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An Elliptic Equation With Spike Solutions Concentrating at Local Minima of the Laplacian of the Potential

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An elliptic equation with spike solutions concentrating at local minima of the Laplacian of the potential *

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Abstract

We consider the equation $-\epsilon^2\Delta u + V(z)u = f(u)$ which arises in the study of nonlinear Schrödinger equations. We seek solutions that are positive on $\mathbb{R}^N$ and that vanish at infinity. Under the assumption that f satisfies super-linear and sub-critical growth conditions, we show that for small ϵ there exist solutions that concentrate near local minima of V. The local minima may occur in unbounded components, as long as the Laplacian of V achieves a strict local minimum along such a component. Our proofs employ variational mountain-pass and concentration compactness arguments. A penalization technique developed by Felmer and del Pino is used to handle the lack of compactness and the absence of the Palais-Smale condition in the variational framework.

1 Introduction

This paper concerns the equation

$$-\epsilon^2\Delta u + V(z)u = f(u) \tag{1.1}$$

on $\mathbb{R}^N$ with $N \geq 1$, where $f(u)$ is a "superlinear" type function such as $f(u) = u^p$, $p > 1$. Such an equation arises when searching for standing wave solutions of the nonlinear Schrödinger equation (see [3]). For small positive ϵ, we seek "ground states," that is, positive solutions u with $u(z) \to 0$ as $|z| \to \infty$. Floer and Weinstein ([6]) examined the case $N = 1$, $f(u) = u^3$ and found that for small ϵ, a ground state u_ϵ exists which concentrates near a non-degenerate critical point of V. Similar results for $N > 1$ were obtained by Oh in [10]-[12]. In [3], del Pino and Felmer found that if V has a strict local minimum, then for small ϵ, (1.1) has a ground state concentrating near that minimum. A strict local minimum occurs when there exists a bounded, open set $\Lambda \subset \mathbb{R}^N$

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with $\inf_\Lambda V < \inf_{\partial\Lambda} V$. They extended their results in [4] to the more general case where V has a "topologically stable" critical point, that is, a critical point obtained via a topological linking argument (see [4] for a precise formulation). Such a critical point persists under small perturbations of V. Examples are a strict local extremum and a saddle point. This very strong result is notable because the critical points of V in question need not be non-degenerate or even isolated. Similar results have been obtained by Li [8], and earlier work of Rabinowitz [13] is also interesting. The recent results of [1] and [9] also permit V to have degenerate critical points.

A common feature of all the papers above is that V must have a non-degenerate, or at least topologically stable, set of critical points. Therefore it is natural to try to remove this requirement. That we must assume some conditions on V is shown by Wang's counterexample [15] - if V is nondecreasing and nonconstant in one variable (e.g. $V(x_1, x_2, x_3) = 2 + \tan^{-1}(x_1)$), then no ground states exist. In [14] the author showed that ground states to (1.1) exist under the assumption that V is almost periodic, together with another mild assumption. Those assumptions did not guarantee that V had a topologically stable critical point.

Aside from periodicity or recurrence properties of V, another approach is to impose conditions on the derivatives of V. That is the approach taken here. We will assume that V has a (perhaps unbounded) component of local minima, along which ΔV achieves a strict local minimum. More specifically, assume f satisfies the following:

(F1) $f \in C^1(\mathbb{R}^+, \mathbb{R})$

(F2) $f'(0) = 0 = f(0)$.

(F3) $\lim_{q\to\infty} f(q)/q^s = 0$ for some $s > 1$, with $s < (N+2)/(N-2)$ if $N \geq 3$.

(F4) For some $\theta > 2$, $0 < \theta F(q) \leq f(q)q$ for all $q > 0$, where $F(\xi) \equiv \int_0^\xi f(t)\,dt$.

(F5) The function $q \mapsto f(q)/q$ is increasing on $(0, \infty)$.

Assumptions (F1)-(F5) are the same as in [3] and are satisfied by $f(q) = q^s$, for example, if $1 < s < (N+2)/(N-2)$. Assume that V satisfies the following:

(V1) $V \in C^2(\mathbb{R}^n, \mathbb{R})$

(V2) $D^\alpha V$ is bounded and Lipschitz continuous for $|\alpha| = 2$.

(V3) $0 < V_- \equiv \inf_{\mathbb{R}^N} V < \sup_{\mathbb{R}^N} V \equiv V^+ < \infty$

(V4) There exists a bounded, nonempty open set $\Lambda \subset \mathbb{R}^N$ and a point $z_0 \in \Lambda$ with $V(z_0) = \inf_\Lambda V \equiv V_0$, and

$$\Delta_0 \equiv \inf\{\Delta V(z) \mid z \in \Lambda,\ V(z) = V_0\} < \inf\{\Delta V(z) \mid z \in \partial\Lambda,\ V(z) = V_0\}$$

Note: A special case of (V4) occurs when Λ is bounded and $V(z_0) < \inf_{\partial\Lambda} V$; this case is treated, under weaker hypotheses, in [3]. A specific example of (V4) is if $N = 2$ and V satisfies (V1)-(V4), with $V(z_1, z_2) = 1 + (z_1^2 - z_2)^2$ for $z_1^2 + z_2^2 \leq 1$. Then $\Delta V(z_1, z_1^2) = 8z_1^2 + 2$ for $z_1^2 + z_2^2 \leq 1$, so we may take $\Lambda = B_1(0,0) \subset \mathbb{R}^2$ and $z_0 = (0,0)$. Then V has a component of local minima that includes the parabolic arc $\{z_2 = z_1^2\} \cap B_1(0,0)$, along which ΔV has a minimum of 2 at $(0,0)$, with $\Delta V > 2$ at the two endpoints of the arc.

