Apr 1st, 8:00 AM

Sensitivity and Optimization Theory for Launch Vehicle Attitude Control System Synthesis

William A. Walter  
*University of Florida - GENE SYS*

Fred O. Simions  
*University of Florida - GENE SYS*

Follow this and additional works at: [https://commons.erau.edu/space-congress-proceedings](https://commons.erau.edu/space-congress-proceedings)

Scholarly Commons Citation  
[https://commons.erau.edu/space-congress-proceedings/proceedings-1969-6th-v1/session-10/1](https://commons.erau.edu/space-congress-proceedings/proceedings-1969-6th-v1/session-10/1)

This Event is brought to you for free and open access by the Conferences at Scholarly Commons. It has been accepted for inclusion in The Space Congress® Proceedings by an authorized administrator of Scholarly Commons. For more information, please contact commons@erau.edu.
INTRODUCTION

In the design of a launch vehicle attitude control system two factors of major concern are the bending moments experienced during flight and the terminal drift. Excursions in these system variables are produced by winds, the most important of which occur during the period surrounding maximum dynamic pressure. Conventional design approaches assume a linear control law and determine gain values and shaping networks based on a great many complex and interacting considerations (e.g., bending, sloshing, the nature of winds, separation dynamics, etc.) in order to meet bending moment and drift requirements. The result is a linear control law with gains which are switched between two or more fixed values during the flight.

A number of investigators have treated the problem from the standpoint of determining an optimal control law which will minimize a cost function based on either bending moment, drift, or a combination of these. In many cases, the models include bending dynamics, however, the effects of wind disturbances are not present during the minimization.

In this paper, an attempt is made to evaluate the performance obtainable by parameter optimization and to compare this performance with that obtainable by optimal control of engine deflection angle. The measure of performance used is

\[ J = \int_{t_0}^{t_f} \left[ \frac{1}{2} c_0 \alpha^2 + k_2 M_B^2 \right] dt \]  

(1)

where \( Z_0(t_f) \) is the terminal drift and \( M_B(t) \) the bending moment. The minimization is carried out with the vehicle subjected to a statistically derived wind profile. A simple, rigid body, constant coefficient model is treated by methods which are directly applicable to more complex system descriptions which include time varying gains and bending dynamics. The approach considered is well suited to iterative analog or hybrid computation and an example implementation is described.

RIGID BODY EQUATIONS OF MOTION

The launch vehicle rigid-body geometry is shown in Fig. 1. Here the \( X, Z \) coordinates provide an attitude reference system with origin at the vehicle center of gravity \( X_{CG} \). The angle between this reference and the actual body coordinates \( X, Z \) is the pitch angle \( \phi \). The vehicle velocity \( V \), and wind velocity \( V_W \) combine to give the air flow velocity relative to the vehicle. \( V_F \). \( \alpha \) is the corresponding angle of attack. The aerodynamic drag and lift forces \( D \) and \( N'\alpha \) act at the vehicle center of pressure \( X_{CP} \). \( \beta \) is the angular displacement of the gimbaled thrust force \( R \) which acts at the engine gimbal point \( X \). Under the assumption that all angles remain small, the equations of motion are obtained in the usual manner.

\[ \dot{Z}_0 = k_1 \frac{Z_0}{X_{CP}} - k_2 \alpha + k_3 \beta \]  

(2)

where \( k_1 = (X_{CP} - X_{CG})N'/I_{XX} \), \( k_2 = \frac{R}{m} \), \( k_3 = \frac{R}{m} \), and \( m \) is the total vehicle mass.

The angular relationship

\[ \alpha = c_4 + k_4 Z_0 \]  

(4)

is evident in Fig. 1, where \( k_4 = 1/V \). The bending moment imposed by loads on the vehicle structure can be expressed as

\[ M_B = M_B^0 \alpha + M_B^\beta \beta \]  

(5)

where the coefficients \( M_B^0 \) and \( M_B^\beta \) depend upon time into the flight and position along the length of the vehicle.
PARAMETER OPTIMIZATION

One of the first steps in attitude control system design by conventional methods is the selection of a control scheme. This is also required for parameter optimization, and the basic attitude/attitude rate scheme

$$\beta = a_0 \Phi + a_1 \Phi$$  \hspace{1cm} (6)

is initially employed.

