An Elliptic Equation With Spike Solutions Concentrating at Local Minima of the Laplacian of the Potential

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An elliptic equation with spike solutions concentrating at local minima of the Laplacian of the potential *

Gregory S. Spradlin

Abstract

We consider the equation $-\epsilon^2 \Delta u + V(z)u = f(u)$ which arises in the study of nonlinear Schrödinger equations. We seek solutions that are positive on $\mathbb{R}^N$ and that vanish at infinity. Under the assumption that $f$ satisfies super-linear and sub-critical growth conditions, we show that for small $\epsilon$ there exist solutions that concentrate near local minima of $V$. The local minima may occur in unbounded components, as long as the Laplacian of $V$ achieves a strict local minimum along such a component. Our proofs employ variational mountain-pass and concentration compactness arguments. A penalization technique developed by Felmer and del Pino is used to handle the lack of compactness and the absence of the Palais-Smale condition in the variational framework.

1 Introduction

This paper concerns the equation

$$-\epsilon^2 \Delta u + V(z)u = f(u)$$

on $\mathbb{R}^N$ with $N \geq 1$, where $f(u)$ is a “superlinear” type function such as $f(u) = u^p$, $p > 1$. Such an equation arises when searching for standing wave solutions of the nonlinear Schrödinger equation (see [3]). For small positive $\epsilon$, we seek “ground states,” that is, positive solutions $u$ with $u(z) \to 0$ as $|z| \to \infty$. Floer and Weinstein ([6]) examined the case $N = 1$, $f(u) = u^3$ and found that for small $\epsilon$, a ground state $u_\epsilon$ exists which concentrates near a non-degenerate critical point of $V$. Similar results for $N > 1$ were obtained by Oh in [10]-[12]. In [3], del Pino and Felmer found that if $V$ has a strict local minimum, then for small $\epsilon$, (1.1) has a ground state concentrating near that minimum. A strict local minimum occurs when there exists a bounded, open set $\Lambda \subset \mathbb{R}^N$

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with $\inf_{\Lambda} V < \inf_{\partial \Lambda} V$. They extended their results in [4] to the more general case where $V$ has a “topologically stable” critical point, that is, a critical point obtained via a topological linking argument (see [4] for a precise formulation). Such a critical point persists under small perturbations of $V$. Examples are a strict local extremum and a saddle point. This very strong result is notable because the critical points of $V$ in question need not be non-degenerate or even isolated. Similar results have been obtained by Li [8], and earlier work of Rabinowitz [13] is also interesting. The recent results of [1] and [9] also permit $V$ to have degenerate critical points.

A common feature of all the papers above is that $V$ must have a non-degenerate, or at least topologically stable, set of critical points. Therefore it is natural to try to remove this requirement. That we must assume some conditions on $V$ is shown by Wang’s counterexample [15] - if $V$ is nondecreasing and nonconstant in one variable (e.g. $V(x_1, x_2, x_3) = 2 + \tan^{-1}(x_1)$), then no ground states exist. In [14] the author showed that ground states to (1.1) exist under the assumption that $V$ is almost periodic, together with another mild assumption. Those assumptions did not guarantee that $V$ had a topologically stable critical point.

Aside from periodicity or recurrence properties of $V$, another approach is to impose conditions on the derivatives of $V$. That is the approach taken here. We will assume that $V$ has a (perhaps unbounded) component of local minima, along which $\Delta V$ achieves a strict local minimum. More specifically, assume $f$ satisfies the following:

(F1) $f \in C^1(\mathbb{R}^+, \mathbb{R})$

(F2) $f'(0) = f(0)$.

(F3) $\lim_{q \to \infty} f(q)/q^s = 0$ for some $s > 1$, with $s < (N + 2)/(N - 2)$ if $N \geq 3$.

(F4) For some $\theta > 2$, $0 < \theta F(q) \leq f(q)q$ for all $q > 0$, where $F(\xi) \equiv \int_0^\xi f(t) dt$.

(F5) The function $q \mapsto f(q)/q$ is increasing on $(0, \infty)$.

Assumptions (F1)-(F5) are the same as in [3] and are satisfied by $f(q) = q^s$, for example, if $1 < s < (N + 2)/(N - 2)$. Assume that $V$ satisfies the following:

(V1) $V \in C^2(\mathbb{R}^n, \mathbb{R})$

(V2) $D^\alpha V$ is bounded and Lipschitz continuous for $|\alpha| = 2$.

(V3) $0 < V_- \equiv \inf_{\mathbb{R}^N} V < \sup_{\mathbb{R}^N} V \equiv V^+ < \infty$

(V4) There exists a bounded, nonempty open set $\Lambda \subset \mathbb{R}^N$ and a point $z_0 \in \Lambda$ with $V(z_0) = \inf_{\Lambda} V \equiv V_0$, and

$$\Delta_0 \equiv \inf\{\Delta V(z) \mid z \in \Lambda, \ V(z) = V_0\} < \inf\{\Delta V(z) \mid z \in \partial \Lambda, \ V(z) = V_0\}$$
Note: A special case of (V4) occurs when $\Lambda$ is bounded and $V(z_0) < \inf_{\partial \Lambda} V$; this case is treated, under weaker hypotheses, in [3]. A specific example of (V4) is if $N = 2$ and $V$ satisfies (V1)-(V4), with $V(z_1, z_2) = 1 + (z_1^2 - z_2)^2$ for $z_1^2 + z_2^2 \leq 1$. Then $\Delta V(z_1, z_1^2) = 8z_1^2 + 2$ for $z_1^2 + z_2^2 \leq 1$, so we may take $\Lambda = B_1(0,0) \subset \mathbb{R}^2$ and $z_0 = (0,0)$. Then $V$ has a component of local minima that includes the parabolic arc $\{z_2 = z_1^2\} \cap B_1(0,0)$, along which $\Delta V$ has a minimum of 2 at $(0,0)$, with $\Delta V > 2$ at the two endpoints of the arc.