We prove the following:

Theorem 1.1 *Let V and f satisfy* (V1)-(V4) *and* (F1)-(F5). *Then there exists $\epsilon_0 > 0$ such that if $\epsilon \leq \epsilon_0$, then (1.0) has a positive solution u_ϵ with $u_\epsilon(z) \to 0$ as $|z| \to \infty$. u_ϵ has exactly one local maximum (hence, global maximum) point $z_\epsilon \in \Lambda$, where Λ is as in* (V4). *There exist $\alpha, \beta > 0$ with $u_\epsilon(z) \leq \alpha \exp(-\frac{\beta}{\epsilon}|z - z_\epsilon|)$ for $\epsilon \leq \epsilon_0$. Furthermore, with V_0 and Δ_0 as in (V4), $V(z_\epsilon) \to V_0$ and $\Delta V(z_\epsilon) \to \Delta_0$ as $\epsilon \to 0$.*

For small ϵ, u_ϵ resembles a "spike," which is sharper for smaller ϵ. The spike concentrates near a local minimum of V where ΔV has a strict local minimum. The proof of Theorem 1.1 employs the techniques of [3], with some refinements necessary because V does not necessarily achieve a *strict* local minimum. Section 2 introduces the penalization scheme developed by Felmer and del Pino, and continues with the beginning of the proof of Theorem 1.1. These beginning arguments are taken practically verbatim from [3], but are included, since the machinery of the penalization technique is used in the remainder of the proof. The reader is invited to consult [3] for more complete proofs. Section 3 contains the completion of the proof, which is original. This part contains delicate computations involving ΔV.

2 The penalization scheme

Extend f to the negative reals by defining $f(q) = 0$ for $q < 0$. Let F be the primitive of f, that is, $F(q) = \int_0^q f(t)\,dt$. Define the functional I_ϵ on $W^{1,2}(\mathbb{R}^N)$ by

$$I_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2}(\epsilon^2|\nabla u|^2 + V(z)u^2) - F(u)\,dz.$$

I_ϵ is a C^1 functional, and there is a one-to-one correspondence between positive critical points of I_ϵ and ground states of (1.1). It is well known that I_ϵ and similar functionals in related problems fail the Palais-Smale condition. That is, a "Palais-Smale sequence," defined as a sequence (u_m) with $I_\epsilon(u_m)$ convergent and $I'_\epsilon(u_m) \to 0$ as $m \to \infty$, need not have a convergent subsequence. To get around this difficulty, we formulate a "penalized" problem, with a corresponding "penalized" functional satisfying the Palais-Smale condition, by altering f outside of Λ.

Let θ be as in (F4). Choose k so $k > \theta/(\theta - 2)$. Let V_- be as in (V3) and $a > 0$ be the value at which $f(a)/a = V_-/k$. Define $\tilde{f}$ by

$$\tilde{f}(s) = \begin{cases} f(s) & s \le a; \\ sV_-/k & s > a, \end{cases} \tag{2.1}$$

$g(\cdot\,, s) = \chi_\Lambda f(s) + (1 - \chi_\Lambda)\tilde{f}(s)$, and $G(z, \xi) = \int_0^\xi g(z, \tau)\,d\tau$. Although not continuous, g is a Carathéodory function. For $\epsilon > 0$, define the penalized functional J_ϵ on $W^{1,2}(\mathbb{R}^N)$ by

$$J_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2}(\epsilon^2|\nabla u|^2 + V(z)u^2) - G(z, u)\,dz. \tag{2.2}$$

A positive critical point of J_ϵ is a weak solution of the "penalized equation"

$$-\epsilon^2 \Delta u + V(z)u = g(z, u), \tag{2.3}$$

that is, a C^1 function satisfying (2.3) wherever g is continuous. It is proven in [3] that J_ϵ satisfies all the hypotheses of the Mountain Pass Theorem of Ambrosetti and Rabinowitz ([2]), including the Palais-Smale condition. Therefore J_ϵ has a critical point u_ϵ, with the mountain pass critical level $c(\epsilon) = J_\epsilon(u_\epsilon)$. $c(\epsilon)$ is defined by the following minimax: let the set of paths $\Gamma_\epsilon = \{\gamma \in C([0, 1], W^{1,2}(\mathbb{R}^N)) \mid \gamma(0) = 0,\ J_\epsilon(\gamma(1)) < 0\}$, and

$$c(\epsilon) = \inf_{\gamma\in\Gamma_\epsilon} \max_{\theta\in[0,1]} J_\epsilon(\gamma(\theta)).$$

As shown in ([3]), because of (F4), $c(\epsilon)$ can be characterized more simply as

$$c(\epsilon) = \inf_{u\in W^{1,2}(\mathbb{R}^N)\setminus\{0\}} \sup_{\tau>0} J_\epsilon(\tau u).$$

The functions $g(z, q)$ and $f(q)$ agree whenever $z \in \Lambda$ or $q < a$. Therefore if u is a weak solution of (2.3) with $u < a$ on $\Lambda^{\mathbf{C}} \equiv \mathbb{R}^N \setminus \Lambda$, then u solves (1.1). Our plan is to find a positive critical point u_ϵ of J_ϵ, which is a weak solution of (2.3), then show that $u_\epsilon(z) < a$ for all $z \in \Lambda^{\mathbf{C}}$.