With the substitution of (4) and (6) into (2) and (3), and the definition of state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \Phi \\ \Phi \\ Z_0 \\ Z_0 \end{bmatrix}$$  \hspace{1cm} (7)

the equations of motion can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(c_1 + c_2 a_0) & -c_2 a_1 & c_1 k_4 & x_2 \\ 0 & 0 & 0 & 1 \\ (k_1 + k_2 + k_3 a_0) & k_3 a_1 & 0 & -k_2 k_4 \end{bmatrix} x + \begin{bmatrix} 0 \\ -c_1 \\ 0 \\ k_2 \end{bmatrix} a_W$$  \hspace{1cm} (8)

which is of the form $\dot{x} = f(x, t, a)$, where $x$ is an $n$ dimensional state vector and $a$ is an $m$ dimensional parameter vector. (Here, $a^T = [a_0, a_1]$, where the prime indicates transposition). For specified aerodynamic coefficients, the parameter optimization problem is that of determining the values for the control gains, $a_0$ and $a_1$, which minimize the performance index

$$J(a) = J_1(x(t_f)) + \int_{t_0}^{t_f} V(x, t, a) dt$$  \hspace{1cm} (9)

where $J_1[x(t_f)] = \frac{1}{2} x_1^2(t_f) - k x_3^2(t_f)$  \hspace{1cm} (10)

and $V(x, t, a) = k_2 x_2^2 x_4^2$  \hspace{1cm} (11)

The minimization is to be carried out for a statistically derived wind input, $a_W(t)$, based on a 95 per cent probability wind speed envelope with a 99 per cent probability wind buildup and superimposed gust. The duration of the process is over a twenty second time interval which includes the period of maximum dynamic pressure. Minimization is achieved by iterative computation of the gradient of $J(a)$ in parameter space, $V_{aJ}$, and adjustment of parameters in the direction of largest negative gradient. The gradient may be expressed as (see Appendix)

$$V_{aJ} = \left[ \frac{\partial J}{\partial a_0} \right]_a(t_f) \int_{t_0}^{t_f} \left[ \frac{\partial \mathbf{V}_{aJ}}{\partial a} \right]_a(t_f) dt$$

where $[\xi]'$ is a matrix whose $i$th row is the $i$th sensitivity vector of the system, i.e.,

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{bmatrix}' = \begin{bmatrix} \frac{\partial x_1}{\partial a_1} & \frac{\partial x_2}{\partial a_1} & \cdots & \frac{\partial x_n}{\partial a_1} \\ \frac{\partial x_1}{\partial a_2} & \frac{\partial x_2}{\partial a_2} & \cdots & \frac{\partial x_n}{\partial a_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial a_m} & \frac{\partial x_2}{\partial a_m} & \cdots & \frac{\partial x_n}{\partial a_m} \end{bmatrix}$$

From (10), (11) and (12) $V_{aJ}$ for the launch vehicle is

$$\begin{bmatrix} 2J \frac{\partial}{\partial a_0} \\ 2J \frac{\partial}{\partial a_1} \end{bmatrix} = \begin{bmatrix} 3x_1 & 3x_2 & 3x_3 & 3x_4 \\ 3a_0 & 3a_0 & 3a_0 & 3a_0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2J \frac{\partial}{\partial a_0} \\ 2J \frac{\partial}{\partial a_1} \end{bmatrix} = \begin{bmatrix} 3x_1 & 3x_2 & 3x_3 & 3x_4 \\ 3a_1 & 3a_1 & 3a_1 & 3a_1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Through the use of (13) the iterative parameter optimization may be implemented on either a digital computer, or an analog computer with digital logic control. For either implementation the same basic evaluations are required for each iteration.

The iterative process is now described and illustrated for an analog implementation. Note in (13) that no sensitivity vector other than $\xi_i$ is required for the evaluation of $2J/2a_i$. This allows the successive evaluation of each of the partial derivatives as indicated in Fig. 2, and permits a considerable reduction in the required number of analog components.

A single iteration consists of the following:

1. Evaluate the system states as functions of time, and from these form $M_B(t)$ as shown in Fig. 3(a).

2. Evaluate the sensitivity vector $\xi_i$ as a function of time as shown in Fig. 3(b) (see Appendix).
iii. Form the products and integration as a function of time as required in (13) and shown in Fig. 3(d).

iv. Store the product and integral values at the terminal time as shown in Fig. 3(d).