We prove the following:

**Theorem 1.1** Let $V$ and $f$ satisfy (V1)-(V4) and (F1)-(F5). Then there exists $\epsilon_0 > 0$ such that if $\epsilon \leq \epsilon_0$, then (1.0) has a positive solution $u_\epsilon$ with $u_\epsilon(z) \to 0$ as $|z| \to \infty$. $u_\epsilon$ has exactly one local maximum (hence, global maximum) point $z_\epsilon \in \Lambda$, where $\Lambda$ is as in (V4). There exist $\alpha, \beta > 0$ with $u_\epsilon(z) \leq \alpha \exp(-\frac{1}{2}|z-z_\epsilon|)$ for $\epsilon \leq \epsilon_0$. Furthermore, with $V_0$ and $\Delta_0$ as in (V4), $V(z_\epsilon) \to V_0$ and $\Delta V(z_\epsilon) \to \Delta_0$ as $\epsilon \to 0$.

For small $\epsilon$, $u_\epsilon$ resembles a “spike,” which is sharper for smaller $\epsilon$. The spike concentrates near a local minimum of $V$ where $\Delta V$ has a strict local minimum. The proof of Theorem 1.1 employs the techniques of [3], with some refinements necessary because $V$ does not necessarily achieve a strict local minimum. Section 2 introduces the penalization scheme developed by Felmer and del Pino, and continues with the beginning of the proof of Theorem 1.1. These beginning arguments are taken practically verbatim from [3], but are included, since the machinery of the penalization technique is used in the remainder of the proof. The reader is invited to consult [3] for more complete proofs. Section 3 contains the completion of the proof, which is original. This part contains delicate computations involving $\Delta V$.

## 2 The penalization scheme

Extend $f$ to the negative reals by defining $f(q) = 0$ for $q < 0$. Let $F$ be the primitive of $f$, that is, $F(q) = \int_0^q f(t) \, dt$. Define the functional $I_\epsilon$ on $W^{1,2}(\mathbb{R}^N)$ by

$$I_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2}(\epsilon^2 |\nabla u|^2 + V(z)u^2) - F(u) \, dz.$$ 

$I_\epsilon$ is a $C^1$ functional, and there is a one-to-one correspondence between positive critical points of $I_\epsilon$ and ground states of (1.1). It is well known that $I_\epsilon$ and similar functionals in related problems fail the Palais-Smale condition. That is, a “Palais-Smale sequence,” defined as a sequence $(u_m)$ with $I_\epsilon(u_m)$ convergent and $I_\epsilon'(u_m) \to 0$ as $m \to \infty$, need not have a convergent subsequence. To get around this difficulty, we formulate a “penalized” problem, with a corresponding “penalized” functional satisfying the Palais-Smale condition, by altering $f$ outside of $\Lambda$. 
Let \( \theta \) be as in (F4). Choose \( k \) so \( k > \theta/(\theta - 2) \). Let \( V_\ast \) be as in (V3) and \( a > 0 \) be the value at which \( f(a)/a = V_\ast/k \). Define \( \tilde{f} \) by
\[
\tilde{f}(s) = \begin{cases} f(s) & s \leq a; \\ sV_\ast/k & s > a, \end{cases}
\]
(2.1)
\[g(z,s) = \chi_A f(s) + (1 - \chi_A) \tilde{f}(s), \]
and \( G(z,\xi) = \int_0^\xi g(z,\tau) \, d\tau. \) Although not continuous, \( g \) is a Carathéodory function. For \( \epsilon > 0 \), define the penalized functional \( J_\epsilon \) on \( W^{1,2}(\mathbb{R}^N) \) by
\[
J_\epsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2} \epsilon^2 \|\nabla u\|^2 + V(z)u^2 - G(z,u) \, dz. \tag{2.2}
\]
A positive critical point of \( J_\epsilon \) is a weak solution of the “penalized equation”
\[-\epsilon^2 \Delta u + V(z)u = g(z,u),\tag{2.3}\]
that is, a \( C^1 \) function satisfying (2.3) wherever \( g \) is continuous. It is proven in [3] that \( J_\epsilon \) satisfies all the hypotheses of the Mountain Pass Theorem of Ambrosetti and Rabinowitz ([2]), including the Palais-Smale condition. Therefore \( J_\epsilon \) has a critical point \( u_\ast \), with the mountain pass critical level \( c(\epsilon) = J_\epsilon(u_\ast) \).

\( c(\epsilon) \) is defined by the following minimax: let the set of paths \( \Gamma_\epsilon = \{ \gamma \in C([0,1],W^{1,2}(\mathbb{R}^N)) \mid \gamma(0) = 0, \ J_\epsilon(\gamma(1)) < 0 \} \), and
\[
c(\epsilon) = \inf_{\gamma \in \Gamma_\epsilon} \max_{\theta \in [0,1]} J_\epsilon(\gamma(\theta)).
\]
As shown in ([3]), because of (F4), \( c(\epsilon) \) can be characterized more simply as
\[
c(\epsilon) = \inf_{u \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}} \sup_{\tau > 0} J_\epsilon(\tau u).
\]
The functions \( g(z,q) \) and \( f(q) \) agree whenever \( z \in \Lambda \) or \( q < a \). Therefore if \( u \) is a weak solution of (2.3) with \( u < a \) on \( \Lambda^C = \mathbb{R}^N \setminus \Lambda \), then \( u \) solves (1.1). Our plan is to find a positive critical point \( u_\ast \) of \( J_\epsilon \), which is a weak solution of (2.3), then show that \( u_\ast(z) < a \) for all \( z \in \Lambda^C \).

