples are stall flutter and aileron buzz.
Airplane dynamics, which was previously considered a distant relative of aeroelasticity, has now strengthened its ties to flutter and other aeroelastic phenomenon. At present flutter analysis is gaining a lot of prominence in the design of control systems, the writing of computational fluid dynamic (CFD) methods for unsteady aerodynamics, and the design of composite materials that are stiff enough to increase the flutter envelope.

Transonic flight is also a very troublesome area for aeroelastic analysis due to the nonlinearity of the flow from the shock waves. Transonic as well as supersonic flight has become a daily occurrence, and therefore aeroelastic analysis is becoming a fundamental process in the design process of high speed aircraft.

Despite the higher emphasis in aerodynamic applications, civil engineering has also had its share of disasters from aeroelastic instabilities; for instance the Tacoma Narrows Bridge. Aeroelasticity therefore does play a role in other parts of the industry for structures which are constantly experiencing fluid forces and oscillations.

### 2.2 Previous Research

Initial flutter analysis was started in 1916 by Lanchester et al. in connection with the asymmetrical flutter of a Handley Page bomber. A rapid increase into the development of the nature of flutter took place in the following two decades with the establishment of the nonstationary airfoil theories. Various scientists such as Ackermann, Wagner, Glauert published numerical results obtained for certain reduced frequencies. In 1934, Theodorsen's exact solution of a harmonically oscillating wing with a flap was published and the range of reduced frequencies was unlimited. Up to this time the few cases of flutter were due only to the wing. In the later part of the 1930's different types of aircraft were manufactured owing to the arms build up associated with World War II and it was during this time that numerous cases of flutter occurred, not only with wings, but also with tail surfaces mainly due to increased velocities.
Simple rules of flutter prevention from statistical data were provided by Küssner and Roxbee Cox; many methods of analysis were discussed and details of aerodynamic forces for control surfaces were published. The two dimensional problem of airfoil flutter with two degrees freedom no longer involved any difficulty. Two dimensional problems with three degrees of freedom—airfoils with flaps—were treated satisfactorily. Flutter analysis started becoming a more specialized field and emphasis was put on theoretical research. Wind-tunnel tests in this period indicated that, aerodynamically, the ‘strip theory’ gives reasonable accuracy for calculating the critical speed, at least in the incompressible range for wings of moderate aspect ratio [1]

2.3 Current Research Efforts

At the end of World War II, airplane speeds increased toward the transonic and supersonic speeds. Complex wing shapes and mass loadings, for instance forward swept or delta wings have also added to the greater demand for faster and more reliable methods of calculating the flutter susceptibility of the aircraft.

Some of the current techniques being employed are, the “moving block and randomdec” applications which allow the testing of subcritical damping and frequencies of in-flight vehicles [7]; transient excitation and data processing techniques employing the fast Fourier transforms; and numerous wind tunnel investigations of transonic and supersonic speeds of various aircraft.

Most of the CFD codes emerging are also incorporating unsteady aerodynamic effects of structures. Control systems which act as active flutter inhibitors are also being widely used especially in supersonic aircraft to damp out flutter vibrations in a wide range of flight regimes.
Chapter 3

Theory

This chapter is divided into 4 sections. The first section outlines the assumptions that were made and the limitations of the different methods used in the theoretical analysis. Section 3.2 derives the equations of motion of the wing system. The important parameters needed to characterize this motion are obtained from experimental data as well as theoretical modeling. Section 3.3 uses these equations of motion and shows the formulation of the flutter speed and frequencies. The last section compares the results obtained and shows the effect of varying different parameters.

3.1 Limitations and Assumptions

The formulation of the flutter equations does include some assumptions which will be outlined;

1. The assumption of small disturbances is used. This leads to a linearization of the equations of motion for the system. Very little is known about the nonlinear case [1]; however experimental evidence shows that the linearized theory of flutter represents closely the real situation.

2. Potential flow is used for the aerodynamic terms, and thus does not account for flutter due to elements such as flow separation, stalling conditions, shock, and various types of time lag effects between the flow pattern and the motion [3].

3. The structural behavior is assumed to be such that the elastic axis may be considered straight.
4. Only one half of the wing is evaluated. The material characteristics and motions are considered symmetrical.

3.2 Formulation of the Equations of Motion

a. Influence Coefficients of the Wing.
Influence coefficients are based on the principle that for adiabatic systems the work done by a conservative force is independent of path and can be expressed as the difference in potential energy, \( V \) between the initial position and the final position of the system. Figure 3.1 shows the systematic addition of forces to determine the influence coefficients.

\[
V_{0 \rightarrow 1} = \frac{1}{2} f_{11} x_1
\]

\[
V_{1 \rightarrow 2} = f_{21} x_2 + \frac{1}{2} f_{22} x_2
\]

Figure 3.1 Systematic loading of a beam from (a) to (d)
The total potential energy can be expressed as

\[ V = \frac{1}{2} f_{11} x_1 + \frac{1}{2} f_{22} x_2 + f_{31} x_3 + f_{32} x_3 + \frac{1}{2} f_{33} x_3 \]  

(3.4)

In summary the influence coefficient method for determining the elements of an n-degree of freedom system is as follows:

1. Assign a unit displacement of \( x_1 \) maintaining \( x_2, x_3, \ldots, x_n \) in their static equilibrium position. Calculate the system of forces required to maintain this as an equilibrium position.
2. Continue this procedure to find all columns of the stiffness matrix.
3. Reciprocity implies the stiffness matrix must be symmetric.

**b. Mass modeling of the wing**

Figure 3.2 shows a Catia model of the interior structure of the LAS wing.
The wing was divided into 12 sections which are 15 inches apart as shown by the white vertical planes. The mass and inertia properties of the different sections were computed with the aid of CATIA software after specifying the material properties of the different wing components. The fabric of the wing was included separately by multiplying the area of each section by the mass per area density of the composite fabric (a trilam cloth).

c. **Free transverse vibrations of the wing**

Consider the LAS wing modeled as a beam with a motor exciter at the tip as shown;

![Figure 3.3 Pendulum and Motor on wing semi-span](image1)

![Figure 3.4 Pendulum and Motor forces](image2)

The rotation of the motor induces motion in the system along the plane of the rotation. However in the present analysis, the motion shall be limited to the vertical direction only even though the rotating pendulum produces a horizontal component of
force. As an added convenience the time origin is taken as $t = 0$ so that the unbalanced force applied to the system is $m_p g + m_p \omega^2 r (\sin \Omega t)$

Figure 3.5 Model of complete system

Let $x$ be the displacement of the non rotating mass from the static equilibrium position. Considering the free undamped vibration of the system the equation of motion can be written as,

$$[M] \{x_n\} + [K] \{x_n\} = \{0\}$$

(3.5)

Where

- $M_w$ is Mass of wing and motor
- $K$ is the wing stiffness matrix

Assume a solution of the form

$$x_n = x_n e^{i\omega t}$$

Where $n$ represents the number of nodes in the system. Equation (3.5) becomes,
\[-\omega^2[M] + [k]\{x\}e^{\text{out}} = 0 \quad (3.6)\]

The non-trivial solution of equation (3.6) exists when,

\[\det\left[k - \omega^2[M]\right] = 0 \quad (3.7)\]

A polynomial equation of the \(n \times n\) matrix is obtained and it is a function of the natural frequencies, \(\omega_n^2\). The LAS wing is modeled as a 4 lumped mass system and it will therefore have a 4 by 4 matrix in the equation of motion and subsequently 4 natural frequencies.

It should also be noted that these frequencies are the coupled mass frequencies in bending deflection. They are considered uncoupled when comparing them between the bending and torsional frequencies.

d. Forced Undamped System

The equation of motion can be written as;

\[
[M_w + m_m]\{x_n\} + [k]\{x_n\} = \{F(t)\} \quad (3.8)
\]

Where,

- \(m_w\) = Mass of wing
- \(m_p\) = Mass of pendulum
- \(m_m\) = Mass of motor
- \(n\) = number of degrees of freedom

\(F(t)\) is the sinusoidal force input of the motor,

\[
\rightarrow \quad F(t) = m_p g + m_p \omega^2 r \sin(\Omega t) \quad \text{assuming phase angle is zero} \quad (3.9)
\]
The full expression can thus be written as,

\[
[M_w + m_m] \{x_n\} + [k]\{x_n\} = \{m_p g + m_p \omega^2 r(\sin \Omega t)\} \tag{3.10}
\]

\[
[M]\{x_n\} + [k]\{x_n\} = \{F(t)\} \quad \rightarrow M = M_w + m_m + m_p \tag{3.11}
\]

Assume solution of the form

\[
\{x_n\} = \{x\} e^{i\omega t}
\]

and

\[
\{F(t)\} = \{F\} e^{i\omega t}
\]

where

\( \bar{x} \) and \( \bar{F} \) are constants to be determined when the natural frequencies are calculated.