For $\epsilon > 0$, let u_ϵ be a critical point of J_ϵ with $J_\epsilon(u_\epsilon) = c(\epsilon)$. Maximum principle arguments show that u_ϵ must be positive. Define the "limiting functional" I_0 by

$$I_0(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V_0 u^2) - F(u) \tag{2.4}$$

and

$$\underline{c} = \inf_{u\in W^{1,2}(\mathbb{R}^N)\setminus\{0\}} \sup_{\tau>0} I_0(\tau u). \tag{2.5}$$

The equation corresponding to (2.4) is

$$-\Delta u + V_0 u = f(u) \tag{2.6}$$

We will prove Theorem 1.1 by proving the following proposition:

Proposition 2.1 *Let $\epsilon > 0$. If u_ϵ is a positive solution of (2.3) satisfying $J_\epsilon(u_\epsilon) = c(\epsilon)$, then*

(i) $\lim_{\epsilon\to 0} \max_{z\in\partial\Lambda} u_\epsilon = 0$.

(ii) *For all ϵ sufficiently small, u_ϵ has only one local maximum point in Λ (call it z_ϵ), with* $\lim_{\epsilon\to 0} V(z_\epsilon) = V_0$

(iii) $\lim_{\epsilon\to 0} \Delta V(z_\epsilon) = \Delta_0$.

Proof of Theorem 1.1: Assuming Proposition 2.1, there exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, $u_\epsilon < a$ on $\partial\Lambda$. In [3] it is shown that if we multiply (2.3) by $(u_\epsilon - a)_+$ and integrate by parts, it follows that $u_\epsilon < a$ on $\Lambda^{\mathbf{C}}$, so u_ϵ solves (1.1). By the definition of a in (2.1), and the maximum principle, u_ϵ has no local maxima outside of Λ, so u_ϵ has exactly one local maximum point z_ϵ, which occurs in Λ.

Define v_ϵ by translating u_ϵ from z_ϵ to zero and dilating it by ϵ, that is,

$$v_\epsilon(z) = u_\epsilon(z_\epsilon + \epsilon z).$$

Then v_ϵ is a weak (C^1) solution of the "translated and dilated" equation

$$-\Delta v_\epsilon + V(z_\epsilon + \epsilon z)v_\epsilon = g(z_\epsilon + \epsilon z, v_\epsilon).$$

Let $\epsilon_j \to 0$. Along a subsequence (called (z_{ϵ_j})), $z_{\epsilon_j} \to \bar{z} \in \overline{\Lambda}$, with $V(\bar{z}) = V_0$ and $\Delta V(\bar{z}) = \Delta_0$.

Along a subsequence, v_{ϵ_j} converges locally uniformly to a function v^0. Pick $R > 0$ so $v^0 < a$ on $\mathbb{R}^N \setminus B_R(0)$. For large enough ϵ, $v_\epsilon < a$ on $\partial B_R(0)$. By the maximum principle arguments of [3], for small ϵ, v_ϵ decays exponentially, uniformly in ϵ. $\diamondsuit$

The proof of Proposition 2.1 will follow if we can prove the following statement.

Proposition 2.2 *If $\epsilon_n \to 0$ and $(z_n) \subset \overline{\Lambda}$ with $u_{\epsilon_n}(z_n) \geq b > 0$, then*

(i) $\lim_{n\to\infty} V(z_n) = V_0$.

(ii) $\lim_{n\to\infty} \Delta V(z_n) = \Delta_0$.

It is proven in [3] that u_ϵ has exactly one local maximum point z_ϵ for small ϵ. Since u_ϵ solves (2.3), the maximum principle implies that $u_\epsilon(z_\epsilon)$ is bounded away from zero. Thus Proposition 2.2 and (V4) give Proposition 2.1(ii)-(iii).

To prove Proposition 2.2, let b and (z_n) be as above. First we repeat the argument of [3] to show that $V(z_n) \to V_0$: suppose this does not happen. Then, along a subsequence, $z_n \to \bar{z} \in \overline{\Lambda}$ with $V(\bar{z}) > V_0$. Define v_n by translating u_{ϵ_n} from z_n to 0 and dilating by ϵ_n; that is,

$$v_n(z) = u_{\epsilon_n}(z_n + \epsilon_n z). \tag{2.7}$$

v_n solves the "translated and dilated" penalized equation

$$-\Delta v_n + V(z_n + \epsilon_n z)v_n = g(z_n + \epsilon_n z, v_n) \tag{2.8}$$

on $\mathbb{R}^N$, with $v_n(z) \to 0$ and $\nabla v_n(z) \to 0$ as $|z| \to \infty$. As shown in [3], (v_n) is bounded in $W^{1,2}(\mathbb{R}^N)$, so by elliptic estimates, (v_n) converges locally along a subsequence (also denoted (v_n)) to $v^0 \in W^{1,2}(\mathbb{R}^N)$. Define χ_n by $\chi_n(z) = \chi_\Lambda(z_n + \epsilon_n z)$, where χ_Λ is the characteristic function of Λ. χ_n converges weakly in L^p over compact sets to a function χ, for any $p > 1$, with $0 \le \chi \le 1$. Define

$$\bar{g}(z, s) = \chi(z)f(s) + (1 - \chi(z))\tilde{f}(s)$$

Then v^0 satisfies

$$-\Delta v + V(\bar{z})v = \bar{g}(z, v) \tag{2.9}$$

on $\mathbb{R}^N$. Define $\bar{G}(z, s) = \int_0^s \bar{g}(z, t)\,dt$. Associated with (2.9) we have the limiting functional $\bar{J}(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V(\bar{z})u^2) - \bar{G}(z, u)\,dz$. v^0 is a positive critical point of $\bar{J}$.