For \( \epsilon > 0 \), let \( u_\ast \) be a critical point of \( J_\epsilon \) with \( J_\epsilon(u_\ast) = c(\epsilon) \). Maximum principle arguments show that \( u_\ast \) must be positive. Define the "limiting functional" \( I_0 \) by
\[
I_0(u) = \int_{\mathbb{R}^N} \frac{1}{2} (\|\nabla u\|^2 + V_0 u^2) - F(u) \tag{2.4}
\]
and
\[
\mathcal{E} = \inf_{u \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}} \sup_{\tau > 0} I_0(\tau u). \tag{2.5}
\]
The equation corresponding to (2.4) is
\[-\Delta u + V_0 u = f(u) \tag{2.6}\]

We will prove Theorem 1.1 by proving the following proposition:
Proposition 2.1 Let $\epsilon > 0$. If $u_\epsilon$ is a positive solution of (2.3) satisfying $J_\epsilon(u_\epsilon) = c(\epsilon)$, then

(i) $\lim_{\epsilon \to 0} \max_{z \in \partial \Lambda} u_\epsilon = 0$.

(ii) For all $\epsilon$ sufficiently small, $u_\epsilon$ has only one local maximum point in $\Lambda$ (call it $z_\epsilon$), with $\lim_{\epsilon \to 0} V(z_\epsilon) = V_0$

(iii) $\lim_{\epsilon \to 0} \Delta V(z_\epsilon) = \Delta_0$.

Proof of Theorem 1.1: Assuming Proposition 2.1, there exists $\epsilon_0 > 0$ such that for $\epsilon < \epsilon_0$, $u_\epsilon < a$ on $\partial \Lambda$. In [3] it is shown that if we multiply (2.3) by $(u_\epsilon - a)_+$ and integrate by parts, it follows that $u_\epsilon < a$ on $\Lambda^c$, so $u_\epsilon$ solves (1.1). By the definition of $a$ in (2.1), and the maximum principle, $u_\epsilon$ has no local maxima outside of $\Lambda$, so $u_\epsilon$ has exactly one local maximum point $z_\epsilon$, which occurs in $\Lambda$

Define $v_\epsilon$ by translating $u_\epsilon$ from $z_\epsilon$ to zero and dilating it by $\epsilon$, that is,

$$v_\epsilon(z) = u_\epsilon(z_\epsilon + \epsilon z).$$

Then $v_\epsilon$ is a weak ($C^1$) solution of the “translated and dilated” equation

$$-\Delta v_\epsilon + V(z_\epsilon + \epsilon z)v_\epsilon = g(z_\epsilon + \epsilon z, v_\epsilon).$$

Let $\epsilon_j \to 0$. Along a subsequence (called $(z_{\epsilon_j})$), $z_{\epsilon_j} \to \bar{z} \in \Lambda$, with $V(\bar{z}) = V_0$ and $\Delta V(\bar{z}) = \Delta_0$.

Along a subsequence, $v_{\epsilon_j}$ converges locally uniformly to a function $v^0$. Pick $R > 0$ so $v^0 < a$ on $\mathbb{R}^N \setminus B_R(0)$. For large enough $\epsilon$, $v_\epsilon < a$ on $\partial B_R(0)$. By the maximum principle arguments of [3], for small $\epsilon$, $v_\epsilon$ decays exponentially, uniformly in $\epsilon$.

The proof of Proposition 2.1 will follow if we can prove the following statement.

Proposition 2.2 If $\epsilon_n \to 0$ and $(z_n) \subset \Lambda$ with $u_{\epsilon_n}(z_n) \geq b > 0$, then

(i) $\lim_{n \to \infty} V(z_n) = V_0$.

(ii) $\lim_{n \to \infty} \Delta V(z_n) = \Delta_0$.

It is proven in [3] that $u_\epsilon$ has exactly one local maximum point $z_\epsilon$ for small $\epsilon$. Since $u_\epsilon$ solves (2.3), the maximum principle implies that $u_\epsilon(z_\epsilon)$ is bounded away from zero. Thus Proposition 2.2 and (V4) give Proposition 2.1(ii)-(iii).

To prove Proposition 2.2, let $b$ and $(z_n)$ be as above. First we repeat the argument of [3] to show that $V(z_n) \to V_0$: suppose this does not happen. Then, along a subsequence, $z_n \to \bar{z} \in \Lambda$ with $V(\bar{z}) > V_0$. Define $v_n$ by translating $u_{\epsilon_n}$ from $z_n$ to 0 and dilating by $\epsilon_n$; that is,

$$v_n(z) = u_{\epsilon_n}(z_n + \epsilon_n z).$$

(2.7)
$v_n$ solves the “translated and dilated” penalized equation

$$-\Delta v_n + V(z_n + \epsilon_n z)v_n = g(z_n + \epsilon_n z, v_n) \quad (2.8)$$
on $\mathbb{R}^N$, with $v_n(z) \to 0$ and $\nabla v_n(z) \to 0$ as $|z| \to \infty$. As shown in [3], $(v_n)$ is bounded in $W^{1,2}(\mathbb{R}^N)$, so by elliptic estimates, $(v_n)$ converges locally along a subsequence (also denoted $(v_n)$) to $v^0 \in W^{1,2}(\mathbb{R}^N)$. Define $\chi_n$ by $\chi_n(z) = \chi_A(z_n + \epsilon_n z)$, where $\chi_A$ is the characteristic function of $\Lambda$. $\chi_n$ converges weakly in $L^p$ over compact sets to a function $\chi$, for any $p > 1$, with $0 \leq \chi \leq 1$. Define

$$\bar{g}(z, s) = \chi(z)f(s) + (1 - \chi(z))\bar{f}(s)$$

Then $v^0$ satisfies

$$-\Delta v + V(\bar{z})v = \bar{g}(z, v) \quad (2.9)$$
on $\mathbb{R}^N$. Define $\bar{G}(z, s) = \int_{-\infty}^{s} \bar{g}(z, t) dt$. Associated with (2.9) we have the limiting functional $\bar{J}(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V(\bar{z})u^2) - \bar{G}(z, u) dz$. $v^0$ is a positive critical point of $\bar{J}$.