Substituting into equation (3.11)

\[
[-\omega^2 [M] + [k]\{x\} e^{i\omega t} = \{F\} e^{i\omega t} \tag{3.12}
\]

\[
\rightarrow [-\omega^2 [M] + [k]\{x\} = \{F\} \tag{3.13}
\]

The LAS wing is modeled as a lumped mass of 4 sections and equation 3.13 becomes;

\[
\begin{bmatrix}
-m_{11} & -m_{12} & -m_{13} & -m_{14} \\
-m_{21} & -m_{22} & -m_{23} & -m_{24} \\
-m_{31} & -m_{32} & -m_{33} & -m_{34} \\
-m_{41} & -m_{42} & -m_{43} & -m_{44}
\end{bmatrix}
\begin{bmatrix}
-k_{11} & -k_{12} & -k_{13} & -k_{14} \\
-k_{21} & -k_{22} & -k_{23} & -k_{24} \\
-k_{31} & -k_{32} & -k_{33} & -k_{34} \\
-k_{41} & -k_{42} & -k_{43} & -k_{44}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
-F_1 \\
0 \\
0 \\
0
\end{bmatrix} \tag{3.14}
\]

This can then be expanded to,
The equation is then solved for \( \{x\} \) to give the nodal positions of the wing as a function of frequency.

\[
\begin{bmatrix}
  -\omega^2 m_{11} + k_{11} & -\omega^2 m_{11} + k_{11} & -\omega^2 m_{11} + k_{11} & -\omega^2 m_{11} + k_{11} \\
  -\omega^2 m_{21} + k_{21} & -\omega^2 m_{22} + k_{22} & -\omega^2 m_{23} + k_{23} & -\omega^2 m_{24} + k_{24} \\
  -\omega^2 m_{31} + k_{31} & -\omega^2 m_{32} + k_{32} & -\omega^2 m_{33} + k_{33} & -\omega^2 m_{34} + k_{34} \\
  -\omega^2 m_{41} + k_{41} & -\omega^2 m_{42} + k_{42} & -\omega^2 m_{43} + k_{43} & -\omega^2 m_{44} + k_{44}
\end{bmatrix}
\begin{bmatrix}
  -x_1 \\
  -x_2 \\
  -x_3 \\
  -x_4
\end{bmatrix}
= \begin{bmatrix}
  \tilde{F}_1 \\
  0 \\
  0 \\
  0
\end{bmatrix}
\tag{3.15}
\]

\[
\begin{bmatrix}
  -x_1 \\
  -x_2 \\
  -x_3 \\
  -x_4
\end{bmatrix}
= \begin{bmatrix}
  -\omega^2 m_{11} + k_{11} & -\omega^2 m_{11} + k_{11} & -\omega^2 m_{11} + k_{11} & -\omega^2 m_{11} + k_{11} \\
  -\omega^2 m_{21} + k_{21} & -\omega^2 m_{22} + k_{22} & -\omega^2 m_{23} + k_{23} & -\omega^2 m_{24} + k_{24} \\
  -\omega^2 m_{31} + k_{31} & -\omega^2 m_{32} + k_{32} & -\omega^2 m_{33} + k_{33} & -\omega^2 m_{34} + k_{34} \\
  -\omega^2 m_{41} + k_{41} & -\omega^2 m_{42} + k_{42} & -\omega^2 m_{43} + k_{43} & -\omega^2 m_{44} + k_{44}
\end{bmatrix}^{-1}
\begin{bmatrix}
  \tilde{F}_1 \\
  0 \\
  0 \\
  0
\end{bmatrix}
\tag{3.16}
\]

### 3.3 Flutter Solutions using Rayleigh-Ritz Method

Recent trends towards high speed wings with more efficient and lighter structures have made flutter considerations more critical in the design of the modern airplane [3]. It is generally recognized that in a wing with large sweepback the interaction between bending and twisting can greatly affect the modes and frequencies of vibration. Local variations in weight and mass moment of inertia, as is the case in the LAS wing, make the choice of two-dimensional parameters very difficult. It is for this reason that the analytical solution of the wing's flutter will involve the three dimensional parameters.
Steps in three dimensional flutter;

i. Make a suitable choice of generalized coordinates

ii. Formulate the kinetic energy

iii. Formulate the strain energy

iv. Formulate the aerodynamic terms

v. Derive the equations of motion and solve for the flutter determinant

The airfoil section below shows the geometrical points that are used for reference.

*Figure 3.6 Airfoil section view with notations*

*Figure 3.7 Notations used for a swept cantilever wing*
Where,

\( b \) semi chord of wing measured perpendicular to elastic axis

\( a_h b \) nondimensional distance between elastic axis and mid-chord and is positive if the elastic axis is aft of mid-chord.

\( x_a b \) nondimensional distance from elastic axis to local center of gravity measured perpendicular to elastic axis and is positive if the center of gravity is aft of the elastic axis

\( h \) local vertical translational displacement of wing at elastic axis

\( \alpha \) angle of attack

\( \Lambda \) Sweep angle

\( x, y \) airplane axes of pitch and roll

\( x_{\text{bar}}, y_{\text{bar}} \) Elastic coordinate axes

Other terms that will be constantly used are;

\( m \) mass per unit span

\( S_y \) static mass moment per unit span about elastic axis and is positive when the center of gravity is aft.

\( I_y \) mass moment of inertia per unit span about the elastic

**a. Make a Suitable Choice of Generalized Coordinates**

Generalized coordinates are any set of independent coordinates equal in number to the degrees of freedom of the system. For example, the coordinates of a spherical pendulum can be represented by two independent angles. Hence the two angles are generalized coordinates, and the spherical pendulum represents a system of two degrees of freedom.
The same pendulum can also be described by the three rectangular coordinates, \( x, y, z \), which exceed the degrees of freedom. The coordinates \( x, y, z \) are also not independent because they are related by the length of the pendulum, that is:

\[
x^2 + y^2 + z^2 = l^2
\]

Generalized coordinates are used in representing the deformation of a structure and is equivalent to imposing certain constraints on the elastic body. The structure can thus be described by dynamic modes of oscillation. This simplified model can yield sufficiently accurate results provided that the semi rigid modes were properly chosen. The following notation shall be used to represent the modes with respect to time and deformations.

\[
h(y, t) = \bar{h}(t)f(y)
\]

\[
\alpha(y, t) = \bar{\alpha}(t)\phi(y)
\]

where

- \( y \) is a position along the wing semispan
- \( \bar{h}(t) \) and \( \bar{\alpha}(t) \) are unknown functions of time
- and \( f(y) \) and \( \phi(y) \) are assumed functions of \( y \)

There are some principles that have proved valuable in the past for selecting the motions which will probably make major contributions to a given type of flutter mode [2]

i. Almost every airplane has a central plane of symmetry. It follows that all structural and rigid-body oscillations can be separated with respect to this plane.

ii. The second principle is that the degrees of freedom which compose a flutter mode usually couple strongly with one another; the coefficients of the various generalized coordinates are of the same order of magnitude in all equations of motion.

iii. A degree of freedom with a large frequency compared with the expected flutter frequency doesn’t have to be considered; It is known that the wing
is capable of oscillations in many different modes when immersed in a
flow of a given speed and density. At, or near, the critical flutter speed, the
other oscillation modes, if accidentally excited, are relatively heavily
damped and quickly die out. Thus, if the assumed semi rigid mode closely
approximates the actual displacements which occur in flutter at the critical
speed, little error will result from neglecting other possible modes.

As a general guide, normal modes should be employed whenever the structure or mass
distribution is very new that there is no basis of previous experience in using a simpler
technique [2]. The shape functions can be roughly determined if one does not have
complete information of the system.

For instance, a straight uniform cantilever beam can be closely approximated by the
assumed mode

\[
\begin{align*}
  f(y) &= \left(\frac{y}{l}\right)^2 \\
  \phi(y) &= \left(\frac{y}{l}\right)
\end{align*}
\]

or

\[
\begin{align*}
  f(y) &= \cosh ky - \cos ky - 0.734(\sinh ky - \sin ky) \quad \text{Where \(k\) is a constant} \\
  \phi(y) &= \sin \frac{ny}{2l}
\end{align*}
\]

All which satisfy the conditions for a cantilever beam:

\[
\begin{align*}
  f(0) &= 0 \quad \phi(0) = 0 \\
  \frac{\partial^2}{\partial y^2} f(0) &= 0 \quad \frac{\partial^2}{\partial y^2} \phi(0) = 0
\end{align*}
\]

These modes are shown graphically below for unit span and assumed values of deflection
Common Modes Shapes for a Cantilever Beam in Bending

\[ f(y) = \cosh(ky) - \cos(ky) - 0.734(\sinh(ky) - \sin(ky)) \]

\[ f(y) = \left(\frac{y}{L}\right)^2 \]

\[ f(y) = \cosh(y) - \cos(y) - 0.734(\sinh(y) - \sin(y)) \]

Figure 3.8 Common Mode Shapes for a Cantilever Beam in Bending

Common Modes Shapes for a Cantilever Beam in Torsion

\[ f(y) = \left(\frac{y}{L}\right) \]

\[ f(y) = \left(\frac{y}{2L}\right) \]

Figure 3.9 Common Mode Shapes for a Cantilever Beam in Torsion
In the case of the LAS wing, the structure and mass distribution of the wing is known, thus the shape functions can be defined. The system has also been subjected to dynamic tests which give a second check for these functions.

b. **Formulate the Kinetic Energy**

Consider the wing section in side view as shown in Figure 3.10. This section is free to translate and rotate about the local and elastic axis. Then a simplified model of the motion can be used as depicted in Figure 3.11;

\[ K'(y) = \frac{1}{2} m(y)v_{cg}^2(y) + \frac{1}{2} I_\alpha(y)\omega^2(y) \]  \hspace{1cm} (3.19)
from kinematics

\[ h_{cg} = h + r \theta \]

\[ \therefore v_{cg} = \dot{h}_{cg} \Rightarrow v_{cg} = \dot{h} + r \dot{\alpha} \]

\[ a_{cg} = \ddot{h}_{cg} \Rightarrow a_{cg} = \ddot{h} + r \ddot{\alpha} \]