Define J_n to be the "translated and dilated" penalized functional corresponding to (2.8), that is,

$$J_n(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V(z_n + \epsilon_n z)u^2) - G(z_n + \epsilon_n z, u)\,dz.$$

Clearly $J_n(v_n) = \epsilon_n^{-N} J_{\epsilon_n}(u_{\epsilon_n})$. In [3] it is proven that

$$\liminf_{n\to\infty} J_n(v_n) \ge \bar{J}(v^0). \tag{2.10}$$

Also, by letting w be a ground state for (2.6) with $I_0(w) = \underline{c}$ (the mountain pass value for I_0, defined in (2.5) and using w as a test function for J_n, it is proven that $\underline{c} \ge \liminf_{n\to\infty} J_n(v_n)$. Thus $\bar{J}(v^0) \le \underline{c}$. Therefore, as shown in [3], $V(\bar{z}) \le V_0$. This contradicts our assumption. Thus $V(z_n) \to V_0$. All the above is the same as was proven in [3]. Next, we must show that $\Delta V(z_n) \to \Delta_0$. That is the focus of the next section.

3 The effect of the Laplacian

Proving $\Delta V(z_n) \to \Delta_0$ is a subtle and delicate problem. Making ϵ_n approach 0 is equivalent to dilating V, which has the effect of making local minima of V behave more like global minima. This assists in finding solutions to (1.1). However, making ϵ_n small *reduces* the effect of differences in ΔV. For this reason, Theorem 1.1 is not only difficult to prove, but is not intuitively compelling, either.

It is known ([7]) that a "least energy solution" of (2.6), that is, a solution w with $I_0(w) = \underline{c}$, must be radially symmetric. We will need to exploit this fact. In order to do this, we will need to work with the maximum points of u_{ϵ_n} instead of merely the (z_n) as given in Proposition 2.2. We need the following concentration-compactness result, which states that the sequence (u_{ϵ_n}) of "mountain-pass type solutions" of (2.3) does not "split":

Lemma 3.1 *If $(z_n) \subset \overline{\Lambda}$, $(y_n) \subset \mathbb{R}^N$, and $b > 0$ with $u_{\epsilon_n}(z_n) > b$ and $u_{\epsilon_n}(y_n) > b$ for all n, then $((z_n - y_n)/\epsilon_n)$ is bounded.*

Proof: define $v_n(z) = u_{\epsilon_n}(z_n + \epsilon_n z)$ as in (2.7). Suppose the lemma is false. Then, along a subsequence, $|y_n - z_n|/\epsilon_n \to \infty$. Let $x_n = (y_n - z_n)/\epsilon_n$. $(\|v_n\|)$ is bounded in $W^{1,2}(\mathbb{R}^N)$ and $|x_n| \to \infty$, so we may pick a sequence $(R_n) \subset \mathbb{N}$ with $R_n \to \infty$, $|x_n| - R_n \to \infty$, and $\|v_n\|_{W^{1,2}(B_{R_n+1}(0)\setminus B_{R_n-1}(0))} \to 0$ as $n \to \infty$. Define cutoff functions $\varphi_n^{1,2} \in C^\infty(\mathbb{R}^N, [0,1])$ satisfying $\varphi_1 \equiv 1$ on $B_{R_n-1}(0)$, $\varphi_1 \equiv 0$ on $B_{R_n}(0)^{\mathbf{C}}$, $\varphi_2 \equiv 1$ on $B_{R_n+1}(0)^{\mathbf{C}}$, $\varphi_2 \equiv 0$ on $B_{R_n}(0)$, and $\|\nabla\varphi_1\|_{L^\infty(\mathbb{R}^N)} < 2$, $\|\nabla\varphi_2\|_{L^\infty(\mathbb{R}^N)} < 2$. Set $v_n^1 = \varphi_n^1 v_n$ and $v_n^2 = \varphi_n^2 v_n$, and $\bar{v}_n = v_n^1 + v_n^2 = (\varphi_n^1 + \varphi_n^2)v_n$.

Choose $T_n > 0$ so $J_n(T_n\bar{v}_n) = 0$. We claim that T_n is well-defined, and bounded in n. Note that the existence of T_n must be checked for the penalized functional J_n, because of the replacement of F with G. By elliptic estimates, there exists an open set $U \subset \mathbb{R}^N$ such that along a subsequence, $v_n^1 > b/2$ on U and $U \subset (\Lambda - z_n)/\epsilon_n \equiv \{z \in \mathbb{R}^N \mid z_n + \epsilon_n z \in \Lambda\}$. Let a be as in (2.1). For $t > 2a/b$ and $z \in U$, $t\bar{v}_n(z) > tb/2 > a$, so $G(z_n + \epsilon_n z, t\bar{v}_n) = F(t\bar{v}_n) > F(bt/2)$. Therefore, for $t > 2a/b$,

$$\begin{aligned} J_n(t\bar{v}_n) &= t^2 \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla\bar{v}_n|^2 + V(z_n + \epsilon_n z)\bar{v}_n^2)\,dz - \int_{\mathbb{R}^N} G(z_n + \epsilon_n z, t\bar{v}_n)\,dz \\ &\le \frac{t^2}{2}(1 + V^+)\|\bar{v}_n\|^2_{W^{1,2}(\mathbb{R}^N)} - \int_U F(t\bar{v}_n) \\ &\le \frac{t^2}{2}(1 + V^+)\|\bar{v}_n\|^2_{W^{1,2}(\mathbb{R}^N)} - \lambda(U)F(tb/2), \end{aligned}$$

where λ indicates the Lebesgue measure. By (F4), there exists $C > 0$ such that for $t > 2a/b$, $F(tb/2) > Ct^\theta$. Therefore, for $t > 2a/b$,

$$J_n(t\bar{v}_n) \le \frac{t^2}{2}(1 + V^+)\|\bar{v}_n\|^2_{W^{1,2}(\mathbb{R}^N)} - Ct^\theta. \tag{3.1}$$

Since $(\bar{v}_n)$ is bounded in $W^{1,2}(\mathbb{R}^N)$, this gives the existence and boundedness of (T_n).