Define $J_n$ to be the “translated and dilated” penalized functional corresponding to (2.8), that is,

$$J_n(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V(z_n + \epsilon_n z)u^2) - G(z_n + \epsilon_n z, u) dz.$$ 

Clearly $J_n(v_n) = \epsilon_n^{-N} J_n(u_{e_n})$. In [3] it is proven that

$$\lim_{n \to \infty} \inf J_n(v_n) \geq \bar{J}(v^0). \quad (2.10)$$

Also, by letting $w$ be a ground state for (2.6) with $I_0(w) = \varrho$ (the mountain pass value for $I_0$, defined in (2.5) and using $w$ as a test function for $J_n$, it is proven that $\varrho \geq \liminf_{n \to \infty} J_n(v_n)$. Thus $\bar{J}(v^0) \leq \varrho$. Therefore, as shown in [3], $V(\bar{z}) \leq V_0$. This contradicts our assumption. Thus $V(z_n) \to V_0$. All the above is the same as was proven in [3]. Next, we must show that $\Delta V(z_n) \to \Delta_0$. That is the focus of the next section.

3 The effect of the Laplacian

Proving $\Delta V(z_n) \to \Delta_0$ is a subtle and delicate problem. Making $\epsilon_n$ approach 0 is equivalent to dilating $V$, which has the effect of making local minima of $V$ behave more like global minima. This assists in finding solutions to (1.1). However, making $\epsilon_n$ small reduces the effect of differences in $\Delta V$. For this reason, Theorem 1.1 is not only difficult to prove, but is not intuitively compelling, either.

It is known ([7]) that a “least energy solution” of (2.6), that is, a solution $w$ with $I_0(w) = \varrho$ must be radially symmetric. We will need to exploit this fact. In order to do this, we will need to work with the maximum points of $u_{e_n}$ instead of merely the $(z_n)$ as given in Proposition 2.2. We need the following concentration-compactness result, which states that the sequence $(u_{e_n})$ of “mountain-pass type solutions” of (2.3) does not “split”:
Lemma 3.1 If \((z_n) \subset \overline{\Lambda}, (y_n) \subset \mathbb{R}^N\), and \(b > 0\) with \(u_{\epsilon_n}(z_n) > b\) and \(u_{\epsilon_n}(y_n) > b\) for all \(n\), then \((z_n - y_n)/\epsilon_n\) is bounded.

Proof: define \(v_n(z) = u_{\epsilon_n}(z + \epsilon_n z)\) as in (2.7). Suppose the lemma is false. Then, along a subsequence, \(|y_n - z_n|/\epsilon_n \to \infty\). Let \(x_n = (y_n - z_n)/\epsilon_n\). \((\|v_n\|)\) is bounded in \(W^{1,2}(\mathbb{R}^N)\) and \(|x_n| \to \infty\), so we may pick a sequence \((R_n) \subset \mathbb{N}\) with \(R_0 \to \infty\), \(|x_n| - R_n \to \infty\), and \(\|v_n\|_{W^{1,2}(B_{R_{n+1}}(0) \setminus B_{R_n}(0))} \to 0\) as \(n \to \infty\). Define cutoff functions \(\varphi_{i,2}^n \in C^\infty(\mathbb{R}^N, [0, 1])\) satisfying \(\varphi_1 \equiv 1\) on \(B_{R_n-1}(0)\), \(\varphi_2 \equiv 1\) on \(B_{R_n}(0)\), \(\varphi_2 \equiv 0\) on \(B_{R_n}(0)\), and \(\|\nabla \varphi_1\|_{L^\infty(\mathbb{R}^N)} < 2\), \(\|\nabla \varphi_2\|_{L^\infty(\mathbb{R}^N)} < 2\). Set \(v_1^n = \varphi_1^n v_n\) and \(v_2^n = \varphi_2^n v_n\), and \(\bar{v}_n = v_1^n + v_2^n = (\varphi_1^n + \varphi_2^n) v_n\).

Choose \(T_n > 0\) so \(J_n(T_n \bar{v}_n) = 0\). We claim that \(T_n\) is well-defined, and bounded in \(n\). Note that the existence of \(T_n\) must be checked for the penalized functional \(J_n\), because of the replacement of \(F\) with \(G\). By elliptic estimates, there exists an open set \(U \subset \mathbb{R}^N\) such that along a subsequence, \(v_1^n > b/2\) on \(U\) and \(U \subset (\Lambda - z_n)/\epsilon_n = \{ z \in \mathbb{R}^N | z_n + \epsilon_n z \in \Lambda\}\). Let \(a\) be as in (2.1). For \(t > 2a/b\) and \(z \in U\), \(t \bar{v}_n(z) > tb/2 > a\), so \(G(z_n + \epsilon_n z, t \bar{v}_n) = F(t \bar{v}_n) > F(tb/2)\). Therefore, for \(t > 2a/b\),

\[
J_n(t \bar{v}_n) = t^2 \int_{\mathbb{R}^N} \frac{1}{2} (|\nabla \bar{v}_n|^2 + V(z_n + \epsilon_n z) \bar{v}_n^2) dz - \int_{\mathbb{R}^N} G(z_n + \epsilon_n z, t \bar{v}_n) dz
\leq \frac{t^2}{2} (1 + V^+) \|\bar{v}_n\|_{W^{1,2}(\mathbb{R}^N)}^2 - \int_U F(t \bar{v}_n)
\leq \frac{t^2}{2} (1 + V^+) \|\bar{v}_n\|_{W^{1,2}(\mathbb{R}^N)}^2 - \lambda(U) F(tb/2),
\]

where \(\lambda\) indicates the Lebesgue measure. By (F4), there exists \(C > 0\) such that for \(t > 2a/b\), \(F(tb/2) > Ct^d\). Therefore, for \(t > 2a/b\),

\[
J_n(t \bar{v}_n) \leq \frac{t^2}{2} (1 + V^+) \|\bar{v}_n\|_{W^{1,2}(\mathbb{R}^N)}^2 - Ct^d.
\]

Since \((\bar{v}_n)\) is bounded in \(W^{1,2}(\mathbb{R}^N)\), this gives the existence and boundedness of \((T_n)\).