Repeat (i) through (iv) for each partial derivative of \( J \). (For evaluation of \( \frac{\partial J}{\partial \alpha} \) the D/A relay is in position \( s^+ \), and for evaluation of \( \frac{\partial J}{\partial \alpha} \) the position \( s^- \) is used.) Finally,

v. Increment the parameters by the amounts

\[
\Delta a_i = -\frac{\partial J}{\partial a_i}
\]

as shown in Fig. 3(d).

A digital logic system is required for:

1. control of the D/A relay which selects the sensitivity vector generated,
2. control of the track-store units, TS1 and TS2, which store the partial derivatives \( \frac{\partial J}{\partial \alpha} \) and \( \frac{\partial J}{\partial \alpha} \) at the terminal time, and
3. control of integrators \( I_1 \) and \( I_2 \) which effect a parameter adjustment during the initial condition period which follows the storage of all partial derivatives for the iteration. The electronic switches used in the generation of wind profile discontinuities also require the application of logic control signals.

In addition to the direct form of \( V_a^j \) given in (13) an adjoint form could be used as a basis for the iterative procedure. However, the small number of adjustable parameters in the present problem, along with the storage requirements of the adjoint approach, make the direct form desirable.
PARAMETER OPTIMIZATION FOR A TYPICAL VEHICLE

The aerodynamic coefficient values
\[ c_1 = -0.0967 \text{ } \text{1/sec}^2 \]
\[ c_2 = 1.097 \text{ } \text{1/sec}^2 \]
\[ k_1 = 0.3368 \text{ } \text{meters/sec}^2 \text{ deg} \]
\[ k_2 = 0.1356 \text{ } \text{meters/sec}^2 \text{ deg} \]
\[ k_3 = 0.2841 \text{ } \text{meters/sec}^2 \text{ deg} \]
\[ k_4 = 0.1419 \text{ } \text{deg/sec/meter} \]
\[ M_g = -1.87 \times 10^8 \text{ } \text{kilopound meters/rad} \]
\[ M_B = -5.35 \times 10^8 \text{ } \text{kilopound meters/rad} \]

were used in the system simulation of Fig. 3. The wind profile to which the model was subjected is
\[ a_w = \begin{cases} 0.389t + 0.2e^{-72(t-10.8)} & 0 \leq t < 15 \\ 11.7 & 15 \leq t < 16 \\ 0.7 & t \geq 16 \end{cases} \]  
(A)

First, separate minimization of terminal drift and the bending moment integral were performed. Simulations were effectuated for a variety of initial parameter values. Figures 4 and 5 display the dynamic behavior and iterative parameter adjustment for the initial values, \( a_0 = 0.5 \) and \( a_1 = 0.4 \). The large initial adjustment in \( a_0 \) and \( a_1 \), and the subsequent drift to larger and larger values seen in these figures were experienced for all initial values of \( a_0 \) and \( a_1 \) examined. (Note the region in which the time scale is compressed and many iterations are performed.) The parameters do not settle to fixed values since the individual cost functions as shown in Figures 6 and 7 are not convex functions in the parameter space considered. This fact, however, does not severely limit the value of this approach, since other considerations restrict the region of parameter space suitable for operation. For example, restrictions on the control system undamped natural frequency \( \omega_n \) and damping ratio \( \xi \), imposed by the natural frequencies associated with other degrees of freedom, such as bending and sloshing, restrict \( a_0 \) and \( a_1 \). Typical limits produced by such considerations are \( 0.1 \leq \omega_n \leq 2 \) and \( 0.4 \leq \xi \leq 0.8 \) which would in turn require \( 0.07 \leq a_0 \leq 13 \) and \( 0.07 \leq a_1 \leq 29 \) in the above example. Incorporation of the elastic properties of the vehicle during the minimization, is currently of considerable interest.

OPTIMUM CONTROL OF LAUNCH VEHICLE ATTITUDE

Optimum control of the engine deflection angle, \( \beta(t) \), is now considered. In contrast to the parameter optimization approach, no control law is assumed a priori. This freedom from restrictions on the controller structure, permits the determination of the "best" value of \( \beta(t) \) at each instant of time insofar as minimization of the cost function (1) is concerned, for the wind disturbance \( a_w \).