Where

- \( h_{cg} \) is vertical height of the system center of gravity
- \( v_{cg} \) is vertical velocity of the center of gravity
- \( I_{ia} \) is the mass moment of inertia about the elastic axis
- \( K_{a} \) and \( K_{h} \) are the torsion and translation spring stiffness respectively

The kinetic equation thus becomes

\[ K'(y) = \frac{1}{2} m(y) \dot{h}^2(y) + \frac{1}{2} I_{ia}(y) \dot{\alpha}^2(y) + S_{a}(y) \dot{h}(y) \dot{\alpha}(y) \]

where \( S_{a} = \frac{1}{2} m r \) and \( I_{ia} = \frac{1}{2} m r^2 \) \hspace{1cm} (3.20)

Substituting equations (3.20) above and integrating with respect to \( y \) over the span

\[ K = \frac{1}{2} mh^2 + \frac{1}{2} I_{ia} \alpha^2 + S_{a} \dot{h} \alpha \]

where

- \( \bar{m} = \int_{0}^{l} m(y) \phi^2(y) dy \) generalized mass
- \( \bar{I}_{a} = \int_{0}^{l} I_{a}(y) \phi^2(y) dy \) generalized mass moment of inertia
- \( \bar{S}_{a} = \int_{0}^{l} S_{a}(y) \phi(y) dy \) generalized static mass

\[ f(y) = 0.0972 \times (\sin(0.0127y) - \sinh(0.0079787y)) \]

\[ f(y) = \frac{x}{l} \]
c. **Formulate the Strain Energy**

In general

\[ V = \frac{1}{2} \int_{\text{vol}} \{\sigma\} \{\varepsilon\} d_{\text{vol}} \]  

\[ V = \frac{1}{2} \int_{\text{vol}} \frac{M_z y}{I_o} \times \frac{M_z y}{EI_o} d_{\text{vol}} \]  

\[ V = \frac{1}{2} \int_{\text{Area}} \int_{\text{length}} \frac{M_z^2 y^2}{EI_o^2} dAdx \Rightarrow \frac{1}{2} \int_{\text{length}} \frac{M_z^2}{EI_o^2} \left( \int_{\text{Area}} y^2 dA \right) dx \]  

\[ \text{but } I = \left( \int_{\text{Area}} y^2 dA \right) \]

\[ \therefore V = \frac{1}{2} \int \frac{M_z^2}{EI_o} dx \]  

The bending moment \( M_z \) is a function of the change in slope of the beam’s elastic curve. For small slopes,

\[ M_z = EI_o \left( \frac{\partial^2 y}{\partial x^2} \right) \]

Substituting into equation (3.25)

\[ V = \frac{1}{2} \int_0^l \left( EI_o \left( \frac{\partial^2 y}{\partial x^2} \right)^2 \right) dx \]
The strain energy due to shear is considerably smaller than the bending and torsion strain energy and is consequently ignored.

The total strain energy is

\[
V = \frac{1}{2} \int_0^l EI_0(y) \left[ \frac{\partial^2 h(y,t)}{\partial y^2} \right]^2 dy + \frac{1}{2} \int_0^l GJ(y) \left[ \frac{\partial \alpha(y,t)}{\partial y} \right]^2 dy
\]

(3.26)

where

- \( E \) is the modulus of elasticity
- \( G \) is the modulus of shear
- \( J \) is the polar moment of inertia
- \( I_0 \) is the cross section moment of inertia

which can be written as

\[
V = \frac{1}{2} K_h h^2 + \frac{1}{2} K_\alpha \alpha^2
\]

(3.27)

Let,

\[
K_h = \int_0^l EI_0(y) \left[ \frac{\partial^2 f(y)}{\partial y^2} \right]^2 dy
\]

(3.28)

\[
K_\alpha = \int_0^l GJ(y) \left[ \frac{\partial \phi(y)}{\partial y} \right]^2 dy
\]

(3.29)

and \( K_h \) and \( K_\alpha \) are the uncoupled stiffness and may be written as,

\[
K_h = m \omega_h^2 \quad \& \quad K_\alpha = I_\alpha \omega_\alpha^2
\]
\[ V = \frac{1}{2} \bar{m} \omega_h^2 h^2 + \frac{1}{2} \bar{I}_\alpha \omega_\alpha^2 \alpha^2 \]  

(3.30)

Where \( \bar{m} \) and \( \bar{I}_\alpha \) are expressed in generalized coordinates.

**d. Calculate the Aerodynamic Terms**

The main forces that are considered in flutter calculation are the lift and pitching moment of the wing. The aerodynamic terms are divided into two sections. The first section is treated under the assumption of quasi-steady aerodynamic derivatives while the second section is based on the unsteady airfoil method. The quasi-steady aerodynamic method introduces very many simplifications which make it easy to go through a detailed analysis. It is also worth mentioning that the results of the quasi-steady assumption may find practical applications for low speed airplanes [1], such as the LAS wing system.

The unsteady airfoil theory on the other hand is so complicated that a great deal of numerical work is required in the solution, and the main analytical features are masked by the calculative complications. The unsteady airfoil theory method, unlike the quasi steady method, also considers the lift from non circulatory origin. The non-circulatory force, is due to “apparent mass” forces whose origin is not associated with the creation of vorticity.

**i. Quasi-steady Aerodynamic Forces**

To make a simplified analysis the quasi-steady assumption is used; The aerodynamic characteristics of an airfoil whose motion consists of variable linear and angular motions are equal, at any instant of time, to the characteristics of the same airfoil moving with constant linear and angular velocities equal to the actual instantaneous values [1]. The inclination of the flow-velocity vector to the profile is also taken to be constant and equal to the actual instantaneous inclinations.
Considering a flat plate in a stream whose velocity at infinity is $U$. Let the plate have a vertical translation $h$ and a rotation $\alpha$ about an axis location at $x_o$ as shown below in Figure 3.12. The figure also shows the vortex distribution along the airfoil whereby the vortices are being shed at the trailing edge of the wing and carried downstream by the flow.

![Figure 3.12 Unsteady flow over a two dimensional airfoil](image)

The main assumption made to this unsteady flow is that the aerodynamic characteristics of an airfoil whose motion consists of variable linear and angular motions are equal, at any instant of time, to the characteristics of the same airfoil moving with constant linear and angular velocities equal to the actual instantaneous values. The translation $h$ is considered positive down and rotation $\alpha$, is positive nose up.

From reference [1], the lift and moment coefficients are;

\[
C_L = \frac{dC_L}{d\alpha} \left[ \alpha + \frac{1}{U} \frac{dh}{dt} + \frac{1}{U} \left( \frac{3}{4} c - x_o \right) \frac{d\alpha}{dt} \right] \quad (3.31)
\]

\[
(C_M)_{l.e.} = -\frac{c \pi}{8U} \frac{d\alpha}{dt} - \frac{1}{4} C_L \quad \text{about the leading edge} \quad (3.32)
\]
considering the lift coefficient, the first term in the main brackets $\alpha$ is the angle contribution to lift due to vortex concentration at the $\frac{1}{4}$ chord point. The second and third terms are the effects of an angle change due to an induced velocity at the $\frac{1}{4}$ chord point.

Consider Figure 3.13 below of a flat plate that translates and rotates about a point e.a.;

![Figure 3.13 Airfoil translation and rotation due to circulation flow](image)

The vertical velocity of the $\frac{3}{4}$ chord point due to translation is given by $\frac{dh}{dt}$ while the vertical velocity due to rotation is given by $\left(\frac{3}{4} - x\right) \frac{d\alpha}{dt}$ assuming small angle rotations. The two velocities are added together and divided by the velocity of flow, $U$ to give the induced angle on the airfoil.
The moment coefficient shows that the resultant lift acts at the $\frac{1}{4}$ chord point, while there is an additional term which is a damping couple $\frac{c\pi}{8U} \frac{d\alpha}{dt}$ which is proportional to the angular velocity.

The lift and moment of a unit span section about the elastic axis can thus be given by

$$L_{e.a.} = \frac{\rho U^2}{2} c \left\{ \frac{dC_L}{d\alpha} \left[ \alpha + \frac{1}{U} \frac{dh}{dt} + \frac{1}{U} \left( \frac{3}{4} c - x_o \right) \frac{d\alpha}{dt} \right] \right\}$$

(3.33)

While the moment is;

$$M_{e.a.} = \frac{\rho U^2}{2} c^2 C_M$$

$$= \frac{\rho U^2}{2} c^2 \left[ (C_M)_{e.a.} + \frac{x_o}{c} C_L \right]$$

(3.34)

**ii. Unsteady aerodynamic forces on an airfoil**

In the unsteady airfoil method the aerodynamic forces on an oscillating airfoil in three degrees of freedom are mainly solved by the solution of certain definite integrals. These integrals have been classically identified as Bessel functions of the first and second kind and of zero and first order [3]. This theory is based on potential flow and the Kutta condition and is therefore very similar to the conventional wing section theory relating to the steady case.