Since \(J_n(T_n \bar{v}_n) = J_n(T_n v_1^n) + J_n(T_n v_2^n) = 0\), we may pick \(i_n \in \{1, 2\}\) with \(J_n(T_n v_{i_n}^n) \leq 0\). By (F5) and (2.1), the map \(t \mapsto J_n(t v_{i_n}^n)\) increases from zero at \(t = 0\), achieves a positive maximum, then decreases to \(-\infty\). We will see more of this in a moment. Thus there exists a unique \(t_n \in (0, T_n)\) with \(J_n(t_n v_{i_n}^n) = \max_{t > 0} J_n(t v_{i_n}^n)\). We claim that \(t_n\) and \(T_n - t_n\) are both bounded away from zero for large \(n\): by \((f_1) - (f_4)\) and (2.1), \(J_n(w) \geq \frac{1}{\theta} \min(1, V^-) \|w\|_{W^{1,2}(\mathbb{R}^N)}^2 - o(\|w\|_{W^{1,2}(\mathbb{R}^N)}^2)\) uniformly in \(n\), so \(\max_{t > 0} J_n(t v_{i_n}^n)\) is bounded away from zero, uniformly in \(n\). It is easy to show that \(J_n\) is Lipschitz on bounded subsets of \(W^{1,2}(\mathbb{R}^N)\), uniformly in \(n\). Since \((T_n)\) is bounded, this implies that \(t_n\) and \(T_n - t_n\) are both bounded away from zero for large \(n\).

By definition of \(v_n\) as a “mountain-pass type critical point” of \(J_n\), we have

\[
\max_{t > 0} J_n(t v_{i_n}^n) \geq \max_{t > 0} J_n(t v_n).
\]
Using the facts that $\|v_n - \bar{v}_n\|_{W^{1,2}(\mathbb{R}^N)} \to 0$ as $n \to \infty$, and $(T_n)$ is bounded, we have

$$\liminf_{n \to \infty} J_n(t_n v_n^{i_n}) = \liminf_{n \to \infty} \max_{t > 0} J_n(t v_n^{i_n}) \geq \liminf_{n \to \infty} \max_{t > 0} J_n(t v_n) = \liminf_{n \to \infty} \max_{t > 0} J_n(t \bar{v}_n) = \liminf_{n \to \infty} J_n(t_n \bar{v}_n) = \liminf_{n \to \infty} (J_n(t_n v_n^{i_n}) + J_n(t_n v_n^{3-i_n})) \geq \liminf_{n \to \infty} J_n(t_n v_n^{i_n}) + \liminf_{n \to \infty} J_n(t_n v_n^{3-i_n}).$$

Now $J_n(T_n v_n^{3-i_n}) = -J_n(T_n v_n^{i_n}) \geq 0$ and $t_n < T_n$, so $J_n(t_n v_n^{3-i_n}) \geq 0$. By (3.2), $\liminf_{n \to \infty} J_n(t_n v_n^{3-i_n}) \leq 0$. Therefore $J_n(t_n v_n^{3-i_n}) \to 0$ as $n \to \infty$.

Since $J_n(w) \geq \frac{1}{2} \min(1,V_-)||w||^2_{W^{1,2}(\mathbb{R}^N)} - o(||w||^2_{W^{1,2}(\mathbb{R}^N)})$ uniformly in $n$, there exists $d \in (0, \liminf_{n \to \infty} t_n)$ such that $\liminf_{n \to \infty} J_n(d v_n^{3-i_n}) > 0$. Since $d < t_n$ and $J_n(d v_n^{3-i_n}) > J_n(t_n v_n^{3-i_n})$ for large $n$, the map $t \mapsto J_n(t v_n^{3-i_n})$ achieves a maximum at some $t'_n \in (0, t_n)$, and that maximum is bounded away from zero.

Summarizing the important facts about the mapping $t \mapsto J_n(t v_n^{3-i_n})$, we have shown that there exists $\rho > 0$ such that for large $n$,

(i) $0 < t'_n < t_n < T_n$

(ii) $(T_n)$ is bounded.

(iii) $(T_n - t_n)$ is bounded away from zero.

(iv) $J_n(t'_n v_n^{3-i_n}) > \rho > 0$

(v) $J_n(t_n v_n^{3-i_n}) \to 0$

(vi) $J_n(T_n v_n^{3-i_n}) \geq 0$

From (i)-(vi) it is apparent that at some $t'_n > t'_n$, the mapping $t \mapsto J_n(t v_n^{3-i_n})$ is at once decreasing and concave upward. But this is impossible: let $n \in \mathbb{N}$ and $w \in W^{1,2}(\mathbb{R}^N) \setminus \{0\}$. Define $\psi(t) = J_n(t w)$ for $t > 0$. Then

$$\psi'(t) = t \int_{\mathbb{R}^N} |\nabla w|^2 + V(z + \epsilon_n z)w^2 \, dz - \int_{\mathbb{R}^N} g(z + \epsilon_n z, tw)w \, dz$$

$$= t \left[ \int_{\mathbb{R}^N} |\nabla w|^2 + V(z + \epsilon_n z)w^2 \, dz - \int_{\{w \neq 0\}} \frac{g(z + \epsilon_n z, tw)}{tw}w^2 \, dz \right].$$