Since $J_n(T_n\bar{v}_n) = J_n(T_n v_n^1) + J_n(T_n v_n^2) = 0$, we may pick $i_n \in \{1,2\}$ with $J_n(T_n v_n^{i_n}) \le 0$. By (F5) and (2.1), the map $t \mapsto J_n(tv_n^{i_n})$ increases from zero at $t = 0$, achieves a positive maximum, then decreases to $-\infty$. We will see more of this in a moment. Thus there exists a unique $t_n \in (0, T_n)$ with $J_n(t_n v_n^{i_n}) = \max_{t>0} J_n(tv_n^{i_n})$. We claim that t_n and $T_n - t_n$ are both bounded away from zero for large n: by $(f_1) - (f_4)$ and (2.1), $J_n(w) \ge \frac{1}{\theta}\min(1, V_-)\|w\|^2_{W^{1,2}(\mathbb{R}^N)} - o(\|w\|^2_{W^{1,2}(\mathbb{R}^N)})$ uniformly in n, so $\max_{t>0} J_n(tv_n^{i_n})$ is bounded away from zero, uniformly in n. It is easy to show that J_n is Lipschitz on bounded subsets of $W^{1,2}(\mathbb{R}^N)$, uniformly in n. Since (T_n) is bounded, this implies that t_n and $T_n - t_n$ are both bounded away from zero for large n.

By definition of v_n as a "mountain-pass type critical point" of J_n, we have

$$\max_{t>0} J_n(tv_n^{i_n}) \ge \max_{t>0} J_n(tv_n).$$

Using the facts that $\|v_n - \bar{v}_n\|_{W^{1,2}(\mathbb{R}^N)} \to 0$ as $n \to \infty$, and (T_n) is bounded, we have

$$\begin{aligned} \liminf_{n\to\infty} J_n(t_n v_n^{i_n}) &= \liminf_{n\to\infty} \max_{t>0} J_n(t v_n^{i_n}) \\ &\geq \liminf_{n\to\infty} \max_{t>0} J_n(t v_n) \\ &= \liminf_{n\to\infty} \max_{t>0} J_n(t \bar{v}_n) \\ &= \liminf_{n\to\infty} J_n(t_n \bar{v}_n) \\ &= \liminf_{n\to\infty} (J_n(t_n v_n^{i_n}) + J_n(t_n v_n^{3-i_n})) \\ &\geq \liminf_{n\to\infty} J_n(t_n v_n^{i_n}) + \liminf_{n\to\infty} J_n(t_n v_n^{3-i_n}). \end{aligned} \tag{3.2}$$

Now $J_n(T_n v_n^{3-i_n}) = -J_n(T_n v_n^{i_n}) \geq 0$ and $t_n < T_n$, so $J_n(t_n v_n^{3-i_n}) \geq 0$. By (3.2), $\liminf_{n\to\infty} J_n(t_n v_n^{3-i_n}) \leq 0$. Therefore $J_n(t_n v_n^{3-i_n}) \to 0$ as $n \to \infty$.

Since $J_n(w) \geq \frac{1}{\theta}\min(1, V_-)\|w\|^2_{W^{1,2}(\mathbb{R}^N)} - o(\|w\|^2_{W^{1,2}(\mathbb{R}^N)})$ uniformly in n, there exists $d \in (0, \liminf_{n\to\infty} t_n)$ such that $\liminf_{n\to\infty} J_n(d v_n^{3-i_n}) > 0$. Since $d < t_n$ and $J_n(d v_n^{3-i_n}) > J_n(t_n v_n^{3-i_n})$ for large n, the map $t \mapsto J_n(t v_n^{3-i_n})$ achieves a maximum at some $t'_n \in (0, t_n)$, and that maximum is bounded away from zero.

Summarizing the important facts about the mapping $t \mapsto J_n(t v_n^{3-i_n})$, we have shown that there exists $\rho > 0$ such that for large n,

(i) $0 < t'_n < t_n < T_n$

(ii) (T_n) is bounded.

(iii) $(T_n - t_n)$ is bounded away from zero.

(iv) $J_n(t'_n v_n^{3-i_n}) > \rho > 0$

(v) $J_n(t_n v_n^{3-i_n}) \to 0$

(vi) $J_n(T_n v_n^{3-i_n}) \geq 0$

From (i)-(vi) it is apparent that at some $t^*_n > t'_n$, the mapping $t \mapsto J_n(t v_n^{3-i_n})$ is at once decreasing and concave upward. But this is impossible: let $n \in \mathbb{N}$ and $w \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}$. Define $\psi(t) = J_n(tw)$ for $t > 0$. Then

$$\begin{aligned} \psi'(t) &= t\int_{\mathbb{R}^N} |\nabla w|^2 + V(z_n + \epsilon_n z) w^2\,dz - \int_{\mathbb{R}^N} g(z_n + \epsilon_n z, tw) w\,dz \\ &= t\left[\int_{\mathbb{R}^N} |\nabla w|^2 + V(z_n + \epsilon_n z) w^2\,dz - \int_{\{w\neq 0\}} \frac{g(z_n + \epsilon_n z, tw)}{tw} w^2\,dz\right]. \end{aligned}$$

By (F5) and (2.1), $t \mapsto g(z_n + \epsilon_n z, tw)/(tw)$ is nondecreasing, so if $\psi'(t)$ ever becomes negative, ψ' is increasing for all time t after that, and the graph of ψ is concave down. Therefore the behavior of $J_n(t v_n^{3-i_n})$ as described in (i)-(vi) is impossible, and Lemma 3.1 is proven. $\diamondsuit$