The lift and moment is thus expressed with respect to the elastic axis of the wing.
\[ L_e = -\pi pb^3 \omega^2 \left( \frac{h}{L_h} + \alpha \left[ L_o - \left( \frac{1}{2} + a_h \right) L_h \right] \right) \]  

\[ M_{e,a} = -\pi pb^4 \omega^2 \left( \frac{h}{b} \left[ M_h - \left( \frac{1}{2} + a_h \right) L_h \right] + \alpha \left[ M_o - \left( \frac{1}{2} + a_h \right) (L_o + M_h) + \left( \frac{1}{2} + a_h \right)^2 L_h \right] \right) \]  

For a swept wing there are two methods of applying aerodynamic theory; one focusing on sections normal to the elastic axis as will be done for the LAS wing and the other for sections in the flight direction. The lift and moment terms are multiplied by the cosine of the sweep angle and are expressed by:

\[ L_e = -\pi pb^3 \omega^2 \cos \Lambda \left( \frac{h}{L_h} + \alpha \left[ L_o - \left( \frac{1}{2} + a_h \right) L_h \right] \right) \]  

\[ M_{e,a} = -\pi pb^4 \omega^2 \cos \Lambda \left( \frac{h}{b} \left[ M_h - \left( \frac{1}{2} + a_h \right) L_h \right] + \alpha \left[ M_o - \left( \frac{1}{2} + a_h \right) (L_o + M_h) + \left( \frac{1}{2} + a_h \right)^2 L_h \right] \right) \]  

\( L_h, L_o, M_h \) and \( M_o \) are aerodynamic coefficients defined as in Appendix A2

For the aerodynamic terms let the generalized lift and pitching moment be represented by \( Q_h \) and \( Q_\alpha \) respectively, therefore

\[ Q_h(t) = -\int_0^t L_e(y,t)f(y)dy \] Generalized lift force

\[ Q_\alpha(t) = -\int_0^t M_e(y,t)\phi(y)dy \] Generalized pitching moment

substituting equations (3.37) and (3.38) into \( Q_h \) and \( Q_\alpha \)
\[ Q_h = \pi \rho br^3 \omega^2 \cos \Lambda \left( A_{hh} \frac{h}{br} + A_{hh} \alpha \right) \]  \[ Q_\alpha = \pi \rho br^4 \omega^2 \cos \Lambda \left( A_{aa} \frac{h}{br} + A_{aa} \alpha \right) \]

where \( br \) is the semi chord at reference section,

\[ A_{hh} = \int_0^L \left( \frac{b}{br} \right)^2 f^2(y) L_h(y) dy \]
\[ A_{ha} = \int_0^L \left( \frac{b}{br} \right)^3 f(y) \phi(y) \left[ L_a - \left( \frac{1}{2} + a_h \right) L_h \right] dy \]
\[ A_{ah} = \int_0^L \left( \frac{b}{br} \right)^3 f(y) \phi(y) \left[ M_h - \left( \frac{1}{2} + a_h \right) L_h \right] dy \]
\[ A_{aa} = \int_0^L \left( \frac{b}{br} \right)^4 \phi^2(y) \left[ M_a - \left( \frac{1}{2} + a_h \right) (L_a + M_h) + \left( \frac{1}{2} + a_h \right)^2 L_h \right] dy \]

\[ T = \frac{1}{2} m h \omega^2 + \frac{1}{2} I_a \alpha^2 + S_a h \alpha \] \[ V = \frac{1}{2} m \omega^2 h^2 + \frac{1}{2} L_a \omega_a^2 \alpha^2 \]

\section{Combine Terms and Solve the Flutter Determinant}

from equations (3.21) and (3.34)

the kinetic and strain equations are given by

\[ T = \frac{1}{2} m h \omega^2 + \frac{1}{2} I_a \alpha^2 + S_a h \alpha \] \[ V = \frac{1}{2} m \omega^2 h^2 + \frac{1}{2} L_a \omega_a^2 \alpha^2 \]
While the forcing terms are the aerodynamic lift and moment equations (4.43) and (4.44), of the form,

\[ Q_h = \pi \rho br^3 \omega^2 \cos \Lambda \left( A_{hh} \frac{\ddot{h}}{br} + A_{ha} \dot{\alpha} \right) \]  
(3.43)

\[ Q_a = \pi \rho br^4 \omega^2 \cos \Lambda \left( A_{ah} \frac{\ddot{h}}{br} + A_{aa} \dot{\alpha} \right) \]  
(3.44)

using the Lagrange equations of motion, (Appendix A1)

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial q_i} \right) + \frac{\partial}{\partial q_i} (V) = Q \quad (q_i = \ddot{h}, \dot{\alpha}) \]

the equations of motion are

\[ \bar{m} \ddot{h} + \bar{S}_a \alpha + \bar{m} \omega_h^2 \bar{h} = \pi \rho br^3 \omega^2 \cos \Lambda \left( A_{hh} \frac{\ddot{h}}{br} + A_{ha} \dot{\alpha} \right) \]  
(3.45)

\[ \bar{S}_a \ddot{h} + \bar{I}_a \alpha + \bar{I}_a \omega_a^2 \alpha = \pi \rho br^4 \omega^2 \cos \Lambda \left( A_{ah} \frac{\ddot{h}}{br} + A_{aa} \dot{\alpha} \right) \]  
(3.46)

At critical flutter, the wing motion has been observed to be simple harmonic in nature. Complex representation can therefore be used to represent the motion;

\[ \bar{h}(t) = h_0 e^{i\omega t} \quad \bar{\alpha}(t) = \alpha_0 e^{i\omega t} \]  
(3.47)
Equations (3.47) are substituted into the equations of motion (3.45) and (3.46), and the factor $e^{ia}$ is cancelled to obtain the time free equations. Additionally, the structural damping is assumed to be proportional to the elastic restoring force.

The detailed mechanism of damping in structures concerned with in aeroelasticity is yet unknown. Hence, a choice of the particular form of damping is open to question [1].

Given an idealized equation of motion,

$$\ddot{x} + 2\xi \omega x + \omega^2 x = \frac{F}{m}$$

The damping coefficient can be represented approximately by;

$$g \approx 2\xi$$

In the presence of flutter where the motion is sinusoidal the effect of structural damping can be accounted for simply by replacing the terms $hK_h$ and $\alpha K_\alpha$ with the terms $hK_h (1 + ig_h)$ and $\alpha K_\alpha (1 + ig_\alpha)$ [1].

Where

The terms $g_h$ and $g_\alpha$ are the damping coefficients and $i = \sqrt{-1}$

Thus the damping is considered to be in phase with the velocity and proportional to the elastic restoring forces.

The resulting equations are

$$-\omega^2 (m \dot{h} + S_\alpha \dot{\alpha}) + (1 + ig_h) m \omega^2 \dot{h} = \pi \rho br^3 \omega^2 \cos \Lambda \left( A_{hh} \frac{\dot{h}}{br} + A_{h\alpha} \dot{\alpha} \right)$$  \hspace{1cm} (3.48)

$$-\omega^2 (S_\alpha \dot{h} + I_\alpha \dot{\alpha}) + (1 + ig_\alpha) I_\alpha \omega^2 \dot{\alpha} = \pi \rho br^4 \omega^2 \cos \Lambda \left( A_{ah} \frac{\dot{h}}{br} + A_{a\alpha} \dot{\alpha} \right)$$  \hspace{1cm} (3.49)
The terms are grouped as multiples of \( \frac{h}{br} \) and \( \alpha \). Equation (3.48) is then divided by \( \pi \rho r^3 \omega^2 l \) while equation (3.49) is divided by \( \pi \rho r^4 \omega^2 l \), making the results dimensionless.

\[
\begin{align*}
\frac{-h}{br} \left\{ \frac{m}{\pi \rho b_r^2} \left[ 1 - \left(1 + ig_h \right) \left( \frac{\omega_h}{\omega} \right)^2 \right] + \cos \Lambda A_{hh} \right\} - \frac{-S}{\pi \rho b_r^3} + \cos \Lambda A_{h\alpha} = 0 \quad (3.50)
\end{align*}
\]

\[
\begin{align*}
\frac{-h}{br} \left\{ \frac{S}{\pi \rho b_r^3} + \cos \Lambda A_{oh} \right\} + \frac{-S}{\pi \rho b_r^4} \left[ 1 - \left(1 + ig_h \right) \left( \frac{\omega_h}{\omega} \right)^2 \right] + \cos \Lambda A_{hh} = 0 \quad (3.51)
\end{align*}
\]

An extra modification is made to the equations using the scheme used by Smilg and Wasserman [4]. The damping coefficient, \( g \) is regarded as one of the unknowns and is combined together with the other unknown \( \frac{\omega_o}{\omega} \) to form a single complex combination

\[
Z = \left( \frac{\omega_o}{\omega} \right)^2 \left[ 1 + ig \right]
\]

The determinant of the equations becomes,

\[
\begin{vmatrix}
\frac{-m}{\pi \rho b_r^2} \left[ 1 - \left( \frac{\omega_h}{\omega} \right)^2 Z \right] + \cos \Lambda A_{hh} & - \frac{S}{\pi \rho b_r^3} + \cos \Lambda A_{h\alpha} \\
- \frac{S}{\pi \rho b_r^3} + \cos \Lambda A_{oh} & \frac{\omega_o}{\omega} \left[ 1 - z \right] + \cos \Lambda A_{\alpha\alpha}
\end{vmatrix} = 0 \quad (3.52)
\]
The resulting quadratic equation which is a function of $Z$ is evaluated and the result is two complex roots; one of the roots corresponds to bending and the other root corresponds to torsion.