By (F5) and (2.1), $t \mapsto g(z + \epsilon_n z, tw)/(tw)$ is nondecreasing, so if $\psi'(t)$ ever becomes negative, $\psi'$ is increasing for all time $t$ after that, and the graph of $\psi$ is concave down. Therefore the behavior of $J_n(t v_n^{3-i_n})$ as described in (i)-(vi) is impossible, and Lemma 3.1 is proven. \diamond
As mentioned before, it will be advantageous to work with the maxima of \((u_{\epsilon_n})\). Choose \((y_n) \subset \mathbb{R}^N\) with
\[
u_{\epsilon_n}(y_n) = \max_{\mathbb{R}^N} u_{\epsilon_n}.
\]
We will prove
\[
\Delta V(y_n) \to \Delta_0. \tag{3.3}
\]
By Lemma 3.0, \(((y_n - z_n)/\epsilon_n)\) is bounded, so \(y_n - z_n \to 0\). Thus (3.3) gives Proposition 2.2(ii), completing the proof of Theorem 1.1. \(\diamondsuit\)

Along a subsequence, \(y_n \to \bar{y} \in \mathcal{A}\). By Proposition 2.2(i), \(V(\bar{y}) = V_0\). Since is not apparent that \(\bar{y} \in \Lambda\), we must proceed carefully. We will redefine the \(v_n\)'s like in (2.7), by translating \(u_{\epsilon_n}\) to 0 and dilating it. That is,
\[
v_n(z) = u_{\epsilon_n}(y_n + \epsilon_n z) \tag{3.4}
\]
Then \(v_n\) is a positive weak solution, vanishing at infinity, of the “penalized, dilated, and translated” PDE
\[
-\Delta v + V(y_n + \epsilon_n z)v = g(y_n + \epsilon_n z, v).
\]
Like before, \((v_n)\) converges locally uniformly to a function \(v_0\). We claim that \(v_0\) is actually a ground state maximizing at 0 of the autonomous limiting equation (2.6). Proof: As before, define \(\chi_n\) by \(\chi_n(z) = \chi(y_n + \epsilon z)\). As before, along a subsequence, \(\chi_n\) converges weakly in \(L^p\), for any \(p > 1\), on compact subsets of \(\mathbb{R}^N\) to a function \(\chi\) with \(0 \leq \chi \leq 1\). Define \(\bar{g}\) by
\[
\tilde{g}(z, s) = \chi(z)f(s) + (1 - \chi(z))\tilde{f}(s).
\]
By the argument of Proposition 2.2, taken from [3], \((v_n)\) converges locally along a subsequence to \(v_0\), a ground state of \(-\Delta v + V_0 v = \bar{g}(z, v)\). The functional corresponding to this equation is \(\tilde{J}(u) = \int_{\mathbb{R}^N} \frac{1}{2}(|\nabla u|^2 + V_0 u^2) - G(z, u) \, dz\), where \(G(z, s) = \int_0^s \tilde{g}(z, t) \, dt\). As before, in (2.10), \(C \geq \liminf_{n \to \infty} J_n(v_n) \geq \tilde{J}(v_0)\), where \(C\) is from (2.5). \(\tilde{J} \geq I_0\), where \(I_0\) is the “autonomous” limiting functional from (2.4), so
\[
C \leq \max_{t > 0} I_0(tv_0) \leq \max_{t > 0} \tilde{J}(tv_0) \leq C\]
and \(v_0\) is actually a ground state of (2.6). \(\diamondsuit\)

Not only does \((v_n)\) converge locally to \(v_0\), but it satisfies the following lemma.

**Lemma 3.2** With \((v_n)\) as in (3.4), for any subsequence of \((v_n)\) there is a radially symmetric ground state \(v_0\) of (2.6) such that \(v_n \to v_0\) uniformly along a subsequence and the \(v_n\)'s decay exponentially, uniformly in \(n\).
Proof: If one establishes uniform convergence, the uniform exponential decay follows readily, using a standard maximum principle argument found in [3]. Suppose the convergence is not uniform. Then there exist a subsequence of \((v_n)\) and a sequence \((x_n) \subset \mathbb{R}^N\) with \(|x_n| \to \infty\) and \(\lim_{n \to \infty} v_n(x_n) = 0\). Let \(d > 0\) with \(d < v_0(0)\) and \(d < \lim_{n \to \infty} v_n(x_n)\). For large \(n\), \(d < v_n(0) = u_{e_n}(z_n)\) and \(d < v_n(x_n) = u_{e_n}(z_n + e_n x_n)\). Letting \(w_n = z_n + e_n x_n\), we obtain \((w_n - z_n)/\epsilon_n = (x_n)\), which is unbounded, violating Lemma 3.1.

To show \(\Delta V(y_n) \to \Delta_0\), we again argue indirectly. Suppose otherwise. Then, along a subsequence, \(y_n \to \bar{y} \in \mathbb{A}\) with

\[
\Delta V(\bar{y}) > \Delta_0. \tag{3.5}
\]

For \(x \in \mathbb{R}^N\), define the translation operator \(\tau_x\) by \(\tau_x u(z) = u(z - x)\), that is, \(\tau_x u\) is \(u\) translated by \(x\). Assume for convenience, and without loss of generality, that

\(0 \in \Lambda, V(0) = V_0, \text{ and } \Delta V(0) = \Delta_0.\)

We will prove that for large \(n\),

\[
\sup_{t > 0} J_{e_n}(t \tau_{-y_n} u_{e_n}) < J_{e_n}(u_{e_n}) = \sup_{t > 0} J_{e_n}(t u_{e_n}), \tag{3.6}
\]

recalling the definition of \(J_e\) in (2.2), and how \(v_n\) is defined from \(u_{e_n}\) in (3.4). That is, translating \(t u_{e_n}\) back to the origin reduces the value of \(J_{e_n}(t v_n)\) because \(V\) has lesser concavity at the origin. This occurs even though shrinking \(\epsilon\) reduces the difference in concavity. (3.6) contradicts the definition of \(u_{e_n}\).