As mentioned before, it will be advantageous to work with the maxima of (u_{ϵ_n}). Choose $(y_n) \subset \mathbb{R}^N$ with

$$u_{\epsilon_n}(y_n) = \max_{\mathbb{R}^N} u_{\epsilon_n}.$$

We will prove

$$\Delta V(y_n) \to \Delta_0. \tag{3.3}$$

By Lemma 3.0, $((y_n - z_n)/\epsilon_n)$ is bounded, so $y_n - z_n \to 0$. Thus (3.3) gives Proposition 2.2(ii), completing the proof of Theorem 1.1. $\diamondsuit$

Along a subsequence, $y_n \to \bar{y} \in \overline{\Lambda}$. By Proposition 2.2(i), $V(\bar{y}) = V_0$. Since is not apparent that $\bar{y} \in \Lambda$, we must proceed carefully. We will redefine the v_n's like in (2.7), by translating u_{ϵ_n} to 0 and dilating it. That is,

$$v_n(z) = u_{\epsilon_n}(y_n + \epsilon_n z). \tag{3.4}$$

Then v_n is a positive weak solution, vanishing at infinity, of the "penalized, dilated, and translated" PDE

$$-\Delta v + V(y_n + \epsilon_n z)v = g(y_n + \epsilon_n z, v).$$

Like before, (v_n) converges locally uniformly to a function v_0. We claim that v_0 is actually a ground state maximizing at 0 of the autonomous limiting equation (2.6). Proof: As before, define χ_n by $\chi_n(z) = \chi(y_n + \epsilon z)$. As before, along a subsequence, χ_n converges weakly in L^p, for any $p > 1$, on compact subsets of $\mathbb{R}^N$ to a function χ with $0 \le \chi \le 1$. Define $\bar{g}$ by

$$\bar{g}(z, s) = \chi(z)f(s) + (1 - \chi(z))\tilde{f}(s).$$

By the argument of Proposition 2.2, taken from [3], (v_n) converges locally along a subsequence to v_0, a ground state of $-\Delta v + V_0 v = \bar{g}(z, v)$. The functional corresponding to this equation is $\bar{J}(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V_0 u^2) - \bar{G}(z, u)\, dz$, where $\bar{G}(z, s) = \int_0^s \bar{g}(z, t)\, dt$. As before, in (2.10), $\underline{c} \ge \liminf_{n\to\infty} J_n(v_n) \ge \bar{J}(v_0)$, where $\underline{c}$ is from (2.5). $\bar{J} \ge I_0$, where I_0 is the "autonomous" limiting functional from (2.4), so

$$\underline{c} \le \max_{t>0} I_0(tv_0) \le \max_{t>0} \bar{J}(tv_0) \le \underline{c},$$

and v_0 is actually a ground state of (2.6). $\diamondsuit$

Not only does (v_n) converge locally to v_0, but it satisfies the following lemma.

Lemma 3.2 *With (v_n) as in (3.4), for any subsequence of (v_n) there is a radially symmetric ground state v_0 of (2.6) such that $v_n \to v_0$ uniformly along a subsequence and the v_n's decay exponentially, uniformly in n.*

Proof: If one establishes uniform convergence, the uniform exponential decay follows readily, using a standard maximum principle argument found in [3]. Suppose the convergence is not uniform. Then there exist a subsequence of (v_n) (denoted (v_n)) and a sequence $(x_n) \subset \mathbb{R}^N$ with $|x_n| \to \infty$ and $\lim_{n\to\infty} v_n(x_n) > 0$. Let $d > 0$ with $d < v_0(0)$ and $d < \lim_{n\to\infty} v_n(x_n)$. For large n, $d < v_n(0) = u_{\epsilon_n}(z_n)$ and $d < v_n(x_n) = u_{\epsilon_n}(z_n + \epsilon_n x_n)$. Letting $w_n = z_n + \epsilon_n x_n$, we obtain $((w_n - z_n)/\epsilon_n) = (x_n)$, which is unbounded, violating Lemma 3.1.

To show $\Delta V(y_n) \to \Delta_0$, we again argue indirectly. Suppose otherwise. Then, along a subsequence, $y_n \to \bar{y} \in \overline{\Lambda}$ with

$$\Delta V(\bar{y}) > \Delta_0 \,. \tag{3.5}$$

For $x \in \mathbb{R}^N$, define the translation operator τ_x by $\tau_x u(z) = u(z-x)$, that is, $\tau_x u$ is u translated by x. Assume for convenience, and without loss of generality, that

$$0 \in \Lambda,\; V(0) = V_0,\; \text{and } \Delta V(0) = \Delta_0.$$

We will prove that for large n,

$$\sup_{t>0} J_{\epsilon_n}(t\tau_{-y_n/\epsilon_n} u_{\epsilon_n}) < J_{\epsilon_n}(u_{\epsilon_n}) = \sup_{t>0} J_{\epsilon_n}(tu_{\epsilon_n}), \tag{3.6}$$

recalling the definition of J_ϵ in (2.2), and how v_n is defined from u_{ϵ_n} in (3.4). That is, translating tu_{ϵ_n} back to the origin reduces the value of $J_{\epsilon_n}(tv_n)$ because V has lesser concavity at the origin. This occurs even though shrinking ϵ reduces the difference in concavity. (3.6) contradicts the definition of u_{ϵ_n}.