According to the definition of $Z$, the real part is $\left( \frac{\omega_n}{\omega} \right)^2$. The imaginary part is $\left( \frac{\omega_n}{\omega} \right)^2 g$.

Hence $g = \frac{\text{Im}(Z)}{\text{Re}(Z)}$.

The flutter frequency is then calculated by the formula

$$\omega = \sqrt{\frac{\omega_n^2}{Z_n}} \quad \text{and} \quad U = \frac{\omega b}{K} \quad (3.53)$$
Chapter 4
Experimental Procedure

This Chapter consists of 3 sections. The first section, 4.1 lists the various parameters that were necessary in establishing the dynamic stability of the wing. Section 4.2 gives a brief description of different instruments used in the data collection process. Section 4.3 describes the experimental process and outlines how the flutter parameters were calculated from the various tests.

4.1 Testing Parameters

Ground test were conducted to determine the variables needed to characterize the flutter speed and frequency. The main parameters to be identified include:

i. Wing stiffness matrix using influence coefficients
ii. Natural frequencies using dynamic excitation
iii. Structural damping of the system
iv. Aerodynamic Coefficients
v. Geometry and mass of the wing

4.2 Equipment Used

The main instruments used in this thesis were;

i. Potentiometers and strain gauges: The potentiometers were used to measure the displacements and rotations of the wing with an accuracy of
0.0001 of an inch. Figure 4.1 below shows an image of one. The main base is mounted on a stationary structure while the string or cable is attached to the vibrating wing.

![Figure 4.1 Potentiometer/String Pot](image)

The strain gauges were also used to define the relative bending and torsion of various parts of the wing structure.

ii. Load cell transducers: The load cells were mounted between the quad pod and the airplane spar cap fittings. The quad pod is the main link between the LAS wing and the host aircraft; therefore any loads from the LAS wing were monitored using these transducers. There were 4 load cells used, each having a 10,000 lb capacity.

iii. Video cameras: Two video cameras were mounted on the LAS wing to observe any indications of flutter behavior. One of the cameras was attached on the kingpost while the other was fixed on the control bar.

iv. Motor with rotating unbalance: The motor and rotating unbalance are used to induce sinusoidal oscillations on the LAS wing tip. The unbalance mass has a weight of 1.6 ounces, which gives distinct amplitudes for varied frequencies.

v. Data acquisition system: The data acquisition system was mainly a desktop computer using a Labview interface to collect data from all the
instruments mentioned above. The computer was installed in the Cessna 337 to enable real time data collection.

vi. Weights: An assortment of various weights was used to measure the displacement and stiffness of the LAS wing, due to the application of the mass forces.

4.3 Sequence of Testing

a. **Wing Stiffness Measurements**

In aeroelasticity the most convenient scheme of describing the elastic properties of a structure is to specify its influence coefficients [1]. The calculation of the influence functions for a structure other than a simple beam may be very difficult, but it is a prerequisite for aeroelastic analyses.

The procedure used in defining the influence matrix involves applying loads at predefined nodal points and measuring all corresponding displacements. For the bending mode analysis the LAS wing was divided into 4 parts as shown in Figure 4.2, thus making it a 4 degree of freedom (DOF) system.

![Figure 4.2](image-url)  
*Figure 4.2  Semi-span wing divided into 4 members and 5 nodes*
An initial mass of 15 pounds was placed on node 1 and the displacement on all 5 nodes was recorded. A second mass of 25 lbs was applied on node 2 and the same measurements on all 5 nodes were recorded. The same procedure was repeated by applying 25 lb loads on node 3 and node 4 consecutively. The same measurements were done for the unloading process to verify that the loading process was entirely elastic.

An initial loading of the wing is shown below:

![Figure 4.3 Static deflection measurements on the wing](image)

A table of the values recorded is shown below:

Table 3.1 Table of loading deflections on the nodal points

<table>
<thead>
<tr>
<th>Displacements (in)</th>
<th>Nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>node</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.38</td>
</tr>
<tr>
<td>2</td>
<td>1.69</td>
</tr>
<tr>
<td>3</td>
<td>1.08</td>
</tr>
<tr>
<td>4</td>
<td>1.13</td>
</tr>
<tr>
<td>5</td>
<td>0.00</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Load applied (lbs)</th>
<th>15</th>
<th>25</th>
<th>25</th>
<th>25</th>
<th>25</th>
</tr>
</thead>
</table>
Characterization of the torsion matrix was not as easy. Due to the relatively higher stiffness in torsion as compared to bending, the wing was modeled as a 1 DOF system rotating about the elastic axis which is the leading edge tube.

Measurement of the wing’s twist was a more complex task, mainly because of two reasons;

First, the wing is very stiff in torsion and thus only small angles can be measured; the possibility of error is very high.

Secondly, the coupling between bending and torsion of the wing increases the possibility of error because the angle is calculated from relative displacements between leading and trailing edge deflections.

For these reasons, the torsional stiffness was treated theoretically as a stepped tube and the material properties of aluminum were applied.

b. Measurement of Natural Frequencies Using Dynamic Excitation

The bending and torsion frequencies were measured using a rotating mass system attached to the end of the wing as shown below;

![Motor with a pendulum at wing tip](image)

*Figure 4.4 Motor with a pendulum at wing tip
Configuration 1.*
The deflections at the nodal points were measured using linear potentiometers (string pots), and the data was collected at a 30 Hz sampling frequency. The rotation speed of the pendulum was varied from 3 -12.8 Hz in 20 non-uniform increments by varying the current through the motor system. This experiment was run in two setup configurations:

The first configuration is the one shown above in Figure 4.4 and this orientation was chosen so as to obtain frequency data predominantly due to bending modes. The small eccentricity of the bending axis introduced a small but negligible amount of torsion in the wing, which was highly damped by the trailing fabric and batten mass. This is because the circular rotation of the pendulum is along the axis of the wing’s leading edge axis. From the data obtained, the natural frequency was determined at the point where the oscillations of the wing became resonant; the highest amplitudes obtained from the potentiometer data.

An example of the data collected from the potentiometers is shown below. Figure 4.5 shows a global chart of all the data collected for the different motor speeds. The second one, Figure 4.6 represents the time scale at which the amplitude is resonant.

![Amplitude Vs Time chart for varying motor speeds](image-url)
The amplitude data was also observed in the frequency domain using a power spectral density (PSD) algorithm in Matlab. The PSD describes how the power or variance of a time series is distributed with frequency. Mathematically it is defined as the Fast Fourier Transform of the autocorrelation sequence of the time series [14]. A PSD plot of the same data shown on Figure 4.6 above is shown below (Figure 4.7).
Figure 4.7  PSD data of wing at a motor frequency of 3.88 Hz

The second configuration which is shown below induces a coupling between torsion and bending modes. The process of isolating the torsional frequencies from the bending was more challenging and a fast Fourier algorithm (see appendix) was used to model all the dominant frequencies in the system.
Two potentiometers were placed, one at the leading edge and the other at the trailing edge of the wing. The difference in displacements was calculated and hence the angle change with time was obtained. The charts below show this:

Figure 4.9 Angle vs. time chart for varying forced frequencies
Figure 4.10  Five second sample of first resonant frequency

It is obvious that the amplitude distribution consists of superimposed waveforms and to analyze this amplitude distribution, a PSD algorithm is used to isolate all the dominant waveforms in this data. The corresponding PSD is shown below;
Power Spectral Density of Amplitude

Coupling between First Torsion and Second Bending Natural Frequency
8.35 Hz

Second Torsion Natural Frequency
17.2 Hz

Frequency (Hz)

Power Spectral Density

Figure 4.11 PSD data of wing at a motor frequency of 8.37 Hz

This data obtained from the dynamic excitation of the system serves as a standard for confirming the results that will be obtained using the stiffness and mass matrix calculations of the wing.

c. Structural Damping of the System

Unlike mass and stiffness, damping cannot be determined by static tests. The energy dissipation exhibited by a device may be the result of air resistance, or from electronic defects in the material of which the device is made [8].
A convenient method used to determine the amount of damping present in the system is to measure the rate of decay of free oscillations. For instance, given a record of decaying oscillations as shown below:

![Amplitude vs. Time for an under damped System](image)

**Figure 4.12  Amplitude vs. Time for an under damped System**

Where,

\[ T = \text{period of oscillation} \]

For a 1 DOF system the equation of motion of the system is

\[ m \ddot{x} + c \dot{x} + kx = 0 \]

Dividing by \( m \)

\[ \Rightarrow \dot{x} + \frac{c}{m} \dot{x} + \frac{k}{m} x = 0 \]

Define \( \omega_n = \sqrt{\frac{k}{m}} \)

And \( \zeta = \text{damping ratio} = \frac{\text{actual damping value}}{\text{critical damping value}} = \frac{b}{2 \sqrt{km}} \)

\[ \therefore \dot{x} + 2 \zeta \omega_n x + \omega_n^2 x = 0 \]
To determine the damping ratio, \( \zeta \) we measure amplitude, \( x_1 \) at time \( t = t_1 \) and amplitude \( x_n \) at time \( t = t_1 + (n - 1)T \)