Substitution of \( a \) from (4) into (2) and (3) gives
\[ \phi = c_1 \phi + c_1 k_4 \phi^2 - c_1 a_w - c_2 \beta \]
and
\[ z = (k_1 + k_2) \phi - k_2 k_4 \phi^2 + k_2 a_w + k_3 \beta \]
for the rigid body equations of motion. For the definition of state variables in (7) the linear model is
\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -c_1 & 0 & c_1 k_4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -c_1 a_w \\ 0 \end{bmatrix} \]
\[ \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k_2 \\ k_3 \end{bmatrix} \]
which is of the form \( \dot{x} = f(x, u, t) \), where \( u \) is in general an \( r \) dimensional vector. In this case \( r \) is equal to unity and \( u = \beta(t) \).

The Maximum Principle of Pontryagin is now used to determine the optimal control \( u = m(t) \) so that the cost function (equation (1) rearranged)
\[ J = \int_0^T \left[ 2 \frac{dz}{dt} + k_M M_B \frac{dM_B}{dt} \right] dt \]
(18) is minimized, subject to (17). The Hamiltonian for this problem is
\[ H = \begin{pmatrix} x_3 x_4 + k_2 M_B^2 \\ x_2 \\ -c_1 x_1 + c_1 k_4 x_4 - c_1 a_w - c_2 \beta \\ x_4 \\ (k_1 + k_2) x_1 - k_2 k_4 x_4 + k_2 a_w + k_3 \beta \end{pmatrix} \]
(19)

where
\[ k_M = M_1 x_1 - M_2 x_4 + M_3 a_w + M_4 \beta \]
\[ M_g = k_M^g \]
and
\[ M_B = k_M^B \]
Fig. 4 Parameter Optimization of $Z^2(t_f)$ Only

Fig. 5 Parameter Optimization of $\int_0^{t_f} M_B^2 dt$ only
Application of the necessary conditions for an extremum results in the equations

\[
\begin{bmatrix}
(M_\alpha^2 x_1 - M_\alpha^2 k_4 x_4 + M_\alpha^2 a_W \\
+ M_\alpha M_\beta - c_1 p_2 + (k_1 + k_2) p_4)
\end{bmatrix} = 0
\]

and

\[
\begin{bmatrix}
x_2 = \lambda_1 x_1 + \lambda_2 x_4 + \lambda_3 p_2 + \lambda_4 p_4 + \lambda_5 a_W \\
\lambda_6 x_1 + \lambda_7 x_4 + \lambda_8 p_2 + \lambda_9 p_4 + \lambda_9 a_W
\end{bmatrix}
\]

where

\[
\lambda_1 = -c_1 + c_2 M_\alpha / M_\beta \\
\lambda_2 = c_1 k_4 - c_2 M_\alpha k_4 / M_\beta \\
\lambda_3 = -c_2^2 / M_\beta^2 \\
\lambda_4 = c_2^2 k_3 / M_\beta^3 \\
\lambda_5 = k_1 k_2 - k_3 M_\alpha / M_\beta \\
\lambda_6 = k_4 + (k_1 + k_2) M_\alpha / M_\beta \\
\lambda_7 = -k_2^2 / M_\beta^2 \\
\lambda_8 = k_2 - k_3 M_\alpha / M_\beta
\]

The terminal time is fixed at twenty seconds as in the previous section, and no terminal constraints are imposed on the state vector. Therefore, the transversality conditions require that the vector of adjoint variables, \( \lambda \), be zero at the terminal time.

A number of methods are available for solution of this two-point boundary value problem. The approach described below is a special case of the boundary condition iteration method, and has the attribute of simplicity. Equation (23) can be viewed as a linear system of the form

\[
\begin{bmatrix}
\frac{\dot{x}}{p} \\
\frac{\dot{\lambda}}{\lambda}
\end{bmatrix} = A \begin{bmatrix}
\frac{\dot{x}}{p} \\
\frac{\dot{\lambda}}{\lambda}
\end{bmatrix} + Bu,
\]

with the solution

\[
\begin{bmatrix}
x(t) \\
p(t)
\end{bmatrix} = \phi(t, t_0) \begin{bmatrix}
x_0 \\
p_0
\end{bmatrix} + \int_{t_0}^{t} \phi(t, \tau) Bu(\tau) d\tau.
\]
where $\Phi(t,t_0)$ is the state transition matrix. The problem is simply to select that $p(0) = P_0$ which makes $p(t_f) = 0$. The matrix $\Phi(t_f,t_0)$ in partitioned form displays the dependence of the terminal values $p(t_f)$ on the initial values $P_0$. The homogenous part of the solution is

$$
\begin{bmatrix}
X(t_f) \\
E(t_f)
\end{bmatrix} =
\begin{bmatrix}
\Phi_{11}(t_f,t_0) & \Phi_{12}(t_f,t_0)
\\
\Phi_{21}(t_f,t_0) & \Phi_{22}(t_f,t_0)
\end{bmatrix}
\begin{bmatrix}
P_0 \\
E_0
\end{bmatrix}
(26)
$$