Pick \(T > 1\) large enough so that for large \(n\), \(J_n(T v_n) = \epsilon_n^{-N} J_{e_n}(T u_{e_n}) < 0\). This is possible by the argument of (3.1). Now (3.6) is equivalent to

\[
\sup_{0 \leq t \leq T} J_{e_n}(t \tau_{-y_n} u_{e_n}) < \sup_{0 \leq t \leq T} J_{e_n}(t u_{e_n}).
\]

To prove the above, it will suffice to prove the stronger fact that for large \(n\), for all \(t \in (0, T)\),

\[J_{e_n}(t u_{e_n}) > J_{e_n}(t \tau_{-y_n} u_{e_n}).\]

Now, along a subsequence, \(v_n \to v_0\) uniformly, so by the definition of \(v_n\) as a dilation of \(\tau_{-y_n} u_{e_n}\) (3.4), \(u_{e_n} \to 0\) uniformly on \(\mathbb{R}^N \setminus \Lambda\) as \(n \to \infty\). Thus for large \(n\) and \(0 \leq t \leq T\), the definition of \(G\) gives

\[G(z, t \tau_{-y_n} u_{e_n}(z)) = F(t \tau_{-y_n} u_{e_n}(z)) \text{ for all } z \in \mathbb{R}^N,\]

so

\[
J_{e_n}(t u_{e_n}) - J_{e_n}(t \tau_{-y_n} u_{e_n}) = \int_{\mathbb{R}^N} \frac{1}{2} t^2 \left( |\nabla u_{e_n}(z)|^2 + V(z) u_{e_n}(z)^2 \right) - G(z, t u_{e_n}(z)) \, dz
\]

\[= \left[ \int_{\mathbb{R}^N} \frac{1}{2} t^2 \left( |\nabla \tau_{-y_n} u_{e_n}(z)|^2 + V(z \tau_{-y_n} u_{e_n}(z)^2) \right) - F(t \tau_{-y_n} u_{e_n}(z)) \, dz \right]
\]

\[\geq \frac{1}{2} t^2 \int_{\mathbb{R}^N} V(z)(u_{e_n}(z)^2 - u_{e_n}(z + y_n)^2) \, dz.
\]
\[ + \int_{\mathbb{R}^N} F(tu_n(z) + y_n) - F(tu_n(z)) \, dz \]
\[ = \frac{1}{2} t^2 \int_{\mathbb{R}^N} (V(z + y_n) - V(z))u_n(z + y_n)^2 \, dz \]
\[ = \frac{1}{2} t^2 \varepsilon_n \int_{\mathbb{R}^N} (V(y_n + \varepsilon_n z) - V(\varepsilon_n z))u_n(\varepsilon_n z + y_n)^2 \, dz \]
\[ = \frac{1}{2} t^2 \varepsilon_n \int_{\mathbb{R}^N} (V(y_n + \varepsilon_n z) - V(\varepsilon_n z))v_n(z)^2 \, dz. \]

For \( n = 1, 2, \ldots \), define \( h_n : \mathbb{R} \to \mathbb{R} \) by
\[
h_n(t) = \int_{\mathbb{R}^N} (V(y_n + tz) - V(tz))v_n^2 \, dz.
\]

Since \( h_n(\varepsilon_n) = \int_{\mathbb{R}^N} (V(y_n + \varepsilon_n z) - V(\varepsilon_n z))v_n^2 \), we must prove that for large \( n \)
\[
h_n(\varepsilon_n) > 0. \tag{3.7}
\]

Assume without loss of generality that \( \Lambda \) was chosen so that there exists \( \rho > 0 \) with
\[
\inf_{N_{\rho}(\Lambda)} V = V_0, \tag{3.8}
\]

where \( N_{\rho}(\Lambda) = \{ x \in \mathbb{R}^N \mid \exists y \in \Lambda \text{ with } |y - x| < \rho \} \). We will prove the following facts about \( h_n \):

**Lemma 3.3** For some \( \beta > 0 \), for large \( n \),

(i) \( h_n \in C^2(\mathbb{R}^+, \mathbb{R}) \)

(ii) \( h_n(0) \geq 0 \)

(iii) \( |h_n'(0)|^2 \leq o(1)h_n(0) \)

(iv) \( h_n''(0) > \beta \)

(v) \( h_n'' \) is locally Lipschitz on \( \mathbb{R}^+ \), uniformly in \( n \).

Here \( o(1) \to 0 \) as \( n \to \infty \). Before proving Lemma 3.3, let us prove how it gives (3.7). By (iv)-(v), there exists \( d > 0 \) such that for large \( n \) and \( 0 \leq t \leq d \), \( h_n''(t) > \beta/2 \). For \( t \in [0, d] \), a Taylor’s series expansion shows that for large \( n \)
\[
h_n(t) \geq h_n(0) + h_n'(0)t + \frac{\beta}{4}t^2 \equiv l_n(t). \tag{3.9}
\]
If \( h_n(0) = 0 \), then by Lemma 3.3(iii), \( h_n'(0) = 0 \), so (3.9) implies that \( h_n(t) > 0 \)
for all \( t \in (0, d) \), giving (3.7) if \( n \) is large enough that \( \varepsilon_n < d \). If \( h_n(0) > 0 \), then
by elementary calculus, \( l_n \) attains a minimum value at \( t = -2h_n'(0)/\beta \), and the minimum value is
\[
\min_{\mathbb{R}} l_n = l_n(-2h_n'(0)/\beta) = h_n(0) - h_n'(0)^2/\beta \geq (1 - o(1))h_n(0),
\]
where \( o(1) \to 0 \) as \( n \to \infty \). For large \( n \), if \( h_n(0) > 0 \) then \( l_n(t) > 0 \) for all \( t \in \mathbb{R} \), so \( h_n(t) > 0 \) for all \( t \in (0, d) \) for large \( n \), implying (3.7) if \( n \) is large enough so that \( \varepsilon_n < d \).
Proof of Lemma 3.3 Statement (ii) is trivial, since $h_n(0) = (V(y_n) - V_0)\int_{\mathbb{R}^n} v_n^2$, and since $z_n \in X$ and $y_n - z_n \to 0$, (3.8) implies $V(y_n) \geq V_0$ for large $n$. (i) and (v) follow from Leibniz’s Rule, $(V_1) - (V_2)$, and the fact that the $v_n$’s decay exponentially, uniformly in $n$. For $j = 1, 2$,

$$h_n^{(j)}(t) = \int_{\mathbb{R}^N} \sum_{|\alpha|=j} (D^{\alpha}V(y_n + tz) - D^{\alpha}V(tz))z^{\alpha}v_n(z)^2\,dz.$$ 