Also the decay in amplitude can be represented as the ratio of the exponential multiplying factors at times \( t_1 \) and \( t_1 + T \).

\[
\frac{x_1}{x_2} = \frac{e^{-\zeta \omega_0 t_1}}{e^{-\zeta \omega_0 (t_1 + T)}} = \frac{1}{e^{-\zeta \omega_0 T}} = e^{\zeta \omega_0 T}
\]

Similarly
\[
\frac{x_1}{x_n} = \frac{e^{-\zeta \omega_0 t_1}}{e^{-\zeta \omega_0 (t_1 + (n-1)T)}} = e^{(n-1)\zeta \omega_0 T}
\]

The logarithm of the ratio of succeeding amplitudes is called the logarithmic decrement, thus,

\[
\text{Logarithmic decrement} = \ln \frac{x_1}{x_2} = \frac{1}{n-1} \left( \ln \frac{x_1}{x_2} \right) = \zeta \omega_d T
\]

\[
= \zeta \omega_n \frac{2\pi}{\omega_d} = \frac{2\pi \zeta}{\sqrt{1 - \zeta^2}}
\]

\[
\therefore \zeta = \frac{1}{n-1} \left( \ln \frac{x_1}{x_n} \right)
\]

\[
\sqrt{4\pi^2 + \left[ \frac{1}{n-1} \left( \ln \frac{x_1}{x_n} \right) \right]^2}
\]

This equation however is not ideal for the flexible wing which is so highly damped that the oscillations fade out almost instantaneously and in most cases irregular due to both coulomb and viscous damping present in the system. For instance the Figure below shows an example of the data collected from a free vibration test where the wing is pulled down from the tip and then released back to its original position.
Because the wing’s oscillation dies out very quickly another approach is to use the phase angle difference between the wing and the 90° out of phase frequency of the wing. This value is observed at 8.9532Hz. The maximum response of the wing is at 3.88 Hz.

At steady state \( x = x_0 \sin(\Omega t - \phi) \)

Where

\[
\tan \phi = \frac{2\zeta \omega \Omega}{\omega^2 - \Omega^2} = \frac{2\zeta r}{1 - r^2}
\]

\( \Omega \) = driving frequency

\( \omega \) = natural frequency

and \( r = \frac{\Omega}{\omega} \)

Max response occurs when \( \Omega = \omega \sqrt{1 - 2\zeta^2} \)
if $\phi = 90^\circ \quad \tan \phi = \infty$ and $1 - r^2 = 0$

$\therefore r = 1 \quad \Rightarrow \Omega = 3.88 \text{Hz} = 24.38 \text{rad/s}$

Max amplitude response is at 3.88 Hz $\therefore r = \frac{3.846}{3.88} = 0.991$

$r^2 = 1 - 2\zeta^2 \quad \Rightarrow \quad 0.983 = 1 - 2\zeta^2$

$\therefore \zeta = 0.0934$

d. Aerodynamic Coefficient Measurements

In aeroelasticity we are concerned primarily with two components of force and one component of moment that act on a body. These are:

1. Lift which is the force perpendicular to the direction of the motion
2. Drag is the force in the direction of motion and is considered positive when the force acts in the downstream direction.
3. Pitching Moment, defined as the moment about an axis perpendicular to both the direction of motion and the lift vector.

It's worth noting that some important characteristic quantities of airfoils, such as,

a. Profile drag coefficient, $C_{DO}$
b. Maximum lift coefficient $C_{L_{\text{max}}}$
c. Moment coefficient at zero lift $C_{mo}$
d. Angle of zero lift $\alpha_{0}$,

are only of minor importance in aeroelasticity [1]. They do not appear in most of the problems. On the other hand, the question of span wise lift distribution corresponding to a twisted airfoil is of great importance in aeroelasticity. For mathematical simplicity, the so
called strip theory is used as a first approximation. In this approach, one assumes that the local lift coefficient $C_l$ is proportional to the local geometric angle of attack $\alpha$

\[ C_l = a\alpha \]  
Where a is a constant

The aerodynamic terms are the most difficult to characterize [3] and this is especially the case when dealing with a membrane swept wing. The complications presented by the configuration of the LAS wing are threefold in nature.

First, the wing is positioned above the airplane wing and therefore interference effects between the two wings are a high possibility.

Second, the nature of the membrane wing presents an almost impossible fluid dynamics problem because the airfoil shape of the wing can take up very many forms depending of the pressure loading as well as the internal structure support. To simplify this problem, some assumptions are made supported by experimental data.

Last, successful prediction of the flutter of swept wings depends, to an even greater degree than in the absence of sweep [2], on accurate knowledge of the structural and aerodynamic properties of the system. The state of the art, particularly on the aerodynamic side for tapered wings, is not so far advanced as for straight wings.

Data from the LAS wing was collected from several taxi runs and the forces were resolved into the various components. The main parameters that were collected included the lift, drag, pitching and rolling moments, angle of attack and speed. The data was reduced from several tests at varying speeds and angle of attack. The Figure 4.14 below shows three curves of $C_L$ vs $alpha$ for the LAS wing and a dotted trend line.

The lift curve slopes are different from one another because of the configuration changes on the sail. The initial data collected is shown in the $C_L$ vs $alpha$ curve for New Smyrna.
The wing was free in both pitch and roll. The other two curves are data points collected at Daytona Beach, where the wing was only limited to movement in pitch. However the main variable that seems to be consistent is the slope of the lift curve slope before the wing stalls.

![CL Alpha Curve for Sail](image)

*Figure 4.14  $C_L$ vs alpha curve slopes for the LAS wing*

The trend line $C_{La}$ is:

$$C_{La} = \frac{1.5 - 0.9}{(22 - 12)} \times 57.3$$

$$= 3.438$$
To verify this value of $C_{L,a}$ against theory, an airfoil section of the LAS wing was drawn in JavaFoil, which is a Martin Hepperle code. The potential flow analysis of this code is done with a higher order panel method (linear varying vorticity distribution) [15]. The software takes a set of airfoil coordinates and calculates the local inviscid flow velocity along the surface of the airfoil, for any desired angle of attack. Figure 4.15 shows the airfoil shape as it is modeled from the coordinate points entered into JavaFoil. The potential flow analysis is then done on the airfoil shape and the local lift and moment curve slopes are obtained for various Reynolds numbers (in our case between $3 \times 10^6$ and $4 \times 10^6$).

![Figure 4.15 Airfoil section of LAS wing as drawn in JavaFoil](image)
The local section $C_{l_\alpha}$ is calculated from chart

$$C_{l_\alpha} = \frac{1.3 - 0.3}{10/57.3} = 5.73$$

the maximum $C_l$ value is $18^\circ$

A Prandtl algorithm was then used to calculate the 3 dimensional $C_L$ and to account for the significant washout (20 degrees) on the LAS wing. This gave the following lift distribution as shown in Figure 4.17. The algorithm also determines the maximum centerline angle of attack to be $28^\circ$ which is very common in hang glider and trike wings.

The origin is taken as the root and non-dimensional span value 1 as the tip.
The lift across the span is distributed as shown above. A $C_L$ vs. alpha plot is then obtained as shown below.

Figure 4.17  LAS wing lift distribution along semi-span using Prandtl lifting line theory

Figure 4.18  $C_L$ vs. Alpha plot from the Prandtl distribution
The C_{L_{\alpha}} value obtained from this distribution is 4.41 which represents a 28.3\% error from the experimental data. This difference may primarily be due to the complex nature of the LAS wing which continuously changes shape during the loading stage of the flight regime. It is however interesting to observe the effect of the washout at lower angles of attack such as 8^\circ, 10^\circ and 12^\circ as seen in Figure 4.19. At low angles of attack the Prandtl lift line theory shows that the tips produce negative lift.

For the flutter analysis the experimental data was chosen (C_L = 3.438) because it represented the real values obtained from flight testing the wing.

![CL distribution along Semi Span](image)

*Figure 4.19 LAS wing lift distribution along semi-span for lower angles of attack*

The pitching moment of the LAS wing is calculated purely from the aerodynamic lift and damping couple which is proportional to the angular velocity. The aerodynamic center is assumed to be the ¼ chord of the wing. It is worth mentioning that the LAS wing does not experience any pitch moment coupling from the airplane. This is because of the simple one bolt connection of the wing to the quad pod.
Chapter 5

Results

5.1 Flutter Analysis Results

The procedures outlined above are put into algorithm form using Matlab. The initial procedure of the program is to read the excel data for the aerodynamic coefficients. The kinetic and strain energy are computed and the program solves for the determinant for varying values of k, the reduced frequency. The results of this simulation are shown in the following graphs;
Flutter is initiated when the damping disappears. From the graph it is seen that the wing will go into flutter at a speed of 176 knots. This is the value at which the damping for the torsion is zero. At this point the oscillations of the wing will continually increase due to an initial disturbance and in this case flutter. Generally the bending mode is always stable and this is also the case for the LAS wing.
The LAS system is a continually changing design and therefore an evaluation of the significance of different parameter changes will help to predict the impact of flutter on the system. Figure 5.2 below shows the effect of changing the elastic axis.

![Figure 5.2 Effect of varying the elastic axis position](image)

As was mentioned earlier, the elastic axis position is defined as the ratio;

\[ a_h = \frac{\text{distance between the midchord and the elastic axis}}{\text{semichord, } b} \]

The elastic position for the LAS wing is ahead of the mid chord and it is thus negative. From the graph this shows that the more forward the elastic axis is the higher the flutter speed will be if all other variables stay constant.
The effect of increasing or decreasing the torsional stiffness of the wing was also studied. The LAS system torsion stiffness is shown at 60 radians per second frequency.