Pick $T > 1$ large enough so that for large n, $J_n(Tv_n) = \epsilon_n^{-N} J_{\epsilon_n}(Tu_{\epsilon_n}) < 0$. This is possible by the argument of (3.1). Now (3.6) is equivalent to

$$\sup_{0\le t\le T} J_{\epsilon_n}(t\tau_{-y_n} u_{\epsilon_n}) < \sup_{0\le t\le T} J_{\epsilon_n}(tu_{\epsilon_n}).$$

To prove the above, it will suffice to prove the stronger fact that for large n, for all $t \in (0, T)$,

$$J_{\epsilon_n}(tu_{\epsilon_n}) > J_{\epsilon_n}(t\tau_{-y_n} u_{\epsilon_n}).$$

Now, along a subsequence, $v_n \to v_0$ uniformly, so by the definition of v_n as a dilation of $\tau_{-y_n} u_{\epsilon_n}$ ((3.4)), $u_{\epsilon_n} \to 0$ uniformly on $\mathbb{R}^N \setminus \Lambda$ as $n \to \infty$. Thus for large n and $0 \le t \le T$, the definition of G gives $G(z, t\tau_{-y_n} u_{\epsilon_n}(z)) = F(t\tau_{-y_n} u_{\epsilon_n}(z)$ for all $z \in \mathbb{R}^N$, so

$$\begin{aligned}
&J_{\epsilon_n}(tu_{\epsilon_n}) - J_{\epsilon_n}(t\tau_{-y_n} u_{\epsilon_n}) \\
&= \int_{\mathbb{R}^N} \frac{1}{2} t^2 \left(|\nabla u_{\epsilon_n}(z)|^2 + V(z) u_{\epsilon_n}(z)^2\right) - G(z, tu_{\epsilon_n}(z))\, dz \\
&\quad -\Big[\int_{\mathbb{R}^N} \frac{1}{2} t^2 \left(|\nabla \tau_{-y_n} u_{\epsilon_n}(z)|^2 + V(z)\tau_{-y_n} u_{\epsilon_n}(z)^2\right) - F(t\tau_{-y_n} u_{\epsilon_n}(z))\, dz\Big] \\
&\ge \frac{1}{2} t^2 \int_{\mathbb{R}^N} V(z)(u_{\epsilon_n}(z)^2 - u_{\epsilon_n}(z + y_n)^2)\, dz
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N} F(tu_{\epsilon_n}(z+y_n)) - F(tu_{\epsilon_n}(z))\,dz \\
=\;& \frac{1}{2}t^2\int_{\mathbb{R}^N}(V(z+y_n)-V(z))u_{\epsilon_n}(z+y_n)^2\,dz \\
=\;& \frac{1}{2}t^2\epsilon_n^N\int_{\mathbb{R}^N}(V(y_n+\epsilon_n z)-V(\epsilon_n z))u_{\epsilon_n}(\epsilon_n z+y_n)^2\,dz \\
=\;& \frac{1}{2}t^2\epsilon_n^N\int_{\mathbb{R}^N}(V(y_n+\epsilon_n z)-V(\epsilon_n z))v_n(z)^2\,dz\,.
\end{aligned}
$$

For $n = 1, 2, \ldots$, define $h_n : \mathbb{R} \to \mathbb{R}$ by

$$h_n(t) = \int_{\mathbb{R}^N}(V(y_n+tz)-V(tz))v_n^2\,dz.$$

Since $h_n(\epsilon_n) = \int_{\mathbb{R}^N}(V(y_n+\epsilon_n z)-V(\epsilon_n z))v_n^2$, we must prove that for large n,

$$h_n(\epsilon_n) > 0\,. \tag{3.7}$$

Assume without loss of generality that Λ was chosen so that there exists $\rho > 0$ with

$$\inf_{N_\rho(\Lambda)} V = V_0, \tag{3.8}$$

where $N_\rho(\Lambda) = \{x \in \mathbb{R}^N \mid \exists y \in \Lambda \text{ with } |y-x| < \rho\}$. We will prove the following facts about h_n:

Lemma 3.3 *For some $\beta > 0$, for large n,*

(i) $h_n \in C^2(\mathbb{R}^+, \mathbb{R})$

(ii) $h_n(0) \geq 0$

(iii) $|h_n'(0)|^2 \leq o(1)h_n(0)$

(iv) $h_n''(0) > \beta$

(v) h_n'' *is locally Lipschitz on $\mathbb{R}^+$, uniformly in n.*

Here $o(1) \to 0$ as $n \to \infty$. Before proving Lemma 3.3, let us prove how it gives (3.7). By (iv)-(v), there exists $d > 0$ such that for large n and $0 \leq t \leq d$, $h_n''(t) > \beta/2$. For $t \in [0, d]$, a Taylor's series expansion shows that for large n,

$$h_n(t) \geq h_n(0) + h_n'(0)t + \frac{\beta}{4}t^2 \equiv l_n(t). \tag{3.9}$$

If $h_n(0) = 0$, then by Lemma 3.3(iii), $h_n'(0) = 0$, so (3.9) implies that $h_n(t) > 0$ for all $t \in (0, d)$, giving (3.7) if n is large enough that $\epsilon_n < d$. If $h_n(0) > 0$, then by elementary calculus, l_n attains a minimum value at $t = -2h_n'(0)/\beta$, and the minimum value is

$$\min_{\mathbb{R}} l_n = l_n(-2h_n'(0)/\beta) = h_n(0) - h_n'(0)^2/\beta \geq (1 - o(1))h_n(0),$$

where $o(1) \to 0$ as $n \to \infty$. For large n, if $h_n(0) > 0$ then $l_n(t) > 0$ for all $t \in \mathbb{R}$, so $h_n(t) > 0$ for all $t \in (0, d)$ for large n, implying (3.7) if n is large enough so that $\epsilon_n < d$.