At the terminal time, for $E_0 = 0$, let the complete solution, (25), be $p(t_f)$. The initial adjoint vector

$$
P_0 = -\Phi^{-1}(t_f,t_0)E_0(t_f)
(27)
$$
when added to the above response produces $p(t_f) = 0$ as required.

A computer solution may be effected as follows:

1. Evaluate $p(t_f)$ for the prescribed initial state vector $x_0$ and disturbance input $a_W$, with $E_0 = 0$ (or other convenient value).

2. Evaluate the state transition submatrix $\Phi_{22}(t_f,t_0)$. The $i$th column is obtained by setting $E_0$ equal to a unit vector in the $p_i$ direction, setting $x_0 = 0$ and $a_W = 0$, and evaluating $p(t_f)$.

3. Evaluate the desired initial adjoint vector from (27).

4. Evaluate the state behavior and all output variables of interest for this value of $E_0$ with the prescribed initial state $x_0$ and disturbance input, $a_W$ applied.

**OPTIMUM CONTROL OF ATTITUDE FOR A TYPICAL VEHICLE**

An investigation was undertaken to determine the optimum control input, $\beta(t)$, for the same vehicle model that was used for parameter optimization. Identical vehicle parameters and wind input $a_W$ were used. A major problem was encountered in the determination of weighting factors which would produce satisfactory dynamic response for all system variables. For the range of $k$ considered, the terminal drift was brought within one meter of the desired reference, at the expense of unacceptable excursions in other system variables.

The linear differential equations, (23), form an unstable set, and computational errors become a problem when the length of the optimization interval is many time constants. To offset this difficulty, double precision arithmetic was used in numerical solutions for this problem.

**CONCLUDING REMARKS**

Methods have been presented for parameter optimization of a launch vehicle attitude control system of fixed structure, and for determination of the associated optimum performance. The latter is provided for the purpose of comparison. The methods considered are well suited for analog, hybrid, or digital computation and are directly applicable to more complex system models which may include elastic properties and time varying gains. Optimization is for a single statistically derived but deterministic disturbance input.

Parameter optimization for a control law based on the attitude/attitude-rate control scheme with an additive angle of attack signal is currently under investigation, along with a study of weighting factor properties in the present problem.

**APPENDIX**

For the system

$$\dot{x} = f(x,t,a) \quad \left. x \right|_{t = t_0} = x_0,$
(28)
$$
where $x$ is an $n$-dimensional state vector $a$ is an $m$-dimensional vector of time invariant adjustable parameters,

we wish to determine $a^*$ which minimizes the performance index

$$J(a) = J_1(x(t_f)) + \int_{t_0}^{t_f} V(x,t,a) \, dt.$$

Here, the initial time $t_0$, final time $t_f$, and initial state $x_0$ are fixed and do not depend on $a$, also the final state $x(t_f,a)$ is free.

Iterative optimization may be effected by the gradient of $J(a)$ in parameter space, $V_a$, and adjusting parameters in the direction of the largest negative gradient.

Consider the first partial derivative of $J$ with respect to the parameter $a_j$ for some fixed $a$

$$\frac{\partial J}{\partial a_j} = \left[ \frac{\partial x(t_f)}{\partial a_j} \right]^{\prime} \frac{\partial J}{\partial x(t_f)} + \int_{t_0}^{t_f} \left[ \frac{\partial x(t_f)}{\partial a_j} \right]^{\prime} \frac{\partial V}{\partial x} + \frac{\partial V}{\partial a_j} \, dt
(30)$$
where the prime (') indicates transposition, or

$$\frac{\partial \mathbf{J}}{\partial a_j} = \mathbf{E}_j^T(t_f) \mathbf{V}_x(t_f) J^T \int_{t_0}^{t_f} \left[ \mathbf{K}' \mathbf{V}_x \frac{\partial \mathbf{V}}{\partial a_j} \right] dt \quad (31)$$

where $\mathbf{E}_j = \frac{\partial \mathbf{x}}{\partial a_j}$ is an $n$-dimensional column vector of sensitivities of the system.