![Graph showing Damping factor g vs Airspeed U for varying torsional stiffness](image)

**Figure 5.3 Varying the torsional stiffness of the wing**

From the graph, varying of the torsional stiffness has a significant increase in the flutter speed of the wing with all other factors remaining the same.

Bending stiffness generally has an insignificant increase in flutter speed [3] and Figure 5.4 below helps to show this. The LAS wing has a bending frequency of 32 radians per second.
The following two figures show pictures taken from an external camera mounted on the kingpost of the aircraft. The selected shots were taken at a speed of 53 knots and the wing was at an angle of attack of $14^0$. The first view (Figure 5.5) shows the wing before any vibration modes are initiated. The second view (Figure 5.6) shows three pictures taken at 7Hz, of the vibration overlaid on top of each other.
Figure 5.5  LAS semi-wing view in non-vibratory mode

Figure 5.6  LAS semi-wing view in vibratory mode
5.2 Mass Modeling of the System

The table below shows the distribution of weights from the root (Keel) of the wing to the tip as well as other corresponding parameters used in the calculation of flutter speeds and frequencies.

Table 5.1 Mass and Inertia Distribution along Semispan of LAS wing

<table>
<thead>
<tr>
<th>Semi Span</th>
<th>W (lbs)</th>
<th>fabric (lbs)</th>
<th>Total Weight (lbs)</th>
<th>Mass (slugs/ft)</th>
<th>cg (ft)</th>
<th>Sy (ft)</th>
<th>Iy (ft)</th>
<th>b (ft)</th>
<th>ah (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Root</td>
<td>1</td>
<td>1.667</td>
<td>5.538</td>
<td>0.115</td>
<td>3.073</td>
<td>0.352</td>
<td>1.082</td>
<td>4.077</td>
<td>-0.977</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2.917</td>
<td>3.345</td>
<td>0.098</td>
<td>2.315</td>
<td>0.227</td>
<td>0.525</td>
<td>3.754</td>
<td>-0.975</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>4.167</td>
<td>2.890</td>
<td>0.085</td>
<td>2.015</td>
<td>0.172</td>
<td>0.347</td>
<td>3.549</td>
<td>-0.974</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>5.417</td>
<td>4.146</td>
<td>0.116</td>
<td>1.929</td>
<td>0.224</td>
<td>0.431</td>
<td>3.326</td>
<td>-0.972</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>6.667</td>
<td>4.033</td>
<td>0.112</td>
<td>1.588</td>
<td>0.178</td>
<td>0.283</td>
<td>3.075</td>
<td>-0.970</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>7.917</td>
<td>3.889</td>
<td>0.107</td>
<td>1.269</td>
<td>0.136</td>
<td>0.172</td>
<td>2.850</td>
<td>-0.967</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>9.167</td>
<td>3.767</td>
<td>0.103</td>
<td>0.945</td>
<td>0.097</td>
<td>0.092</td>
<td>2.625</td>
<td>-0.964</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>10.417</td>
<td>2.033</td>
<td>0.060</td>
<td>0.416</td>
<td>0.025</td>
<td>0.010</td>
<td>2.386</td>
<td>-0.961</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>11.667</td>
<td>2.230</td>
<td>0.064</td>
<td>0.788</td>
<td>0.050</td>
<td>0.040</td>
<td>2.139</td>
<td>-0.956</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>12.917</td>
<td>1.414</td>
<td>0.043</td>
<td>0.326</td>
<td>0.014</td>
<td>0.005</td>
<td>1.880</td>
<td>-0.950</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>14.167</td>
<td>1.386</td>
<td>0.041</td>
<td>0.227</td>
<td>0.009</td>
<td>0.002</td>
<td>1.590</td>
<td>-0.941</td>
</tr>
<tr>
<td>Tip</td>
<td>12</td>
<td>16.404</td>
<td>3.473</td>
<td>0.051</td>
<td>0.181</td>
<td>0.009</td>
<td>0.002</td>
<td>1.298</td>
<td>-0.928</td>
</tr>
<tr>
<td>Total</td>
<td>38.144</td>
<td>5.055</td>
<td>43.199</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5.2 Free undamped vibration

The natural frequency of the system is not affected by the damping or the forcing function and can therefore be represented as;

\[
\begin{bmatrix}
0.2186 & 0 & 0 & 0 \\
0 & 0.155 & 0 & 0 \\
0 & 0 & 0.1285 & 0 \\
0 & 0 & 0 & 0.4192
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
+ 
\begin{bmatrix}
3.657 & 2.866 & -3.231 & 2.692 \\
-3.231 & 2.744 & 21.477 & 1.474 \\
2.692 & -11.060 & 1.474 & -4.894
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \{0\}
\]

(20)

Assuming a solution of the form

\[x_n = x_ne^{\lambda t}\]
Equation 20 can be written as,

\[
\begin{pmatrix}
-\omega^2 & 0.2186 & 0 & 0 & 0 \\
0 & 0.155 & 0 & 0 & 0 \\
0 & 0 & 0.1285 & 0 & 0 \\
0 & 0 & 0 & 0.4192 & 0 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix}
+ \begin{pmatrix}
3.657 & 2.866 & -3.231 & 2.692 \\
-3.231 & 2.744 & 21.477 & 1.474 \\
2.692 & -11.060 & 1.474 & -4.894 \\
\end{pmatrix}
\begin{pmatrix}
ex \\
e^{-\omega t} \\
e^{i\omega t} \\
e^{-i\omega t} \\
\end{pmatrix}
= 0
\]

(21)

The non trivial solution of equation (21) exists when,

\[
\begin{vmatrix}
3.657 - 0.2186\omega^2 & 2.866 & -3.231 & 2.692 \\
2.866 & -13.472 - 0.155\omega^2 & 2.744 & -11.060 \\
-3.231 & 2.744 & 21.477 - 0.1285\omega^2 & 1.474 \\
2.692 & -11.060 & 1.474 & -4.894 - 0.4192\omega^2 \\
\end{vmatrix}
= 0
\]

The resulting polynomial of the matrix is of fourth order meaning it will have 4 roots.

The polynomial is

\[
0.0030181x^4 - 0.23722x^3 - 52.785x^2 + 1049.3x - 4856.7
\]

The roots of this polynomial are

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequencies (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.8116</td>
</tr>
<tr>
<td>2</td>
<td>4.2112</td>
</tr>
<tr>
<td>3</td>
<td>0 + 10.5433i</td>
</tr>
<tr>
<td>4</td>
<td>13.0665</td>
</tr>
</tbody>
</table>
5.3 Forced undamped system

The forcing function of the motor is,

\[ F(t) = m_p g + m_p \omega^2 r (\sin \omega t) \]

assuming phase angle is zero

\[ \therefore F(t) = 0.1 + 0.00978 \omega^2 (\sin \omega t) \]

The equation of motion can thus be written as;

\[
\begin{bmatrix}
0.3615 & 0 & 0 & 0 \\
0 & 0.155 & 0 & 0 \\
0 & 0 & 0.1285 & 0 \\
0 & 0 & 0 & 0.4192
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
+ \begin{bmatrix}
3.657 & 2.866 & -3.231 & 2.692 \\
-3.231 & 2.744 & 21.477 & 1.474 \\
2.692 & -11.060 & 1.474 & -4.894
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
0.1 + 0.00978 \omega^2 (\sin \omega t) \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[ x_n = x_n e^{i\omega t} \]

and

\[ F = Fe^{i\omega t} \]

Substituting into equation (5)

\[
\begin{bmatrix}
3.657 - 0.3615 \omega^2 & 2.866 & -3.231 & 2.692 \\
2.866 & -13.472 - 0.155 \omega^2 & 2.744 & -11.060 \\
-3.231 & 2.744 & 21.477 - 0.1285 \omega^2 & 1.474 \\
2.692 & -11.060 & 1.474 & -4.894 - 0.4192 \omega^2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= \begin{bmatrix}
0.1 + 0.00978 \omega^2 (\sin \omega t) \\
0 \\
0 \\
0
\end{bmatrix}
\]

The equation is then solved for \( \begin{bmatrix} x \end{bmatrix} \) to give the nodal positions of the wing as a function of frequency.
\[
\begin{bmatrix}
-x_1 \\
-x_2 \\
-x_3 \\
-x_4 \\
\end{bmatrix} = \begin{bmatrix}
0.1 + 0.00978 \omega^2 (\sin \alpha) \\
0 \\
0 \\
0 \\
\end{bmatrix} \begin{bmatrix}
3.657 - 0.3615 \omega^2 & 2.866 & -3.231 & 2.692 \\
2.866 & -13.472 - 0.155 \omega^2 & 2.744 & -11.060 \\
-3.231 & 2.744 & 21.477 - 0.1285 \omega^2 & 1.474 \\
2.692 & -11.060 & 1.474 & -4.894 - 0.4192 \omega^2 \\
\end{bmatrix}^{-1}
\]
Chapter 6
Conclusions

The analytical methods used to derive the flutter speeds used a Rayleigh type analysis. The results from these methods show a flutter speed of 176 knots, showing that the wing oscillations experienced at 50 knots are a result of other aerodynamic and structural mechanisms.