Proof of Lemma 3.3 Statement (ii) is trivial, since $h_n(0) = (V(y_n) - V_0)\int_{\mathbb{R}^N} v_n^2$, and since $z_n \in \overline{\Lambda}$ and $y_n - z_n \to 0$, (3.8) implies $V(y_n) \geq V_0$ for large n. (i) and (v) follow from Leibniz's Rule, $(V_1)-(V_2)$, and the fact that the v_n's decay exponentially, uniformly in n. For $j = 1, 2$,

$$h_n^{(j)}(t) = \int_{\mathbb{R}^N} \sum_{|\alpha|=j} (D^\alpha V(y_n + tz) - D^\alpha V(tz)) z^\alpha v_n(z)^2 \, dz.$$

Since (V2) holds, v_n decays exponentially, uniformly in n, $y_n \to \bar{y}$, and v_0 is radially symmetric, we have

$$\begin{aligned}
h_n''(0) &= \int_{\mathbb{R}^N} \sum_{|\alpha|=2} (D^\alpha V(y_n) - D^\alpha V(0)) z^\alpha v_n(z)^2 \, dz \\
&\to \int_{\mathbb{R}^N} \sum_{|\alpha|=2} (D^\alpha V(\bar{y}) - D^\alpha V(0)) z^\alpha v_0(z)^2 \, dz \\
&= \int_{\mathbb{R}^N} \sum_{i=1}^{N} (D^{ii} V(\bar{y}) - D^{ii} V(0)) z_i^2 v_0(z)^2 \, dz \\
&= \int_{\mathbb{R}^N} \sum_{i=1}^{N} (D^{ii} V(\bar{y}) - D^{ii} V(0)) \frac{1}{N} |z|^2 v_0(z)^2 \, dz \\
&= \frac{1}{N} (\Delta V(\bar{y}) - \Delta V(0)) \int_{\mathbb{R}^N} |z|^2 v_0(z)^2 \, dz > 0
\end{aligned}$$

by assumption (3.5). Since Lemma 3.3(v) holds, we have Lemma 3.3(iv).

To prove Lemma 3.3(iii), we will need the following calculus lemma:

Lemma 3.4 *Let $U \subset \mathbb{R}^N$ and $r > 0$. Let $V \in C^2(N_r(U), \mathbb{R})$ with $\inf_{N_r(U)} V \equiv V_0 > -\infty$, $|\nabla V|$ bounded on $N_r(U)$, and $D^2 V$ Lipschitz on $N_r(U)$. Then there exists $C > 0$ with*

$$|\nabla V(z)|^2 \leq C(V(z) - V_0) \tag{3.10}$$

for all $z \in U$.

Proof: let $B > 0$ with $|D^2 V(z)\xi \cdot \xi| \leq B$ for all $\xi \in \mathbb{R}^N$ with $|\xi| = 1$. Also let B be big enough so

$$B > |\nabla V(z)|/r$$

for all $z \in U$. Pick $z \in U$. If $|\nabla V(z)| = 0$, then (3.10) is obvious. Otherwise, let $d = |\nabla V(z)|/B < r$. Define $\varphi(t) = V(z - t\nabla V(z)/|\nabla V(z)|)$ for $t \in [0, d]$. φ is C^2, $\varphi(0) = V(z)$, and $\varphi'(0) = -|\nabla V(z)|$. By choice of B and the fact that $B_d(z) \subset N_r(U)$, $|\varphi''(t)| \leq B$ for all $t \in [0, d]$. Taylor's theorem gives

$$\varphi(d) - \varphi(0) = \varphi'(0)d + \varphi''(\xi)\frac{d^2}{2} \leq -|\nabla V(z)|d + Bd^2/2 = -\frac{|\nabla V(z)|^2}{2B}.$$

Also $\varphi(d) \geq V_0$ because $B_d(z) \subset N_r(U)$. Therefore,

$$\frac{|\nabla V(z)|^2}{2B} \leq \varphi(0) - \varphi(d) \leq V(z) - V_0 \,.$$

Lemma 3.4 is proven. $\diamondsuit$

To prove Lemma 3.3(iii), first note that, by the radial symmetry of v_0, the uniform exponential decay of v_n, and the uniform convergence $v_n \to v_0$,

$$\begin{aligned}
|h_n'(0)| &= |(\nabla V(y_n) - \nabla V(0)) \cdot \int_{\mathbb{R}^N} z v_n^2 \, dz| \\
&= |\nabla V(y_n) \cdot \int_{\mathbb{R}^N} z v_n^2 \, dz| \\
&= |\nabla V(y_n) \cdot \int_{\mathbb{R}^N} z v_0^2 \, dz + \quad \nabla V(y_n) \cdot \int_{\mathbb{R}^N} z(v_n^2 - v_0^2) \, dz| \\
&= |\nabla V(y_n) \cdot \int_{\mathbb{R}^N} z(v_n^2 - v_0^2) \, dz| \\
&\leq |\nabla V(y_n)| \, |\int_{\mathbb{R}^N} z(v_n^2 - v_0^2) \, dz| \\
&\leq o(1)|\nabla V(y_n)|,
\end{aligned}$$

so Lemma 3.4 implies

$$\begin{aligned}
|h_n'(0)|^2 &\leq o(1)|\nabla V(y_n)|^2 \leq o(1)(V(y_n) - V_0) \\
&\leq o(1)(V(y_n) - V_0) \int_{\mathbb{R}^N} v_n^2 \\
&= o(1)h_n(0),
\end{aligned}$$

since $\int_{\mathbb{R}^N} v_n^2$ is bounded away from zero. Lemma 3.3(iii) is proven. Thence follow (3.7), (3.3), Proposition 2.2, and Theorem 1.1.

Remarks: Besides the results cited in the introduction, many important results for equations of type (1.1) have been found recently. For instance, the work in [3]-[5] suggests that Theorem 1.1 could be strengthened by working on a smaller domain than $\mathbb{R}^N$, or by weakening the hypotheses on V. It is natural to try to extend Theorem 1.1 to cases where V is not C^2, or to the case where the second derivatives of V do not provide a condition like (V4), but higher-order derivatives do.

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