These partial derivatives for $j=1,2,\ldots,m$ form the $m$-dimensional column vector

$$\mathbf{V}_x = \frac{\partial J}{\partial a}$$

where

$$\mathbf{E} = \left[ \mathbf{E}_1^T \mathbf{E}_2^T \cdots \mathbf{E}_m^T \right]^T$$

From equations (30), (31), and (32) it is evident that the sensitivity vectors $\mathbf{E}_j$ play a central role in the calculation of $\mathbf{V}_x$. An ordinary linear differential equation useful in the calculation of the $j$th sensitivity vector may be obtained from equation (28) by expanding $\mathbf{f}$ in a Taylor series.

$$\mathbf{x}^0 + \mathbf{\dot{x}} = \mathbf{f}^0(t, a^0) + \frac{\partial \mathbf{f}}{\partial x_1} \mathbf{\Delta x}_1 + \frac{\partial \mathbf{f}}{\partial x_2} \mathbf{\Delta x}_2 + \cdots + \frac{\partial \mathbf{f}}{\partial x_n} \mathbf{\Delta x}_n$$

$$+ \frac{\partial \mathbf{f}}{\partial a_1} \mathbf{\Delta a}_1 + \frac{\partial \mathbf{f}}{\partial a_2} \mathbf{\Delta a}_2 + \cdots + \frac{\partial \mathbf{f}}{\partial a_m} \mathbf{\Delta a}_m$$

$$= \mathbf{f}^0(t, a^0) + \sum_{i=1}^{n} \frac{\partial \mathbf{f}}{\partial x_i} \mathbf{\Delta x}_i + \sum_{j=1}^{m} \frac{\partial \mathbf{f}}{\partial a_j} \mathbf{\Delta a}_j$$

where only linear terms are shown, and $\mathbf{x}^0$ is the nominal trajectory for $a=a^0$, with all partial derivatives evaluated along this nominal trajectory.

Subtraction of $\mathbf{x}^0 \mathbf{f}^0(t, a^0)$ from both sides and division by $\Delta a_j$ gives

$$\frac{\mathbf{\Delta x}}{\Delta a_j} = \frac{\partial \mathbf{f}}{\partial x_1} \frac{\mathbf{\Delta x}}{\Delta a_j} + \frac{\partial \mathbf{f}}{\partial x_2} \frac{\mathbf{\Delta x}}{\Delta a_j} + \cdots + \frac{\partial \mathbf{f}}{\partial x_n} \frac{\mathbf{\Delta x}}{\Delta a_j} + \frac{\partial \mathbf{f}}{\partial a_1} \mathbf{\Delta a}_1 + \frac{\partial \mathbf{f}}{\partial a_2} \mathbf{\Delta a}_2 + \cdots + \frac{\partial \mathbf{f}}{\partial a_m} \mathbf{\Delta a}_m$$

$$= \frac{\partial \mathbf{f}}{\partial a_j} \quad (34)$$

where $\Delta a_j = 0$, for $i \neq j$.

Thus

$$\frac{\partial \mathbf{x}}{\Delta a_j} = \mathbf{E}_j \frac{\partial \mathbf{f}}{\partial a_j} + \frac{\partial \mathbf{f}}{\partial a_j} \quad (35)$$

or

$$\mathbf{E}_j = \mathbf{E}_j + \frac{\partial \mathbf{f}}{\partial a_j} \quad (36)$$

where

$$\mathbf{E}_j = [\frac{\partial \mathbf{f}}{\partial x_1}, \frac{\partial \mathbf{f}}{\partial x_2}, \ldots, \frac{\partial \mathbf{f}}{\partial x_n}]$$

is the Jacobian matrix, and

$$\mathbf{E}_j(t_0, a) = 0, \ j = 1, 2, \ldots, m$$

since the initial condition $\mathbf{x}_0$ does not depend on $a$.

Equation (36) gives rise to a convenient method for the simultaneous computation of all sensitivity coefficients associated with a single parameter's variation. For linear systems the original system model provides the sensitivity coefficients when forced by $\frac{\partial \mathbf{f}}{\partial a_j}$ in place of the original forcing vector.

REFERENCES