One of the main reasons for these oscillations may be due the freedom in pitch motion of the LAS wing. Unlike most conventional aircraft the LAS wing is attached to the host aircraft by a single bolt. The keel is modeled as a fixed support and does not account for the added freedom of the wing in pitch.

Another reason for the oscillations may be due to the fact that the theoretical results are based on the classical approach of flutter calculation which assumes clean potential flow around the wing, which is different from stall flutter. Stall flutter, which is the flutter in which the airfoil sections are in stalled flow during at least part of each cycle of oscillation, may be inducing these oscillations at high angles of attack. Since aircraft rarely come close to stall speeds when flying at the maximum velocities and dynamic pressures for which they are designed, the problem is not a serious one on wings and tails [2].

Another hypothesis for the oscillations of the wing during the early flight tests was prop wash effect in producing a periodic excitation to the LAS wing. This presumption was tested by running the flight test with only the rear propeller. The same aeroelastic oscillations of the wing tips were still evident even with the absence of prop wash from the forward propeller.
The aeroelastic oscillations of the wing were also noted to be more pronounced at New Smyrna airport as opposed to the Daytona airport. One of the big differences between the two airports is that New Smyrna is smaller and enclosed by tall trees and vegetation. The airflow on the runway will tend to be more turbulent and thus a higher Reynolds number. The Reynolds number is a big factor when considering stall flutter and may be a main contributor to the dynamic oscillations of the wing at low speeds.

The main variables which play a vital role in the calculation of flutter speeds and frequencies are the natural frequencies, both bending and torsional, the mass distribution of the system and the aerodynamic coefficients of lift and pitching moment. Computation of the inertial and stiffness portions of the flutter equations is straightforward. Well over 90% of the thought and labor is usually devoted to the generalized aerodynamic forces. Considering the troubles which can arise even from strip theory, it is not surprising that three dimensional effects are so rarely included in practice [2]. This could be another source of discrepancy between the theoretical and experimental results; the LAS wing's aerodynamics is very difficult to calculate not only because of the higher oscillation characteristics of the entire wing but mainly because of the airfoil deformation in flight.

The LAS wing is currently certified by the producer for a maximum speed of 86 knots. The basis for this speed therefore seems to be mostly structural and based on its loading capacity at speeds above this speed. It is therefore very encouraging that the theoretical flutter speed has a margin of safety of approximately 2.

From the theoretical analysis and comparisons that were done, it is evident the LAS wing is free from flutter during the normal speeds that it is designed for and the critical speed that the pilot should keep in mind is the V.N.E speed of the LAS wing of 86 knots.
Chapter 7

Recommendations

A foolproof method for avoiding flutter would consist in making the structure very rigid [1] (for instance, twice as rigid as it would be made according to static calculations) and in perfectly balancing the control surfaces. However, an airplane conceived along these basic lines obviously would not be feasible due to the immense weight. One must look for compromises; keeping in mind that flutter must be avoided, without adding weight.

Theory shows that the influence of the bending rigidity on the critical speed is slight, and experience confirms the theory regarding this point [3]. The LAS wing shows a very slight change in flutter speed for increased bending stiffness. It is satisfactory to avoid increasing the bending rigidity in future designs of the LAS wing in the prevention of flutter.

In contrast to bending rigidity, the torsional rigidity is of fundamental importance as was seen from (figure 4.14) which shows a substantial increase in the flutter speed with increased torsional rigidity. One should therefore provide the highest possible degree of it. For future designs the best way of attaining the highest amount of torsional rigidity is to increase the leading edge tube diameter which increasing the polar moment of inertia.

In order to obtain sufficient rigidity, it is necessary to avoid as far as possible, discontinuities in the covering of the wing or avoid the cutouts which diminish the torsional rigidity. For the LAS wing it is thus important to make sure that the zippers, which are mainly used for wing inspection, are always closed and that the seams on the fabric are still intact to ensure the torsional rigidity is not diminished during flight.
Any forms of monocoque construction such as the tubular-spar, box-spar are highly recommended as opposed to independent spar constructions [2] in increasing the torsional stiffness. The LAS wing has numerous battens, which mainly provide shear strength to the wing. However, excluding for the wing tip batten, all the other battens are not structurally joined to the leading edge tube. It would highly be recommended that a form of attachment is considered in future designs so as to elevate the torsional stiffness.

The distribution of masses is critical in flutter control. Any increase in rigidity necessary for avoiding flutter will be accompanied by an increase in weight. However, in most occasions a prudent distribution of masses, without increase, will give the same result. The essential rule from this viewpoint is to place, as far as possible, the entire weight toward the front [9]. This is seen to be another driving factor for the high flutter speed of the LAS wing which has most of the weight closer to the leading edge.

For structures not intended to be lifting devices such as suspension bridges, it is generally beneficial for aeroelastic stability to design the structure so as to have the least projected frontal area against the wind. Decreasing the projected area decreases the magnitude of the aerodynamic forces. This follows from the fact that the aerodynamic forces are (proportional to the vorticity strength, which in turn is proportional to the profile drag. A reduction in projected frontal area reduces the profile drag, and hence reduces the effective aerodynamic force [1].

Servo mechanisms have also gained a lot of applicability in active control of flutter vibrations.

A basis of future research would be to investigate the effects of free pitch movements of the wing as opposed to the classical motions of torsion and bending of the rigid wing. This research would help to identify the inherent instability of leaving the wing free in pitch.
Appendix

A1. Lagrange’s equation of motion

Proof

For conservative forces; \( \mathbf{F} = -\nabla V \), where \( V \equiv Force\ Potential \)

Newton’s 2\(^{nd}\) Law; \( \sum \mathbf{F} = ma \)

\[
\Rightarrow -\frac{\partial V}{\partial x} = m \frac{\partial^2 x}{\partial t^2}
\]

\[
\Rightarrow -\frac{\partial V}{\partial x} = m \frac{\partial}{\partial t} \left[ \frac{1}{2} \frac{\partial}{\partial x} \left( x^2 \right) \right]
\]

\[
-\frac{dv}{dx} = \frac{d}{dt} \left[ \frac{d}{dx} \left( \frac{1}{2} mv^2 \right) \right]
\]

\[
-\frac{dv}{dx} = \frac{d}{dt} \left[ \frac{d}{dx} \left( k \right) \right]
\]

adding one term on each side whose derivative is zero

\[
\frac{\partial}{\partial x} (k - v) = \frac{d}{dt} \left[ \frac{\partial}{\partial x} (k - v) \right]
\]

defining, \( L = k - v \)
\[-d \left( \frac{\partial L}{\partial x} \right) + \frac{\partial}{dx}(L) = 0 \]

For a system subjected to a non-conservative force \( Q \) and also in which damping is derived from a dissipation function \( \infty \), the Lagrangian equation becomes,

\[-d \left( \frac{\partial L}{\partial x} \right) + \frac{\partial}{dx}(L) + \frac{\partial}{dx}(Q) = Q \]

### A2. Complex Circulation Functions

\[ L_h = 1 - 2i \frac{\nu}{b \omega} (F + iG) \]

\[ L_\alpha = \frac{1}{2} - i \left( \frac{\nu}{b \omega} \right) \left[ 1 + 2(F + iG) \right] - 2 \left( \frac{\nu}{b \omega} \right)^2 (F + iG) \]

\[ M_h = \frac{1}{2} \]

\[ M_\alpha = \frac{3}{8} - i \left( \frac{\nu}{b \omega} \right) \]

where \( i = \sqrt{-1} \)

\[ C(k) = F(k) + iG(k) \]

\[ F = \frac{J_1 (J_1 + Y_0) + Y_1 (Y_1 - J_0)}{(J_1 + Y_0)^2 + (Y_1 - J_0)^2} \]

\[ G = -\frac{Y_1 Y_0 + J_1 J_0}{(J_1 + Y_0)^2 + (Y_1 - J_0)^2} \]
where $J_0$, $J_1$, $Y_0$, $Y_1$ are standard Bessel functions of the first and second kinds, of argument $k$.

These aerodynamic integrals, $(L_h, L_a, M_h$ and $M_a)$ are ideal for a straight wing and an additional modification has to be made for the LAS wing which is tapered. On a tapered wing these integrals vary along the span for a particular choice of reduced frequency,

$$K = \frac{v b_r}{U}.$$ Thus for each $K_R$, the integrals must be interpolated from tables for a series of values of $k$, and the integrals are then evaluated by a trapezoidal rule. However, these aerodynamic coefficients in incompressible flow can be closely approximated by polynomials in $1/k$. Using notation from [3] the following polynomials and subsequent table (4.1) are used;

$$L_h = 1 + \frac{K_R}{K} K_2(L_h) = 1 + \frac{b_r}{b} K_2(L_h) \quad (4.39)$$

$$L_a = 0.5 + \frac{b_r}{b} K_2(L_a) + \left( \frac{b_r}{b} \right)^2 K_3(L_a) \quad (4.40)$$

$$M_h = 0.5 \quad (4.41)$$

$$M_a = 0.375 + \frac{b_r}{b} K_2(M_a) \quad (4.42)$$
Table R.1 Aerodynamic coefficients of Incompressible Tapered Airfoils [3]

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<th>( v/b\omega )</th>
<th>( K_2(L_h) )</th>
<th>( K_2(L_a) )</th>
<th>( K_3(L_a) )</th>
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