Since (V2) holds, $v_n$ decays exponentially, uniformly in $n$, $y_n \to \bar{y}$, and $v_0$ is radially symmetric, we have

$$h_n''(0) = \int_{\mathbb{R}^N} \sum_{|\alpha|=2} (D^{\alpha}V(y_n) - D^{\alpha}V(0))z^{\alpha}v_n(z)^2\,dz$$

$$\to \int_{\mathbb{R}^N} \sum_{|\alpha|=2} (D^{\alpha}V(\bar{y}) - D^{\alpha}V(0))z^{\alpha}v_0(z)^2\,dz$$

$$= \int_{\mathbb{R}^N} \sum_{i=1}^{N} (D^{ii}V(\bar{y}) - D^{ii}V(0))z^2v_0(z)^2\,dz$$

$$= \int_{\mathbb{R}^N} \sum_{i=1}^{N} (D^{ii}V(\bar{y}) - D^{ii}V(0))\frac{1}{N}|z|^2v_0(z)^2\,dz$$

$$= \frac{1}{N}(\Delta V(\bar{y}) - \Delta V(0))\int_{\mathbb{R}^N} |z|^2v_0(z)^2\,dz > 0$$

by assumption (3.5). Since Lemma 3.3(v) holds, we have Lemma 3.3(iv).

To prove Lemma 3.3(iii), we will need the following calculus lemma:

Lemma 3.4 Let $U \subseteq \mathbb{R}^N$ and $r > 0$. Let $V \in C^2(N_r(U), \mathbb{R})$ with $\inf_{N_r(U)} V = V_0 > -\infty$, $|\nabla V|$ bounded on $N_r(U)$, and $D^2 V$ Lipschitz on $N_r(U)$. Then there exists $C > 0$ with

$$|\nabla V(z)|^2 \leq C(V(z) - V_0)$$

(3.10)

for all $z \in U$.

Proof: let $B > 0$ with $|D^2 V(z) \xi \cdot \xi| \leq B$ for all $\xi \in \mathbb{R}^N$ with $|\xi| = 1$. Also let $B$ be big enough so

$$B > |\nabla V(z)|/r$$

for all $z \in U$. Pick $z \in U$. If $|\nabla V(z)| = 0$, then (3.10) is obvious. Otherwise, let $d = |\nabla V(z)|/B < r$. Define $\varphi(t) = V(z - t\nabla V(z)/|\nabla V(z)|)$ for $t \in [0, d]$. $\varphi$ is $C^2$, $\varphi(0) = V(z)$, and $\varphi'(0) = -|\nabla V(z)|$. By choice of $B$ and the fact that $B_d(z) \subset N_r(U)$, $|\varphi''(t)| \leq B$ for all $t \in [0, d]$. Taylor’s theorem gives

$$\varphi(d) - \varphi(0) = \varphi'(0)d + \frac{\varphi''(\xi)}{2}d^2 \leq -|\nabla V(z)|d + Bd^2/2 = -\frac{|\nabla V(z)|^2}{2B}.$$
Also $\varphi(d) \geq V_0$ because $B_d(z) \subset N_r(U)$. Therefore,
\[
\frac{|\nabla V(z)|^2}{2B} \leq \varphi(0) - \varphi(d) \leq V(z) - V_0.
\]
Lemma 3.4 is proven. 

To prove Lemma 3.3(iii), first note that, by the radial symmetry of $v_0$, the uniform exponential decay of $v_n$, and the uniform convergence $v_n \to v_0$,
\[
|h_n'(0)| = |(\nabla V(y_n) - \nabla V(0)) \cdot \int_{\mathbb{R}^N} z v_n^2 \, dz| \\
= |\nabla V(y_n) \cdot \int_{\mathbb{R}^N} z v_n^2 \, dz| \\
= |\nabla V(y_n) \cdot \int_{\mathbb{R}^N} z v_0^2 \, dz + \nabla V(y_n) \cdot \int_{\mathbb{R}^N} z (v_n^2 - v_0^2) \, dz| \\
\leq |\nabla V(y_n)| \int_{\mathbb{R}^N} z (v_n^2 - v_0^2) \, dz \\
\leq o(1)|\nabla V(y_n)|,
\]
so Lemma 3.4 implies
\[
|h_n'(0)|^2 \leq o(1)|\nabla V(y_n)|^2 \leq o(1)(V(y_n) - V_0) \\
\leq o(1)(V(y_n) - V_0) \int_{\mathbb{R}^N} v_n^2 \\
= o(1)h_n(0),
\]
since $\int_{\mathbb{R}^N} v_n^2$ is bounded away from zero. Lemma 3.3(iii) is proven. 
Thence follow (3.7), (3.3), Proposition 2.2, and Theorem 1.1.

Remarks: Besides the results cited in the introduction, many important results for equations of type (1.1) have been found recently. For instance, the work in [3]-[5] suggests that Theorem 1.1 could be strengthened by working on a smaller domain than $\mathbb{R}^N$, or by weakening the hypotheses on $V$. It is natural to try to extend Theorem 1.1 to cases where $V$ is not $C^2$, or to the case where the second derivatives of $V$ do not provide a condition like (V4), but higher-order derivatives do.

References


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